HM 1

Exercise 1:

1) A rectangle is defined as $S = \{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i , i = 1, \dots, n\}$

Let $x, y \in S$ and $\lambda \in [0,1]$

$\lambda x + (1-\lambda) y \in S$?

we have $\forall i \in [\![1, n]\!]$ $\begin{cases} \alpha_i \leq x_i \leq \beta_i \\ \alpha_i \leq y_i \leq \beta_i \end{cases}$

Then $\lambda \alpha_i + (1-\lambda)\alpha_i \leq \lambda x_i + (1-\lambda) y_i \leq \lambda \beta_i + (1-\lambda)\beta_i$

Then $\alpha_i \leq \lambda x_i + (1-\lambda) y_i \leq \beta_i$ for all $i = 1, \dots, n$

Thus $\lambda x + (1-\lambda) y \in S$

Conclusion: S is convex.

2) $S = \{ (x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$

Let's take $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in S

s.t $x_1 x_2 \geq 1$ and $y_1 y_2 \geq 1$

Let's show that $z = \lambda x + (1-\lambda) y \in S$ where $0 \leq \lambda \leq 1$

$$z = (\underbrace{\lambda x_1 + (1-\lambda) y_1}_{z_1}, \underbrace{\lambda x_2 + (1-\lambda) y_2}_{z_2})$$

We have $z_1 \times z_2 = (\lambda x_1 + (1-\lambda) y_1) \times (\lambda x_2 + (1-\lambda) y_2)$

$$\Rightarrow z_1 z_2 = \lambda^2 \underbrace{x_1 x_2}_{\geq 1} + \underbrace{\lambda(1-\lambda) x_1 y_2 + \lambda(1-\lambda) y_1 x_2}_{\lambda(1-\lambda)(x_1 y_2 + y_1 x_2)} + (1-\lambda)^2 \underbrace{y_1 y_2}_{\geq 1}$$

$y_1 y_2 \geq 1$ and $x_1 x_2 \geq 1$ give that: $x_1 y_2 \geq y_2/x_2$

and $y_1 x_2 \geq \frac{x_2}{y_2}$

Then $y_1 x_2 + x_1 y_2 \geq \frac{y_2}{x_2} + \frac{x_2}{y_2}$

$$\geq \left(\frac{\sqrt{y_2}}{\sqrt{x_2}} - \frac{\sqrt{x_2}}{\sqrt{y_2}}\right)^2 + 2 \geq 2$$

Then $z_1 z_2 \geq \lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2$

$$\geq \cancel{\lambda^2} + \cancel{2\lambda} - 2 + 1 - \cancel{2\lambda} + \cancel{\lambda^2} = 1$$

Conclusion : S is convex

3) $A = \{ x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S \}$ where $S \subset \mathbb{R}^n$

Let's take $x \in A$, then $\|x - x_0\|_2 \leq \|x - y\|_2$ for all $y \in S$

$\Rightarrow (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y)$

$\Rightarrow \cancel{x^T x} - x x_0 - x_0^T x + x_0^T x_0 \leq x^T x - x y - y^T x - y^T y$

$$\Rightarrow \underbrace{(y - x_0)^T x \leq \frac{y^T y + x_0^T x_0}{2}}$$

An equation of a half space written as $a^T x \leq b$ s.t:

$$\begin{cases} a = y - x_0 \\ b = \dfrac{y^T y + x_0^T x_0}{2} \end{cases}$$

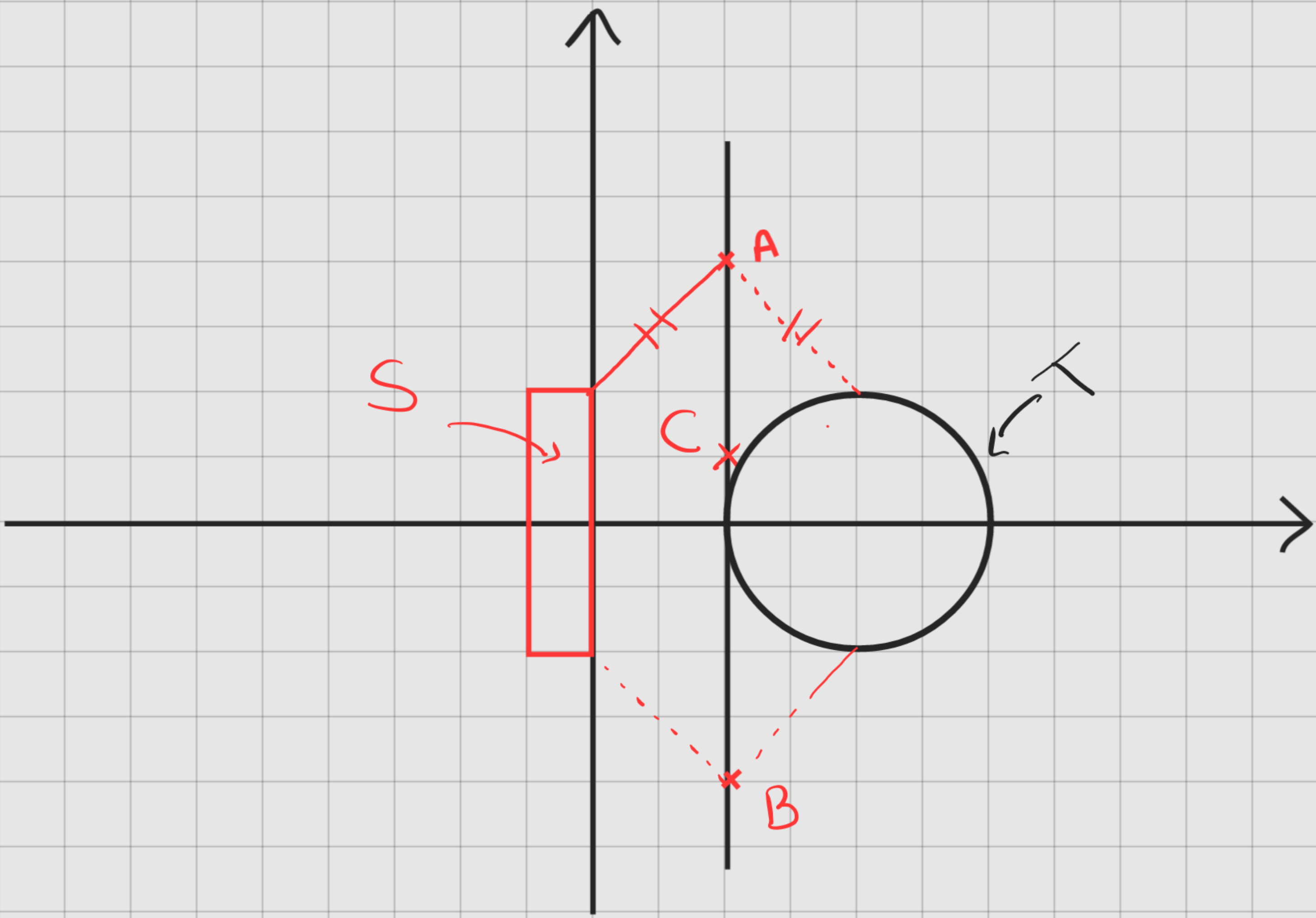
Thus, $A = \bigcap_{y \in S} \{ x \mid a^T x \leq b \}$ is convex as it is an intersection of halfspaces.

4) Let $S = \{ x \mid dist(x, S) \le dist(x, T) \}$

where $S, T \subseteq \mathbb{R}^n$

$$\begin{cases} dist(x, S) = \inf \{ \|x - z\|_2 \mid z \in S \} \end{cases}$$

Consider the case $n = 2$:



A and B are in the set A (because $dist(A, S) = dist(A, T)$ and $dist(B, T) = dist(B, S)$)

But C which is in $[A, B]$ is not (because $dist(C, S) > dist(C, T)$.

Thus A is not convex.

5) $S = \{x \mid x + S_2 \subseteq S_1\}$ where $S_1, S_2 \subseteq \mathbb{R}^n$ with

S_1 is a convex set.

S can be written as: $S = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\}$

$$= \bigcap_{y \in S_2} \{x \mid x \in S_1 - y\}$$

$$= \bigcap_{y \in S_2} \{S_1 - y\}$$

S_1 is a convex set, then for all $x_1, x_2 \in S_1$ and $\lambda \in [0,1]$,

we have $\lambda x_1 + (1-\lambda) x_2 \in S_1$

then $\lambda x_1 + (1-\lambda) x_2 - y \in S_1 - y$

then $\underbrace{\lambda(x_1 - y)}_{\in \{S_1 - y\}} + (1-\lambda)\underbrace{(x_2 - y)}_{\in \{S_1 - y\}} \in S_1 - y$

Thus $\{S_1 - y\}$ is convex

The intersection of convex sets is a convex set.

Conclusion: $S = \bigcap_{y \in S_2} \{S_1 - y\}$ is a convex set.

Exercise 2:

1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

*) $\operatorname{dom} f = \mathbb{R}^2_{++}$: a convex set.
*) f is twice differentiable. (becouse it is linear)

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = x_2 \\ \dfrac{\partial f}{\partial x_2} = x_1 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 f}{\partial^2 x_1} = 0 \\ \dfrac{\partial^2 f}{\partial^2 x_2} = 0 \end{cases} \text{ and } \begin{cases} \dfrac{\partial^2 f}{\partial x_1 \partial x_2} = 1 \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_2} = 1 \end{cases}$$

$$\Rightarrow \nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1 is a trivial eigen value of this matrix

the other eigen value verifies $\lambda_2 + 1 = 0 \Rightarrow \lambda_2 = -1$

Then the Hessian matrix is not positive definite.

$\Rightarrow$ we conclude that f is not convexe and is not concave

.) let $S_\alpha = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$.

In the first exercise, we prove that S_1 is convex

this result can be extended to all $\alpha \in \mathbb{R}$ to have S_α convex

So, f is quasiconcave (and not quasiconvex)

2) $f(x_1, x_2) = \dfrac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$

We compute the Hessian matrix:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \dfrac{2}{x_1^3 x_2} & \dfrac{1}{x_1^2 x_2^2} \\ \\ \dfrac{1}{x_1^2 x_2^2} & \dfrac{2}{x_1 x_2^3} \end{bmatrix}$$

*) the first principal minor of this matrix is $\dfrac{2}{x_1^3 x_2}$, and the second one is $\dfrac{2}{x_1 x_2^3}$.

we have $x_1, x_2 > 0$, then the hessian matrix is positive definite (according to Sylvester criteria).

Then f is convex

*) Now, let's take $S_\alpha = \{x \mid f(x) \leq \alpha\}$

$$= \left\{x_1, x_2 \in \mathbb{R}_{++} \mid \frac{1}{x_1 x_2} \leq \alpha\right\}$$

·) if $\alpha \leq 0$: $S_\alpha = \emptyset$ a convex set

$\cdot)$ if $\alpha > 0$: $S_\alpha = \left\{ x_1, x_2 \in \mathbb{R}_{++} \mid x_1 x_2 \geq \frac{1}{\alpha} \right\}$

According to the previous function, S_α is convex

Then f is quasiconvex.

3) $f(x_1, x_2) = \dfrac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & -1/x_2^2 \\ -\dfrac{1}{x_2^2} & \dfrac{2x_1}{x_2^3} \end{bmatrix}$$

$\ast)$ We have
$$\begin{cases} \det\left(\nabla^2 f(x_1, x_2)\right) = -\dfrac{1}{x_2^4} < 0 \\ \operatorname{tr}\left(\nabla^2 f(x_1, x_2)\right) = \dfrac{2x_1}{x_2^3} > 0 \end{cases}$$

Then, the eigen values of the hessian matrix have different sign

Thus, f is not a convex function nor a concave one.

$\cdot)$ The set $S_\alpha = \left\{ (x_1, x_2) \in \mathbb{R}^2_{++} \mid \frac{x_1}{x_2} \leq \alpha \right\}$ defines a halfspaces

Then, f is quasilinear (quasiconcave and quasiconvex)

4)

$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$ on $\mathbb{R}^2_{++}$

Let's compute the hessian matrix:

-) $\dfrac{\partial^2 f}{\partial x_1^2} = \alpha(\alpha-1)\, x_1^{\alpha-2}\, x_2^{1-\alpha}$

-) $\dfrac{\partial^2 f}{\partial x_2^2} = -\alpha(\alpha-1)\, x_1^{\alpha}\, x_2^{-\alpha-1}$

-) $\dfrac{\partial^2 f}{\partial x_1 \partial x_2} = \dfrac{\partial^2 f}{\partial x_2 \partial x_1} = -\alpha(1-\alpha)\, x_1^{\alpha-1}\, x_2^{-\alpha-1}$

$$\nabla^2 f(x_1, x_2) = \underbrace{-(1-\alpha)\alpha\, x_1^{\alpha}\, x_2^{1-\alpha}}_{\leq 0} \underbrace{\begin{bmatrix} \frac{1}{x_1^2} & \frac{-1}{x_1 x_2} \\ \frac{-1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix}}_{\text{this matrix is positive definite}}$$

Because $\alpha \in [0,1]$ and $x_1, x_2 > 0$

Then the eigen values of the hessian matrix are negative
$\Rightarrow$ f is concave (and quasiconcave).

Exercice 3:

1) $f(X) = tr(X^{-1})$ on $dom\, f = S^n_{++}$

Let $X, Y \in dom\, f$ and $\lambda \in [0,1]$

$X = P^{-1} D P$ s.t $D = diag(\underbrace{\lambda_1, \dots, \lambda_n}_{\text{eigen values} > 0})$, P orthonormal matrix

$X^{-1} = P^{-1} D^{-1} P$

$tr(X^{-1}) = \displaystyle\sum_{i=1}^{n} \frac{1}{\lambda_i}$ $\Rightarrow$ this is a sum of convex functions

$$f(x) = \frac{1}{x}$$

Thus f is convex

2) $f(X, y) = y^T X^{-1} y$ on $\text{dom} f = S^n_{++} \times \mathbb{R}$

According to the second course:

$$y^T A^{-1} y = \sup_x \{ 2 y^T x - x^T Q x \}$$

Note $g(X, y) = \underbrace{2 y^T x - x^T X x}$

This is a linear function for both variables X and y

Then g is a convex function in (X, y)

Knowing that the supremum of a convex function is convex

Thus, $f(X, y) = y^T X^{-1} y$ is convex

3) $f(X) = \sum_{i=1}^{n} \sigma_i(X)$ on $\text{dom} f = S^n,$

We know that $\sigma_i(X) = \sup_{\substack{\|u\|_2 = 1 \\ \|v\|_2 = 1}} u^T X v$ where u, v unit vectors

$$f(X) = \sum_{i=1}^{n} \sigma_i(X) = \sum_{i=1}^{n} \underbrace{\sup_{\substack{\|u\|_2 = 1 \\ \|v\|_2 = 1}}}\, u^T X v$$

It is the supremum of a linear function of u and v

And the supremum of linear functions are convex

Then f is convex.

Optional exercices:

1) $K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0\}$

Three conditions should be verified:

$$\begin{cases} K_{m+} \text{ is closed (contains its boundary)} \\ K_{m+} \text{ is solid (has non empty interior)} \\ K_{m+} \text{ is pointed (contains no line)} \end{cases}$$

1. K_{m+} is closed

$$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0\}$$

$$= \bigcap_{i=1}^{n-1} \underbrace{\{x_i \geqslant x_{i+1}\}}_{\text{halfspace}} \cap \underbrace{\{x_n \geqslant 0\}}_{\text{halfspace}}$$

$\Rightarrow K_{m+}$ is an intersection of halfspaces

Then K_{m+} is closed

2 - K_{m+} is solid

the point $x = (2^n, 2^{n-1}, \dots, 1) \in K_{m+}$

$\Rightarrow K_m^+ \neq \emptyset$

Then K_{m+} is solid.

3 - K_{m+} is pointed :

$x \in K_m^+ \Rightarrow x_1 \geq x_2 \geq \dots \geq x_n \geq 0 \Rightarrow x_i \geq 0 \; \forall i \in [\![1, n]\!]$

$-x \in K_{m+} \Rightarrow -x_1 \geq -x_2 \geq \dots \geq -x_n \geq 0 \Rightarrow -x_i \geq 0 \; \forall i \in [\![1, n]\!]$

$\Rightarrow x_i = 0 \quad \forall i \in [\![1, n]\!]$

K_{m+} is pointed

2) The dual cone is defined as:

$$K_{m+}^* = \left\{ y \mid y^T x \geq 0 \quad \forall x \in K_{m+} \right\}$$

$$= \left\{ y \mid \sum_{i=1}^{n} y_i x_i \geq 0 \quad \forall x \in K_{m+} \right\}$$

We have $x \in K_m^+$, then $x_1 \geq \dots \geq x_n \geq 0$

The inequality of Cauchy schwartz gives, for $y \in K_{m+}^*$ and $x \in K_{m+}$

$$0 \leq \sum_{i=1}^{n} y_i x_i \leq \left(\sum_{i=1}^{n} y_i \right) \underbrace{\left(\sum_{i=1}^{n} x_i \right)}_{\leq 2a_n}$$

Then $\sum_{i=1}^{n} y_i \geq 0$.

Thus: $K^*_{m+} = \left\{ y \mid \sum_{i=1}^{n} y_i \geq 0 \text{ for all } x \in K_{m+} \right\}$.

Exercice 5 : The conjugate of a function

1) $f(x) = \max_{i=1,\dots,n} x_i$ on $\mathbb{R}^n$

The conjugate of a function is defined as:

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \left\{ y^T x - f(x) \right\}$$
$$= \sup_{x \in \mathbb{R}^n} \left\{ y^T x - \max_{i=1,\dots,n} x_i \right\}$$

$$\sum_{i=1}^{n} y_i x_i - x_k = \sum_{i \neq k} y_i x_i + x_k (y_k - 1)$$