Solution Of Fuzzy Multi-objective Linear Fractional Programming Problem

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Abstract. This article presents the study of find out the solution of fuzzy multi-objective linear fractional programming (FMOLFP) problem. In this approach, by using Charnes and Cooper method, the FMOLFP problem is transformed to fuzzy multi-objective linear fractional programming (FMOLFP) problem by suitable transformation. The reduced problem is then solved by using min operator and average operator model and hence find out the fuzzy-efficient solution of the problem. One numerical example is presented to demonstrate the approach.

Keywords: Multi-objective linear programming, Multi-objective linear fractional programming, Fuzzy set theory, Min operator,Fuzzy-efficient solution.

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1 Introduction

In mathematical optimization, linear-fractional programming (LFP) is a generalization of linear programming (LP). Whereas the objective function in a linear program is a linear function, the objective function in a linear-fractional program is a ratio of two linear functions. A linear program can be regarded as a special case of a linear-fractional program in which the denominator is the constant function one. In literature [1], fractional programming problem was studied extensively in the middle of 1960s and early 1970s of the last century. In many practical applications like stock problems, ore-blending problems, shipping schedule problems, optimal policy for a Markovian chains, sensitivity of linear programming (LP) problem, optimization of ratio criterion gives more insight into the situations than the optimization of each criterion [5].

Multi-objective linear fractional programming (MOLFP) problem is used to describe the problem associated with multiple objectives where objective functions are written in fractional formulas. These type of problems has been used in production planning, inventory management, financial and banking sector, etc.

The concept of 'Decision making in fuzzy environment' was first studied by Bellman and Zadeh [27]. Zimmermann [13] first proposed the fuzzy linear programming (FLP)problem. Charnes and Cooper [1] solved a programming problem with linear fractional functionals by resolving it into two linear programming (LP) problem. By suitable transformation, Chakraborty and Gupta [21] have transformed MOLFP problem to formulate an equivalent MOLP problem under fuzzy set theoretic approach. Furthere more, there are a few studies [16],[26], [6],[11] on MOLFP problem. Zimmermann [13] proposed the min-operator model to the multi-objective linear programming (MOLP)problem. Luhandjula [19] proved some properties of min operator but the solution obtained by min operator doesn't give compensatory and efficient solution [10],[29]. To overcome this difficulty, Lee and Li [10] have proposed two-phase approach to get more efficient and satisfactory result. Guu et.al [28] have applied two phase approach in MOLFP problem while Chen and Chou [15] proposed a fuzzy approach to integrate the min operator, average operator and two-phase methods. In 1997, Guu and Wu [29] have solved two phase approach with positive weighted coefficients, not necessarily equal, for solving the MOLP gives an efficient solution and they [30] proposed a two-phase method to improve the solution yielded by the max-min operator when solving the linear programming with imprecision parameters. Zimmermann and Zysno [14] observed from an experiment that most of the real world problems are neither non-compensatory (min-operator) nor full compensatory (average operator). In 2001, Wu and Guu have presented a compromise model between non-compensatory (min-operator) nor full compensatory (average operator) for obtaining fuzzy efficient solution of MOLP problem. To solve FLPP, Werner's [3] proposed membership functions for the fuzzy objective and applied the concept of max-min operator [13]. If the decision maker modify membership functions of the objective and constraints then Werner's method gives the compromise fuzzy efficient solutions.
In this paper, we have studied MOLFP problem under fuzzy environment where MOLFP problem is converted

to MOLPP by suitable transformation. Min-operator and average operator model are used to solve MOLP problem and hence find the fuzzy-efficient solution of the problem.
The paper is organised as follows:-
Section 2 outlines the definitions of LFP amd MOLFP. In section 3, we have discussed the transformation of MOLFP to MOLP, use of min operator and average operator model in transformed MOLP and to find out the fuzzy efficient solution. Section 4 demonstrates numerical example to illustrate the approach. Section 5 discusses the conclusion of this paper.

2 Definitions and Preliminaries

Definition 2.1. [23] Linear Fractional Programming- The general format of linear fractional programming (LFP) may be written as:

$$Max\, Z(x) = \frac{cx+\alpha}{dx+\beta},$$

subject to the constraints:

$$x \in S = \{x|Ax = b, x \geq 0\}$$

where $A \in R^{m\times n}$, $b \in R^m$, $x \in R^n$, $c, d \in R^n$, $\alpha, \beta \in R$ and S is a non-empty and bounded set.

Definition 2.2. [23] Multi-objective Linear Fractional Programming Problem- The general format of a multi-objective linear fractional programming problem which is stated as follows-

$$\text{Max Z(x)}=\{Z_1(x), Z_2(x), ..., Z_n(x)\}$$

subject to the constraints:

$$x \in S = \{x \in R^n : Ax \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, x \geq 0\}$$
$$\text{with } b \in R^m, A \in R^{m\times n}$$

$$\text{and} \quad Z_i(x) = \frac{c_ix+\alpha_i}{d_ix+\beta_i} = \frac{N_i(x)}{D_i(x)};$$

where c_i, $d_i \in R^n$ and α_i, β_i are constants and S $\neq \phi$.

3 Methodology

3.1 Transformation of MOLFP problem to MOLP problem

Consider the following fuzzy multi-objective linear fractional programming problem as follows-

$$\text{Max } Z(x) = \{Z_1(x), Z_2(x), ..., Z_n(x)\}$$

subject to the constraints:

$$x \in S = \{x \in R^n : (Ax)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \check{b_j}, x \geq 0\} \tag{3.1.1}$$

with $b \in R^m, A \in R^{m\times n}$ and

$$Z_i(x) = \frac{c_ix+\alpha_i}{d_ix+\beta_i} = \frac{N_i(x)}{D_i(x)};$$

for all i=1,2,...,n and j=1,2,...,m. where c_i, $d_i \in R^n$ and α_i, β_i are constants.

By using Charne's and Cooper method [1], substituting y=tx in (3.1.1), we get,

$$Max \quad f_i(y,t) = c_iy + \alpha_it$$

subject to the constraints:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} 0,$$
$$d_i y + \beta_i t = 1$$

$$\text{where y,} t \geq 0.$$

The above MOLFP problem is equivalent to the following MOLP problem as follows:

$$Max\, f_i(y,t) = c_i y + \alpha_i t$$

subject to the constraint:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} 0, \tag{3.1.2}$$

$$d_i y + \beta_i t = 1$$
$$\text{where y,} t \geq 0$$

where $f_i(y,t)$, i=1,2,.....,n are affine functions, fuzzy resources $\check{p}_j$ is in $[p_j, p_j + q_j]$ with given q_j (without loss of generality we assume that $0 < q_j < \infty$) for each j.
For the crisp objective function of problem (3.1.2), Werners [3],[4] proposed a max-min operator method which is similar to Zimmermann [13]. The possible range $[f_i^0, f_i^1]$ for the i^{th} objective function can be obtained as follows-

$$f_i^0 = Max f_i$$

subject to the constraints:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} p_j, \tag{3.1.3}$$

$$\text{where y, } t \geq 0$$
$$d_i y + \beta_i t = p_j$$

and

$$f_i^1 = Max f_i$$

subject to the constraints:

$$(Ay - bt)_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} p_j + q_j, \tag{3.1.4}$$

$$d_i y + \beta_i t = p_j + q_j$$
$$\text{where y, } t \geq 0.$$

A non-decreasing linear membership function for the i^{th} objective function is defined as follows:

$$\mu_{0i}(y,t) = \begin{cases} 1 & , if\, f_i(y,t) > f_i^1 \\ \frac{f_i(y,t) - f_i^0}{f_i^1 - f_i^0} & , if\, f_i^0 \leq f_i(y,t) \leq f_i^1 \\ 0 & , if\, f_i(y,t) < f_i^0 \end{cases} \tag{3.1.5}$$

A non-increasing linear membership function for the j^{th} fuzzy constraint is defined as follows:

$$\mu_j(y,t) = \begin{cases} 1 & , if\, (Ay - bt)_j < p_j \\ \frac{(p_j + q_j) - (Ay - bt)_j}{q_j} & , if\, p_j \leq (Ay - bt)_j \leq p_j + q_j \\ 0 & , if\, (Ay - bt)_j > p_j + q_j \end{cases} \tag{3.1.6}$$

When all the membership functions corresponding to objective functions and constraints are known, the FMOLFP problem is solved by following approach-

Let λ_i be the minimum acceptability of the i^{th} objective and λ_{n+j} be the minimum acceptability of the j^{th} constraint.

$$\text{Let } \lambda = min\{\lambda_i, \lambda_{n+j}\} \qquad \forall i, j$$

Using the max-min principle of Bellman and Zadeh [27] and introducing the variable λ adopts the following formulation:

$$Max\lambda$$

subject to the constraints :

$$\begin{aligned} 1 \geq \mu_{0i}(y,t) \geq \lambda \geq 0, \forall i = 1,2,...,n \\ 1 \geq \mu_j(y,t) \geq \lambda \geq 0, \forall j = 1,2,....,m \\ \lambda \in [0,1], y,t \geq 0. \end{aligned} \tag{3.1.7}$$

If there exists an unique optimal solution of model (3.1.7), then it is a fuzzy-efficient solution to the FMOLP problem (3.1.2). However if the uniqueness of a solution is not satisfied, the fuzzy-efficient is not guaranteed for all solutions of model (3.1.7), but atleast one of the multiple optimal solution is fuzzy-efficient [4]. In order to produce a fuzzy-efficient solution several approaches have been proposed in the literature. Guu and Wu [29],[30] proposed the step in which they use the additive criterion to aggregate the fuzzy goals.

Jimenez and Bilbao [22] proposed an extension to the Guu and Wu [29],[30] and Dubois and Fortemps [9] approaches.

In order to achieve this, they proposed a modified technique for generating a pareto-optimal solution to the MOLFP problem (3.1.1) which preserves the property of being a fuzzy-efficient solution to the model (3.1.5) and (3.1.6). Solving the model (3.1.7), optimal value λ^* can be obtained. This λ^* denotes that the satisfaction level for all membership functions can be simultaneously obtain.

3.2 To obtain Pareto-optimal solutions in case some goals are fully achieved:

Within the scope of the multi-criteria decision making (MCDM) theory, the Pareto-optimality is a necessary condition in order to guarantee the rationality of a decision. Therefore a "reasonable" solution to a MOLFP problem should be Pareto-optimal and as FMOLFP is an approach which is used in order to be able to solve MOLFP problems, any solution of a FMOLFP problem should be also pareto-optimal.

Definition 3.1 [22]: A decision plan $x^* \in S$ is said to be a Pareto-optimal solution to the MOLP problem (3.1.2) iff there doesn't exist another $x \in S$ such that

$f_k(x) \leq f_k(x^*)$ for all k (k=1,2,....,N)
and $f_l(x) < f_l(x^*)$ for atleast one l.

Definition 3.2 [3],[4]: A decision plan $x^* \in S$ is said to be fuzzy-efficient solution to the FMOLFP problem (3.1.7) if and only if there does not exist another $x \in S$ such that

$\mu_k(f_k(x)) \geq \mu_k(f_k(x^*))$ for all k (k=1,2,....,N)
and $\mu_l(f_l(x)) > \mu_l(f_l(x^*)$ for atleast one l.

It is obvious that any fuzzy-efficient solution x^* to FMOLFP problem such that $f_i(y,t) \in (p_j, p_j + q_j)$ for all j, is a Pareto-optimal solution to the FMOLFP problem (3.1.2). But if membership degree is 1, the fuzzy-efficiency does not guarantee Pareto-optimality which is given in the following theorem-

Theorem 3.1. *[22]: Let x^* be a fuzzy-efficient solution to the FMOLFP problem (3.1.2) such that $\mu_l(f_l)(x^*) = 1$ for some l, i.e., $f_l(x^*) \leq p_l$, then it could be the case that x^* is not a Pareto-optimal solution. This is due to the fact that on the left of p_l the membership function μ_l is constantly equal to 1.*

3.2.1 To obtain a fuzzy-efficient solution:

Guu and Wu [29],[30] proposed two-phase method where phase-I gives max-min operator i.e., it solves model (3.1.7). In that case if x^* be the optimal solution obtained in phase-I and if x^* is an unique optimal solution then it is fuzzy-efficient and the procedure stops.
If x^* is not an unique optimal solution, then go on to the phase-II which is as follows:-

$$Max \sum_{i=1}^{n} \lambda_i$$

subject to the constraints:

$$\begin{aligned} 1 \geq \mu_{0i} f_i(y,t) \geq \lambda_i \geq \mu_{0i}^* f_i(y,t), i = 1,2,...,n \\ 1 \geq \mu_j f_i(y,t) \geq \lambda_i \geq \mu_j^* f_i(y,t), j = 1,2,...,m \\ y,t \geq 0. \end{aligned} \tag{3.2.1}$$

Guu and Wu [30] showed that any optimal solution x^{**} to problem (3.2.1) is a fuzzy-efficient solution to the FMOLFP problem (3.1.2).

3.2.2 To obtain Pareto-optimal solution

When there are several objective functions it can be a difficult task for the decision maker to determine coherent aspiration levels, so it is possible that some goals are easily achieved,i.e.,(say) $\mu_j(f_j(x^{**})) = 1$ for some j, where x^{**} could be either an optimal solution of model (3.2.1) or a discrimin-optimal solution of Dubois and Fortemps [9]. Due to this, fuzzy -efficiency doesn't guarantee the Pareto-optimality by Theorem 3.1. From that it is seen that satisfaction degree is equal to 1 that means fuzzy constraints are useless. For that we use here conventional goal programming (GP) problem. This type of GP is used to obtain for a Pareto-optimal solution of a MOLPP. This method was proposed by Masud et. al [2] which states that maximizing the sum of the over-achievement of the goals. To obtain the Pareto-optimality the proposed model is-

$$Max \sum_{s=1}^{p} n_s$$

subject to the constraints:

$$\begin{gathered} f_s(y) + n_s = f_s(y^{**}), \text{ s=1,2,......,p} \\ \mu_{0r}(f_i(y,t)) = \mu_{0r}(f_i(y^{**}, t^{**})), r = p+1, ..., k \\ \mu_r(f_i(y,t)) = \mu_r(f_i(y^{**}, t^{**})) \\ \text{where } n_s \geq 0,\ y, t \geq 0\ . \end{gathered} \tag{3.2.2}$$

where the subscripts 's' refers to the objective functions such that $\mu_s(f_s(y^{**}, t^{**})) = 1$ and the subscripts 'r' refers to the objective functions such that $\mu_r(f_r(y^{**}, t^{**})) < 1$. The variables n_s have the same significance as the negative deviation in conventional GP. So this model(3.2.2) is demonstrate that the solution obtained by using this model is Pareto-optimal solution.

Lemma 3.2. *[22] Let x^0 be an optimal solution of the problem (3.2.2), then x^0 is a fuzzy efficient solution to the FMOLP problem (3.1.2) and a Pareto-optimal solution to the MOLFP problem (3.1.1).*

3.3 Framework for solving MOLFP problem with fuzzy goals

The procedure to obtain a Pareto-optimal solution to the MOLFP problem are discussed above which has the additional property of being fuzzy-efficient. The steps for the proposed algorithm can be stated below:

Step 1. Convert MOLFP problem to MOLP problem by using Charnes and Cooper method (by substituting $y = tx)(t > 0)$.

Step 2. For constraints, take fuzzy resources $\check{p}_j$ in $[p_j, p_j + q_j]$ with q_j for each j where $0 < q_j < \infty$.

Step 3. For objective functions, take possible range $[f_i^0, f_i^1]$ for the i^{th} objective functions (model (3.1.3) and (3.1.4)).

Step 4. Write the linear membership functions for the i^{th} objective function (as shown in model (3.1.5)) and that for j^{th} fuzzy constraint (as shown in model (3.1.6)).

Step 5. Let $\lambda = min\{\lambda_i, \lambda_{n+j}\} \quad \forall i, j$ where λ_i be the minimum acceptability of the i^{th} objective and λ_{n+j} be the minimum acceptability of the j^{th} constraint.

Step 6. To find out the fuzzy-efficient solution, use Bellman and Zadeh's max-min principle method [22]-

Case-I. The optimal solution is unique:

1. If all the membership degrees are less than 1 the solution is fuzzy-efficient and also pareto-optimal.

2. If some membership degrees are equal to 1, i.e., if one or more targets are fully achieved, the solution is fuzzy-efficient but it may not be Pareto-optimal.

Case-II. If there exists multiple optimal solutions then go to Step 7 (model (3.1.7)).

Step 7. Solve the problem by 2^{nd} phase or phase-II of the model (3.2.1). Then-

Case-I. If all the membership function are less than 1 the solution is fuzzy efficient and also Pareto-optimal. The algorithm is stops here.

Case-II. If some membership degrees are equal to 1, i.e., if one or more targets are fully achieved, the solution is fuzzy-efficient but it may not be Pareto-optimal. Then Go to Step 8.

Step 8. Maximize the sum of negative deviations, for targets that are fully achieved, without making the values obtained in the previous step worse (model (3.2.2)). The solution is fuzzy-efficient and also Pareto-optimal. The algorithm stops here.

4 Numerical Example

Ex 1: Consider a MOLFP problem with two objective functions as follows: [21]

$$\text{Max } (Z_1(x) = \frac{-3x_1+2x_2}{x_1+x_2+3}, Z_2(x) = \frac{7x_1+x_2}{5x_1+2x_2+1})$$

subject to the constraints:

$$\begin{gathered} x_1 - x_2 \geq 1, \\ 2x_1 + 3x_2 \leq 15, \\ x_1 \geq 3, \\ x_1, x_2 \geq 0. \end{gathered}$$

Solution:
Here, $Z_1(x) < 0$ for each x in the feasible region and $Z_2(x) \geq 0$ for some x in the feasible region. The above MOLFP problem is equivalent to the following MOLP problem-

$$\{f_1(y,t) = y_1 + y_2 + 3t, f_2(y,t) = 7y_1 + y_2\}$$

subject to the constraints:

$$\begin{gathered} -y_1 + y_2 + t \leq 0, \\ 2y_1 + 3y_2 - 15t \leq 0, \\ -y_1 + 3t \leq 0, \\ 5y_1 + 2y_2 + t = 1, \\ 3y_1 - 2y_2 \leq 1, \\ y_1, y_2, t \geq 0. \end{gathered}$$

where the fuzzy resources with the corresponding maximal tolerances are $p_1 = 5$, $p_2 = 15$, $p_3 = 25$, $p_4 = 35$,$p_5 = 50$. Solving by LP package, the range of the objective function are as follows:

$$\begin{gathered} [f_1^0, f_1^1] = [1.32, 5.00] \\ \text{and } [f_2^0, f_2^1] = [1.40, 35.00]. \end{gathered}$$

The membership function of the two objective functions are defined as follows:

$$\mu_{01}(y,t) = \begin{cases} 1 & , f_1(y,t) > 5.00 \\ \frac{f_1(y,t)-1.32}{3.68} & , 1.32 \leq f_1(y,t) \leq 5.00 \\ 0 & , f_1(y,t) < 1.32 \end{cases}$$

and

$$\mu_{02}(y,t) = \begin{cases} 1 & , f_2(y,t) > 35.00 \\ \frac{f_2(y,t)-1.40}{33.6} & , 1.40 \leq f_2(y,t) \leq 35.00 \\ 0 & , f_2(y,t) < 1.40 \end{cases}$$

For each of fuzzy constraints, the non-increasing linear membership functions are written as follows:

$$\mu_1(y,t) = \begin{cases} 1 & , g_1(y,t) < 1 \\ \frac{5-g_1(y,t)}{4} & , 1 \leq g_1(y,t) \leq 5 \\ 0 & , g_1(y,t) > 5 \end{cases}$$

$$\mu_2(y,t) = \begin{cases} 1 & , g_2(y,t) < 1 \\ \frac{15-g_2(y,t)}{14} & , 1 \leq g_2(y,t) \leq 15 \\ 0 & , g_2(y,t) > 15 \end{cases}$$

$$\mu_3(y,t) = \begin{cases} 1 & , g_3(y,t) < 1 \\ \frac{25-g_3(y,t)}{24} & , 1 \leq g_3(y,t) \leq 25 \\ 0 & , g_3(y,t) > 25 \end{cases}$$

$$\mu_4(y,t) = \begin{cases} 1 & , g_4(y,t) < 1 \\ \frac{35-g_4(y,t)}{34} & , 1 \leq g_4(y,t) \leq 35 \\ 0 & , g_4(y,t) > 35 \end{cases}$$

$$\mu_5(y,t) = \begin{cases} 1 & , g_5(y,t) < 1 \\ \frac{50-g_5(y,t)}{49} & , 1 \leq g_4(y,t) \leq 50 \\ 0 & , g_5(y,t) > 50 \end{cases}$$

Step 1: Solve the max-min problem

```
# Create the problem variable to contain the problem data
model = LpProblem("Max-min problem", LpMaximize)
# Create 4 variables
y1 = LpVariable("variable 1", 0, None, LpInteger)
y2 = LpVariable("variable 2", 0, None, LpInteger)
t = LpVariable("variable 3", 0, None, LpInteger)
z = LpVariable("variable 4", 0, None, LpInteger)
# Create maximize objective function
model += z
# Create fourteen  constraints
model += 1 * y1 + 1 * y2 + 3 * t <= 5.00, "C1"
model += 1 * y1 + 1 * y2 + 3 * t - 3.68 * z >= 1.32, "C2"
model += 7 * y1 + 2 * y2 <= 35.00, "C3"
model += 7 * y1 + 2 * y2 - 33.6 * z >= 1.40, "C4"
model += 1 * y1 + 1 * y2 + 1 * t >= 1, "C5"
model += 1 * y1 + 1 * y2 + 1 * t + 4 * z <= 5, "C6"
model += 2 * y1 + 3 * y2 - 15 * t >= 1, "C7"
model += 2 * y1 + 3 * y2 - 15 * t + 14 * z <= 15, "C8"
model += -1 * y1 + 3 * t >= 1, "C9"
model += -1 * y1 + 3 * t + 24 * z <= 25, "C10"
model += 5 * y1 + 2 * y2 + 1 * t >= 1, "C11"
model += 5 * y1 + 2 * y2 + 1 * t + 34 * z <= 35, "C12"
model += 3 * y1 - 2 * y2 >= 1, "C13"
model += 3 * y1 - 2 * y2 + 49 * z <= 50, "C14"
print(v.name, "=", v.varValue)
```

Solving by Python programming, we get, the optimal solution is

$$y^* = (1.58, 1.87, 0.52)$$
$$\therefore x^* = (1.7, 2.05).$$

$$\text{Hence, } Z_1(x^*) = -0.20,\ Z_2(x^*) = 1.06.$$

Here some membership functions are equal to 1 then the solution is fuzzy-efficient but may not be pareto-optimal.

Step 2: For a fuzzy-efficient solution by solving the following problem-

```
model = LpProblem("Max-min problem", LpMaximize)
# Create 6 variables
y1 = LpVariable("variable 1", 0, None, LpInteger)
y2 = LpVariable("variable 2", 0, None, LpInteger)
t = LpVariable("variable 3", 0, None, LpInteger)
z = LpVariable("variable 4", 0, None, LpInteger)
n1 = LpVariable("variable 5", 0, None, LpInteger)
n2 = LpVariable("variable 6", 0, None, LpInteger)
# Create maximize objective function
model += n1 + n2
# Create fourteen  constraints
model += 6 * y1 + 5 * y2 + 1 * n1 >= 12.98, "C13"
model += 2 * y1 + 3 * y2 + 1 * n2 >= 5.9, "C14"
model += 6 * y1 + 5 * y2 <= 53.34, "C1"
model += 6 * y1 + 5 * y2 - 52.13 * z >= 1.21, "C2"
model += 2 * y1 + 3 * y2 <= 24.88, "C3"
model += 2 * y1 + 3 * y2 - 24.33 * z >= 0.55, "C4"
model += 1 * y1 + 2 * y2 - 3 * t >= 0, "C5"
model += 1* y1 + 2 * y2 -3 * t + 5 * z <= 5, "C6"
model += 3 * y1 + 2 * y2 - 6 * t >= 0, "C7"
model += 3 * y1 + 2 * y2 - 6 * t + 15 * z <= 15, "C8"
model += 2 * y1 + 7 * t >= 1, "C9"
model += 2 * y1 + 7 * t + 17 * z <= 18, "C10"
model += 1 * y1 + 1 * y2 + 7 * t >= 1, "C11"
model += 1 * y1 + 1 * y2 + 7 * t + 19 * z <= 20, "C12"
print(v.name, "=", v.varValue)
```

Solving by Python programming, the optimal solution is

$$y^{**} = (1.49, 1.74, 0.48).$$
$$\therefore x^{**} = (3.104, 3.625).$$

$$\text{Hence, } Z_1(x^{**}) = -0.212,\ Z_2(x^{**}) = 1.067.$$

Here some membership functions are equal to 1 then the solution is fuzzy-efficient but not pareto-optimal.

We observe that the decision x^{**} improves the solution x^* because $Z_1(x^{**}) < Z_1(x^*)$, $Z_2(x^{**}) < Z_2(x^*)$.
Step 3: For a Pareto-optimal solution by solving the following problem-

```
# Create 6 variables
y1 = LpVariable("variable 1", 0, None, LpInteger)
y2 = LpVariable("variable 2", 0, None, LpInteger)
t = LpVariable("variable 3", 0, None, LpInteger)
z = LpVariable("variable 4", 0, None, LpInteger)
n1 = LpVariable("variable 5", 0, None, LpInteger)
n2 = LpVariable("variable 6", 0, None, LpInteger)
# Create maximize objective function
model += n1 + n2
# Create sixteen  constraints
model += 1 * y1 + 1 * y2 + 3 * t + 1 * n1 >= 4.67, "C1"
model += 7 * y1 + 1 * y2 + 1 * n2 >= 12.17, "C2"
model += 1 * y1 + 1 * y2 + 3 * t <= 5.00, "C3"
model += 1 * y1 + 1 * y2 + 3 * t - 3.68 * z >= 1.32, "C4"
model += 7 * y1 + 1 * y2 <= 35.00, "C5"
model += 7 * y1 + 2 * y2 - 33.6 * z >= 1.40, "C6"
model += 1 * y1 + 1 * y2 + 1 * t >= 1, "C7"
model += 1 * y1 + 1 * y2 + 1 * t + 4 * z <= 5, "C8"
model += 2 * y1 + 3 * y2 - 15 * t >= 1, "C9"
model += 2 * y1 + 3 * y2 - 15 * t + 14 * z <= 15, "C10"
model += -1 * y1 + 3 * t >= 1, "C11"
model += -1 * y1 + 3 * t + 24 * z <= 25, "C12"
model += 5 * y1 + 2 * y2 + 1 * t >= 1, "C13"
model += 5 * y1 + 2 * y2 + 1 * t + 34 * z <= 35, "C14"
model += 3 * y1 - 2 * y2 >= 1, "C15"
model += 3 * y1 - 2 * y2 + 49 * z <= 50, "C16"
```

Solving by Python programming, the optimal solution is

$$y^0 = (5, 7, 2). \therefore x^0 = (2.5, 3.5).$$

$$\text{Hence, } Z_1(x^0) = -0.055,\ Z_2(x^0) = 1.0243.$$

By using Step-8 (model (3.2.2)), we see that the solution is fuzzy efficient and hence Pareto-optimal.

5 Conclusions

In this paper, we have studied FMOLFP by using min-operator model. By using M.Jimenez and A.Bilbao approach we have found out the fuzzy-efficient solution and also the pareto-optimal solution of the FMOLFP problem. We have shown that if the membership degree is less than 1, then the solution is fuzzy-efficient and also Pareto-optimal. But if the membership degree is 1 or more, then the solution is fuzzy-efficient but not Pareto-optimal. But by using GP approach (model(3.2.2)), we have seen that the solution obtained by this approach gives both fuzzy efficiency and Pareto optimality. Two numerical examples are solved and from that we see that the solution obtained by our proposed algorithm gives fuzzy-efficient solution and hence also Pareto-optimal solution.

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