

## Exercise 1

### Question 1

A rectangle is defined as

$$S = \{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \dots, n\}$$

Let  $x, y \in S$  and  $\lambda \in [0, 1]$ . We want to check if

$$\lambda x + (1 - \lambda)y \in S$$

We have, for all  $i \in \{1, \dots, n\}$ :

$$\alpha_i \leq x_i \leq \beta_i \quad \text{and} \quad \alpha_i \leq y_i \leq \beta_i$$

Then

$$\lambda \alpha_i + (1 - \lambda) \alpha_i \leq \lambda x_i + (1 - \lambda) y_i \leq \lambda \beta_i + (1 - \lambda) \beta_i$$

and for all  $i = 1, \dots, n$ :

$$\alpha_i \leq \lambda x_i + (1 - \lambda) y_i \leq \beta_i$$

Thus

$$\lambda x + (1 - \lambda)y \in S$$

**Conclusion:**  $S$  is convex.

### Question 2

$$S = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$$

Let's take  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  in  $S$ , such that  $x_1 x_2 \geq 1$  and  $y_1 y_2 \geq 1$ .  
Let's show that  $z = \lambda x + (1 - \lambda)y \in S$  where  $0 \leq \lambda \leq 1$ .

$$z = (\underbrace{\lambda x_1 + (1 - \lambda)y_1}_{z_1}, \underbrace{\lambda x_2 + (1 - \lambda)y_2}_{z_2})$$

We have

$$\begin{aligned} z_1 z_2 &= (\lambda x_1 + (1 - \lambda)y_1)(\lambda x_2 + (1 - \lambda)y_2) \\ &= \lambda^2 \underbrace{x_1 x_2}_{\geq 1} + \underbrace{\lambda(1 - \lambda)x_1 y_2 + \lambda(1 - \lambda)y_1 x_2}_{\lambda(1 - \lambda)(x_1 y_2 + y_1 x_2)} + (1 - \lambda)^2 \underbrace{y_1 y_2}_{\geq 1} \end{aligned}$$

We have

$$y_1 y_2 \geq 1 \quad \text{and} \quad x_1 x_2 \geq 1 \implies x_1 y_2 \geq \frac{y_2}{x_2}$$

and

$$y_1 x_2 \geq \frac{x_2}{y_2}$$

Then

$$y_1 x_2 + x_1 y_2 \geq \frac{y_2}{x_2} + \frac{x_2}{y_2} \geq \left( \frac{\sqrt{y_2}}{\sqrt{x_2}} - \frac{\sqrt{x_2}}{\sqrt{y_2}} \right)^2 + 2 \geq 2$$

Then

$$\begin{aligned} z_1 z_2 &\geq \lambda^2 + 2\lambda(1 - \lambda) + (1 - \lambda)^2 \\ &\geq \lambda^2 + 2\lambda - 2 + 1 - 2\lambda + \lambda^2 = 1 \end{aligned}$$

Then  $z \in S$

**Conclusion:**  $S$  is convex.

### Question 3

$$A = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\} \text{ where } S \subset \mathbb{R}^n$$

Let's take  $x \in A$ , then  $\|x - x_0\|_2 \leq \|x - y\|_2$  for all  $y \in S$ .

$$\begin{aligned} &\Rightarrow (x - x_0)^T(x - x_0) \leq (x - y)^T(x - y) \\ &\Rightarrow x^T x - x x_0 - x^T x + x_0^T x_0 \leq x^T x - x y - y^T x + y^T y \\ &\Rightarrow (y - x_0)^T x \leq \frac{y^T y + x_0^T x_0}{2} \end{aligned}$$

An equation of a half-space written as  $a^T x \leq b$  such that:

$$\begin{aligned} a &= y - x_0 \\ b &= \frac{y^T y + x_0^T x_0}{2} \end{aligned}$$

Thus,

$$A = \bigcap_{y \in S} \{x \mid a^T x \leq b\}$$

is convex as it is an intersection of half-spaces.

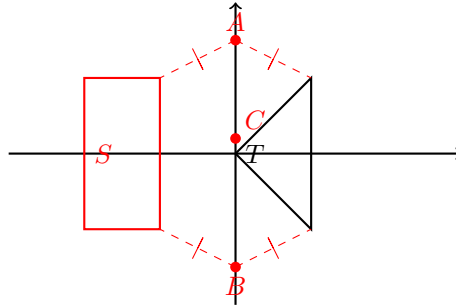
### Question 4

Let

$$S = \{x \mid \text{dist}(x, S) \leq \text{dist}(x, T)\}$$

where  $\begin{cases} S, T \subset \mathbb{R}^n \\ \text{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\} \end{cases}$

Consider the case  $n = 2$ :



$A$  and  $B$  are in the set  $A$  (because  $\text{dist}(A, S) = \text{dist}(A, T)$  and  $\text{dist}(B, T) = \text{dist}(B, S)$ ). But  $C$ , which is in  $[A, B]$ , is not (because  $\text{dist}(C, S) > \text{dist}(C, T)$ ).

**Conclusion:**  $A$  is not convex.

### Question 5

$$S = \{x \mid x + S_2 \subseteq S_1\} \text{ where } S_1, S_2 \subseteq \mathbb{R}^n \text{ with } S_1 \text{ a convex set.}$$

$S$  can be written as:

$$\begin{aligned} S &= \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} \\ &= \bigcap_{y \in S_2} \{x \mid x \in S_1 - y\} \\ &= \bigcap_{y \in S_2} \{S_1 - y\} \end{aligned}$$

$S_1$  is a convex set, then for all  $x_1, x_2 \in S_1$  and  $\lambda \in [0, 1]$ , we have:

$$\lambda x_1 + (1 - \lambda)x_2 \in S_1$$

Then

$$\lambda x_1 + (1 - \lambda)x_2 - y \in S_1 - y$$

Then

$$\lambda \underbrace{(x_1 - y)}_{\in \{S_1 - y\}} + (1 - \lambda) \underbrace{(x_2 - y)}_{\in \{S_1 - y\}} \in S_1 - y$$

Thus,

$$\{S_1 - y\} \text{ is convex}$$

The intersection of convex sets is a convex set.

**Conclusion:**

$$S = \bigcap_{y \in S_2} \{S_1 - y\} \text{ is a convex set.}$$

## Exercise 2

### Question 1

$f(x_1, x_2) = x_1 x_2$  on  $\mathbb{R}_{++}^2$ .

-  $\text{dom } f = \mathbb{R}_{++}^2$ : a convex set.

-  $f$  is twice differentiable (because it is linear).

$$\begin{cases} \frac{\partial f}{\partial x_1} = x_2 \\ \frac{\partial f}{\partial x_2} = x_1 \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 f}{\partial x_1^2} = 0 \\ \frac{\partial^2 f}{\partial x_2^2} = 0 \end{cases} \text{ and } \begin{cases} \frac{\partial^2 f}{\partial x_1 \partial x_2} = 1 \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1 \end{cases}$$

$$\Rightarrow \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda_1 = 1$  is a trivial eigenvalue of this matrix. The other eigenvalue verifies  $\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -1$ . Then the Hessian matrix is not positive definite.

$\Rightarrow$  we conclude that  $f$  is not convex and is not concave

Let  $S_\alpha = \{(x_1, x_2) \in \mathbb{R}_{++}^2 \mid x_1 x_2 \geq \alpha\}$

In the first exercise, we proved that  $S_1$  is convex. This result can be extended to all  $\alpha \in \mathbb{R}$  to have  $S_\alpha$  convex. So,  $f$  is quasiconcave (and not quasiconvex).

### Question 2

Let  $f(x_1, x_2) = \frac{1}{x_1 x_2}$  on  $\mathbb{R}_{++}^2$ .

We compute the Hessian matrix:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$$

- The first principal minor of this matrix is  $\frac{2}{x_1^3 x_2}$ , and the second one is  $\frac{2}{x_2 x_1^3}$ . - We have  $x_1, x_2 > 0$ , then the Hessian matrix is positive definite (according to Sylvester's criteria).

Then  $f$  is convex.

Now, let's take  $S_\alpha = \{x \mid f(x) \leq \alpha\}$

$$S_\alpha = \left\{ (x_1, x_2) \in \mathbb{R}_{++}^2 \mid \frac{1}{x_1 x_2} < \alpha \right\}$$

- If  $\alpha \leq 0$ :

$$S_\alpha = \emptyset, \text{ a convex set}$$

- If  $\alpha > 0$ :

$$S_\alpha = \left\{ (x_1, x_2) \in \mathbb{R}_{++}^2 \mid x_1 x_2 \geq \frac{1}{\alpha} \right\}$$

According to the previous function,  $S_\alpha$  is convex.

Then,  $f$  is quasiconvex.

### Question 3

Let  $f(x_1, x_2) = \frac{x_1}{x_2}$  on  $\mathbb{R}_{++}^2$ .

The Hessian matrix is given by:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

We have:

$$\det(\nabla^2 f(x_1, x_2)) = -\frac{1}{x_2^4} < 0$$

$$\text{tr}(\nabla^2 f(x_1, x_2)) = \frac{2x_1}{x_2^3} > 0$$

Then, the eigenvalues of the Hessian matrix have different signs.

Thus,  $f$  is not a convex function nor a concave one.

The set  $S_\alpha = \left\{ (x_1, x_2) \in \mathbb{R}_{++}^2 \mid \frac{x_1}{x_2} \leq \alpha \right\}$  defines a halfspace.

Then,  $f$  is quasilinear (quasiconcave and quasiconvex).

### Question 4

Let

$$f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}, \quad \text{where } 0 \leq \alpha \leq 1 \text{ on } \mathbb{R}_{++}^2$$

Let's compute the Hessian matrix:

$$\frac{\partial^2 f}{\partial x_1^2} = \alpha(\alpha - 1)x_1^{\alpha-2}x_2^{1-\alpha}$$

$$\frac{\partial^2 f}{\partial x_2^2} = -\alpha(\alpha - 1)x_1^\alpha x_2^{-\alpha-1}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = -\alpha(1 - \alpha)x_1^{\alpha-1}x_2^{-\alpha-1}$$

The Hessian matrix is given by:

$$\nabla^2 f(x_1, x_2) = \underbrace{\begin{bmatrix} -\alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & -\alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha-1} \\ -\alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha-1} & -\alpha(\alpha-1)x_1^\alpha x_2^{-\alpha-1} \end{bmatrix}}_{\leq 0 \text{ because } \alpha \in (0,1) \text{ and } x_1, x_2 > 0} \underbrace{\begin{bmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix}}_{\text{this matrix is positive definite}}$$

Then the eigenvalues of the Hessian matrix are negative, we conclude:

$f$  is concave (and quasiconcave).

### Exercise 3

#### Question 1

$f(X) = \text{tr}(X^{-1})$  on  $\text{dom } f = S_{++}^n$ .

Let  $X, Y \in \text{dom } f$  and  $\lambda \in [0, 1]$ .

$X = P^{-1}DP$ , such that  $D = \text{diag}(\underbrace{\lambda_1, \dots, \lambda_n}_{\text{eigen values } > 0})$ ,  $P$  orthonormal matrix.

$$X^{-1} = P^{-1}D^{-1}P$$

$$\text{tr}(X^{-1}) = \sum_{i=1}^n \frac{1}{\lambda_i} \Rightarrow \text{this is a sum of convex functions}$$

Thus,  $f$  is convex.

## Question 2

$f(X, y) = y^T X^{-1} y$  on  $\text{dom } f = S_{++}^n \times \mathbb{R}^n$   
According to the second course:

$$y^T A^{-1} y = \sup_x \{2y^T x - x^T Q x\}$$

Note  $\underbrace{g(X, y) = 2y^T x - x^T X x}_{\text{This is a linear function for both variables } X \text{ and } y.}$

Then  $g$  is a convex function in  $(X, y)$ . Knowing that the supremum of a convex function is convex,

Thus,  $f(X, y) = y^T X^{-1} y$  is convex.

## Question 3

$f(X) = \sum_{i=1}^n \sigma_i(X)$  on  $\text{dom } f = S^n$ .

We know that  $\sigma_i(X) = \sup_{\|u\|_2=1, \|v\|_2=1} u^T X v$  where  $u, v$  are unit vectors.

$$f(X) = \sum_{i=1}^n \sigma_i(X) = \sum_{i=1}^n \underbrace{\sup_{\|u\|_2=1, \|v\|_2=1} u^T X v}_{\text{It is the supremum of a linear function of } u \text{ and } v.}$$

And the supremum of linear functions are convex.

Thus,  $f$  is convex as it is a sum of linear functions.

## Optional Exercises

### Question 1

$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$

Three conditions should be verified:

$$\begin{cases} K_{m+} \text{ is closed (contains its boundary)} \\ K_{m+} \text{ is solid (has non-empty interior)} \\ K_{m+} \text{ is pointed (contains no line)} \end{cases}$$

- $K_{m+}$  is closed

$$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\} = \bigcap_{i=1}^{n-1} \{x_i \geq x_{i+1}\} \cap \{x_n \geq 0\}$$

$$= \bigcap_{i=1}^{n-1} \text{halfspace} \cap \text{halfspace}$$

$\Rightarrow K_{m+}$  is an intersection of halfspaces

Then  $K_{m+}$  is closed.

- $K_{m+}$  is solid

The point  $x = (2^n, 2^{n-1}, \dots, 1) \in K_{m+}$ .

$$\Rightarrow K_{m+} \neq \emptyset$$

Then  $K_{m+}$  is solid.

- $K_{m+}$  is pointed

If  $x \in K_{m+}$ , then  $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ .

If  $-x \in K_{m+}$ , then  $-x_1 \geq -x_2 \geq \dots \geq -x_n \geq 0$ .

$$\Rightarrow x_i = 0 \quad \forall i \in \{1, \dots, n\}$$

Then  $K_{m+}$  is pointed.

## Question 2

The dual cone is defined as:

$$\begin{aligned} K_{m+}^* &= \{y \mid y^T x \geq 0 \quad \forall x \in K_{m+}\} \\ &= \left\{ y \mid \sum_{i=1}^n y_i x_i \geq 0 \quad \forall x \in K_{m+} \right\} \end{aligned}$$

We have  $x \in K_{m+}$ , then  $x_1 \geq \dots \geq x_n \geq 0$ .

The inequality of Cauchy-Schwarz gives, for  $y \in K_{m+}^*$  and  $x \in K_{m+}$ :

$$0 \leq \sum_{i=1}^n y_i x_i \leq \left( \sum_{i=1}^n y_i \right) \left( \sum_{i=1}^n x_i \right)$$

We have,

$$\left( \sum_{i=1}^n x_i \right) \leq 2x_1$$

If  $x_1 > 0$ , then

$$\left( \sum_{i=1}^n y_i \right) \geq 0 \tag{1}$$

If  $x_1 = 0$ , then the inequality is still satisfied since:

$$\sum_{i=1}^n y_i x_i = 0$$

Thus, we conclude that:

$$K_{m+}^* = \left\{ y \mid \sum_{i=1}^n y_i \geq 0 \text{ for all } x \in K_{m+} \right\}$$