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Equilibrium analysis of edge-heterogeneous binary network games

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ABSTRACT

Recent studies on the equilibrium of binary network games have primarily focused on scenarios characterized by agent heterogeneity, where agents exhibit unique attributes but their interactions with different neighbors remain uniform. In this paper, we investigate the edge-heterogeneous binary network game, a more general framework that incorporates heterogeneity into agent interactions. We establish two sufficient equilibrium conditions under asynchronous best-response dynamics from different perspectives. The first condition requires underlying symmetry in interactions between neighboring agents, integrating and generalizing three classical convergence situations in binary network games. The second condition focuses on network balance, positing that equilibrium is achievable if the coordination value network of a game is structurally balanced. Additionally, for games meeting this condition, we develop a method to predict the final state based on initial state information. These results reveal factors that steer edge-heterogeneous binary network games towards equilibrium, providing valuable insights for controlling such highly nonlinear systems. Lastly, we extend the analysis to higher-order network games and propose an equilibrium condition for edge-heterogeneous 2-order network games.

1. Introduction

In recent years, evolutionary games on complex networks have garnered significant research interest and attention. Complex networks provide the topological structure for interactions in evolutionary games, establishing a mathematical foundation for more precise analysis [1,2]. At the same time, evolutionary games introduce more dynamic and proactive forms of network interactions, showcasing enhanced sophistication and intelligence in systems [3]. Evolutionary games on complex networks have been extensively employed to explore various phenomena across biological and social domains, including collective behavior [4], the emergence of cooperation [5], and rapid diffusion [6].

Binary network games, where each agent's strategy set contains only two elements, represent a fundamental yet essential case of evolutionary games on complex networks. Many prominent games utilize binary strategy sets, such as the Prisoner's Dilemma [7,8], the Snowdrift Game [9], and the Stag-Hunt Game [10]. A key problem in binary network games is determining whether the system reaches an equilibrium state under asynchronous updating. Extensive research has been conducted on this issue. For instance, [11] showed that populations exclusive of coordinators or exclusive of anticoordinators converge under arbitrary network structures. [12] explored the equilibrium of heterogeneous binary network games on ring structures, while [13,14] considered binary network games in well-mixed population with mixed updating rules. However, these studies mainly focus on agent-heterogeneous cases, where each agent has its own payoff matrix and plays pairwise games with all neighbors using the same matrix. This constrains agents to

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interact uniformly with different neighbors. In truth, real-world scenarios often exhibit significant variations in agents' interactions with different neighbors. The convergence conditions for such games remain unclear.

To address this issue, this paper analyzes the equilibrium of edge-heterogeneous binary network games (EHBNGs), a more general form of binary network games. In this framework, heterogeneity is edge-based: each directed edge (i,j) in the network is associated with its own payoff matrix π_{ij}, and agent i plays a pairwise game with its neighbor j based on π_{ij}. The update rule we consider is asynchronous best-response dynamics. "Best-response" implies that an agent updates its strategy to maximize the total utility based on its neighbors' current strategies. It is frequently considered in the study of evolutionary games [15–18] and could be the most prevalent choice in binary network games [19]. "Asynchronous" indicates that only one agent is activated to update its strategy at each moment. The significance of asynchronous best-response dynamics also lies in the fact that the equilibrium state under this dynamics is ensured to be a Nash equilibrium.

In this paper, we propose two sufficient conditions under which the edge-heterogeneous binary network game (EHBNG) can reach an equilibrium state under asynchronous best-response dynamics, for arbitrary network structures. The first condition, referred to as the generalized symmetry condition, requires symmetry in the relationships between neighbors. This condition integrates and generalizes three types of convergence cases in binary network games: agent-heterogeneous coordinating games [11], agent-heterogeneous anticoordinating games [11], and symmetric games [20]. We show that games meeting this condition are potential games [21], thus ensuring convergence. The second condition, termed the structural balance condition, focuses on network balance. It asserts that the game will reach an equilibrium state if its associated coordination value network is structurally balanced. We present a novel dual-game approach to establish this condition. Meanwhile, for games that satisfy the structural balance condition, we develop a technique to predict the final equilibrium, which provides the most comprehensive forecast possible from the initial state. This is achieved by identifying all agents that consistently choose the same strategy in every equilibrium state that can be reached from the initial state. In addition, we extend our analysis to higher-order network games and propose a condition for the equilibrium of the edge-heterogeneous 2-order network games. We validate all the proposed conditions through numerical simulations.

The structure of this paper is organized as follows: Section 2 provides a detailed description of the concepts used in the subsequent discussions. Section 3 details our main results for EHBNGs, including two convergence conditions and a method for predicting the final equilibrium state. Section 4 presents our analysis for edge-heterogeneous 2-order network games, proposing an equilibrium condition. Section 5 is dedicated to presenting the numerical simulations that validate our equilibrium conditions. Finally, Section 6 summarizes the paper and outlines potential directions for future research.

2. Preliminaries

2.1. EHBNG

An EHBNG can be described by a triple tuple $(\mathbb{G}, \mathcal{S}, \Pi)$. In this tuple, $\mathbb{G} = (\mathcal{V}, \mathcal{E}, E)$ is a directed, unweighted, connected network without self-loops, where $\mathcal{V} = \{1, 2, \dots, n\}$ represents the set of agents, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of edges. $\mathcal{S} = \{A, B\}$ is the binary strategy set for every agent, and Π is the set of payoff matrices. For each $(i,j) \in \mathcal{E}$, there exists a 2×2 matrix $\pi_{ij} \in \Pi$. In this paper, we focus on pure strategies. Each agent i chooses a strategy from $\mathcal{S}$ and receives a payoff from each edge (i,j) according to the payoff matrix

$$\pi_{ij} = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \end{array}, \qquad a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}. \tag{1}$$

Denote the population strategy state by the n-dimensional vector $s \in \mathcal{S}^{\mathcal{V}}$. The ith component s_i denotes the strategy of agent i. If the strategy pair $s_i - s_j$ is $A - A$, agent i receives a payoff a_{ij} from edge (i,j); similarly, the strategy pairs $A - B$, $B - A$, and $B - B$ correspond to payoffs b_{ij}, c_{ij}, and d_{ij}, respectively. The total utility for agent i is the sum of payoffs from all its associated edges, expressed as

$$u_i = \sum_{j \in \mathcal{N}_i} (\pi_{ij})_{s_i, s_j},$$

where $\mathcal{N}_i = \{j \mid (i,j) \in \mathcal{E}\}$ is the neighbor set of agent i.

The agent-heterogeneous binary network game (AHBNG) is a special case of the EHBNG. If each agent uses the same payoff matrix across all its neighbors (i.e., $\forall i \in \mathcal{V}$, $\forall j_1, j_2 \in \mathcal{N}_i$, there exists $\pi_{ij_1} = \pi_{ij_2}$), then the EHBNG degenerates into an AHBNG. Compared to the homogeneous binary network game (where all payoff matrices are the same), the AHBNG permits each agent to have its own distinct payoff matrix. However, it cannot model cases where the heterogeneity arises from differences among an agent's neighbors. For example, an agent's strategic interactions with a friend (cooperative relationship) and an enemy (competitive relationship) should inherently involve different payoff matrices. To address such scenarios, the EHBNG framework is necessary. It provides the most general formulation of binary network games under pairwise interactions.

For convenience, we assume in EHBNGs that for any $i, j \in \mathcal{V}$, if the edge (i,j) does not exist, a zero payoff matrix $\pi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is included in the set Π. Then the utility for agent i can be rewritten as

$$u_i = \sum_{j \in \mathcal{V}} (\pi_{ij})_{s_i, s_j}.$$

This simplification also allows the game $(\mathbb{G}, \mathcal{S}, \Pi)$ to be concisely represented as $(\mathcal{V}, \mathcal{S}, \Pi)$.

2.2. Asynchronous best-response dynamics

Under asynchronous best-response dynamics, the game evolves in discrete time $t = 1, 2, \ldots$. At each moment, one agent is activated to switch its strategy. The active agent updates its strategy to the one that maximizes the total utility. Let $f_i : \mathcal{S}^{\mathcal{V}} \to \mathcal{S}$ represent the revision protocol for agent i, where $f_i(s)$ denotes the strategy agent i will choose when activated under state s, and $u_i(s)$ represent the utility of agent i under state s. Denote by $s_{-i} \in \mathcal{S}^{\mathcal{V}\setminus\{i\}}$ the strategy state for all agent in $\mathcal{V}$ except i and denote $s = (s_i, s_{-i}) \in \mathcal{S}^{\mathcal{V}}$, the utility function becomes $u_i(s) = u_i(s_i, s_{-i})$. Then the dynamics can be expressed by the equation:

$$f_i(s) = \begin{cases} A, & u_i(A, s_{-i}) > u_i(B, s_{-i}), \\ B, & u_i(A, s_{-i}) < u_i(B, s_{-i}), \\ z_i(s), & u_i(A, s_{-i}) = u_i(B, s_{-i}), \end{cases} \tag{2}$$

where z_i is the tie breaker that determines the strategy choice when i receives equal utility for choosing A and B. We consider three types of tie breakers: (i) z_i^1, persistently choosing A, formalized as $\forall s \in \mathcal{S}^{\mathcal{V}}, z_i^1(s) = A$; (ii) z_i^2, persistently choosing B, formalized as $\forall s \in \mathcal{S}^{\mathcal{V}}, z_i^2(s) = B$; (iii) z_i^3, maintaining the current choice, formalized as $\forall s \in \mathcal{S}^{\mathcal{V}}, z_i^3(s) = s_i$.

Regarding the activation sequence, we introduce two common assumptions as follows.

Assumption 1 (Persistent activation [11]). For any time t and any agent i, there always exists a time $t' > t$ that i will be activated at t'.

Assumption 1 avoids situations where an agent seeks to change its strategy for higher utility but is never activated. An equivalent statement of Assumption 1 is that every agent would be activated an infinite number of times.

Assumption 2 (Random activation [22]). The activation sequence is a stochastic process that satisfies the following condition: there exists a constant $\kappa > 0$ such that for any agent i and any time t, the probability of agent i being activated at t is greater than κ.

If an activation sequence satisfies Assumption 2, it satisfies Assumption 1 with probability 1. Thus, Assumption 2 can be viewed as a sufficient condition for Assumption 1.

In this paper, the update rules for all games are assumed to follow asynchronous best-response dynamics by default.

2.3. Coordination

The coordination property plays an important role in our equilibrium analysis. It examines how an agent influenced by its neighbors' actions, reflecting an underlying orderliness. We now explore the coordination property in an EHBNG.

Under best-response dynamics, the EHBNG behaves as a threshold model [23]. Denote $s = (s_i, s_{-i})$, the choice of agent i is determined by the utility differential between the two strategies, defined as:

$$\begin{aligned} t_i^A(s) &= u_i(A, s_{-i}) - u_i(B, s_{-i}) \\ &= \sum_{j \in \mathcal{V}} (a_{ij} - c_{ij})\delta_j^A(s) + (b_{ij} - d_{ij})\delta_j^B(s), \end{aligned} \tag{3}$$

where $\delta_j^X(s)$, with $X \in \{A, B\}$, is an indicator function that equals 1 if $s_j = X$, and 0 otherwise.

If $t_i^A(s) > 0$, agent i will choose strategy A when activated. Thus, $t_i^A(s)$ can be viewed as the tendency of agent i to choose strategy A under state s; a higher value indicates a stronger preference. Similarly, $t_i^B(s) = u_i(B, s_{-i}) - u_i(A, s_{-i})$ signifies the tendency of agent i to choose strategy B.

Next, we consider the impact of a neighbor's actions on agent i. Denote by $s_{-i,-j} \in \mathcal{S}^{\mathcal{V}\setminus\{i,j\}}$ the strategy state for all agents in $\mathcal{V}$, except for agents i and j. Denote $s = (s_i, s_j, s_{-i,-j}) \in \mathcal{S}^{\mathcal{V}}$. Assume neighbor j changes its strategy from B to A. The resultant change in agent i's tendency to choose strategy A can be expressed as follows:

$$\begin{aligned} v(s_{-i,-j}) &= t_i^A(s_i, A, s_{-i,-j}) - t_i^A(s_i, B, s_{-i,-j}) \\ &= u_i(A, A, s_{-i,-j}) - u_i(B, A, s_{-i,-j}) \\ &\quad - u_i(A, B, s_{-i,-j}) + u_i(B, B, s_{-i,-j}) \\ &= a_{ij} - c_{ij} - b_{ij} + d_{ij}. \end{aligned}$$

Similarly, when neighbor j switches from A to B, the corresponding change in agent i's tendency to select strategy B is:

$$\begin{aligned} v(s_{-i,-j}) &= t_i^B(s_i, B, s_{-i,-j}) - t_i^B(s_i, A, s_{-i,-j}) \\ &= u_i(B, B, s_{-i,-j}) - u_i(B, A, s_{-i,-j}) \\ &\quad - u_i(A, B, s_{-i,-j}) + u_i(A, A, s_{-i,-j}) \\ &= d_{ij} - c_{ij} - b_{ij} + a_{ij}. \end{aligned}$$

Therefore, we define $w_{ij} = a_{ij} - c_{ij} - b_{ij} + d_{ij}$ as the coordination value for the payoff matrix π_{ij}. If $w_{ij} \geq 0$, we describe agent i as coordinating with neighbor j, implying a positive effect when j switches to strategy X on i's decision to also choose X. Conversely, if $w_{ij} < 0$, agent i is said to be anticoordinating with j, reflecting a negative impact. If for all $i, j \in \mathcal{V}$, there exist $w_{ij} \geq 0$, then the game is known as a super-modular game [24,25].

Another commonly used concept of coordination is *X-coordinating*, which may be more intuitive. We now give this concept and establish its connection to the coordination value w_{ij}.

Definition 1 (X-coordinating [20,22]). In a network game characterized by an agent set $\mathcal{V}$, a finite strategy set $\mathcal{C}$, and a revision protocol $\{f_j \mid j \in \mathcal{V}\}$, an agent $i \in \mathcal{V}$ is termed *X-coordinating*, $X \in \mathcal{C}$, if for any two state vectors $s, r \in \mathcal{C}^{\mathcal{V}}$ satisfying

$$\{j \mid s_j = X\} \subseteq \{j \mid r_j = X\},$$

the following holds:

$$f_i(s) = X \Rightarrow f_i(r) = X.$$

If every agent $i \in \mathcal{V}$ for any $X \in \mathcal{C}$ is *X-coordinating*, the game is classified as a *coordinating network game*. The condition $\{j \mid s_j = X\} \subseteq \{j \mid r_j = X\}$ establishes a partial order relation, which we denote by $s \overset{X}{\leq} r$.

We can interpret the *X-coordinating* property from a dynamical view. Assume that s is the initial state. The condition $f_i(s) = X$ indicates that, under state s, agent i prefers strategy X. After a period of evolution, the system transitions from state s to state r. The condition $s \overset{X}{\leq} r$ implies that, during this process, no agent who initially chose strategy X switched away from it. The X-coordinating property ensures that, in this case, agent i still prefers strategy X under state r. This means that agent i's preference for X persists unless the set of X-choosers shrinks. In other words, agent i's inclination toward strategy X is coordinating with the prevalence of X in the population.

Henceforth, considering a binary strategy set $\mathcal{C} = \{A, B\} = \mathcal{S}$. Denote by $\neg$ and $\wedge$ the logical operators "no" and "and", respectively, an agent i being *A-coordinating* can be expressed as

$$\begin{aligned} & \forall s, r \in \mathcal{S}^{\mathcal{V}}, s \overset{A}{\leq} r \Rightarrow (f_i(s) = A \Rightarrow f_i(r) = A) \\ \iff & \forall s, r \in \mathcal{S}^{\mathcal{V}}, s \overset{A}{\leq} r \Rightarrow \neg(f_i(s) = A \wedge f_i(r) = B)) \\ \iff & \forall s, r \in \mathcal{S}^{\mathcal{V}}, \neg(s \overset{A}{\leq} r \wedge f_i(s) = A \wedge f_i(r) = B). \end{aligned} \tag{4}$$

As $s \overset{A}{\leq} r \iff r \overset{B}{\leq} s$, (4) is equivalent to

$$\forall s, r \in \mathcal{S}^{\mathcal{V}}, \neg(r \overset{B}{\leq} s \wedge f_i(r) = B \wedge f_i(s) = A),$$

which means that agent i is also *B-coordinating*. Consequently, i being *A-coordinating* is equivalent to i being *B-coordinating*.

Consider the *A-coordinating* property between agents. Denote $s = (s_j, s_{-j})$, we say agent i is *A-coordinating* with agent j if the following condition is satisfied:

$$f_i(B, s_{-j}) = A \Rightarrow f_i(A, s_{-j}) = A, \quad \forall s_{-j} \in \mathcal{S}^{\mathcal{V}\setminus\{j\}}.$$

We have the following lemma.

Lemma 1. *Agent i is A-coordinating if and only if for all $j \in \mathcal{V}$, agent i is A-coordinating with j.*

Proof. Necessity is straightforward. For sufficiency, consider any states $s, r \in \mathcal{S}^{\mathcal{V}}$ satisfying $s \overset{A}{\leq} r$. If $\{j \mid r_j = A\} \setminus \{j \mid s_j = A\} = \emptyset$, it follows that $f_i(s) = A \Rightarrow f_i(r) = A$. If there exists a non-empty set, denote it by $\{i_1, i_2, \ldots, i_m\}$. For any state $x \in \mathcal{S}^{\mathcal{V}}$, denote $x = (x_{i_1}, x_{i_2}, \ldots, x_{i_m}, x_{-i_1,-i_2,\ldots,-i_m})$, where $x_{-i_1,-i_2,\ldots,-i_m} \in \mathcal{S}^{\mathcal{V}\setminus\{i_1,i_2,\ldots,i_m\}}$. Then we have $s = (s_{i_1}, s_{i_2}, \ldots, s_{i_m}, s_{-i_1,-i_2,\ldots,-i_m})$. For $k \in \{0, 1, 2, \ldots, m\}$, define $s^k = (A, A, \ldots, A, s_{i_{k+1}}, \ldots, s_{i_m}, s_{-i_1,-i_2,\ldots,-i_m})$, then $s^0 = s$, $s^m = r$. If $f_i(s) = A$ and for all $j \in \mathcal{V}$, agent i is *A-coordinating* with j, then

$$f_i(s^0) = A \Rightarrow f_i(s^1) = A \Rightarrow \ldots \Rightarrow f_i(s^m) = A,$$

implying $f_i(s) = A \Rightarrow f_i(r) = A$. Hence agent i is *A-coordinating*. □

For EHBNGs, we derive the relationship between the coordination value w_{ij} and the *A-coordinating* property.

Proposition 1. *In an EHBNG, if $w_{ij} \geq 0$, then agent i is A-coordinating with agent j.*

Proof. For an arbitrary state s, denote $s = (s_j, s_{-j})$. Assume $f_i(B, s_{-j}) = A$, then there must exists $t_i^A(B, s_{-j}) \geq 0$. Given that $t_i^A(A, s_{-j}) - t_i^A(B, s_{-j}) = w_{ij} \geq 0$, it follows that $t_i^A(A, s_{-j}) \geq 0$. If $t_i^A(A, s_{-j}) > 0$, then $f_i(A, s_{-j}) = A$. Alternatively, if $t_i^A(A, s_{-j}) = t_i^A(B, s_{-j}) = 0$, since $f_i(B, s_{-j}) = A$, we have $z_i \in \{z_i^1, z_i^3\}$. If $z_i = z_i^1$, then $f_i(A, s_{-j}) = A$. If $z_i = z_i^3$, then $f_i(A, s_{-j}) = s_i = f_i(B, s_{-j}) = A$. Hence, $f_i(B, s_{-j}) = A \Rightarrow f_i(A, s_{-j}) = A$, confirming that agent i is *A-coordinating* with agent j. □

When $w_{ij} < 0$, agent i could still be *A-coordinating* with agent j. Since this is a trivial case and not pivotal to our subsequent discussions, here we propose the proposition without providing a proof.

Proposition 2. *In an EHBNG, for an arbitrary state s, denote $s = (s_j, s_{-j})$. When $w_{ij} < 0$, agent i is A-coordinating with agent j if and only if all the following conditions are met:*

1. $\forall s_{-j} \in \mathcal{S}^{\mathcal{V}\setminus\{j\}}$, $t_i^A(B, s_{-j}) \in (-\infty, 0] \cup [-w_{ij}, +\infty)$. *If there exists s_{-j} such that $t_i^A(B, s_{-j}) = 0$, then $z_i = z_i^2$; if there exists s_{-j} such that $t_i^A(B, s_{-j}) = -w_{ij}$, then $z_i = z_i^1$.*
2. $\forall s_{-j} \in \mathcal{S}^{\mathcal{V}\setminus\{j\}}$, $t_i^B(A, s_{-j}) \in (-\infty, 0] \cup [-w_{ij}, +\infty)$. *If there exists s_{-j} such that $t_i^B(A, s_{-j}) = 0$, then $z_i = z_i^1$; if there exists s_{-j} such that $t_i^B(A, s_{-j}) = -w_{ij}$, then $z_i = z_i^2$.*

The coordination value w_{ij} plays a significant role in our discussions. To articulate the overall coordination property in the game, we construct the coordination value matrix $W = [w_{ij}]_{n\times n}$, which will be central to our equilibrium analysis for EHBNGs.

3. Equilibrium conditions for EHBNGs

In this section, we introduce our convergence conditions for EHBNGs. The first part presents the generalized symmetry condition and its conceptualization, while the second part introduces the structural balance condition and develops a method to predict the final equilibrium state for games satisfying this condition.

3.1. Generalized symmetry condition

Inspired by [11] and [20], we formulate the generalized symmetry condition. In this condition, the coordination value is treated as the result of a fusion of agents' intrinsic properties and their interrelationships. The key requirement is the symmetry in the interrelationships.

Theorem 1. *Consider an EHBNG $(\mathcal{V}, \mathcal{S}, \Pi)$ with the coordination value matrix $W = [w_{ij}]_{n\times n}$. Suppose that Assumption 1 holds. If W can be decomposed into the product of a non-negative diagonal matrix and a symmetric matrix, i.e., there exist $Q = [q_{ij}]_{n\times n}$ (with diagonal elements $q_{ii} \in \mathbb{R}_{\geq 0}$ and non-diagonal elements $q_{ij} = 0$) and $E = [e_{ij}]_{n\times n}$ ($e_{ij} = e_{ji}$ and $e_{ij} \in \mathbb{R}$) satisfying $W = QE$, then the game will converge in finite time.*

Proof. Initially, we assume that for each agent i, $q_{ii} > 0$. The utility differential between strategy A and strategy B under state s is given by

$$u_i(A, s_{-i}) - u_i(B, s_{-i}) = \sum_{j\in\mathcal{V}} (a_{ij} - c_{ij})\delta_j^A(s) + (b_{ij} - d_{ij})\delta_j^B(s).$$

For simplicity, we omit s when it does not cause ambiguity and denote $u_i(A, s_{-i})$ and $u_i(B, s_{-i})$ by u_i^A and u_i^B. Since $\delta_j^A = 1 - \delta_j^B$, we derive

$$\begin{aligned} u_i^A - u_i^B &= \sum_{j\in\mathcal{V}} (a_{ij} - c_{ij}) \frac{1 + (\delta_j^A - \delta_j^B)}{2} \\ &\quad + \sum_{j\in\mathcal{V}} (b_{ij} - d_{ij}) \frac{1 - (\delta_j^A - \delta_j^B)}{2} \\ &= \sum_{j\in\mathcal{V}} (a_{ij} - c_{ij} + d_{ij} - b_{ij}) \frac{\delta_j^A - \delta_j^B}{2} \\ &\quad + \sum_{j\in\mathcal{V}} \frac{1}{2}(a_{ij} - c_{ij} - d_{ij} + b_{ij}). \end{aligned}$$

Define the constant θ_i as

$$\theta_i = \sum_{j\in\mathcal{V}} \frac{1}{2q_{ii}}(a_{ij} - c_{ij} - d_{ij} + b_{ij})$$

and the function of state s, r_i^A, as

$$r_i^A = \frac{u_i^A - u_i^B}{q_{ii}} = \sum_{j\in\mathcal{V}} e_{ij} \frac{\delta_j^A - \delta_j^B}{2} + \theta_i.$$

Define $\mathcal{H}$ as the set of the absolute values of r_i^A, excluding zero:

$$\mathcal{H} = \{|r_i^A(s)| \mid i \in \mathcal{V}, s \in \mathcal{S}^{\mathcal{V}}\} \setminus \{0\}.$$

If $\mathcal{H}$ is an empty set, then under any state s, for all agents i, $u_i^A = u_i^B$ holds. When an agent i with tie breaker $z_i \in \{z_i^1, z_i^2\}$ is activated, it will choose its preference strategy. Once all the strategies of agents with tie breakers $z_i \in \{z_i^1, z_i^2\}$ align with their preferences, the game converges.

If $\mathcal{H}$ is not empty, since it contains finite elements, it has a minimum k. Define the potential function for agent i under state s as follows:

$$\phi_i = \begin{cases} \frac{1}{4}\sum_{j\in\mathcal{V}} e_{ij}(\delta_j^A - \delta_j^B) + \theta_i, & \text{if } s_i = A, \\ -\frac{1}{4}\sum_{j\in\mathcal{V}} e_{ij}(\delta_j^A - \delta_j^B) - \theta_i, & \text{if } s_i = B. \end{cases}$$

The potential function for the population under state s is defined as

$$\Phi = \sum_{i\in\mathcal{V}} \phi_i + k\delta_i^{z_i},$$

where

$$\delta_i^{z_i}(s) = \begin{cases} 1, & z_i(s) = s_i, \\ 0, & z_i(s) \neq s_i. \end{cases}$$

If agent i is activated and its strategy is switched from B to A under state s, there must be $r_i^A \geq 0$, and the potential change for each agent would be

- For agent i, $\Delta\phi_i = \frac{1}{2}\sum_{j\in\mathcal{V}} e_{ij}(\delta_j^A - \delta_j^B) + 2\theta_i$;
- For agent $j \in \mathcal{V}\setminus\{i\}$, if $s_j = A$, $\Delta\phi_j = \frac{1}{2}e_{ij}$; if $s_j = B$, $\Delta\phi_j = -\frac{1}{2}e_{ij}$.

Then the change of Φ would be

$$\begin{aligned} \Delta\Phi &= \sum_{j\in\mathcal{V}} e_{ij}(\delta_j^A - \delta_j^B) + 2\theta_i + \Delta(k\delta_i^{z_i}) \\ &= 2r_i^A + \Delta(k\delta_i^{z_i}). \end{aligned}$$

If $r_i^A > 0$, $\Delta\Phi \geq 2k - k = k$. If $r_i^A = 0$, then the strategy of agent i being switched from B to A means that z_i is z_i^1 and thus z_i always chooses A, then the change of Φ would be $\Delta\Phi = 0 + k = k$.

We now know that every time an agent switches its strategy from B to A, the population potential Φ increases by at least k. With a similar discussion, we get that if an agent switches its strategy from A to B, Φ also increases by at least k. Since Φ is upper bounded, agents in the network can only change their strategies a finite number of times. Combined with Assumption 1, we conclude that the game will converge in finite time.

If there exist agents $i \in \mathcal{V}$ such that $q_{ii} = 0$, we divide $\mathcal{V}$ into two subsets: $\mathcal{V}_1 = \{i \mid q_{ii} = 0\}$ and $\mathcal{V}_2 = \{i \mid q_{ii} > 0\}$. For agents in $\mathcal{V}_1$, we have $w_{ij} = 0$ for any $j \in \mathcal{V}$. According to (3), the utility differential $u_i^A - u_i^B$ remains constant across every state s. Thus, when all agents in $\mathcal{V}_1$ have been activated once, they will no longer change their strategies. Then analyzing the game dynamics within population $\mathcal{V}_2$ in a similar way as the situation where all $q_{ii} > 0$, we can conclude that agents in $\mathcal{V}_2$ can only change their strategies a finite number of times. This completes the proof. □

Now we demonstrate the implications of Theorem 1 by relating it to three types of binary network games that have been proven to converge in previous studies.

Example 1 (Agent-heterogeneous coordinating game [11]). In agent-heterogeneous coordinating games, the topological structure is represented by an undirected, unweighted network with adjacent matrix $U = [u_{ij}]_{n\times n}$. If agents i and j are neighbors, $u_{ij} = 1$; otherwise, $u_{ij} = 0$. Each agent i possesses its own payoff matrix

$$\pi_i = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \end{array}, \quad a_i, b_i, c_i, d_i \in \mathbb{R}, \tag{5}$$

satisfying the condition $w_i = a_i - c_i + d_i - b_i \geq 0$. Each agent i plays pairwise games with all its neighbors, deriving a payoff from each edge based on the matrix π_i. The total utility for agent i under state s is calculated as

$$u_i(s) = \sum_{j:u_{ij}=1} (\pi_i)_{s_i,s_j}.$$

In this scenario, the coordination value matrix W can be decomposed into the product of a non-negative diagonal matrix $Q = [q_{ij}]_{n\times n}$, with $q_{ii} = w_i$ for diagonal elements and $q_{ij} = 0$ for non-diagonal elements, and a symmetric matrix $E = U$, aligning with the generalized symmetry condition.

Example 2 (Agent-heterogeneous anticoordinating game [11]). Agent-heterogeneous anticoordinating games follow the same form as agent-heterogeneous coordinating games, except that the coordination value for each agent i satisfies $w_i \leq 0$. In this scenario, the coordination value matrix W can be decomposed into $W = (-Q)(-U)$, aligning with the generalized symmetry condition.

Example 3 (Symmetric game [20]). Symmetric games are a special category of EHBNGs satisfying the property that, for all $i, j \in \mathcal{V}$, $\pi_{ij} = \pi_{ji}$. Consequently, the coordination value matrix W is itself symmetric. This aligns with the generalized symmetry condition, as W can be expressed as $W = IW$, where I is the identity matrix.

Consider the decomposition $W = QE$ in Theorem 1. W is the coordination value matrix, capturing the interaction property on each edge. The decomposition expresses W as the product of Q, which reflects the intrinsic property of each agent, and E, which represents the relationships between agents. In Examples 1 and 2, the heterogeneity only lies in the agent property matrix Q, with the elements in relationship matrix E constrained in $\{0,1\}$ or $\{0,-1\}$. Conversely, in Example 3, the heterogeneity is solely in the relationship matrix E, with the agent property matrix Q being identity matrix. The proposed generalized symmetry condition encompasses both the heterogeneity in matrices Q and E, suggesting that symmetric relationships (i.e. a symmetric matrix E) can lead to equilibrium, irrespective of the heterogeneity present.

3.2. Structural balance condition

The generalized symmetry condition suggests that symmetric relationships can lead a game to equilibrium. To utilize this condition, the exact values of w_{ij} for each edge must be known. However, in some scenarios, obtaining detailed information about the payoff matrix π_{ij} or the coordination value w_{ij} is not feasible. Aim to this situation, we propose the structural balance condition from a more macroscopic perspective, which focuses solely on the coordination properties on the edges without requiring more detailed information.

3.2.1. Condition illustration

First, we provide a brief description of the conception used in the following discussion.

A signed social network [26] can be described by a tuple $(\mathcal{V}, \mathcal{E}, M)$, where $\mathcal{V} = \{1, 2, \dots, n\}$ stands for an agent set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set, and $M = [m_{ij}]_{n\times n} \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ is a signed adjacency matrix. In this matrix, $m_{ij} > 0$ indicates a positive edge on which agent j has a positive influence on agent i, while $m_{ij} < 0$ indicates a negative edge. Since the adjacency matrix M contains all the information about the edge set $\mathcal{E}$, we represent the signed network $\mathbb{G}(\mathcal{V}, \mathcal{E}, M)$ more briefly as $\mathbb{G}(\mathcal{V}, M)$ in the following discussion.

The signed network $\mathbb{G}(\mathcal{V}, M)$ is called structurally balanced [27–29] if the agent set can be partitioned into two disjoint sets $\mathcal{X}$ and $\mathcal{Y}$ such that for any $i, j \in \mathcal{X}$ or any $i, j \in \mathcal{Y}$, $m_{ij} \geq 0$ and for any $i \in \mathcal{X}, j \in \mathcal{Y}$ or any $i \in \mathcal{Y}, j \in \mathcal{X}$, $m_{ij} \leq 0$. Structural balance implies that for every cycle in the graph, the product of edge signs is non-negative, reflecting a harmonious state in social relationships that align with perceptions such as "the enemy of my enemy is my friend" [30,31].

Our second condition demonstrates that if the coordination value network of a game is structurally balanced, the game almost surely converge.

Theorem 2. *Consider an EHBNG $(\mathcal{V}, \mathcal{S}, \Pi)$ with coordination value matrix W. Suppose that Assumption 2 holds. If the signed network $\mathbb{G}(\mathcal{V}, W)$ is structurally balanced, then the game will converge with probability* 1.

It takes several steps to prove Theorem 2. Define Γ as an n-agent EHBNG $(\mathcal{V}, \mathcal{S}, \Pi)$ with coordination value matrix W, such that $\mathbb{G}(\mathcal{V}, W)$ is structurally balanced. Its agent set $\mathcal{V}$ can be divided into two disjoint subsets $\mathcal{X}, \mathcal{Y}$ as aforementioned. First, we construct a dual game $\hat{\Gamma}$, whose population dynamics is determined by Γ and vice versa. This thought bears some resemblance to the concept of the enlarged graph discussed in [32]. Below, we propose the precise mathematical definition of the dual game.

Definition 2 (*Dual game*). Consider two games G and $\hat{G}$ with the same agent set $\mathcal{V}$ and strategy set $\mathcal{S}$. We say $\hat{G}$ is a dual game for G if there exists a bijection $\varphi : \mathcal{S}^{\mathcal{V}} \to \mathcal{S}^{\mathcal{V}}$ between their state vectors that satisfies the following statement:

- In game G, starting from any state vector s^0 and activating agent with any sequence $(i^t)_{t=1}^{\infty}$, the resulting state path is denoted as $(s^t)_{t=0}^{\infty}$, meaning that at time t, agent i^t is activated and the population state switches from s^{t-1} to s^t. Then in game $\hat{G}$, starting from the state vector $\varphi(s^0)$ and activating with the same sequence $(i^t)_{t=1}^{\infty}$, the resulting state path will be $(\varphi(s^t))_{t=0}^{\infty}$.

By this definition, if $\hat{s}$ is an equilibrium state of game $\hat{G}$, then $s = \varphi^{-1}(\hat{s})$ is an equilibrium state of game G. Moreover, if game $\hat{G}$ converges under Assumption 1 or Assumption 2 from any initial state, then so does game G, and its equilibrium state is completely determined by the equilibrium state of game $\hat{G}$ through the bijection φ. Therefore, through the dual game method, we can analyze the equilibrium behavior of the original game G by studying the dual game $\hat{G}$. In particular, for game Γ which satisfies the structural balance condition, we construct a more harmonious dual game $\hat{\Gamma}$, whose coordination value matrix is non-negative.

In game Γ, let the payoff matrix be denoted by $\pi_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}$, then the payoff matrix $\hat{\pi}_{ij}$ in game $\hat{\Gamma}$ is defined as follows:

- for $i, j \in \mathcal{X}$, $\hat{\pi}_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}$;
- for $i, j \in \mathcal{Y}$, $\hat{\pi}_{ij} = \begin{pmatrix} d_{ij} & c_{ij} \\ b_{ij} & a_{ij} \end{pmatrix}$;
- for $i \in \mathcal{X}, j \in \mathcal{Y}$, $\hat{\pi}_{ij} = \begin{pmatrix} b_{ij} & a_{ij} \\ d_{ij} & c_{ij} \end{pmatrix}$;
- for $i \in \mathcal{Y}, j \in \mathcal{X}$, $\hat{\pi}_{ij} = \begin{pmatrix} c_{ij} & d_{ij} \\ a_{ij} & b_{ij} \end{pmatrix}$.

Next, we define the tie breaker for agent i in $\hat{\Gamma}$ as:

$$\hat{z}_i = \begin{cases} z_i^1 & (z_i = z_i^1 \wedge i \in \mathcal{X}) \vee (z_i = z_i^2 \wedge i \in \mathcal{Y}), \\ z_i^2 & (z_i = z_i^1 \wedge i \in \mathcal{Y}) \vee (z_i = z_i^2 \wedge i \in \mathcal{X}), \\ z_i^3 & z_i = z_i^3, \end{cases} \tag{6}$$

where $\wedge$ and $\vee$ denote logical operators “and” and “or”.

Now we prove that $\hat{\Gamma}$ is the dual game of Γ with the state bijection $\varphi : \mathcal{S}^{\mathcal{V}} \to \mathcal{S}^{\mathcal{V}}$ defined as follows: for $i \in \mathcal{X}$, $\varphi(s)_i = s_i$; for $i \in \mathcal{Y}$, $\varphi(s)_i \in \{A, B\} \setminus \{s_i\}$.

Lemma 2. *Consider an arbitrary activation sequence $(i^t)_{t=1}^{\infty}$ and an arbitrary state vector s^0. In game Γ, starting from s^0 and following the sequence $(i^t)_{t=1}^{\infty}$, the corresponding state path is denoted as $(s^t)_{t=0}^{\infty}$. In game $\hat{\Gamma}$, starting from $\hat{s}^0 = \varphi(s^0)$ and following the same sequence $(i^t)_{t=1}^{\infty}$, denote the state path as $(\hat{s}^t)_{t=0}^{\infty}$. Then $\hat{s}^t = \varphi(s^t)$ holds for any $t \in [0, +\infty)$.*

Proof. In game Γ, similar to (3), the utility differential between A and B under state $s = (s_i, s_{-i})$ can be expressed as

$$\begin{aligned} t_i^A(s) &= u_i(A, s_{-i}) - u_i(B, s_{-i}) \\ &= \sum_{j\in\mathcal{V}} (a_{ij} - c_{ij})\delta_j^A(s) + (b_{ij} - d_{ij})\delta_j^B(s) \\ &= \sum_{j\in\mathcal{X}} (a_{ij} - c_{ij})\delta_j^A(s) + (b_{ij} - d_{ij})\delta_j^B(s) \\ &\quad + \sum_{j\in\mathcal{Y}} (a_{ij} - c_{ij})\delta_j^A(s) + (b_{ij} - d_{ij})\delta_j^B(s). \end{aligned} \tag{7}$$

In game $\hat{\Gamma}$, we make a classification discussion: $\square$

Case 1: Agent i in set $\mathcal{X}$. By the definition of the payoff matrix $\hat{\pi}_{ij}$, the utility differential between A and B under state $\varphi(s)$ is given by

$$\begin{aligned} \hat{t}_i^A(\varphi(s)) &= \hat{u}_i(A, \varphi(s)_{-i}) - \hat{u}_i(B, \varphi(s)_{-i}) \\ &= \sum_{j\in\mathcal{X}} (a_{ij} - c_{ij})\delta_j^A(\varphi(s)) + (b_{ij} - d_{ij})\delta_j^B(\varphi(s)) \\ &\quad + \sum_{j\in\mathcal{Y}} (b_{ij} - d_{ij})\delta_j^A(\varphi(s)) + (a_{ij} - c_{ij})\delta_j^B(\varphi(s)). \end{aligned} \tag{8}$$

By the definition of the bijection φ, we get:

$$\begin{aligned} &\delta_j^A(s) = \delta_j^A(\varphi(s)), \delta_j^B(s) = \delta_j^B(\varphi(s)), \quad \text{if } j \in \mathcal{X}; \\ &\delta_j^A(s) = \delta_j^B(\varphi(s)), \delta_j^B(s) = \delta_j^A(\varphi(s)), \quad \text{if } j \in \mathcal{Y}. \end{aligned} \tag{9}$$

According to (7), (8), and (9), we have

$$t_i^A(s) = \hat{t}_i^A(\varphi(s)).$$

Denote by f_i and $\hat{f}_i$ the revision protocols for agent i in game Γ and game $\hat{\Gamma}$, respectively. If $f_i(s) = A$, $t_i^A(s) \geq 0$. This implies

1. If $t_i^A(s) = \hat{t}_i^A(\varphi(s)) > 0$, $\hat{f}_i(\varphi(s)) = A$;
2. If $t_i^A(s) = \hat{t}_i^A(\varphi(s)) = 0$, we have $z_i(s) = A$. Combined with (6), we can get $\hat{z}_i \in \{z_i^1, z_i^3\}$. When $\hat{z}_i = z_i^1$, $\hat{z}_i(\varphi(s)) = A$; when $\hat{z}_i = z_i^3$, we have $z_i = z_i^3$, then $\hat{z}_i(\varphi(s)) = \varphi(s)_i = s_i = z_i(s) = A$. Thus, we have $\hat{f}_i(\varphi(s)) = \hat{z}_i(s) = A$.

Therefore, if $f_i(s) = A$, $\hat{f}_i(\varphi(s)) = A$. A similar discussion yields that if $f_i(s) = B$, $\hat{f}_i(\varphi(s)) = B$. Thus, we conclude

$$f_i(s) = \hat{f}_i(\varphi(s)), \quad \forall i \in \mathcal{X}. \tag{10}$$

Case 2: Agent i in set $\mathcal{Y}$. By the definition of the payoff matrix $\hat{\pi}_{ij} \in \hat{\Pi}$, the utility differential between A and B under state $\varphi(s)$ in game $\hat{\Gamma}$ is

$$\begin{aligned} \hat{t}_i^A(\varphi(s)) &= \hat{u}_i(A, \varphi(s)_{-i}) - \hat{u}_i(B, \varphi(s)_{-i}) \\ &= \sum_{j\in\mathcal{X}} (c_{ij} - a_{ij})\delta_j^A(\varphi(s)) + (d_{ij} - b_{ij})\delta_j^B(\varphi(s)) \\ &\quad + \sum_{j\in\mathcal{Y}} (d_{ij} - b_{ij})\delta_j^A(\varphi(s)) + (c_{ij} - a_{ij})\delta_j^B(\varphi(s)). \end{aligned} \tag{11}$$

According to (7), (9), and (11), we have

$$t_i^A(s) = -\hat{t}_i^A(\varphi(s)).$$

If $f_i(s) = A$, $t_i^A(s) \geq 0$. This implies

1. If $\hat{t}_i^A(\varphi(s)) = -t_i^A(s) < 0$, $\hat{f}_i(\varphi(s)) = B$;
2. If $\hat{t}_i^A(\varphi(s)) = -t_i^A(s) = 0$, we have $z_i(s) = A$. Combined with (6), we can get $\hat{z}_i \in \{z_i^2, z_i^3\}$. when $\hat{z}_i = z_i^2$, $\hat{z}_i(\varphi(s)) = B$; when $\hat{z}_i = z_i^3$, we have $z_i = z_i^3$. Since $z_i(s) = A$, $s_i = A$. Combined with $i \in \mathcal{Y}$, we can get $\hat{z}_i(\varphi(s)) = \varphi(s)_i = B$. Thus, we have $\hat{f}_i(\varphi(s)) = \hat{z}_i(s) = B$.

With a similar discussion, we can get that if $f_i(s) = B$, $\hat{f}_i(\varphi(s)) = A$. Then we have

$$\hat{f}_i(\varphi(s)) \in \{A, B\} \setminus \{f_i(s)\}, \quad \forall i \in \mathcal{Y}. \tag{12}$$

From (10) and (12), we know that for any time t, if $\hat{s}^t = \varphi(s^t)$, then $\hat{s}^{t+1} = \varphi(s^{t+1})$ holds. Given that $\hat{s}^0 = \varphi(s^0)$, through mathematical induction, we complete the proof of Lemma 2.

Now, we can study the game $\hat{\Gamma}$ instead of Γ. Let $\hat{\omega}_{ij}$ denote the coordination value of the payoff matrix $\hat{\pi}_{ij}$ in the game $\hat{\Gamma}$. By the definition of $\hat{\pi}_{ij}$, it follows that for any $i, j \in \mathcal{V}$, $\hat{\omega}_{ij} \geq 0$. Using Lemma 1 and Proposition 1, we can conclude:

Lemma 3. *Game $\hat{\Gamma}$ is a coordinating network game.*

Next, we study the game $\hat{\Gamma}$ using its coordinating property. For the conciseness of expression, we provide two definitions:

Definition 3. Starting from state s, if there exists an activation sequence that leads to state s', then s' is said to be reachable from s.

Definition 4. Let $\mathcal{T}$ be a set of states, and and let X be a strategy in $\{A, B\}$. If there exists a state $s \in \mathcal{T}$ such that $\forall r \in \mathcal{T}, r \overset{X}{\leq} s$, then s is termed the *X-maximum* state in $\mathcal{T}$.

We introduce the conclusions of the binary *coordinating network game* from [25], which will be used in the following discussion. For now, we temporarily view the game $\hat{\Gamma}$ as a general binary *coordinating network game*.

Lemma 4 ([25]). *In coordinating network game $\hat{\Gamma}$, for an arbitrary state $\hat{s}$, denote by $\hat{R}(\hat{s})$ the set of reachable states from $\hat{s}$ and $\hat{E}(\hat{s})$ the set of reachable equilibrium states from $\hat{s}$.* ***(i)*** *$\forall X \in \{A, B\}$, X-maximum state in $\hat{R}(\hat{s})$ exists.* ***(ii)*** *Denote the A-maximum state in $\hat{R}(\hat{s})$ by $\hat{s}^a$ and the B-maximum state in $\hat{R}(\hat{s}^a)$ by $\hat{s}^{ab}$, $\hat{s}^{ab} \in \hat{E}(\hat{s})$;* ***(iii)*** *$\hat{s}^{ab}$ is the A-maximum state in $\hat{E}(\hat{s})$.*

For coherence, we present our proof for Lemma 4. In this proof, we propose an algorithm to derive *X-maximum* state.

Algorithm 1 X-maximizing evolution in game $\hat{\Gamma}$.

Input: the initial state $\hat{s}$, the chosen strategy X
Output: the *X-maximum* state in $\hat{R}(\hat{s})$

```
Define V_x = {j | f̂_j(ŝ) = X, ŝ_j ≠ X}
while V_x ≠ ∅ do
    Starting from ŝ, activate all agents in V_x with an arbitrary sequence and reach a state, recording it as intermediate variable ŝ'
    V_x ← {j | f̂_j(ŝ') = X, ŝ'_j ≠ X}
    ŝ ← ŝ'
end while
return ŝ.
```

Let $\vec{X}_i$ denote the action that agent i is activated and switches its strategy to X from X', where $X \in \{A, B\}$ and $X' \in \{A, B\} \setminus \{X\}$. Note, the action $\vec{X}_i$ can occur under state $\hat{s}$ in game $\hat{\Gamma}$ if and only if there hold $\hat{s}_i = X'$ and $\hat{f}_i(\hat{s}) = X$. Now we prove the lemma.

Proof of Lemma 4. **(i)** Assume $\hat{s}$ is the initial state, it undergoes X-maximizing evolution and reaches $\hat{s}^\circ$. We now prove that $\hat{s}^\circ$ is the X-maximum state in $\hat{R}(\hat{s})$. Therefore, for any given initial state, Algorithm 1 can be used to derive the X-maximum state, thereby guaranteeing its existence. Define action $\vec{X}_i$ where $i \in \{j \mid \hat{s}_j^\circ = X'\}$ as Type I action. If there exists a $\hat{s}' \in \hat{R}(\hat{s})$, such that $\hat{s}' \overset{X}{\leq} \hat{s}^\circ$ is not satisfied, then the evolution path from $\hat{s}$ to $\hat{s}'$ contains at least one Type I action. Denote the first Type I action in this path as $\vec{X}_{i_1}$ (action that agent i_1 switches from X' to X) and the state under which $\vec{X}_{i_1}$ occurred as $\hat{s}^1$, we have $\hat{s}^1 \overset{X}{\leq} \hat{s}^\circ$. Combined with the occurrence of $\vec{X}_{i_1}$ under state $\hat{s}^1$ and $\hat{s}_{i_1}^\circ = X'$, by X-coordinating property, we conclude that action $\vec{X}_{i_1}$ can take place under state $\hat{s}^\circ$, which contradicts the break condition in Algorithm 1.

(ii) If for all $i \in \mathcal{V}$, neither action $\vec{A}_i$ nor action $\vec{B}_i$ can take place under $\hat{s}^{ab}$, then $\hat{s}^{ab} \in \hat{E}(\hat{s})$. The non-occurrence of action $\vec{B}_i$ is guaranteed by Algorithm 1. Regarding action $\vec{A}_i$, it is sufficient to consider i in the set $\{j \mid \hat{s}_j^{ab} = B\}$. This can be divided into two parts:

1. $i \in \{j \mid \hat{s}_j^{ab} = B, \hat{s}_j^a = B\}$. For each i in this set, we have $\hat{f}_i(\hat{s}^a) = B$ since no $\vec{A}_i$ can occur under $\hat{s}^a$. Given that $\hat{s}^a \overset{B}{\leq} \hat{s}^{ab}$, by the A-coordinating property, we have $\hat{f}_i(\hat{s}^{ab}) = B$, which means $\vec{A}_i$ cannot occur under $\hat{s}^{ab}$.
2. $i \in \{j \mid \hat{s}_j^{ab} = B, \hat{s}_j^a = A\}$. For each i in this set, there exists an action $\vec{B}_i$ during the path from $\hat{s}^a$ to $\hat{s}^{ab}$. Denote by $\hat{s}'$ the state under which the action occurred, we have $\hat{f}_i(\hat{s}') = B$. Since from $\hat{s}^a$ to $\hat{s}^{ab}$ only action $\vec{B}$ occurred, there holds $\hat{s}' \overset{B}{\leq} \hat{s}^{ab}$, therefore $\hat{f}_i(\hat{s}^{ab}) = B$, indicating that $\vec{A}_i$ cannot occur under $\hat{s}^{ab}$.

(iii) Assume $\hat{s}^*$ is an arbitrary state in $\hat{E}(\hat{s})$. In the B-maximizing evolution from $\hat{s}^a$ to $\hat{s}^{ab}$, denote each intermediate variable $\hat{s}'$ in Algorithm 1 as $\hat{s}^{ab_1}, \hat{s}^{ab_2}, \ldots$, then the evolution path can be briefly expressed as

$$\hat{s}^a \to \hat{s}^{ab_1} \to \hat{s}^{ab_2} \to \cdots \to \hat{s}^{ab}.$$

Divide $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1 = \{j \mid \hat{s}^a_j = A, \hat{f}_j(\hat{s}^a) = B\}$ and $\mathcal{V}_2 = \mathcal{V} \setminus \mathcal{V}_1$. Given $\hat{s}^a \overset{B}{\leq} \hat{s}^*$, then for all $i \in \mathcal{V}_1, \hat{f}_i(\hat{s}^*) = B$. Since $\hat{s}^* \in \hat{E}(\hat{s})$, we have $\hat{s}^*_i = \hat{f}_i(\hat{s}^*) = B$. Thus, $\{j \mid \hat{s}^*_j = B, j \in \mathcal{V}_1\} = \mathcal{V}_1$, implying $\{j \mid \hat{s}^{ab_1}_j = B, j \in \mathcal{V}_1\} \subseteq \{j \mid \hat{s}^*_j = B, j \in \mathcal{V}_1\}$. From Algorithm 1, we know from $\hat{s}^a$ to $\hat{s}^{ab_1}$, only agents in $\mathcal{V}_1$ change strategies. Therefore, by $\hat{s}^a \overset{B}{\leq} \hat{s}^*$, we have $\{j \mid \hat{s}^{ab_1}_j = B, j \in \mathcal{V}_2\} = \{j \mid \hat{s}^a_j = B, j \in \mathcal{V}_2\} \subseteq \{j \mid \hat{s}^*_j = B, j \in \mathcal{V}_2\}$. Combined with $\{j \mid \hat{s}^{ab_1}_j = B, j \in \mathcal{V}_1\} \subseteq \{j \mid \hat{s}^*_j = B, j \in \mathcal{V}_1\}$, we have $\{j \mid \hat{s}^{ab_1}_j = B\} \subseteq \{j \mid \hat{s}^*_j = B\}$, which is equivalent to $\hat{s}^{ab_1} \overset{B}{\leq} \hat{s}^*$.

With similar discussion, by $\hat{s}^{ab_1} \overset{B}{\leq} \hat{s}^*$ we can get $\hat{s}^{ab_2} \overset{B}{\leq} \hat{s}^*$. Repeat the processes we get

$$\hat{s}^a \overset{B}{\leq} \hat{s}^* \Rightarrow \hat{s}^{ab_1} \overset{B}{\leq} \hat{s}^* \Rightarrow \hat{s}^{ab_2} \overset{B}{\leq} \hat{s}^* \Rightarrow \cdots \Rightarrow \hat{s}^{ab} \overset{B}{\leq} \hat{s}^*.$$

We have demonstrated that for all $\hat{s}^* \in \hat{E}(\hat{s}), \hat{s}^{ab} \overset{B}{\leq} \hat{s}^*$. This complete the proof.

Remark 1. The work in [22] demonstrates that in a *coordinating network game* with a strategy set containing more than two strategies, a reachable equilibrium exists from any initial state. Our proof can be extended to derive this result. Assume the strategy set is $\{X_1, X_2, \ldots, X_m\}$, let the arbitrary initial state s undergoes X_1-maximizing, X_2-maximizing, …, X_m-maximizing evolution, eventually reaching state s°. Following a similar reasoning used in our proof for Lemma 4, it can be concluded that s° is an equilibrium state. This approach to equilibrium has a computational complexity of $O(n^2)$, while the method in [22] has a computational complexity of $O(n^3)$.

With the help of Lemma 2 and Lemma 4 (ii), we can prove Theorem 2.

Proof of Theorem 2. By Lemma 4 (ii), in game $\hat{\Gamma}$, starting from each state, we can find a path that leads to a convergence state. Assume the length of the longest one is m. By Assumption 2, starting from an arbitrary state, the probability that the game converges after m steps is greater than κ^m. Therefore, starting from any state, the probability that the game converges in t steps can be evaluated as follows:

$$P_t \geq 1 - (1 - \kappa^m)^{\left\lfloor \frac{t}{m} \right\rfloor},$$

where $\left\lfloor \frac{t}{m} \right\rfloor$ denotes the largest integer less than or equal to $\frac{t}{m}$. As $t \to \infty$, $P_t \to 1$. Thus, starting from any state, game $\hat{\Gamma}$ converges with probability 1. Combined with Lemma 2, it follows that starting from any state, game Γ converges with probability 1.

Remark 2. Due to the highly nonlinearity induced by cascade effects, EHBNGs generally do not converge. Theorem 2 offers a potential control strategy to guide an EHBNG to equilibrium. By applying the evolutionary game approach from [33], we can identify a partition of the agent set that causes minimum structural conflicts. Incentives can then be introduced to payoff matrix to eliminate these structural conflicts. Theorem 2 ensures that, with this intervention, the game will converge. While this solution is not necessarily optimal, it is broadly feasible, cost-effective, and particularly suitable for games in relatively harmonious populations. Furthermore, it is robust: the structural balance condition ensures that every state reaches equilibrium. Thus, even in the presence of substantial disturbances, the game will still converge after a period of evolution once incentives are applied.

3.2.2. $\mathbb{G}(\mathcal{V}, E)$-class game

Structural balance is a sufficient but not necessary condition for convergence. A counterexample can be found in an EHBNG with three agents that satisfies for all $i \neq j, w_{ij} = -1$. This game converges due to the satisfaction of the generalized symmetry condition, but its coordination value network is not structurally balanced.

However, in scenarios where only the coordination properties between agents can be determined, structural balance becomes more significant. To elucidate this, we provide the following definition.

Definition 5. Let $\mathbb{G}(\mathcal{V}, E)$ represent a directed, unweighted signed network with an agent set $\mathcal{V}$ and an adjacency matrix $E \in \{-1, 1\}^{n\times n}$. An EHBNG is categorized as a $\mathbb{G}(\mathcal{V}, E)$-class game if:

(i) It retains the same agent set $\mathcal{V}$,
(ii) Its coordination value matrix W complies with $\mathrm{sgn}(w_{ij}) = e_{ij}$ for all $i, j \in \mathcal{V}$, where

$$\mathrm{sgn}(x) = \begin{cases} +1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

The matrix E represents a blurred version of W that preserves the information of coordination types (coordinating or anticoordinating) while discarding the exact coordination values. A $\mathbb{G}(\mathcal{V}, E)$-class game refers to a class of games that share the same underlying coordination structure among agents, as depicted by $\mathbb{G}(\mathcal{V}, E)$. To analyze its convergence properties, we propose the following proposition.

Proposition 3. *All $\mathbb{G}(\mathcal{V}, E)$-class games will converge under Assumption 2 if and only if $\mathbb{G}(\mathcal{V}, E)$ is structurally balanced.*

Proof. The sufficiency has been established by Theorem 2. We now demonstrate the necessity by showing that if $\mathbb{G}(\mathcal{V}, E)$ is not structurally balanced, there exists at least one $\mathbb{G}(\mathcal{V}, E)$-class game that may not converge under Assumption 2. Given that $\mathbb{G}(\mathcal{V}, E)$ is structurally unbalanced, there exists a cycle within the network whose product of edge signs is negative. Specifically, there is a sequence of distinct agents $i_0, i_1, \dots, i_k$ such that $e_{i_0 i_1} e_{i_1 i_2} \dots e_{i_{k-1} i_k} e_{i_k i_0} = -1$. For edge (i, j) not in this cycle, set $\pi_{ij} = \begin{pmatrix} e_{ij}/2n & 0 \\ 0 & e_{ij}/2n \end{pmatrix}$; for edge (i, j) in this cycle, set $\pi_{ij} = \begin{pmatrix} e_{ij} & 0 \\ 0 & e_{ij} \end{pmatrix}$. Then, the strategies of $i_0, i_1, \dots, i_k$ are completely determined by their neighbors in this cycle. Consider an arbitrary state s; without loss of generality, assume $s_{i_0} = A$, and activate with sequence $(i_k, i_{k-1}, \dots, i_0)$. In this evolution process, if $e_{i_k i_0} = +1$, i_k will choose A; if $e_{i_k i_0} = -1$, i_k will choose B. And the same occurs in other activation times. Since $e_{i_0 i_1} e_{i_1 i_2} \dots e_{i_{k-1} i_k} e_{i_k i_0} = -1$, the transmitted strategy changes odd times. Thus, when i_0 is activated, it will switch to B. Then we can get that any state s in this game can not be an equilibrium state, thereby proving that the game does not converge. □

Proposition 3 establishes that the structural balance condition is both sufficient and necessary to guarantee the convergence of every member within a $\mathbb{G}(\mathcal{V}, E)$-class game. When we view the system from a macroscopic perspective, where only the coordination types between agents are observable, but not the precise coordination values, we can only distinguish between different $\mathbb{G}(\mathcal{V}, E)$-class games. In this case, the structural balance condition becomes an indispensable criterion for ensuring convergence.

3.2.3. Final state prediction

Under random activation, if the reachable equilibrium states are not unique, it becomes impossible to know which state the population will finally reach at the beginning of the game. However, in many situations, we can still derive some information from the initial state. To illustrate, consider the following example.

Example 4. Consider an EHBNG with 4 agents $\{1, 2, 3, 4\}$. The payoff matrices are defined as follows: $\pi_{12} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, $\pi_{21} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\pi_{23} = \pi_{32} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\pi_{34} = \pi_{43} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, while all other π_{ij} are zero matrices. Assuming the game starts from the state (B, B, B, A), then the reachable equilibrium states are (A, A, B, B) and (A, A, A, A).

Although it is impossible to determine the final state at the beginning of the game, we can infer that in the final state, agents 1 and 2 will always choose strategy A, while agents 3 and 4 may choose either strategy A or strategy B. Therefore, even without complete foresight, we can make partial predictions by identifying agents whose strategies will consistently remain the same across all reachable equilibrium states. However, in games with a large population, this task can be become significantly complex.

Based on Lemma 2, Algorithm 1, and Lemma 4 (iii), we propose a method to achieve this for games that fulfill the structural balance condition. Denote by s an arbitrary initial state in game Γ. We first analyze the problem in the dual game $\hat{\Gamma}$ with initial state $\hat{s} = \varphi(s)$. Let $\hat{s}$ undergo A-maximizing evolution followed by B-maximizing evolution, resulting in the state $\hat{s}^{ab}$. Conversely, let $\hat{s}$ undergo B-maximizing evolution followed by A-maximizing evolution, resulting in the state $\hat{s}^{ba}$. From the proof of Lemma 4, we know that $\hat{s}^{ab}$ and $\hat{s}^{ba}$ are the *A-maximum* state and the *B-maximum* state in $\hat{E}(\hat{s})$, respectively. By the definition of *X-maximum* state, we have the following equation:

$$\{j \mid \forall \hat{s}^* \in \hat{E}(\hat{s}), \hat{s}^*_j = A\} = \{j \mid \hat{s}^{ba}_j = A\}, \tag{13}$$

$$\{j \mid \forall \hat{s}^* \in \hat{E}(\hat{s}), \hat{s}^*_j = B\} = \{j \mid \hat{s}^{ab}_j = B\}. \tag{14}$$

Eqs. (13) and (14) suggest that starting from $\hat{s}$ in game $\hat{\Gamma}$, we can make sure that agents in $\{j \mid \hat{s}^{ba}_j = A\}$ will ultimately choose A, and those in $\{j \mid \hat{s}^{ab}_j = B\}$ will choose B. These sets encompass every agent whose final strategy can be determined at the beginning of the game since for agent $i \notin \{j \mid \hat{s}^{ba}_j = A\} \cup \{j \mid \hat{s}^{ab}_j = B\}$, there exits $\hat{s}^{*1}, \hat{s}^{*2} \in \hat{E}(\hat{s})$ such that $\hat{s}^{*1}_i = A, \hat{s}^{*2}_i = B$. Thus, both the possibility of choosing A and the possibility of choosing B in the final state remain positive.

Returning to the original game Γ, let $E(s)$ be the set of reachable equilibrium states starting from $s = \varphi^{-1}(\hat{s})$ in game Γ. We have the following proposition.

Proposition 4. *Define $\mathcal{X}_a = \{j \mid j \in \mathcal{X}, \hat{s}^{ba}_j = A\}$, $\mathcal{Y}_a = \{j \mid j \in \mathcal{Y}, \hat{s}^{ba}_j = A\}$, $\mathcal{X}_b = \{j \mid j \in \mathcal{X}, \hat{s}^{ab}_j = B\}$, and $\mathcal{Y}_b = \{j \mid j \in \mathcal{Y}, \hat{s}^{ab}_j = B\}$. Then we have*

$$\{j \mid \forall s^* \in E(s), s^*_j = A\} = \mathcal{X}_a \cup \mathcal{Y}_b, \tag{15}$$

$$\{j \mid \forall s^* \in E(s), s^*_j = B\} = \mathcal{X}_b \cup \mathcal{Y}_a. \tag{16}$$

Proof. We only prove (15), and (16) can be derived in a similar manner. $\{j \mid \forall s^* \in E(s), s^*_j = A\} = \bigcap_{s^* \in E(s)} \{j \mid s^*_j = A\}$. From Lemma 2, we have $E(s) = \varphi^{-1}(\hat{E}(\hat{s})) = \{\varphi^{-1}(\hat{s}^*) \mid \hat{s}^* \in \hat{E}(\hat{s})\}$. Thus,

$$\begin{aligned} \bigcap_{s^* \in E(s)} \{j \mid s^*_j = A\} &= \bigcap_{\varphi^{-1}(\hat{s}^*) \in \varphi^{-1}(\hat{E}(\hat{s}))} \{j \mid \varphi^{-1}(\hat{s}^*)_j = A\} \\ &= \bigcap_{\hat{s}^* \in \hat{E}(\hat{s})} \{j \mid \varphi^{-1}(\hat{s}^*)_j = A\} \end{aligned}$$

$$= \bigcap_{\hat{s}^*\in\hat{E}(\hat{s})} \{j \mid \hat{s}^*_j = A, j \in \mathcal{X}\} \cup \{j \mid \hat{s}^*_j = B, j \in \mathcal{Y}\}.$$

For arbitrary $\hat{s}^{*1}, \hat{s}^{*2} \in \hat{E}(\hat{s})$, $\{j \mid \hat{s}^{*1}_j = A, j \in \mathcal{X}\} \cap \{j \mid \hat{s}^{*2}_j = B, j \in \mathcal{Y}\} = \emptyset$, therefore we have

$$\begin{aligned}
&\bigcap_{\hat{s}^*\in\hat{E}(\hat{s})} \{j \mid \hat{s}^*_j = A, j \in \mathcal{X}\} \cup \{j \mid \hat{s}^*_j = B, j \in \mathcal{Y}\} \\
&= \left(\bigcap_{\hat{s}^*\in\hat{E}(\hat{s})} \{j \mid \hat{s}^*_j = A, j \in \mathcal{X}\}\right) \cup \left(\bigcap_{\hat{s}^*\in\hat{E}(\hat{s})} \{j \mid \hat{s}^*_j = B, j \in \mathcal{Y}\}\right) \\
&= \{j \mid \forall \hat{s}^* \in \hat{E}(\hat{s}), \hat{s}^*_j = A; j \in \mathcal{X}\} \cup \{j \mid \forall \hat{s}^* \in \hat{E}(\hat{s}), \hat{s}^*_j = B; j \in \mathcal{Y}\} \\
&= \mathcal{X}_a \cup \mathcal{Y}_b.
\end{aligned}$$

□

In game Γ, denote by s the initial state. At the beginning of the game, we can assert that in the final state, agents in $\mathcal{X}_a \cup \mathcal{Y}_b$ will choose A and those in $\mathcal{X}_b \cup \mathcal{Y}_a$ will choose B. For any other agents, both A and B remain possible. The computational cost of this prediction is $O(n^2)$, which is manageable for typical applications.

4. Equilibrium condition for higher-order network games

In EHBNGs, the utility of each agent can be reduced to the sum of payoffs from pairwise games with its neighbors. However, in many real-world scenarios, an agent's payoff may be collectively influenced by multiple neighbors and cannot be decomposed into simple pairwise interactions, as exemplified by public goods games [34]. Such scenarios necessitate modeling network games using higher-order networks [35].

In this section, we extend the analysis of coordination to higher-order network games, thereby deriving the condition for their equilibrium. We focus on 2-order network games, and conclusions for network games whose order is higher than 2 can be derived from similar discussions.

An edge-heterogeneous binary 2-order network game Γ^2 can be describe by a triple tuple $(\mathbb{G}, \mathcal{S}, \Pi)$. Here, $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ represents a 2-order network, where $\mathcal{V}$ is the set of agents, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{T}_{\mathcal{V}}$ is the set of 2-order hyperedges with $\mathcal{T}_{\mathcal{V}} = \{\{i,j\} \mid i,j \in \mathcal{V}, i \neq j\}$. An element $(i, \{j,k\}) \in \mathcal{E}$ indicates the existence of a hyperedge that from agent i to agents j and k, though which agent i plays a game with agents j and k, receiving a corresponding payoff. The strategy set is $\mathcal{S} = \{A, B\}$, and Π denotes the set of payoff tensors. For each $(i, \{j,k\}) \in \mathcal{E}$, there exists a $2\times2\times2$ tensor π_{ijk}. Agent i receives a payoff from the game on $(i, \{j,k\})$ according to the tensor

$$\pi_{ijk} = \left(\begin{bmatrix} r^{ijk}_{111} & r^{ijk}_{112} \\ r^{ijk}_{121} & r^{ijk}_{122} \end{bmatrix} \quad \begin{bmatrix} r^{ijk}_{211} & r^{ijk}_{212} \\ r^{ijk}_{221} & r^{ijk}_{222} \end{bmatrix} \right). \tag{17}$$

Here, the subscripts 1 and 2 correspond to strategies A and B, respectively. Each entry of the tensor represents the payoff to agent i when the strategies of agents i, j, and k align with the subscripts. For example, r^{ijk}_{212} represents the payoff to agent i when the strategy profile is $s_i = B$, $s_j = A$, and $s_k = B$. Denote by θ an indicator that $\theta(A) = 1, \theta(B) = 2$, the total utility for agent i is then given by:

$$u_i = \sum_{(i,\{j,k\})\in\mathcal{E}} (\pi_{ijk})_{\theta(s_i)\theta(s_j)\theta(s_k)}. \tag{18}$$

Now we consider the equilibrium condition for the 2-order network game Γ^2. Both [22] and [25] have demonstrated that *coordinating network games* converge under Assumption 2 (as also discussed in Section 3 of this paper). Therefore, to ensure convergence, we can find a condition to make Γ^2 a *coordinating network game*. From the analysis in Section 2.3, we can conclude the following lemma.

Lemma 5. *In game Γ^2, denote $s = (s_i, s_j, s_{-i,-j}) \in \mathcal{S}^{\mathcal{V}}$. The coordination value for agent i with respect to agent j is expressed as*

$$\begin{aligned}
v(s_{-i,-j}) = {} & u_i(A, A, s_{-i,-j}) + u_i(B, B, s_{-i,-j}) \\
& - u_i(A, B, s_{-i,-j}) - u_i(B, A, s_{-i,-j}).
\end{aligned} \tag{19}$$

If for all $i, j \in \mathcal{V}$ and all $s_{-i,-j} \in \mathcal{S}^{\mathcal{V}\setminus\{i,j\}}$, $v(s_{-i,-j}) \geq 0$, then the game is a coordinating network game.

We first consider a game on a hyperedge $(i, \{j,k\})$. Denote by (s_i, s_j, s_k) the strategies of the three agents and $p_{ijk}(s_i, s_j, s_k)$ the payoff for agent i under state (s_i, s_j, s_k) in this game. The coordination value for agent i with respect to agent j in this game can be expressed as

$$\begin{aligned}
v(s_k) &= p_{ijk}(A, A, s_k) + p_{ijk}(B, B, s_k) \\
&\quad - p_{ijk}(B, A, s_k) - p_{ijk}(A, B, s_k) \\
&= (r^{ijk}_{111} - r^{ijk}_{211} - r^{ijk}_{121} + r^{ijk}_{221})\, \delta^A_k \\
&\quad + (r^{ijk}_{112} - r^{ijk}_{212} - r^{ijk}_{122} + r^{ijk}_{222})\, \delta^B_k
\end{aligned}$$

$$= w^A_{ij,k}\delta^A_k + w^B_{ij,k}\delta^B_k, \tag{20}$$

where $w^A_{ij,k} = r^{ijk}_{111} - r^{ijk}_{211} - r^{ijk}_{121} + r^{ijk}_{221}$, and $w^B_{ij,k} = r^{ijk}_{112} - r^{ijk}_{212} - r^{ijk}_{122} + r^{ijk}_{222}$. Define $w_{ij,k} = \min\{w^A_{ij,k}, w^B_{ij,k}\}$, $w^p_{ij,k} = |w^A_{ij,k} - w^B_{ij,k}|$ and $X_{ij,k} = \arg\max_X w^X_{ij,k}$. Then (20) can be expressed as

$$v(s_k) = w_{ij,k} + w^p_{ij,k}\delta_k^{X_{ij,k}}. \tag{21}$$

In (21), the coordination value of agent i with respect to agent j on a hyperedge $(i, \{j, k\})$ is decomposed into two components. The term $w^p_{ij,k}\delta_k^{X_{ij,k}}$ indicates a positive synergistic effect exerted by k on the pair (i, j), where the choice of a specific strategy by k enhances i's coordination value with respect to j. The term $w_{ij,k}$ is the basic coordination value of i with respect to j on the hyperedge $(i, \{j, k\})$, which reflects the coordination value when k's synergistic effect is absent.

Returning to the overall network game, the coordination value between agents i and j in Γ^2 is calculated as follows:

$$\begin{aligned} v(s_{-i,-j}) &= u_i(A, A, s_{-i,-j}) + u_i(B, B, s_{-i,-j}) \\ &\quad - u_i(A, B, s_{-i,-j}) - u_i(B, A, s_{-i,-j}) \\ &= \sum_{(i,\{j,k\})\in\mathcal{E}} p_{ijk}(A, A, s_k) + p_{ijk}(B, B, s_k) \\ &\quad - \sum_{(i,\{j,k\})\in\mathcal{E}} p_{ijk}(A, B, s_k) + p_{ijk}(B, A, s_k) \\ &= \sum_{(i,\{j,k\})\in\mathcal{E}} w_{ij,k} + w^p_{ij,k}\delta_k^{X_{ij,k}}. \end{aligned} \tag{22}$$

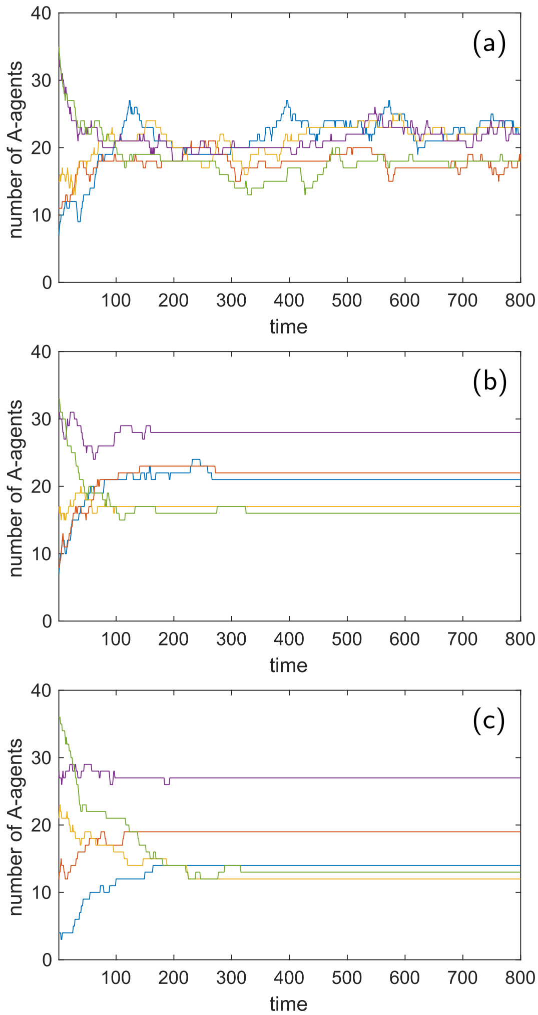

Fig. 1. Simulations for EHBNGs under: (a) no restriction, (b) the generalized symmetry condition, and (c) the structural balance condition.

Define $w_{ij}^2 = \sum_{(i,\{j,k\})\in\mathcal{E}} w_{ij,k}$. The term w_{ij}^2 represents the sum of the basic coordination values from all hyperedges initiated by agent i that include the pair (i,j), while the remaining terms correspond to the collection of positive synergistic effects. When all positive synergistic effects are inactive, the coordination value $v(s_{-i,-j})$ equals w_{ij}^2; therefore, w_{ij}^2 is the minimum value of $v(s_{-i,-j})$. It follows that $w_{ij}^2 \geq 0$ if and only if $v(s_{-i,-j}) \geq 0$ for all states s. Combined with Lemma 5, we can get the following conclusion.

Proposition 5. *In game Γ^2, if for any $i,j\in\mathcal{V}$, there exists $w_{ij}^2 \geq 0$, then game Γ^2 is a coordinating network game, thus it will converge under Assumption 2.*

An interesting case of the condition in Proposition 5 is that for all $\pi_{ijk}\in\Pi$, the following conditions hold:

$$
\begin{aligned}
r_{111}^{ijk} &\geq \overbrace{r_{121}^{ijk}, r_{112}^{ijk}} \geq r_{122}^{ijk},\\
r_{222}^{ijk} &\geq \overbrace{r_{221}^{ijk}, r_{212}^{ijk}} \geq r_{211}^{ijk},
\end{aligned}
$$

where $\overbrace{x_1, x_2}$ implies that both x_1 and x_2 satisfy the condition. This case is referred to as *opponent coordinating* [36], suggesting that an agent earns more payoff if its neighbors adopt the same strategy as it does. By slightly extending the proof in [36], it can be demonstrated that if Γ^2 is opponent coordinating, it will also be a *coordinating network game* under asynchronous imitation dynamics, meaning that if Γ^2 is *opponent coordinating*, it will converge under both asynchronous best-response dynamics and asynchronous imitation dynamics.

5. Simulations

In this section, we present numerical simulations to validate our theoretical results. We consider network games involving 40 agents.

For EHBNGs, we conduct three groups of simulations:

- The first group involves EHBNGs with no restrictions, where each payoff matrix is generated randomly.
- The second group focuses on EHBNGs that satisfy the generalized symmetry condition. We start by randomly generating a non-negative diagonal matrix Q and a symmetric matrix E. We then construct the coordination value matrix $W = QE$ and set up the corresponding payoff matrix based on W.
- The third group examines EHBNGs that satisfy the structural balance condition. We randomly generate a structurally balanced graph $\mathbb{G}(W)$, and construct the corresponding payoff matrix based on W.

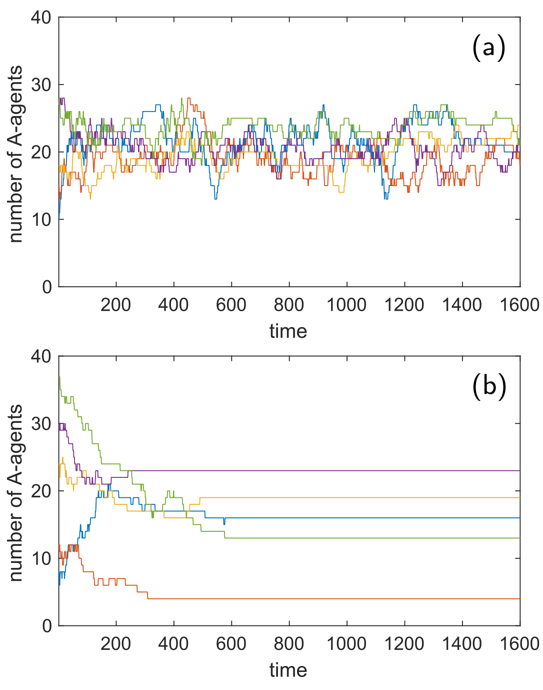

Fig. 2. Simulations for edge-heterogeneous binary 2-order network games under: (a) no restriction, (b) condition in Proposition 5.

For each group, we perform five independent simulations, with each simulation generating a new payoff matrix set, initial state, and activation sequence. The simulation results for EHBNGs are presented in Fig. 1.

For edge-heterogeneous binary 2-order network games, we establish that each agent participates in approximately 10 games on 2-order hyperedges. We conduct two groups of simulations:

- The first group involves games with no restrictions, where the payoff tensors are generated randomly.
- In the second group, we begin by randomly generating a payoff tensor set and then calculate w_{ij}^2 for each pair of agents. If $w_{ij}^2 < 0$, we add a new 2-order hyperedge containing agents i and j to adjust w_{ij}^2 to meet the condition stated in Proposition 5.

Similarly, five independent simulations are performed for each group. The results are presented in Fig. 2.

The simulations demonstrate that unrestricted EHBNGs and edge-heterogeneous binary 2-order network games generally fail to converge. In contrast, when these games meet the conditions proposed in this paper, they reach equilibrium states after a period of evolution.

6. Conclusion

In this paper, we have examined a general form of binary network games, specifically edge-heterogeneous binary network games. We introduce two sufficient conditions for convergence under asynchronous best-response dynamics. The first condition, the generalized symmetry condition, is elucidated through comparisons with agent-heterogeneous coordinating games, agent-heterogeneous anticoordinating games, and symmetric games. We prove it using the potential game method. The second condition, the structural balance condition, posits that equilibrium is achievable if the coordination value network of a game is structurally balanced. We develop a dual-game approach to establish this condition and propose a method to predict the final equilibrium based on initial state information for games satisfying this condition. Furthermore, we study the edge-heterogeneous 2-order network games, providing a condition that ensures the game behaves as a *coordinating network game*, thus guaranteeing convergence.

Future work could explore the following two aspects: (i) For EHBNGs that meet either the generalized symmetry condition or the structural balance condition, further analysis could focus on their convergence time based on the property of coordination value matrix. (ii) For edge-heterogeneous higher-order network games, realistic constraints could be incorporated to derive less stringent equilibrium conditions.

Data availability

No data was used for the research described in the article.

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