Exercise 1

Question 1

A rectangle is defined as

$$S = \{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$$

Let $x, y \in S$ and $\lambda \in [0, 1]$. We want to check if

$$\lambda x + (1 - \lambda)y \in S$$

We have, for all $i \in \{1, \ldots, n\}$:

$$\alpha_i \leq x_i \leq \beta_i \quad \text{and} \quad \alpha_i \leq y_i \leq \beta_i$$

Then

$$\lambda\alpha_i + (1 - \lambda)\alpha_i \leq \lambda x_i + (1 - \lambda)y_i \leq \beta_i + (1 - \lambda)\beta_i$$

and for all $i = 1, \ldots, n$:

$$\alpha_i \leq \lambda x_i + (1 - \lambda)y_i \leq \beta_i$$

Thus

$$\lambda x + (1 - \lambda)y \in S$$

Conclusion: S is convex.

Question 2

$$S = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1x_2 \geq 1\}$$

Let's take $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in S, such that $x_1x_2 \geq 1$ and $y_1y_2 \geq 1$.
Let's show that $z = \lambda x + (1 - \lambda)y \in S$ where $0 \leq \lambda \leq 1$.

$$z = (\underbrace{\lambda x_1 + (1 - \lambda)y_1}_{z1}, \underbrace{\lambda x_2 + (1 - \lambda)y_2}_{z2})$$

We have

$$\begin{aligned} z_1z_2 &= (\lambda x_1 + (1 - \lambda)y_1)(\lambda x_2 + (1 - \lambda)y_2) \\ &= \lambda^2 \underbrace{x_1x_2}_{\geq 1} + \underbrace{\lambda(1 - \lambda)x_1y_2 + \lambda(1 - \lambda)y_1x_2}_{\lambda(1-\lambda)(x_1y_2+y_1x_2)} + (1 - \lambda)^2 \underbrace{y_1y_2}_{\geq 1} \end{aligned}$$

We have

$$y_1y_2 \geq 1 \quad \text{and} \quad x_1x_2 \geq 1 \implies x_1y_2 \geq \frac{y_2}{x_2}$$

and

$$y_1x_2 \geq \frac{x_2}{y_2}$$

Then

$$y_1x_2 + x_1y_2 \geq \frac{y_2}{x_2} + \frac{x_2}{y_2} \geq \left(\frac{\sqrt{y_2}}{\sqrt{x_2}} - \frac{\sqrt{x_2}}{\sqrt{y_2}}\right)^2 + 2 \geq 2$$

Then

$$\begin{aligned} z_1z_2 &\geq \lambda^2 + 2\lambda(1 - \lambda) + (1 - \lambda)^2 \\ &\geq \lambda^2 + 2\lambda - 2 + 1 - 2\lambda + \lambda^2 = 1 \end{aligned}$$

Then $z \in S$
Conclusion: S is convex.

Question 3

$$A = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\} \text{ where } S \subset \mathbb{R}^n$$

Let's take $x \in A$, then $\|x - x_0\|_2 \leq \|x - y\|_2$ for all $y \in S$.

$$\Rightarrow (x - x_0)^T(x - x_0) \leq (x - y)^T(x - y)$$
$$\Rightarrow x^T x - x x_0 - x^T x + x_0^T x_0 \leq x^T x - xy - y^T x - y^T y$$
$$\Rightarrow (y - x_0)^T x \leq \frac{y^T y + x_0^T x_0}{2}$$

An equation of a half-space written as $a^T x \leq b$ such that:

$$a = y - x_0$$

$$b = \frac{y^T y + x_0^T x_0}{2}$$

Thus,

$$A = \bigcap_{y \in S} \{x \mid a^T x \leq b\}$$

is convex as it is an intersection of half-spaces.

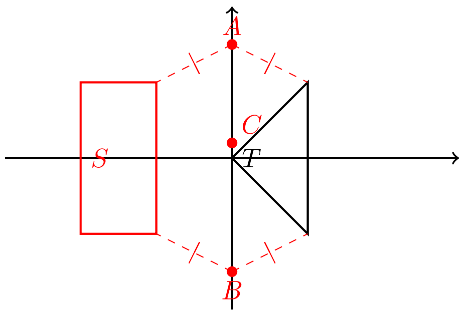
Question 4

Let

$$S = \{x \mid \text{dist}(x, S) \leq \text{dist}(x, T)\}$$

where $\begin{cases} S, T \subset \mathbb{R}^n \\ \text{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\} \end{cases}$

Consider the case $n = 2$:



A and B are in the set A (because $\text{dist}(A, S) = \text{dist}(A, T)$ and $\text{dist}(B, T) = \text{dist}(B, S)$). But C, which is in $[A, B]$, is not (because $\text{dist}(C, S) > \text{dist}(C, T)$).

Conclusion: A is not convex.

Question 5

$$S = \{x \mid x + S_2 \subseteq S_1\} \text{ where } S_1, S_2 \subseteq \mathbb{R}^n \text{ with } S_1 \text{ a convex set.}$$

S can be written as:

$$S = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\}$$
$$= \bigcap_{y \in S_2} \{x \mid x \in S_1 - y\}$$
$$= \bigcap_{y \in S_2} \{S_1 - y\}$$

S_1 is a convex set, then for all $x_1, x_2 \in S_1$ and $\lambda \in [0, 1]$, we have:

$$\lambda x_1 + (1 - \lambda) x_2 \in S_1$$

Then

$$\lambda x_1 + (1-\lambda)x_2 - y \in S_1 - y$$

Then

$$\lambda \underbrace{(x_1 - y)}_{\in\{S_1-y\}} + (1-\lambda)\underbrace{(x_2 - y)}_{\in\{S_1-y\}} \in S_1 - y$$

Thus,

$$\{S_1 - y\} \text{ is convex}$$

The intersection of convex sets is a convex set.
Conclusion:

$$S = \bigcap_{y\in S_2} \{S_1 - y\} \text{ is a convex set.}$$

Exercise 2

Question 1

$f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
- $\text{dom} f = \mathbb{R}^2_{++}$: a convex set.
- f is twice differentiable (because it is linear).

$$\begin{cases} \frac{\partial f}{\partial x_1} = x_2 \\ \frac{\partial f}{\partial x_2} = x_1 \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 f}{\partial x_1^2} = 0 \\ \frac{\partial^2 f}{\partial x_2^2} = 0 \end{cases} \text{ and } \begin{cases} \frac{\partial^2 f}{\partial x_1 \partial x_2} = 1 \\ \frac{\partial^2 f}{\partial x_2 \partial x_2} = 1 \end{cases}$$

$$\Rightarrow \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda_1 = 1$ is a trivial eigenvalue of this matrix. The other eigenvalue verifies $\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -1$.
Then the Hessian matrix is not positive definite.

$$\Rightarrow \text{we conclude that } f \text{ is not convex and is not concave}$$

Let $S_\alpha = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$
In the first exercise, we proved that S_1 is convex. This result can be extended to all $\alpha \in \mathbb{R}$ to have S_α convex.
So, f is quasiconcave (and not quasiconvex).

Question 2

Let $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$.
We compute the Hessian matrix:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$$

- The first principal minor of this matrix is $\frac{2}{x_1^3 x_2}$, and the second one is $\frac{2}{x_2 x_1^3}$. - We have $x_1, x_2 > 0$, then the Hessian matrix is positive definite (according to Sylvester's criteria).
Then f is convex.
Now, let's take $S_\alpha = \{x \mid f(x) \leq \alpha\}$

$$S_\alpha = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid \frac{1}{x_1 x_2} < \alpha\right\}$$

- If $\alpha \leq 0$:

$$S_\alpha = \emptyset, \text{ a convex set}$$

- If $\alpha > 0$:

$$S_\alpha = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \frac{1}{\alpha}\right\}$$

According to the previous function, S_α is convex.
Then, f is quasiconvex.

Question 3

Let $f(x_1, x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$.
The Hessian matrix is given by:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

We have:

$$\det\left(\nabla^2 f(x_1, x_2)\right) = -\frac{1}{x_2^4} < 0$$

$$\operatorname{tr}\left(\nabla^2 f(x_1, x_2)\right) = \frac{2x_1}{x_2^3} > 0$$

Then, the eigenvalues of the Hessian matrix have different signs.
Thus, f is not a convex function nor a concave one.
-The set $S_\alpha = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid \frac{x_1}{x_2} \leq \alpha\right\}$ defines a halfspace.
Then, f is quasilinear (quasiconcave and quasiconvex).

Question 4

Let

$$f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}, \quad \text{where } 0 \leq \alpha \leq 1 \text{ on } \mathbb{R}^2_{++}$$

Let's compute the Hessian matrix:

$$\frac{\partial^2 f}{\partial x_1^2} = \alpha(\alpha - 1)x_1^{\alpha-2}x_2^{1-\alpha}$$

$$\frac{\partial^2 f}{\partial x_2^2} = -\alpha(\alpha - 1)x_1^{\alpha}x_2^{-\alpha-1}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = -\alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha-1}$$

The Hessian matrix is given by:

$$\nabla^2 f(x_1, x_2) = \underbrace{-(1-\alpha)\alpha x_1^\alpha x_2^{1-\alpha}}_{\leq 0 \text{ because } \alpha \in (0,1) \text{ and } x_1, x_2 > 0} \underbrace{\begin{bmatrix} \frac{1}{x_1^2} & \frac{-1}{x_1 x_2} \\ \frac{-1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix}}_{\text{this matrix is positive definite}}$$

Then the eigenvalues of the Hessian matrix are negative, we conclude:

$$f \text{ is concave (and quasiconcave).}$$

Exercise 3

Question 1

$f(X) = \operatorname{tr}(X^{-1})$ on $\operatorname{dom} f = S^n_{++}$.
Let $X, Y \in \operatorname{dom} f$ and $\lambda \in [0, 1]$.
$X = P^{-1}DP$, such that $D = \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_n}_{\text{eigen values } > 0})$, P orthonormal matrix.

$X^{-1} = P^{-1}D^{-1}P$

$$\operatorname{tr}(X^{-1}) = \sum_{i=1}^{n} \frac{1}{\lambda_i} \quad \Rightarrow \text{ this is a sum of convex functions}$$

Thus, f is convex.

Question 2

$f(X, y) = y^T X^{-1} y$ on dom $f = S^n_{++} \times \mathbb{R}^n$
According to the second course:

$$y^T A^{-1} y = \sup_x \{2y^T x - x^T Q x\}$$

Note $\underbrace{g(X, y) = 2y^T x - x^T X x}_{\text{This is a linear function for both variables X and y.}}$

Then g is a convex function in (X, y). Knowing that the supremum of a convex function is convex,

Thus, $f(X, y) = y^T X^{-1} y$ is convex.

Question 3

$f(X) = \sum_{i=1}^n \sigma_i(X)$ on dom $f = S^n$.

We know that $\sigma_i(X) = \sup_{\|u\|_2=1, \|v\|_2=1} u^T X v$ where u, v are unit vectors.

$$f(X) = \sum_{i=1}^n \sigma_i(X) = \sum_{i=1}^n \underbrace{\sup_{\|u\|_2=1, \|v\|_2=1} u^T X v}_{\text{It is the supremum of a linear function of u and v.}}$$

And the supremum of linear functions are convex.

Thus, f is convex as it is a sum of linear functions.

Optional Exercises

Question 1

$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$
Three conditions should be verified:

$$\begin{cases} K_{m+} \text{ is closed (contains its boundary)} \\ K_{m+} \text{ is solid (has non-empty interior)} \\ K_{m+} \text{ is pointed (contains no line)} \end{cases}$$

- **K_{m+} is closed**

$$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\} = \bigcap_{i=1}^{n-1} \{x_i \geq x_{i+1}\} \cap \{x_n \geq 0\}$$

$$= \bigcap_{i=1}^{n-1} \text{halfspace} \cap \text{halfspace}$$

$$\Rightarrow K_{m+} \text{ is an intersection of halfspaces}$$

  Then K_{m+} is closed.

- **K_{m+} is solid**
  The point $x = (2^n, 2^{n-1}, \ldots, 1) \in K_{m+}$.

$$\Rightarrow K_{m+} \neq \emptyset$$

  Then K_{m+} is solid.

- **K_{m+} is pointed**
  If $x \in K_{m+}$, then $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$.
  If $-x \in K_{m+}$, then $-x_1 \geq -x_2 \geq \ldots \geq -x_n \geq 0$.

$$\Rightarrow x_i = 0 \quad \forall i \in \{1, \ldots, n\}$$

  Then K_{m+} is pointed.

Question 2

The dual cone is defined as:

$$K_{m+}^* = \{y \mid y^T x \geq 0 \quad \forall x \in K_{m+}\}$$

$$= \left\{ y \mid \sum_{i=1}^{n} y_i x_i \geq 0 \quad \forall x \in K_{m+} \right\}$$

We have $x \in K_{m+}$, then $x_1 \geq \ldots \geq x_n \geq 0$.
The inequality of Cauchy-Schwarz gives, for $y \in K_{m+}^*$ and $x \in K_{m+}$:

$$0 \leq \sum_{i=1}^{n} y_i x_i \leq \left(\sum_{i=1}^{n} y_i \right) \left(\sum_{i=1}^{n} x_i \right)$$

We have,

$$\left(\sum_{i=1}^{n} x_i \right) \leq 2x_1$$

If $x_1 > 0$, then

$$\left(\sum_{i=1}^{n} y_i \right) \geq 0 \tag{1}$$

If $x_1 = 0$, then the inequality is still satisfied since:

$$\sum_{i=1}^{n} y_i x_i = 0$$

Thus, we conclude that:

$$K_{m+}^* = \left\{ y \mid \sum_{i=1}^{n} y_i \geq 0 \text{ for all } x \in K_{m+} \right\}$$