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Coevolutionary dynamics of multidimensional opinions over coopetitive influence networks[☆]

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ABSTRACT

To better understand opinion dynamics on social networks, especially when antagonistic interactions exist, we propose a novel coevolution model of multidimensional opinions and coopetitive (cooperative–competitive) influence networks. In this model, agents update their opinions according to the designed multidimensional Altafini-type rule. Additionally, the asynchronous evolutionary dynamics of influence networks is formulated based on three well-established sociological mechanisms: symmetry, influence, and person-opinion homophily. Going beyond the limitations of existing models in explaining network structural evolution, we characterize the sign equilibria of the influence dynamics as equivalent to all possible fully connected structurally balanced configurations, and prove that the influence networks will almost surely converge to sign equilibria within finite time. Further, we claim convergence of the opinion dynamics model and systematically analyze the role of the set of logic matrices in determining the limiting opinion distribution. For irreducible logic matrices, agents' opinions on each topic exhibit a bipartite consensus under conditions of structurally balanced logic matrices and no competing logical interdependencies. For reducible logic matrices, the scope of opinions on a topic within an open strongly connected component (SCC) is determined by that of the closed SCCs it connects to. Finally, we provide simulation examples to illustrate the theoretical results.

1. Introduction

Opinion dynamics, an interdisciplinary research hotspot spanning cybernetics, physics, and sociology, has been extensively explored over the past few decades (Bernardo, Altafini, Proskurnikov, & Vasca, 2024; Noorazar, 2020; Proskurnikov & Tempo, 2017). Local interactions among individuals or agents form an influence network, giving rise to complex collective behaviors that are crucial in areas such as group decision-making, marketing strategies, and network construction. Recently, the study of opinion dynamics has received particular interest from the control community, in part for its analogies and parallels with multi-agent systems.

Based on the well-established framework in systems and control, many agent-based models have been developed to describe the generation, propagation, and aggregation of opinions on social networks, where consensus issues have emerged as a leading focus (Jia, MirTabatabaei, Friedkin, & Bullo, 2015; Kang & Li, 2022; Wang, Bernardo, Hong, Vasca, Shi, & Altafini, 2022).

Thus far, numerous distributed protocols for achieving multi-agent consensus have been proposed, with the success of consensus heavily relying on network connectivity (Cao, Morse, & Anderson, 2008; Mao, Wu, Fan, Lü, & Chen, 2025; Wu, Wu, Wang, Mao, Lu, Lü, Zhang, & Lü, 2024; Xia, Cao, & Johansson, 2015). Tracing back to pioneering research, the DeGroot model (DeGroot, 1974) posits that each agent updates its opinions in discrete time by taking convex combinations of its own and its neighbors' opinions, ensuring consensus under mild connectivity conditions. Subsequently, Altafini introduced modifications to the DeGroot model, enabling the inclusion of negative weights and resulting in a more general linear consensus framework (Altafini, 2012). The structure formed by agents interacting through positive and negative edges, indicating friendly and hostile relationships, is commonly modeled as signed networks, also referred to as coopetitive (cooperative–competitive) networks. In such situations, bipartite consensus represents a special and highly regarded convergence state, whereas structural balance proves to be an indispensable property for achieving it (Xia et al., 2015).

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Derived from several seminal works by Heider (Heider, 1944, 1946), structural balance theory characterizes the stable configuration of social relations (Facchetti, Iacono, & Altafini, 2011). By imposing the structural balance assumption on static signed networks, the bipartite consensus has been widely studied under various forms of network connectivity, including strong connectivity and the presence of spanning trees (Lin, Jiao, & Wang, 2018; Xia et al., 2015). In reality, friendly or hostile relationships between agents may change due to differences in agents' states, such as opinions, and any perceived structural imbalance leads to cognitive dissonance, which agents strive to resolve (Cartwright & Harary, 1956). Currently, dynamic structural balance theory has been established to explain the tendency of unbalanced signed networks to evolve towards balanced ones (Jia, Friedkin, & Bullo, 2016; Mei, Cisneros-Velarde, Chen, Friedkin, & Bullo, 2019), as supported by empirical studies on real-world signed networks (Szell, Lambiotte, & Thurner, 2010). Nonetheless, existing models frequently fail to converge (Marvel, Kleinberg, Kleinberg, & Strogatz, 2011; Traag, Dooren, & Leenheer, 2013) or get stuck in a structurally unbalanced state (Antal, Krapivsky, & Redner, 2005; Marvel et al., 2011). Therefore, the first objective of this article is to develop a mathematically rigorous dynamical model for structural evolution that is both tractable and exhibits desirable convergence properties.

Although early models of opinion dynamics typically focus on one specific topic or multiple independent topics, real-world scenarios often involve interdependent topics where the dynamics of topic-specific opinions are entangled (Friedkin, Proskurnikov, Tempo, & Parsegov, 2016). In this context, consensus studies on matrix-weighted networks can offer extensive insights for opinion dynamics analysis (Pan, Shao, Mesbahi, Xi, & Li, 2018; Trinh, Nguyen, Lim, & Ahn, 2018). In Ahn, Tran, Trinh, Ye, Liu, and Moore (2020), a matrix-valued weighted network was employed to characterize the relationships among multiple cross-coupling topics in opinion dynamics. Towards the Friedkin–Johnson model (Friedkin & Johnsen, 1990), a multidimensional extension was made to describe the evolution of agents' opinions on multiple topics in the discrete-time domain, with the logical interdependencies between topics captured by a homogeneous logic matrix (Parsegov, Proskurnikov, Tempo, & Friedkin, 2016). The continuous-time version of Parsegov et al. (2016) was introduced and analyzed in Ye, Trinh, Lim, Anderson, and Ahn (2020). More recently, the modulus consensus for the time-varying Altafini model containing interrelated topics has been investigated, but limited to irreducible logic matrices (Yang, Cao, Yuan, & Wang, 2023). Despite these advances, existing consensus protocols still fall short in capturing more comprehensive features of interdimensional communication among agents, such as the heterogeneity of belief systems[1] and the cascade structure (indicating reducibility) of logic matrices (Jia, Friedkin, & Bullo, 2017; Ye, Liu, Wang, & Anderson, 2019). Accordingly, the second objective of this article is to consider more general matrices as the coupling topology between topics.

Beyond static influence structures, the interplay of opinions and interpersonal appraisals has recently drawn significant interest, inspiring various mathematical models. For instance, Liu, Cui, Chen, Mei, and Gao (2024), Liu, Cui, Mei, Dörfler, and Buss (2020) investigated homophily-driven appraisal dynamics coupled with influence-based opinion updates, establishing the equivalence among social balance, modulus consensus, convergence, and persistence of non-vanishing appraisals. In Mei, Chen, Friedkin, and Dörfler (2022), the SIH and SIOH dynamics were introduced to model local interactions, proving almost-sure convergence to structural balance in non-complete graphs. Furthermore, Disarò and Valcher (2024) extended the Friedkin–Johnsen model by incorporating homophily-based influence matrices, showing that opinion convergence is achieved when the appraisal matrix stabilizes in finite time. While these works provide valuable insights, they primarily center on single-topic settings or restricted network structures, often assuming synchronized appraisal updates. Building upon these insights, our work advances the study of coevolutionary dynamics by incorporating multiple logically interdependent topics, permitting asynchronous appraisals on coopetitive networks, while ensuring almost-sure convergence even in non-complete graphs. This extends the applicability of coevolutionary dynamics to more realistic social scenarios, thus addressing the aforementioned two challenges and making four main contributions as follows.

Firstly, we introduce a multidimensional extension to the Altafini-type update rule for opinion dynamics, surpassing the classic work in Altafini (2012) by integrating the impact of heterogeneous logic matrices on the limiting opinion distribution. Within this interaction framework, agents undergo concurrent influence from both the heterogeneous belief system and the coopetitive influence network.

Secondly, we formulate an asynchronous update model to describe the evolution of influence networks, in which agents asynchronously adjust their appraisals of others using three well-established sociological mechanisms: the symmetry mechanism (Emerson, 1976), the influence mechanism (Friedkin & Johnsen, 2011), and the person-opinion homophily mechanism (Cartwright & Harary, 1956). Implications of these mechanisms are deferred until their formal presentation in the model formulation section.

Thirdly, regarding the influence dynamics, we characterize its sign equilibria as equivalent to all possible fully connected structurally balanced configurations of the influence network. Additionally, we establish the almost-sure convergence of influence networks to a sign equilibrium within finite time, thereby facilitating the study of asymptotic properties in the coevolutionary dynamics.

Finally, we provide a set of results that systematically analyze the limiting opinion distribution of the coevolutionary dynamics in the case of irreducible and reducible heterogeneous logic matrices. We place significant emphasis on analyzing the role of these logic matrices, especially their structure, in determining whether opinions on a given topic will reach a bipartite consensus or not.

The remainder of this article is arranged as follows. Some notations and basic concepts on graph theory are introduced in Section 2. The coevolution model of multidimensional opinions and coopetitive influence networks is formulated in Section 3. Theoretical results regarding the model asymptotic properties are presented in Section 4, with simulation examples provided in Section 5 to support the derived theories. Finally, concluding remarks are drawn in Section 6.

2. Preliminaries

2.1. Notation

The set of real numbers is denoted by $\mathbb{R}$. The sets of n-dimensional real vectors and $n\times n$-dimensional real matrices are denoted by $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively. The vectors of all ones and all zeros are denoted by $\mathbf{1}_n$ and $\mathbf{0}_n \in \mathbb{R}^n$, respectively. In addition, $\boldsymbol{I}_n$ represents the identity matrix, and $\boldsymbol{O}$ represents the zero matrix of compatible dimensions. If $\boldsymbol{x}_n = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, then $\mathrm{diag}(\boldsymbol{x})$ represents the diagonal matrix with diagonal entries $x_1, \dots, x_n$. For a matrix $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, $|\boldsymbol{A}|$ denotes

[1] The set of topics, the functional interdependencies between the topics, and the mechanisms by which agents handle such relationships constitute what Converse termed the "belief system" in Converse (2006).

the matrix $|\boldsymbol{A}| = [|a_{ij}|]$, and $\boldsymbol{A} = \text{blkdiag}(\cdot)$ is used for the block matrix $\boldsymbol{A}$ where all the blocks in $(\cdot)$ lie on the diagonal of $\boldsymbol{A}$. For a nonnegative matrix $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, it is considered row-stochastic if $\boldsymbol{A}\mathbf{1}_n = \mathbf{1}_n$. Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same dimension are denoted by $\boldsymbol{A} \sim \boldsymbol{B}$ if and only if they have the same pattern of zero and nonzero entries. The sign function $\text{sgn}(\cdot) : \mathbb{R}^{n\times n} \to \{-1, 0, 1\}^{n\times n}$ maps a real matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ into a matrix taking values in $\{-1, 0, 1\}$. Specifically, $[\text{sgn}(\boldsymbol{A})]_{ij} = \text{sgn}(a_{ij})$ for all $i, j \in \{1, \ldots, n\}$, where $\text{sgn}(x)$ is the standard sign function defined as follows: $\text{sgn}(x) = 1\,(\text{sgn}(x) = -1)$ if $x > 0\,(x < 0)$, and $\text{sgn}(x) = 0$ if $x = 0$. The Kronecker product is denoted by $\otimes$.

2.2. Graph theory

A weighted signed digraph is described by $\mathcal{G}[\boldsymbol{A}] = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ represents the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of edges. A directed edge from v_i to v_j is denoted as $e_{ij} = (v_i, v_j)$, and self-loops are allowed, i.e., e_{ii} may exist in $\mathcal{E}$. Besides, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ represents the edge weight matrix, where $a_{ij} \neq 0$ if and only if $e_{ji} \in \mathcal{E}$. In particular, $a_{ij} > 0$ $\left(a_{ij} < 0\right)$ if the edge e_{ji} has a positive (negative) weight. A directed path is a finite sequence of directed edges with the form $(v_{h_1}, v_{h_2}), (v_{h_2}, v_{h_3}), \ldots, (v_{h_{r-1}}, v_{h_r})$, where $v_{h_1}, \ldots, v_{h_r} \in \mathcal{V}$ are distinct nodes and $e_{h_{s-1}h_s} \in \mathcal{E}$ for $s = 2, \ldots, r$. Node v_i is reachable from node v_j if there exists at least one directed path from v_j to v_i. If every node in $\mathcal{G}[\boldsymbol{A}]$ can be reached from any other node by traversing a directed path, then $\mathcal{G}[\boldsymbol{A}]$ is said to be strongly connected such that $\boldsymbol{A}$ is irreducible; a matrix is reducible if it is not irreducible.

A digraph $\mathcal{H}[\boldsymbol{A}'] = (\mathcal{V}', \mathcal{E}', \boldsymbol{A}')$ is called a subgraph of $\mathcal{G}[\boldsymbol{A}] = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$. A subgraph $\mathcal{H}[\boldsymbol{A}']$ is a strongly connected component (SCC) of a digraph $\mathcal{G}[\boldsymbol{A}]$ if $\mathcal{H}[\boldsymbol{A}']$ is strongly connected, and no other subgraph of $\mathcal{G}[\boldsymbol{A}]$ that strictly contains $\mathcal{H}[\boldsymbol{A}']$ is strongly connected. An SCC without incoming edges is said to be closed, while one with incoming edges is said to be open. A directed cycle is a directed path that has the same endpoints. The length of a directed cycle is the number of directed edges in it. If the length of each directed cycle in $\mathcal{G}[\boldsymbol{A}]$ cannot be divided by an integer $k > 1$, then the directed graph is considered aperiodic. Actually, any graph with a self-loop is aperiodic. A directed cycle is considered positive if it contains an even number of negative edges, and negative otherwise.

Definition 1 (*Altafini, 2012*)**.** A signed digraph $\mathcal{G}[\boldsymbol{A}]$ is termed structurally balanced if it admits a bipartition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of the nodes, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $a_{ij} \geq 0$ for any $v_i, v_j \in \mathcal{V}_{s_1} (s_1 \in \{1, 2\})$ and $a_{ij} \leq 0$ for any $v_i \in \mathcal{V}_{s_1}, v_j \in \mathcal{V}_{s_2} (s_1 \neq s_2, s_1, s_2 \in \{1, 2\})$; if $\mathcal{G}[\boldsymbol{A}]$ does not meet these conditions, it is termed structurally unbalanced.

3. Model formulation

In this section, we present a coevolution model that incorporates the evolution rules of both multidimensional opinions and coopetitive influence networks.

3.1. Evolution of multidimensional opinions

Consider a social network consisting of n agents who simultaneously express their opinions on m logically interdependent topics, with agent and topic index sets being denoted as $\mathcal{I} = \{1, \ldots, n\}$ and $\mathcal{T} = \{1, \ldots, m\}$, respectively. According to the standard definition of an opinion (Friedkin et al., 2016), $\boldsymbol{x}_i(t) = [x_i^1(t), \ldots, x_i^m(t)]^T \in [-1, 1]^m$ represents the opinions of agent i on m topics at time t, where $x_i^p(t)$ represents the opinion of agent i on topic p with $p \in \mathcal{T}$. In details, the opinion value $x_i^p(t)$ reflects the attitude of agent i towards topic p, where $x_i^p(t) > 0$ implies that agent i supports topic p, $x_i^p(t) < 0$ implies that agent i rejects topic p, and $x_i^p(t) = 0$ means that agent i holds a neutral attitude towards topic p. The magnitude of $x_i^p(t)$ denotes the strength of attitude, with $|x_i^p(t)| = 1$ representing the maximal support or rejection.

The multidimensional opinion dynamics of agent i can be described by the equation:

$$\boldsymbol{x}_i(t+1) = \boldsymbol{C}_i\boldsymbol{x}_i(t) + \epsilon \boldsymbol{u}_i(t). \tag{1}$$

Here, the logic matrix $\boldsymbol{C}_i \in \mathbb{R}^{m\times m}$ symbolizes the logical interdependence between the m topics captured by agent i, where the (p, q)th entry $c_{pq,i}$ signifies the influence of topic q on topic p. Additionally, $\boldsymbol{u}_i(t) \in \mathbb{R}^m$ is the control input of agent i at time t, and $\epsilon > 0$ is the step-size parameter to adjust the step length of the displacement of $\boldsymbol{C}_i\boldsymbol{x}_i(t)$ along the update direction $\boldsymbol{u}_i(t)$.

The interpersonal influence among the n agents is supposed to be time-varying and described by a weighted signed digraph $\mathcal{G}[\boldsymbol{W}(t)]$. Inspired by the Altafini-type update rule (Altafini, 2012), the distributed protocol is designed as

$$\boldsymbol{u}_i(t) = \sum_{j\in\mathcal{I}\setminus\{i\}} |w_{ij}(t)|\boldsymbol{C}_i(\text{sgn}(w_{ij}(t))\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)), \tag{2}$$

where $w_{ij}(t)$ represents the influence that agent j has on agent i at time t. In the context of social influence networks, $w_{ij}(t) > 0$ implies that agent i holds a positive appraisal of agent j, signifying the positive influence of agent j on agent i, whereas $w_{ij}(t) < 0$ implies that agent j holds a negative appraisal of agent i. Additionally, agent i has no appraisal of agent j when $w_{ij}(t) = 0$. For convenience, the terms "influence" and "appraisal" will be used interchangeably, provided that there is no ambiguity.

The logic matrices are deemed heterogeneous, meaning that there are at least two agents i and j with $\boldsymbol{C}_i \neq \boldsymbol{C}_j$ for $i, j \in \mathcal{I}$. Indeed, a central focus of this article is to explore how the structure of the logic matrices, particularly the heterogeneity in their entries, influences the evolution of opinions. Let $\boldsymbol{x}(t) = [\boldsymbol{x}_1^T(t), \ldots, \boldsymbol{x}_n^T(t)]^T \in [-1, 1]^{nm}$ and $\tilde{\boldsymbol{C}} = \text{blkdiag}(\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n) \in \mathbb{R}^{nm\times nm}$. By substituting (2) into (1), model (1) can be written in a compact form

$$\boldsymbol{x}(t+1) = (\tilde{\boldsymbol{C}} - \epsilon \boldsymbol{L}(t))\boldsymbol{x}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t), \tag{3}$$

where $\boldsymbol{A}(t) \triangleq (\tilde{\boldsymbol{C}} - \epsilon \boldsymbol{L}(t)) \in \mathbb{R}^{nm\times nm}$ is a state matrix, and the signed matrix-weighted Laplacian is naturally defined as $\boldsymbol{L}(t) = [\boldsymbol{L}_{ij}(t)]_{n\times n}$ with its entries $\boldsymbol{L}_{ij}(t) \in \mathbb{R}^{m\times m}$ satisfying

$$\boldsymbol{L}_{ij}(t) = \begin{cases} -w_{ij}(t)\boldsymbol{C}_i & i \neq j, \\ \sum_{k\in\mathcal{I}\setminus\{i\}} |w_{ik}(t)|\boldsymbol{C}_i & i = j. \end{cases} \tag{4}$$

Given a logic matrix $\boldsymbol{C}_i$ with $i \in \mathcal{I}$ and a signed influence matrix $\boldsymbol{W}(t)$, the opinions $\boldsymbol{x}_i(t)$ of agent i evolve according to the opinion dynamics model (1). To ensure the rationality of model (1), two assumptions are imposed as follows, which will hold throughout this article.

Assumption 1 (*Ye et al., 2019*)**.** For all $i \in \mathcal{I}$ and $p \in \mathcal{T}$, $|\boldsymbol{C}_i|$ is row-stochastic with the diagonal entries $c_{pp,i} > 0$. Each eigenvalue of the logic matrix $\boldsymbol{C}_i$ is either 1 or has a modulus less than 1. If an eigenvalue of $\boldsymbol{C}_i$ is 1, then it is semisimple. Besides, for every $i, j \in \mathcal{I}$, it holds that $\boldsymbol{C}_i \sim \boldsymbol{C}_j$.

Assumption 2. The initial influence network $\mathcal{G}[\boldsymbol{W}(0)]$ is connected, and the matrix $|\boldsymbol{W}(t)|$ is row-stochastic with the diagonal entries $w_{ii}(t) \in (0, 1)$ for all $i \in \mathcal{I}$ and $t \geq 0$. Additionally, there exists a positive number $\beta > 0$ such that the nonzero entries of $\boldsymbol{W}(t)$ satisfy $|w_{ij}(t)| \geq \beta$ for all $i, j \in \mathcal{I}$ and $t \geq 0$.

Referring to Ye et al. (2019), the part on eigenvalues of the logic matrix $\boldsymbol{C}_i$ in Assumption 1 ensures the eventual consistency of agent i's belief system. The row-sum constraints of $\boldsymbol{C}_i$ in Assumption 1 and $\boldsymbol{W}(t)$ in Assumption 2 guarantee that $x_i^p(t) \in [-1, 1]$ for all $t \geq 0$ if $x_i^p(0) \in [-1, 1]$ holds for all $i \in \mathcal{I}$ and $p \in \mathcal{T}$. For ease of analysis, the constraint $\boldsymbol{C}_i \sim \boldsymbol{C}_j$ for every $i, j \in \mathcal{I}$ ensures that all agents hold the same view on the dependencies between topics, although the weights and signs assigned may differ. In the modeling context, it is reasonable to assume an initially connected influence network, and the requirement of strictly positive diagonals $w_{ii}(0) \in (0, 1)$ ensures that every agent can be influenced by itself and at least one other agent within the group. Throughout the rest of this article, the well-posedness of these assumptions is clear at any time. For shorthand, we use $\boldsymbol{W}(0) = \boldsymbol{W}_0$.

Remark 1. In retrospect of the logic matrices $\boldsymbol{C}_i\, \forall i \in \mathcal{I}$, which characterize the functional interdependencies between topics within the belief systems, heterogeneity can arise from various factors, including variations in agents' education, occupation, or political affiliation. For instance, agents affiliated with different political parties, such as Democrats and Republicans in the U.S., may assign notably different weights (including the signs) in $\boldsymbol{C}_i$ when discussing topics related to societal issues. Starting from this rational interpretation, we extend previous studies confined to homogeneous logic matrices (Parsegov et al., 2016; Ye et al., 2020) and offer deeper insights into understanding the impact of heterogeneity on the evolution of opinions.

3.2. Evolution of coopetitive influence networks

In what follows, we formulate an asynchronous update model to characterize the evolution of influence networks through local interaction mechanisms. The detailed evolution rules are described in four steps, utilizing the intermediate variables $\tilde{w}_{ij}$, $\tilde{r}_{ij}$, and $\hat{r}_{ij}$.

Step 1: It is assumed that the agent's self-appraisal is independent of the topic being discussed, given that agents are typically reluctant to change their self-cognition. For simplicity, we use $\omega_i \in (0, 1)$ to denote the self-appraisal of agent i, thus $w_{ii}(t) \equiv \omega_i$ for all $i \in \mathcal{I}$ and $t \geq 0$. With this assumption, the diagonals of the influence matrix can be separated as

$$\boldsymbol{W}(t) = \operatorname{diag}(\boldsymbol{\omega}) + (\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\omega}))\boldsymbol{R}(t), \tag{5}$$

where $\boldsymbol{R}(t)$ represents the relative influence matrix with zero diagonals.

Step 2: We begin by considering the impact of network structure on the evolution of influence weights in terms of two well-established sociological mechanisms, namely symmetry (Emerson, 1976) and influence (Friedkin & Johnsen, 2011). At each time step t, an off-diagonal entry $w_{pq}(t)$ of $\boldsymbol{W}(t)$ is randomly selected, where $p \neq q$, and updated as follows:

(a1) With probability $\varrho_1(t)$, the symmetry mechanism is triggered: If $e_{pq}(t) \in \mathcal{E}[\boldsymbol{W}(t)]$, then $\tilde{w}_{pq}(t+1) = w_{qp}(t)$; otherwise, $\tilde{w}_{pq}(t+1) = w_{pq}(t)$;

(a2) With probability $\varrho_2(t)$, the influence mechanism is triggered: Let $\mathcal{N}_{pq}(t) = \{l \in \mathcal{I} \setminus \{p, q\} : w_{pl}(t) \neq 0, w_{lq}(t) \neq 0\}$. If $\mathcal{N}_{pq}(t) \neq \emptyset$, then an agent k is randomly selected from $\mathcal{N}_{pq}(t)$, and $\tilde{w}_{pq}(t+1) = \operatorname{sgn}(w_{kq}(t))w_{pk}(t)$; otherwise, $\tilde{w}_{pq}(t+1) = w_{pq}(t)$.

Here, $\varrho_1(t), \varrho_2(t) \in \{0, 1\}$ are randomly picked binary parameters such that $\varrho_1(t)+\varrho_2(t) = 1$, determining the triggered mechanism for each update. From (5), it is easy to check that $w_{ij}(t) = (1 - \omega_i)r_{ij}(t)$ for $i \neq j$, and $w_{ij}(t) = \omega_i$ for $i = j$. Thus, the above update procedures can be equivalently rewritten as

(b1) The symmetry mechanism is triggered in the case of $\varrho_1(t) = 1$: If $e_{pq}(t) \in \mathcal{E}[\boldsymbol{R}(t)]$, then $\tilde{r}_{pq}(t+1) = (1 - \omega_q)r_{qp}(t)/(1 - \omega_p)$; otherwise, $\tilde{r}_{pq}(t+1) = r_{pq}(t)$;

(b2) The influence mechanism is triggered in the case of $\varrho_2(t) = 1$: Given the specified set $\mathcal{N}_{pq}(t)$ and agent k, if $\mathcal{N}_{pq}(t) \neq \emptyset$, then $\tilde{r}_{pq}(t+1) = \operatorname{sgn}(r_{kq}(t))r_{pk}(t)$; otherwise, $\tilde{r}_{pq}(t+1) = r_{pq}(t)$.

Step 3: We further consider the impact of agents' opinions on the evolution of influence weights in terms of the person-opinion homophily mechanism (Cartwright & Harary, 1956). As a result, the selected agent p continues to adjust the relative interpersonal appraisal of agent q by taking the opinion dynamics into account. Typically, the degree of friendliness (resp., hostility) between agents decreases (resp., increases) with the increase of their opinion differences. In this scenario, the updated intermediate relative appraisal can be expressed as

$$\begin{aligned}\hat{r}_{pq}(t+1) = {} & \tilde{r}_{pq}(t+1)g_1[\pi_{pq}(t+1)]\chi_{\{\tilde{r}_{pq}(t+1)>0\}} \\ & + \tilde{r}_{pq}(t+1)g_2[\pi_{pq}(t+1)]\chi_{\{\tilde{r}_{pq}(t+1)<0\}},\end{aligned} \tag{6}$$

where $g_1[x] \triangleq e^{-x}$ and $g_2[x] \triangleq 1 - e^{-x} + \sigma$, as weight adjustment terms, have no effect on the existence of interpersonal appraisals, with $0 < \sigma \ll 1$. Let $\pi_{pq}(t) \triangleq \|\boldsymbol{x}_p(t) - \boldsymbol{x}_q(t)\|_2$ represent the similarity of opinions held by agents p and q on all m topics at time t. Additionally, $\chi_{\{\cdot\}}$ is an indicator function defined as $\chi_{\{\cdot\}} = 1$ if the predicate $\{\cdot\}$ is true, and $\chi_{\{\cdot\}} = 0$ otherwise.

Step 4: Let $\hat{r}_{ij}(t+1) = r_{ij}(t), (i, j) \in (\mathcal{I} \times \mathcal{I}) \setminus \{(p, q)\}$. The objective here is to ensure the row-stochastic property of $|\boldsymbol{W}(t)|$ for any $t \geq 0$. Based on (5), $|\boldsymbol{R}(t)|$ is row-stochastic with zero diagonals since $|\boldsymbol{W}(t)|$ is row-stochastic. Therefore, the normalized relative influence matrix reads

$$r_{ij}(t+1) = \frac{\hat{r}_{ij}(t+1)}{\sum_{k\in\mathcal{I}}|\hat{r}_{ik}(t+1)|}. \tag{7}$$

The update of appraisals is not only affected by the connection state, but is also closely related to the opinions expressed. Clearly, $r_{ii}(t) = 0, \forall i \in \mathcal{I}, t \geq 0$. From time t to $t+1$, only the appraisals of the selected agent p towards others are updated, while the appraisals of the other agents remain unchanged. Furthermore, in the case of $r_{pq}(t) = 0$, agents are allowed to maintain their appraisals of others at time t, provided that the symmetry (resp., influence) mechanism in Step 2 is triggered, satisfying $e_{pq}(t) \notin \mathcal{E}[\boldsymbol{R}(t)]$ (resp., $\mathcal{N}_{pq}(t) = \emptyset$).

Integrating the above arguments, the evolutionary dynamics of influence networks can be formalized by

$$\boldsymbol{W}(t+1) = \mathcal{F}[\boldsymbol{W}(t), \boldsymbol{x}(t+1)], \tag{8}$$

where $\mathcal{F}[\cdot]$ denotes the mapping function of the influence matrix, with its explicit form defined in Steps 1–4.

The mechanisms of symmetry, influence, and person-opinion homophily constitute the three key components of our proposed asynchronous update model for influence networks. Specifically, the symmetry mechanism and the person-opinion homophily mechanism suggest that agents tend to be friendly towards those who reciprocate their friendliness and those who share similar opinions, respectively. Additionally, the influence mechanism suggests that an agent's appraisal of another agent is influenced by the appraisals of their common neighbors. These mechanisms are rooted in empirical research in social psychology (Cartwright & Harary, 1956; Emerson, 1976; Friedkin & Johnsen, 2011) and have been extensively integrated into previous models, as evidenced by Liu et al. (2020), Mei, Chen, et al. (2022).

Remark 2. The random asynchronous update scheme is a standard setup in social dynamics modeling (Proskurnikov & Tempo, 2017). Analogous to asynchronously sampling all potential edges, excluding self-loops, the selection of any common neighbor within a triad can also be executed asynchronously when the influence mechanism is triggered in Step 2. This approach not only captures the limited information acquisition ability of agents but also enhances the interpretability and tractability of the theoretical analysis.

In formulating the update rules for influence networks, the introduction of the person-opinion homophily mechanism in Step 3 intertwines the evolution of opinion and influence weights: agents update their opinions based on the state matrix $\boldsymbol{A}(t)$, while the influence weights are reassigned according to the current influence matrix $\boldsymbol{W}(t)$ and the updated opinion vector $\boldsymbol{x}(t+1)$. To conclude this section, we define the coevolution model below.

Definition 2 (*Coevolution Model*)**.** Let $\boldsymbol{x}(t) \in \mathbb{R}^{nm}$ denote the opinion vector of n agents discussing m topics and $\boldsymbol{W}(t) \in \mathbb{R}^{n\times n}$ denote the influence matrix at time t. With the state matrix $\boldsymbol{A}(t)$ defined in (3) and the mapping function $\mathcal{F}[\cdot]$ explicitly defined in Steps 1–4, the opinion and influence coevolution system is given by:

(i) Opinion evolution: $\boldsymbol{x}(t+1) = \boldsymbol{A}(t)\boldsymbol{x}(t)$,
(ii) Influence evolution: $\boldsymbol{W}(t+1) = \mathcal{F}[\boldsymbol{W}(t), \boldsymbol{x}(t+1)]$.

4. Main results

The main results are structured in two parts. First, we theoretically analyze the topological evolution of influence networks. Then, we mathematically investigate the asymptotic opinion behavior on multiple interdependent topics, addressing both cases of irreducible and reducible logic matrices.

4.1. Topological properties of influence networks

For the coevolution model in Definition 2, it is challenging to completely separate opinion evolution from interpersonal influence. To explore the asymptotic properties of coevolutionary dynamics, we initially focus on analyzing the evolution of influence networks from a topological perspective, regardless of opinion evolution details.

When investigating signed digraphs, attention naturally turns to their structural balance properties.

Lemma 1. *Consider an influence network evolving via the dynamics (8). Once $\mathcal{G}[\boldsymbol{W}(t)]$ reaches structural balance at some time $\tau_1 \geq 0$, it will remain structurally balanced with the same bipartition of agents for all $t \geq \tau_1$.*

Proof. Assume that $\mathcal{G}[\boldsymbol{W}(t)]$ first reaches structural balance at time $t = \tau_1 (\tau_1 \geq 0)$, so there exists a bipartition $\mathcal{I}_1 = \{1, \ldots, \kappa\}$ and $\mathcal{I}_2 = \{\kappa+1, \ldots, n\}$, $1 \leq \kappa \leq n$ of $\mathcal{I}$ such that the edges between $\mathcal{I}_1$ and $\mathcal{I}_2$ are all negative, and the edges within each set are all positive. Furthermore, we have $r_{ij}(\tau_1) \geq 0$ for all $i, j \in \mathcal{I}_1$ or $i, j \in \mathcal{I}_2$.

Then at time $t = \tau_1 + 1$, by applying Step 2 in Section 3.2, we obtain $\tilde{r}_{ij}(t+1) \geq 0$. Because $\omega_i \in (0, 1)$ for all $i \in \mathcal{I}$, the diagonal matrix $(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\omega})) = \mathrm{diag}(\boldsymbol{1}_n - \boldsymbol{\omega})$ has strictly positive diagonal entries, implying that $\boldsymbol{R}(t)$ has the same pattern of zero/nonzero entries as $\boldsymbol{W}(t)$ except for its diagonal. Hence, if $\tilde{r}_{ij}(\tau_1+1) > 0$, then there must exist either a directed path of length 1 from agent i to agent j, or a directed path of length 2 from agent j to agent i in $\mathcal{G}[\boldsymbol{W}(\tau_1)]$.

Similarly, if $r_{ij}(\tau_1) \leq 0$ for $i \in \mathcal{I}_1, j \in \mathcal{I}_2$ or $i \in \mathcal{I}_2, j \in \mathcal{I}_1$, then we obtain $r_{ij}(\tau_1 + 1) \leq 0$. By iterating this argument for $t = \tau_1 + 2, \tau_1 + 3, \ldots$, it follows that the signs of $r_{ij}(t)$ remain unchanged for all $i, j \in \mathcal{I}$. Therefore, for any $t \geq \tau_1$, $\mathcal{G}[\boldsymbol{W}(t)]$ keeps the structurally balanced state with the same bipartition $\{\mathcal{I}_1, \mathcal{I}_2\}$ as $\mathcal{G}[\boldsymbol{W}(\tau_1)]$. ■

Based on Lemma 1, the structural balance state achieved by influence networks remains invariant along the coevolutionary dynamics. To articulate this invariance and advance further, we introduce the concept of sign equilibrium as follows.

Definition 3. An influence matrix $\boldsymbol{W}(t^*)$, or equivalently an influence network $\mathcal{G}[\boldsymbol{W}(t^*)]$, is said to be a sign equilibrium of the influence dynamics (8) if, at time t^*, $\mathcal{G}[\boldsymbol{W}(t)]$ achieves a state of structural balance with a fully connected configuration.

The evolution of influence networks is a stochastic process, where randomness stems from the update sequence, intricately linked to the selected edge and the triggered update mechanisms at each time step. In the following lemma, we establish the almost-sure convergence of influence networks towards sign equilibria.

Lemma 2. *Consider the coevolutionary dynamics given by Definition 2. Suppose that Assumption 2 holds. Then, the evolutionary trajectory $\boldsymbol{W}(t)$ almost surely converges to a sign equilibrium within finite time.*

Proof. Given any initial opinion vector $\boldsymbol{x}(0) \in \mathbb{R}^{nm}$, we construct a state sequence

$$\mathcal{M} = \Big\{\mathrm{sgn}(\boldsymbol{W}_0) \in \{-1, 0, 1\}^{n\times n} : \forall \boldsymbol{W}_0 \in [-1, 1]^{n\times n} \text{ that satisfies Assumption 2}\Big\}.$$

Based on Definition 2, it is established that $\mathrm{sgn}(\boldsymbol{W}(t)) \in \mathcal{M}$ for any $t \geq 0$, signifying the invariance of $\mathcal{M}$ along the coevolutionary dynamics. To demonstrate that $\boldsymbol{W}(t)$ reaches sign equilibria of the influence dynamics within finite time, it suffices to show that for any $\mathrm{sgn}(\hat{\boldsymbol{W}}_0) \in \mathcal{M}$, one can manually construct an update sequence, along which the trajectory $\hat{\boldsymbol{W}}(t)$ converges to a sign equilibrium within finite time. Such an update sequence can be constructed as follows.

Referring to the dynamics (8), an immediate result follows: $\mathcal{G}[\hat{\boldsymbol{W}}(t)]$ will evolve to a fully connected state as $t \to \infty$, regardless of whether it reaches a structural balance state. One may assume that $\mathcal{G}[\hat{\boldsymbol{W}}(t)]$ achieves structural balance at time τ_1 and becomes fully connected at time τ_2. If $\tau_1 \leq \tau_2$, then the desired conclusion is directly justified. However, our focus lies in investigating the scenario where $\tau_1 > \tau_2$, meaning that $\mathcal{G}[\hat{\boldsymbol{W}}(t)]$ evolves to be fully connected without reaching a sign equilibrium at time τ_2. For any $t \geq \tau_2$, we consider the following two update cases: (i) if there exist agents $i, j \in \mathcal{I}$ such that $\mathrm{sgn}(\hat{w}_{ij}(t)) = -1$ and $\mathrm{sgn}(\hat{w}_{ji}(t)) = 1$, we update the edge (j, i) using the symmetry mechanism: $\hat{w}_{ij}(t+1) = \hat{w}_{ji}(t)$; (ii) if $\mathrm{sgn}(\hat{w}_{ij}(t)) = \mathrm{sgn}(\hat{w}_{ji}(t))$ holds for all $i, j \in \mathcal{I}$ but $\mathcal{G}[\hat{\boldsymbol{W}}(t)]$ is not structurally balanced, then there exists an agent $k \in \mathcal{I}\setminus\{i, j\}$ such that $\mathrm{sgn}(\hat{w}_{ij}(t)) = \mathrm{sgn}(\hat{w}_{ji}(t)) = -1$ and $\mathrm{sgn}(\hat{w}_{ik}(t)\hat{w}_{kj}(t)) = 1$. In this case, we update the edge (j, i) using the influence mechanism, resulting in $\hat{w}_{ij}(t+1) = \hat{w}_{ik}(t)\hat{w}_{kj}(t)$. Evidently, if neither case (i) nor case (ii) holds at some time τ_1, then $\mathcal{G}[\hat{\boldsymbol{W}}(\tau_1)]$ reaches the structural balance state such that $\hat{\boldsymbol{W}}(\tau_1)$ is a sign equilibrium of the influence dynamics.

Regarding the existence of the finite time τ_1, we define $\varphi(t) = \sum_{i,j\in\mathcal{I}} \chi_{\{\hat{w}_{ij}(t)<0\}}$ as the number of negative edges in $\mathcal{G}[\hat{\boldsymbol{W}}(t)]$ for $t \geq \tau_2$. In view of the update cases (i) and (ii), we observe that $\varphi(t)$ monotonically decreases until $\mathcal{G}[\hat{\boldsymbol{W}}(t)]$ reaches a structural balance state. Together with $\varphi(0) \in [0, n(n-1)]$, it follows that the update process will terminate within finite time steps, indicating that $\mathcal{G}[\hat{\boldsymbol{W}}(t)]$ can achieve structural balance at finite time τ_1. Additionally, the trajectory $\mathrm{sgn}(\hat{\boldsymbol{W}}(t))$ corresponds to a finite-state Markov chain over the state sequence $\mathcal{M}$, with the absorbing states equivalent to the sign equilibria of the influence dynamics. By applying Theorem 11.3 in Grinstead and Snell

(1997), we infer that there exists a finite time $\tau_{\max} = \max\{\tau_1, \tau_2\}$ such that every trajectory starting from $\mathcal{M}$ almost surely converges to the sign equilibria. In other words, $\mathcal{G}[\boldsymbol{W}(t)]$ eventually evolves to a fully connected structurally balanced configuration. This completes the proof. ■

Given any influence network $\mathcal{G}[\boldsymbol{W}_0]$ satisfying Assumption 2, we establish the almost-sure convergence of the influence dynamics by demonstrating the existence of at least one finite update sequence leading to a sign equilibrium. Unlike previous models that assume complete graphs for social balance analysis (Marvel et al., 2011; Mei et al., 2019), our framework requires only initial connectivity, making it more applicable to realistic settings. The resulting stable configurations are particularly relevant to real-world scenarios where a group of agents engage in simultaneous discussions on multiple logically interdependent topics (Jia et al., 2015; Tian & Wang, 2018).

The almost-sure convergence to global structural balance in our model is primarily ensured by the flexible connectivity structure and asynchronous appraisal update mechanisms, preventing trajectories from becoming trapped in jammed states. Monte Carlo simulations, described in the Appendix, further corroborate this result. This stands in sharp contrast to Mei, Chen, et al. (2022), where fixed, non-complete appraisal networks with bilateral edges and finite weight sets can lead to local balance under certain initial conditions and update rules, thereby obstructing global balance. Nevertheless, if the influence network in our model were initially disconnected or if the homophily mechanism were extended beyond merely adjusting interpersonal weights (see Step 3 in Section 3.2) to allow for tie reversals, more intricate equilibrium behaviors — such as local or non-trivial balance — could potentially emerge. Exploring such extensions presents a promising avenue for future research, offering deeper insights into the broader dynamical patterns of influence networks.

4.2. Asymptotic properties of coevolutionary dynamics

Denote the opinions of all agents on topic k at time t by $\boldsymbol{y}_k(t) = [y_k^1(t), \ldots, y_k^n(t)]^T = [x_1^k(t), \ldots, x_n^k(t)]^T$. Basically, for a given $i \in \mathcal{I}$ and $k \in \mathcal{T}$, $y_k^i(t)$ indicates the opinion of agent i on topic k at time t, i.e., $x_i^k(t)$. Let $\boldsymbol{\Psi}_{kj} = \operatorname{diag}(c_{kj,1}, \ldots, c_{kj,n}) \in \mathbb{R}^{n\times n}$ for all $k, j \in \mathcal{T}$. It follows that

$$\boldsymbol{y}_k(t+1) = \boldsymbol{W}(t)\boldsymbol{y}_k(t) + \epsilon \boldsymbol{v}_k(t), \tag{9}$$

where the control input $\boldsymbol{v}_k \in \mathbb{R}^n$ of agents on topic k is given by

$$\boldsymbol{v}_k(t) = \sum_{q\in\mathcal{T}\setminus\{k\}} |\boldsymbol{\Psi}_{kq}|\boldsymbol{W}(t)(\operatorname{sgn}(\boldsymbol{\Psi}_{kq})\boldsymbol{y}_q(t) - \boldsymbol{y}_k(t)). \tag{10}$$

Considering $\boldsymbol{y}(t) = [\boldsymbol{y}_1^T(t), \ldots, \boldsymbol{y}_m^T(t)]^T \in [-1, 1]^{nm}$, the opinion evolution model (3) can be converted to

$$\boldsymbol{y}(t+1) = (\boldsymbol{I}_m \otimes \boldsymbol{W}(t) - \epsilon\bar{\boldsymbol{L}}(t))\boldsymbol{y}(t) = \boldsymbol{B}(t)\boldsymbol{y}(t), \tag{11}$$

where $\boldsymbol{B}(t) \triangleq (\boldsymbol{I}_m \otimes \boldsymbol{W}(t) - \epsilon\bar{\boldsymbol{L}}(t)) \in \mathbb{R}^{nm\times nm}$ is a state matrix, and $\bar{\boldsymbol{L}}(t) = [\bar{\boldsymbol{L}}_{pq}(t)]_{m\times m}$ is the signed matrix-weighted Laplacian with its entries $\bar{\boldsymbol{L}}_{pq}(t) \in \mathbb{R}^{n\times n}$ satisfying

$$\bar{\boldsymbol{L}}_{pq}(t) = \begin{cases} -\boldsymbol{\Psi}_{pq}\boldsymbol{W}(t) & p \neq q, \\ \sum_{k\in\mathcal{T}\setminus\{p\}} |\boldsymbol{\Psi}_{pk}|\boldsymbol{W}(t) & p = q. \end{cases} \tag{12}$$

From the above, one knows that the block matrix entries $\boldsymbol{B}_{pq}(t) = \epsilon\boldsymbol{\Psi}_{pq}\boldsymbol{W}(t)$ when $p \neq q$, and otherwise, $\boldsymbol{B}_{pp}(t) = (\boldsymbol{I}_n - \epsilon\sum_{k\in\mathcal{T}\setminus\{p\}} |\boldsymbol{\Psi}_{pk}|)\boldsymbol{W}(t)$.

The associated graph $\mathcal{G}[\boldsymbol{A}(t)]$ of matrix $\boldsymbol{A}(t)$ in model (3) is essentially the same as the associated graph $\mathcal{G}[\boldsymbol{B}(t)]$ of matrix $\boldsymbol{B}(t)$ in model (11), except for a node reordering. A key motivation to investigate $\mathcal{G}[\boldsymbol{B}(t)]$ and the dynamical model (11) arises from the desirable property exhibited by the influence matrix $\boldsymbol{W}(t)$, namely, its finite-time convergence to a state of strongly connected structural balance. This property greatly facilitates the examination of connectivity as well as the structural balance or imbalance state of $\mathcal{G}[\boldsymbol{B}(t)]$, given the initial influence matrix $\boldsymbol{W}_0$ and the logic matrices $\boldsymbol{C}_i\, \forall i \in \mathcal{I}$.

Evidently, the dynamical behavior of model (11) is governed by the position of the eigenvalues of $\boldsymbol{B}(t)$. To ensure convergence of model (11), we impose the following assumption on the step-size parameter ϵ, and describe the spectral properties of $\boldsymbol{B}(t)$ immediately afterward.

Assumption 3. The step-size parameter ϵ satisfies $0 < \epsilon < 1/(1-\rho_{\min})$, wherein $\rho_{\min}$ is a positive number defined as $\rho_{\min} = \min\{c_{pp,i} : i \in \mathcal{I}, p \in \mathcal{T}\}$.

Lemma 3. *Suppose that Assumptions 1–3 hold. Then $\boldsymbol{B}(t)$ in model (11) has positive diagonals and $|\boldsymbol{B}(t)|$ is row-stochastic. Furthermore, the eigenvalues of $\boldsymbol{B}(t)$ are either the unit 1 or inside the open unit circle.*

Proof. Given $\boldsymbol{B}(t) = \boldsymbol{I}_m \otimes \boldsymbol{W}(t) - \epsilon\bar{\boldsymbol{L}}(t)$ and $0 < \epsilon < 1/(1-\rho_{\min})$, we have $b_{(k-1)n+i,(k-1)n+i}(t) = (1-\epsilon(1-c_{pp,i}))w_{ii}(t) \geq (1-\epsilon(1-\rho_{\min}))w_{ii}(t) > 0$ for all $i \in \mathcal{I}$ and $k \in \mathcal{T}$, indicating that the diagonal entries of $\boldsymbol{B}(t)$ are all strictly positive. Let $\bar{\boldsymbol{\Psi}} = \operatorname{blkdiag}(\boldsymbol{\Psi}_{11}, \ldots, \boldsymbol{\Psi}_{mm}) \in \mathbb{R}^{nm\times nm}$. By the definition of the signed Laplacian matrix, we observe that

$$\begin{aligned} |\boldsymbol{B}(t)|\mathbf{1}_{nm} &= |\boldsymbol{I}_m \otimes \boldsymbol{W}(t) - \epsilon(\boldsymbol{I}_{nm} - \bar{\boldsymbol{\Psi}})(\boldsymbol{I}_m \otimes \boldsymbol{W}(t))|\mathbf{1}_{nm} \\ &\quad + \epsilon(\boldsymbol{I}_{nm} - \bar{\boldsymbol{\Psi}})(\boldsymbol{I}_m \otimes |\boldsymbol{W}(t)|)(\mathbf{1}_m \otimes \mathbf{1}_n) \\ &= (\boldsymbol{I}_{nm} - \epsilon(\boldsymbol{I}_{nm} - \bar{\boldsymbol{\Psi}}))(\boldsymbol{I}_m \otimes |\boldsymbol{W}(t)|)(\mathbf{1}_m \otimes \mathbf{1}_n) \\ &\quad + \epsilon(\boldsymbol{I}_{nm} - \bar{\boldsymbol{\Psi}})(\mathbf{1}_m \otimes \mathbf{1}_n) \\ &= \mathbf{1}_{nm}, \end{aligned}$$

with equalities derived from the row-stochastic property of $|\boldsymbol{C}_i|$ and $|\boldsymbol{W}(t)|$ in Assumptions 1 and 2, as well as the value constraint on ϵ in Assumption 3. Thus, $|\boldsymbol{B}(t)|$ is row-stochastic. Applying the Gershgorin circle theorem, all eigenvalues of $\boldsymbol{B}(t)$ lie within the union of the nm discs given by $\mathcal{D}_i = \{d \in \mathbb{C} : |d - b_{ii}| \leq \sum_{j\neq i}|b_{ij}|\}, i \in \{1, \ldots, nm\}$. Based on the row-sum constraint and positive diagonal properties of $\boldsymbol{B}(t)$, we deduce that its eigenvalues are either unit 1 or within the open unit circle. ■

Remark 3. As obtained from Lemma 3, the network dynamics (11) can be viewed as the discrete-time Altafini model in Altafini (2012). However, a notable distinction arises in the investigation of (11) that the negative edges in $\mathcal{G}[\boldsymbol{B}(t)]$ are not solely attributed to antagonistic interpersonal interactions in $\mathcal{G}[\boldsymbol{W}(t)]$, but also to negative logical interdependencies in $\mathcal{G}[\boldsymbol{C}_i]$. Hence, both the signed influence matrix $\boldsymbol{W}(t)$ and the heterogeneous logic matrices $\boldsymbol{C}_i\, \forall i \in \mathcal{I}$ contribute to shaping the dynamical behavior.

Guided by Section 4.1, we now place significant emphasis on analyzing the role of logic matrices, particularly their structure, in determining the limiting opinion distribution. This analysis encompasses both cases of irreducible and reducible logic matrices. Before proceeding, we introduce two crucial definitions from Liu, Chen, Başar, and Belabbas (2017) and Ye et al. (2019).

Definition 4. An influence network contains agents with competing logical interdependencies on topic $p \in \mathcal{T}$ if there exist agents $i, j \in \mathcal{I}$ such that the nonzero entries $c_{pq,i}$ and $c_{pq,j}$ of $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$, respectively, are with opposite signs for some $q \in \mathcal{T}\setminus\{p\}$.

Definition 5. Model (11) reaches a modulus consensus on topic $k \in \mathcal{T}$ if there exists a constant $\alpha_k \in [0, 1]$ such that $\lim_{t\to\infty} |\boldsymbol{y}_k(t)| = \alpha_k \mathbf{1}_n$. Specifically, model (11) reaches a bipartite consensus on topic k when $\alpha_k \neq 0$; otherwise, it reaches a trivial consensus on topic k.

For the sake of analysis, it is assumed that $\mathcal{G}[\boldsymbol{W}(t)]$ becomes structurally balanced at time τ_1 and fully connected at time τ_2, as stated in Lemma 2. Denote $\tau_3 (\leq \tau_2)$ as the time when $\mathcal{G}[\boldsymbol{W}(t)]$ reaches a strongly connected state. Let $\tau_{\max} = \max\{\tau_1, \tau_2\}$ and $\tau = \max\{\tau_1, \tau_3\}$.

Below, we present the primary convergence result for the coevolutionary dynamics given by (8) and (11).

Theorem 1. *For a population of n agents who exchange and update opinions on m logically interdependent topics, the opinion vector $\boldsymbol{y}(t)$ evolves via (11) and the influence matrix $\boldsymbol{W}(t)$ evolves via (8). Suppose that Assumptions 1–3 hold and all the matrices $\boldsymbol{C}_i\,\forall i \in \mathcal{I}$ are irreducible. Then, there exists some $\boldsymbol{y}^* \in [-1, 1]^{nm}$ such that $\lim_{t\to\infty} \boldsymbol{y}(t) = \boldsymbol{y}^*$ holds with exponentially fast convergence for any initial vector $\boldsymbol{y}(0) \in [-1, 1]^{nm}$.*

Proof. Based on the theorem hypothesis concerning $\boldsymbol{C}_i \sim \boldsymbol{C}_j\,\forall i, j \in \mathcal{I}$, let $\bar{\boldsymbol{C}}$ be a nonnegative row-stochastic matrix with the same zero and nonzero pattern of entries as $\boldsymbol{C}_i\,\forall i \in \mathcal{I}$. In other words, $\bar{\boldsymbol{C}} \sim \boldsymbol{C}_i\,\forall i \in \mathcal{I}$. The signed Laplacian matrix corresponding to $\bar{\boldsymbol{C}}$ is denoted as $\boldsymbol{L}_{\bar{\boldsymbol{C}}} \in \mathbb{R}^{m\times m}$, where $[\boldsymbol{L}_{\bar{\boldsymbol{C}}}]_{pq} = -\bar{c}_{pq}$ if $p \neq q$, and $[\boldsymbol{L}_{\bar{\boldsymbol{C}}}]_{pq} = \sum_{k\in\mathcal{T}\setminus\{p\}} |\bar{c}_{pk}|$ otherwise, for all $p, q \in \mathcal{T}$. As a result, the graph $\mathcal{G}[(\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\bar{\boldsymbol{C}}}) \otimes \boldsymbol{W}(t)]$ has the same set of nodes and edges as $\mathcal{G}[\boldsymbol{B}(t)]$, although they may differ in terms of edge weights and signs. This implies that these two graphs possess the same connectivity properties. Besides, it is obtained from Lemma 2 that the sequence $\{\mathcal{G}[\boldsymbol{W}(t)]\}_{t=0}^{\infty}$ is uniformly jointly strongly connected, as there exists a positive integer t_2 such that $\bigcup_{t_1}^{t_2} \mathcal{G}[\boldsymbol{W}(t)]$ is jointly strongly connected for $t_1 \geq \tau_3$. For positive integers t_1 and t_2, there exists a union $\bigcup_{t_1}^{t_2} \mathcal{G}[\boldsymbol{B}(t)]$. In the following analysis, the relationship between $\bigcup_{t_1}^{t_2} \mathcal{G}[\boldsymbol{W}(t)]$ and $\bigcup_{t_1}^{t_2} \mathcal{G}[\boldsymbol{B}(t)]$ will be examined.

Since the union of graphs can be regarded as a simplified graph by removing unnecessary edges that do not affect the connectivity of the entire networks, the simplified graph $\mathcal{G}[\bar{\boldsymbol{B}}]$ possesses the same connectivity properties as the simplified graph $\mathcal{G}[(\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\bar{\boldsymbol{C}}}) \otimes \bar{\boldsymbol{W}}]$. It is derived from the uniformly jointly strongly connected sequence $\{\mathcal{G}[\boldsymbol{W}(t)]\}_{t=0}^{\infty}$ that the simplified graph $\mathcal{G}[\bar{\boldsymbol{W}}]$ is strongly connected. Moreover, under Assumption 3, $\mathcal{G}[\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\bar{\boldsymbol{C}}}]$ has the same zero and nonzero pattern of entries as $\mathcal{G}[\bar{\boldsymbol{C}}]$. Therefore, it shares the same connectivity properties as $\mathcal{G}[\bar{\boldsymbol{C}}]$. In addition, both $\mathcal{G}[\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\bar{\boldsymbol{C}}}]$ and $\mathcal{G}[\bar{\boldsymbol{W}}]$ have positive main diagonal entries, indicating that both graphs are aperiodic. Based on Lemma 1 in Ye et al. (2019), one obtains that $\mathcal{G}[(\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\bar{\boldsymbol{C}}}) \otimes \bar{\boldsymbol{W}}]$ is aperiodic, and consistent with $\mathcal{G}[\bar{\boldsymbol{C}}]$ in terms of connectivity properties. That is to say, $\mathcal{G}[\bar{\boldsymbol{B}}]$ is aperiodic, and consistent with $\mathcal{G}[\bar{\boldsymbol{C}}]$ in terms of connectivity properties. By leveraging existing results on the proof of convergence property for logic matrices (Ye et al., 2019), we can conclude that $\lim_{t\to\infty} \boldsymbol{y}(t) = \boldsymbol{y}^*$ exponentially fast for any initial condition $\boldsymbol{y}(0) \in [-1, 1]^{nm}$, where $\boldsymbol{y}^* \in [-1, 1]^{nm}$ represents the steady-state opinion distribution. ■

It is shown in Theorem 1 that model (11) always converges when the logic matrices $\boldsymbol{C}_i\,\forall i \in \mathcal{I}$ are irreducible. However, in cases where $\boldsymbol{C}_i\,\forall i \in \mathcal{I}$ are reducible, the opinions of agents on different topics may not always converge, but will eventually fall within a certain bound. An example illustrating this is provided in Section 5.

Having established the convergence result for irreducible logic matrices, we now present the following theorem to illustrate the impact of $\boldsymbol{C}_i$ on whether the opinions of all agents on a given topic converge to a bipartite consensus.

Theorem 2. *Let the hypotheses in Theorem 1 hold. Suppose that Assumptions 1–3 hold and $\boldsymbol{C}_i$ for all $i \in \mathcal{I}$ are irreducible. Then, the following statements hold:*

(1) If there are no competing logical interdependencies, and $\mathcal{G}[\boldsymbol{C}_i]\,\forall i \in \mathcal{I}$ are structurally balanced, the opinions of agents in $\mathcal{G}[\boldsymbol{B}(t)]$ on all m topics will achieve the same state of bipartite consensus exponentially fast for almost all initial conditions.

(2) If either (i) $\mathcal{G}[\boldsymbol{C}_i]\,\forall i \in \mathcal{I}$ are structurally unbalanced, or (ii) there are agents with competing logical interdependencies, the opinions of agents in $\mathcal{G}[\boldsymbol{B}(t)]$ on all m topics will achieve a state of trivial consensus exponentially fast for almost all initial conditions.

Proof. Based on the convergence result of Theorem 1, we provide the proof of bipartite and trivial consensus criteria for model (11) as follows.

Statement (1): Since there are no agents with competing logical interdependencies, meaning that the (p, q)th entry $c_{pq,i}$ of the logic matrix $\boldsymbol{C}_i$ has the same sign for all $i \in \mathcal{I}$, we disregard the magnitudes of edge weights for $\mathcal{G}[\boldsymbol{C}_i]$ and focus solely on its structural balance property. Thus, we denote $\mathcal{G}[\boldsymbol{C}_i]\,\forall i \in \mathcal{I}$ as $\mathcal{G}[\boldsymbol{C}]$. Furthermore, since $\mathcal{G}[\boldsymbol{C}_i]\,\forall i \in \mathcal{I}$ are structurally balanced, the set of nodes of $\mathcal{G}[\boldsymbol{C}]$ can be partitioned into $\mathcal{T}_1 = \{v_{c_1}, \ldots, v_{c_\iota}\}$ and $\mathcal{T}_2 = \{v_{c_{\iota+1}}, \ldots, v_{c_m}\}$, where $1 \leq \iota \leq m$. According to Lemma 2, we also denote $\mathcal{G}[\boldsymbol{W}(t)]$ as $\mathcal{G}[\boldsymbol{W}]$, whose set of nodes is bipartitioned into $\mathcal{I}_1 = \{v_{w_1}, \ldots, v_{w_\kappa}\}$ and $\mathcal{I}_2 = \{v_{w_{\kappa+1}}, \ldots, v_{w_n}\}$, where $1 \leq \kappa \leq n$ and $t \geq \tau_1$.

Following the above analysis, the coupling graph $\mathcal{G}[\boldsymbol{B}] = \mathcal{G}[(\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\boldsymbol{C}}) \otimes \boldsymbol{W}]$ involving $\boldsymbol{C}$ and $\boldsymbol{W}$ implies that the structural balance of $\mathcal{G}[\boldsymbol{B}]$ depends on both $\mathcal{G}[\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\boldsymbol{C}}]$ and $\mathcal{G}[\boldsymbol{W}]$. Under Assumption 3, we can ensure the structural balance of $\mathcal{G}[\boldsymbol{I}_m - \epsilon\boldsymbol{L}_{\boldsymbol{C}}]$, which has the same node bipartition as $\mathcal{G}[\boldsymbol{C}]$. Moreover, given that $\mathcal{G}[\boldsymbol{W}]$ is structurally balanced, and referring to the proof of Theorem 2 in Yang et al. (2023), we deduce that there are no negative cycles in $\mathcal{G}[\boldsymbol{B}]$, i.e., $\mathcal{G}[\boldsymbol{B}]$ is structurally balanced. Hence, there exists a finite time τ such that $\mathcal{G}[\boldsymbol{B}(t)]$ is structurally balanced and strongly connected for any $t \geq \tau$.

For the matrix $\boldsymbol{B}(t)$, we define two nonnegative matrices $\boldsymbol{B}^+(t)$ and $\boldsymbol{B}^-(t)$, where the entries of $\boldsymbol{B}^+(t)$ correspond to the positive neighbors of each node, including self-loops, and the entries of $\boldsymbol{B}^-(t)$ correspond to the negative neighbors of each node. It is evident that $\boldsymbol{B}(t) = \boldsymbol{B}^+(t) - \boldsymbol{B}^-(t)$. Denote $y_i^+(t) = y_i(t)$, $y_i^-(t) = -y_i(t)$, and let $\boldsymbol{z}(t) = [y_1^+(t), \ldots, y_{nm}^+(t), y_1^-(t), \ldots, y_{nm}^-(t)]^T$. Then model (11) can be reformulated into

$$\boldsymbol{z}(t+1) = \begin{pmatrix} \boldsymbol{B}^+(t) & \boldsymbol{B}^-(t) \\ \boldsymbol{B}^-(t) & \boldsymbol{B}^+(t) \end{pmatrix} \boldsymbol{z}(t) = \boldsymbol{Q}(t)\boldsymbol{z}(t), \tag{13}$$

where $\boldsymbol{Q}(t)$ is a row-stochastic with all entries nonnegative by Lemma 3, and $t \geq \tau$. Denote the sets of nodes in $\mathcal{G}[\boldsymbol{B}(t)]$ and $\mathcal{G}[\boldsymbol{Q}(t)]$ as $\{v_1, \ldots, v_{nm}\}$ and $\{v_1^+, \ldots, v_{nm}^+, v_1^-, \ldots, v_{nm}^-\}$, respectively. We describe the relationship between $\mathcal{G}[\boldsymbol{B}(t)]$ and $\mathcal{G}[\boldsymbol{Q}(t)]$: if there is a positive edge (v_i, v_j) in $\mathcal{G}[\boldsymbol{B}(t)]$, then there are two directed edges $(v_i^+, v_j^+), (v_i^-, v_j^-)$ in $\mathcal{G}[\boldsymbol{Q}(t)]$; if there is a negative edge (v_i, v_j) in $\mathcal{G}[\boldsymbol{B}(t)]$, then there are two directed edges $(v_i^+, v_j^-), (v_i^-, v_j^+)$ in $\mathcal{G}[\boldsymbol{Q}(t)]$; additionally, the corresponding edges in $\mathcal{G}[\boldsymbol{B}(t)]$ and $\mathcal{G}[\boldsymbol{Q}(t)]$ have the same absolute value of weights. Given that $\mathcal{G}[\boldsymbol{B}(t)]$ is structurally balanced for $t \geq \tau$, the set of nodes in $\mathcal{G}[\boldsymbol{B}(t)]$ can be divided into two parts, namely $\mathcal{V}_{\boldsymbol{B}}^+ = \{v_1, \ldots, v_r\}$ and $\mathcal{V}_{\boldsymbol{B}}^- = \{v_{r+1}, \ldots, v_{nm}\}$, where $1 \leq r \leq nm$. From Lemma 1 in Xia et al. (2015), it is known that $\mathcal{G}[\boldsymbol{Q}(t)]$ contains two disconnected components with node sets $\mathcal{V}_{\boldsymbol{Q}}^+ = \{v_1^+, \ldots, v_r^+, v_{r+1}^-, \ldots, v_{nm}^-\}$ and $\mathcal{V}_{\boldsymbol{Q}}^- = \{v_1^-, \ldots, v_r^-, v_{r+1}^+, \ldots, v_{nm}^+\}$, both of which are strongly connected. Denote the matrix corresponding to the node set $\mathcal{V}_{\boldsymbol{Q}}^+$

as $\boldsymbol{Q}_s(t)$, and let $\boldsymbol{z}_s(t) = [y^+_{s_1}(t), \dots, y^+_{s_r}(t), y^-_{s_{r+1}}(t), \dots, y^-_{s_{nm}}(t)]$. Then the dynamic equation for a subsystem of (13) can be represented as $\boldsymbol{z}_s(t+1) = \boldsymbol{Q}_s(t)\boldsymbol{z}_s(t)$. According to Lemma 2 and Theorem 9.2 in Hendrickx and Blondel (2008), there exists a constant $y^*_s \in [0, 1]$ such that the solution of this subsystem converges exponentially fast to $y^*_s \mathbf{1}_{nm}$. Consequently, it holds that $\lim_{t\to\infty} y_{s_j}(t) = -\lim_{t\to\infty} y_{s_i}(t) = -y^*_s$ for any $v_i \in \mathcal{V}^+_{\boldsymbol{B}}$ and $v_j \in \mathcal{V}^-_{\boldsymbol{B}}$. Since the initial conditions that render the agreed value of each component to be 0 come from a set with zero Lebesgue measure, we conclude that there exists a positive constant $y^* \in (0, 1]$ such that $\lim_{t\to\infty} |\boldsymbol{y}_k(t)| = y^*\mathbf{1}_n$ holds for all $k \in \mathcal{T}$ and almost all initial conditions. This completes the proof of Statement (1).

Statement (2): In this proof, we follow the analysis and notations used in the proof of Statement (1). Recall that $\mathcal{G}[\boldsymbol{I}_m - \epsilon \boldsymbol{L}_{\boldsymbol{C}}]$ shares the same structural balance or imbalance properties as $\mathcal{G}[\boldsymbol{C}]$ under Assumption 3, and $\mathcal{G}[\boldsymbol{W}]$ is structurally balanced. Thus, we infer from the proof of Theorem 3 in Yang et al. (2023) that there exist circles with an odd number of negative edges in $\mathcal{G}[\boldsymbol{B}]$, indicating that $\mathcal{G}[\boldsymbol{B}]$ is structurally unbalanced when one of conditions (i) or (ii) holds for any $t \geq \tau$. Applying Lemma 2 in Xia et al. (2015), we deduce that $\mathcal{G}[\boldsymbol{Q}(t)]$ is strongly connected with the node set $\mathcal{V}_{\boldsymbol{Q}} = \{v^+_1, \dots, v^+_{nm}, v^-_1, \dots, v^-_{nm}\}$. Since $\boldsymbol{Q}(t)$ is a row-stochastic matrix with all entries nonnegative, there exists a constant $y^* \in [0, 1]$ such that the solution of (13) converges exponentially fast to $y^*\mathbf{1}_{nm}$. Moreover, considering $\lim_{t\to\infty}(y^+_i(t)+y^-_i(t)) = 2y^* = 0$ for all $i \in \{1, \dots, nm\}$, it follows that y^* must be 0. Therefore, we conclude that $\lim_{t\to\infty} \boldsymbol{y}_k(t) = \mathbf{0}_n$ holds for any $k \in \mathcal{T}$ and all initial conditions. This completes the proof of Statement (2). ■

The main results in Theorem 2 rely on the assumption that the logic matrix $\boldsymbol{C}_i$ is irreducible for all $i \in \mathcal{I}$. However, in practice, it is often encountered that $\boldsymbol{C}_i$ is reducible, indicating that $\mathcal{G}[\boldsymbol{C}_i]$ is not strongly connected. This implies a cascade structure of logical interdependencies among the topics within the given problem context. Hence, we proceed to explore the asymptotic properties of the coevolution model for reducible $\boldsymbol{C}_i\, \forall i \in \mathcal{I}$.

From an algebraic perspective, the hypothesis $\boldsymbol{C}_i \sim \boldsymbol{C}_j$ in Assumption 1 implies the existence of a common permutation matrix $\boldsymbol{P} \in \mathbb{R}^{m\times m}$ such that, for all $i \in \mathcal{I}$, $\boldsymbol{P}^T\boldsymbol{C}_i\boldsymbol{P}$ is lower block triangular. Thus, the topics $p \in \mathcal{T}$ can be rearranged such that, for each $i \in \mathcal{I}$, one has

$$\boldsymbol{C}_i = \begin{pmatrix} \Theta_{1,i} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \Theta_{21,i} & \Theta_{2,i} & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_{s1,i} & \Theta_{s2,i} & \cdots & \Theta_{s,i} \end{pmatrix}, \tag{14}$$

where $\Theta_{j,i} \in \mathbb{R}^{m_j\times m_j}$ is irreducible for any $j \in \mathcal{S} \triangleq \{1, \dots, s\}$ and m_j are positive integers such that $\sum_{j=1}^{s} m_j = m$. Essentially, each $\Theta_{j,i}$ corresponds to an SCC $\mathcal{G}[\Theta_{j,i}]$ and there are s components in total, with the jth component comprising m_j topics. By Lemma 2, there exists a finite time τ such that for any $t \geq \tau$, $\boldsymbol{W}(t)$ is strongly connected and structurally balanced. Then, when $t \geq \tau$, $\boldsymbol{B}(t)$ in model (11) is naturally expressed as

$$\boldsymbol{B}(t) = \begin{pmatrix} \Xi_1(t) & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \Xi_{21}(t) & \Xi_2(t) & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \Xi_{s1}(t) & \Xi_{s2}(t) & \cdots & \Xi_s(t) \end{pmatrix}. \tag{15}$$

From the decomposition of $\boldsymbol{B}(t)$, it follows that $\Xi_p(t) \in \mathbb{R}^{nm_p\times nm_p}$ is irreducible for any $p \in \mathcal{S}$, and $\boldsymbol{B}(t)$ possesses positive diagonals by Assumption 3. As a result, $\mathcal{G}[\Xi_p(t)]$ is strongly connected and aperiodic when $t \geq \tau$.

If the topic set $\mathcal{T}_p$ corresponds to a closed SCC $\mathcal{G}[\Xi_p(t)]$, implying that $\Xi_{pj}(t) = \boldsymbol{O}$ holds for all $j \neq p$, Theorem 2 establishes that for all $k \in \mathcal{T}_p$, $\lim_{t\to\infty} |\boldsymbol{y}_k(t)| = y^*\mathbf{1}_n$ holds exponentially fast, where $y^* \in [0, 1]$. Conversely, if the topic set $\mathcal{T}_p$ is associated with an open SCC $\mathcal{G}[\Xi_p(t)]$, then the theorem we present below can be employed to determine the limiting state of opinions on a given topic.

Theorem 3. *Let the hypotheses in Theorem 1 hold. Suppose that Assumptions 1–3 hold and $\boldsymbol{C}_i\, \forall i \in \mathcal{I}$ are decomposed as specified in (14). Let $\mathcal{G}[\Theta_{c,i}]$ be a closed SCC of $\mathcal{G}[\boldsymbol{C}_i]$ with m_c topics, and $\mathcal{G}[\Theta_{o,i}]$ be an open SCC of $\mathcal{G}[\boldsymbol{C}_i]$ with m_o topics. Then the following statements hold if $\mathcal{G}[\Theta_{o,i}]$ is structurally balanced for all $i \in \mathcal{I}$ and there are no competing logical interdependencies on these $m_c + m_o$ topics:*

(1) If $\mathcal{G}[\Theta_{o,i}]$ is solely connected to one $\mathcal{G}[\Theta_{c,i}]$, then the opinions of agents on the m_o topics converge exponentially fast to the same modulus consensus state as those on the m_c topics.

(2) If $\mathcal{G}[\Theta_{o,i}]$ is solely connected to multiple $\mathcal{G}[\Theta_{c,i}]$, and the maximum and minimum values of modulus consensus solutions to these $\mathcal{G}[\Theta_{c,i}]$ are denoted by Ω_1 and Ω_2 ($\Omega_1 \geq \Omega_2 \geq 0$), respectively, then the opinions of agents on the m_o topics eventually fall in $[-\Omega_1, -\Omega_2] \cup [\Omega_2, \Omega_1]$.

Proof. **Statement (1):** Since the open SCC $\mathcal{G}[\Theta_{o,i}]$ is solely connected to one closed SCC $\mathcal{G}[\Theta_{c,i}]$, the opinions and influence weights of agents in $\mathcal{G}[\Xi_o(t)]$ are only influenced by $\mathcal{G}[\Xi_c(t)]$ and $\mathcal{G}[\Xi_o(t)]$ for $t \geq \tau$. Then, the dynamic equation within $\mathcal{G}[\Xi_c(t)]$ and $\mathcal{G}[\Xi_o(t)]$ can be formulated as $\boldsymbol{y}_{co}(t+1) = \boldsymbol{B}_{co}(t)\boldsymbol{y}_{co}(t)$, where $\boldsymbol{B}_{co}(t) = \left(\begin{smallmatrix} \Xi_c(t) & \boldsymbol{O} \\ \Xi_{co}(t) & \Xi_o(t) \end{smallmatrix}\right)$ represents the matrix formed by the coupling among n agents and $m_c + m_o$ topics. Denote the node sets corresponding to $\mathcal{G}[\Xi_c(t)]$ and $\mathcal{G}[\Xi_o(t)]$ as $\mathcal{V}_c = \{v_1, \dots, v_{nm_c}\}$ and $\mathcal{V}_o = \{v_{nm_c+1}, \dots, v_{n(m_c+m_o)}\}$, respectively. Given that there are no competing logical interdependencies on these topics, $\mathcal{G}[\Theta_{c,i}]$ possesses the same structural balance or imbalance property for each $i \in \mathcal{I}$. Therefore, we discuss the following two scenarios separately to investigate the properties of the above dynamic equation.

The first scenario assumes that $\mathcal{G}[\Theta_{c,i}]\, \forall i \in \mathcal{I}$ are structurally balanced. According to Lemma 2 and the theorem hypotheses, one easily obtains that $\mathcal{G}[\boldsymbol{B}_{co}(t)]$ is structurally balanced for $t \geq \tau$. Without loss of generality, we assume that there are r_1 nodes of $\mathcal{G}[\Xi_c(t)]$ and r_2 nodes of $\mathcal{G}[\Xi_o(t)]$ in the same part $\mathcal{V}^+_{\boldsymbol{B}_{co}} = \{v_1, \dots, v_{r_1}, v_{nm_c+1}, \dots, v_{nm_c+r_2}\}$, while the others are in another hostile part $\mathcal{V}^-_{\boldsymbol{B}_{co}} = \{v_{r_1+1}, \dots, v_{nm_c}, v_{nm_c+r_2+1}, \dots, v_{n(m_c+m_o)}\}$. Similar to the proof in Theorem 2, by letting $\bar{\boldsymbol{z}}(t) = [\boldsymbol{y}_c(t)^T, \boldsymbol{y}_o(t)^T, -\boldsymbol{y}_c(t)^T, -\boldsymbol{y}_o(t)^T]^T$, the above dynamic equation can be transformed into

$$\bar{\boldsymbol{z}}(t+1) = \begin{pmatrix} \boldsymbol{B}^+_{co}(t) & \boldsymbol{B}^-_{co}(t) \\ \boldsymbol{B}^-_{co}(t) & \boldsymbol{B}^+_{co}(t) \end{pmatrix} \bar{\boldsymbol{z}}(t) = \bar{\boldsymbol{Q}}(t)\bar{\boldsymbol{z}}(t),$$

where $\bar{\boldsymbol{Q}}(t)$ is a nonnegative row-stochastic matrix. Since $\mathcal{G}[\boldsymbol{B}_{co}(t)]$ is structurally balanced for $t \geq \tau$, it is shown that $\mathcal{G}[\bar{\boldsymbol{Q}}(t)]$ contains two disconnected components with the node sets $\mathcal{V}^+_{\bar{\boldsymbol{Q}}} = \{v^+_1, \dots, v^+_{r_1}, v^+_{nm_c+1}, \dots, v^+_{nm_c+r_2}, v^-_{r_1+1}, \dots, v^-_{nm_c}, v^-_{nm_c+r_2+1}, \dots, v^-_{n(m_c+m_o)}\}$ in one component and $\mathcal{V}^-_{\bar{\boldsymbol{Q}}} = \{v^-_1, \dots, v^-_{r_1}, v^-_{nm_c+1}, \dots, v^-_{nm_c+r_2}, v^+_{r_1+1}, \dots, v^+_{nm_c}, v^+_{nm_c+r_2+1}, \dots, v^+_{n(m_c+m_o)}\}$ in the other. Moreover, each component of $\mathcal{G}[\bar{\boldsymbol{Q}}(t)]$ contains a directed spanning tree, owing to the existence of such trees in $\mathcal{G}[\boldsymbol{B}_{co}(t)]$. By employing a similar proof as in Statement (1) of Theorem 2, it can be established that there exists a constant $y^*_{co} \in [0, 1]$ such that $\lim_{t\to\infty} y_{co_i}(t) = -\lim_{t\to\infty} y_{co_j}(t) = y^*_{co}$ holds exponentially fast for any $v_i \in \mathcal{V}^+_{\boldsymbol{B}_{co}}$ and $v_j \in \mathcal{V}^-_{\boldsymbol{B}_{co}}$. Consequently, $\lim_{t\to\infty}|y_{c_j}(t)| = \lim_{t\to\infty}|y_{o_i}(t)| = y^*_{co}$ for any $v_j \in \mathcal{V}_o$ and $v_i \in \mathcal{V}_c$. Therefore, the opinions of agents on the m_o topics converge to the same modulus consensus state as those on the m_c topics exponentially fast.

The other scenario assumes that $\mathcal{G}[\Theta_{c,i}]\,\forall i \in \mathcal{I}$ are structurally unbalanced. It is obtained from Lemma 2 and the theorem hypotheses that $\mathcal{G}[\boldsymbol{B}_{co}(t)]$ is structurally unbalanced for $t \geq \tau$. Hence, we observe that $\mathcal{G}[\bar{\boldsymbol{Q}}(t)]$ contains only one component with the node set $\mathcal{V}_{\bar{\boldsymbol{Q}}} = \{v_1^+, \ldots, v_{n(m_c+m_o)}^+, v_1^-, \ldots, v_{n(m_c+m_o)}^-\}$. According to Theorem 9.2 in Hendrickx and Blondel (2008) and Statement (2) of Theorem 2, we deduce that $\lim_{t\to\infty} y_{c_j}(t) = \lim_{t\to\infty} y_{o_i}(t) = 0$ holds exponentially fast for any $v_j \in \mathcal{V}_o$ and $v_i \in \mathcal{V}_c$. Therefore, the opinions of agents on the m_o topics converge to the same trivial consensus state as those on the m_c topics exponentially fast.

From the above analysis, we conclude that there always exists a constant $y_{co}^* \in [0, 1]$ such that $\lim_{t\to\infty}|y_{c_j}(t)| = \lim_{t\to\infty}|y_{o_i}(t)| = y_{co}^*$ holds for any $v_j \in \mathcal{V}_o$ and $v_i \in \mathcal{V}_c$. The specific value of y_{co}^* depends entirely on the closed SCC $\mathcal{G}[\Xi_c(t)]$. This completes the proof of Statement (1).

Statement (2): Without loss of generality, we assume that the open SCC $\mathcal{G}[\Theta_{o,i}]$ is solely connected to $h \geq 2$ closed SCCs $\{\mathcal{G}[\Theta_{c_k,i}]\}_{k\in\{1,\ldots,h\}}$. Thus, the opinions and influence weights of agents in $\mathcal{G}[\Xi_o(t)]$ are only influenced by $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\ldots,h\}}$ and $\mathcal{G}[\Xi_o(t)]$ for $t \geq \tau$. The dynamic equation within $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\ldots,h\}}$ and $\mathcal{G}[\Xi_o(t)]$ for $t \geq \tau$ can be represented as

$$
\begin{aligned}
\tilde{\boldsymbol{y}}_{co}(t+1) &= \begin{pmatrix} \Xi_{c_1}(t) & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \Xi_{c_2}(t) & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \Xi_{c_h}(t) & \boldsymbol{O} \\ \Xi_{oc_1}(t) & \Xi_{oc_2}(t) & \cdots & \Xi_{oc_h}(t) & \Xi_o(t) \end{pmatrix} \tilde{\boldsymbol{y}}_{co}(t) \\
&= \tilde{\boldsymbol{B}}_{co}(t)\tilde{\boldsymbol{y}}_{co}(t).
\end{aligned} \tag{16}
$$

Let the node sets corresponding to $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\ldots,h\}}$ and $\mathcal{G}[\Xi_o(t)]$ be denoted as $\mathcal{V}_{\bar{c}} = \{v_1, \ldots, v_{nm_{\bar{c}}}\}$ and $\mathcal{V}_{\bar{o}} = \{v_{nm_{\bar{c}}+1}, \ldots, v_{n(m_{\bar{c}}+m_o)}\}$, respectively, where $m_{\bar{c}} = \sum_{k=1}^{h} m_{c_k}$. In the absence of competing logical interdependencies, we can infer from Theorem 2 that the opinions of agents in each $\mathcal{G}[\Xi_{c_k}(t)], k \in \{1, \ldots, h\}$ will eventually achieve a state of bipartite or trivial consensus, depending on whether $\mathcal{G}[\Xi_{c_k}(t)]$ is structurally balanced or unbalanced. Denote Ω_1 and Ω_2 ($\Omega_1 \geq \Omega_2 \geq 0$) as the maximum and minimum modulus consensus values of $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\ldots,h\}}$, respectively. Define $\Omega_1(t) = \max_{v_i\in\mathcal{V}_{\bar{c}}}|\tilde{y}_{co_i}(t)|$, $K_1(t) = \max_{v_j\in\mathcal{V}_{\bar{o}}}|\tilde{y}_{co_j}(t)|$, $\Omega_2(t) = \min_{v_i\in\mathcal{V}_{\bar{c}}}|\tilde{y}_{co_i}(t)|$, and $K_2(t) = \min_{v_j\in\mathcal{V}_{\bar{o}}}|\tilde{y}_{co_j}(t)|$. According to the consensus result in Theorem 2, it follows that $\lim_{t\to\infty}\Omega_1(t) = \Omega_1$ and $\lim_{t\to\infty}\Omega_2(t) = \Omega_2$. Next, we claim that the absolute opinion values of agents in $\mathcal{G}[\Xi_o(t)]$ are asymptotically bounded by $[\Omega_2, \Omega_1]$. This claim can be demonstrated through the following two aspects.

Regarding the first aspect, the statement is as follows: If $K_1(t) > \Omega_1(t)$ holds only for a finite time t, then there exists a time t' such that $K_1(t) \leq \Omega_1(t)$ for all $t \geq t'$, and hence $\lim_{t\to\infty}\sup K_1(t) \leq \lim_{t\to\infty}\Omega_1(t) = \Omega_1$. In this case, the desired conclusion follows. Next, we assume that $K_1(t) > \Omega_1(t)$ holds for an infinite time sequence $t = t_1^*, t_2^*, \ldots$, where $t \geq \tau$. Without loss of generality, we pick the specific time t_1^* for discussion. It is evident from (16) that $\Omega_1(t_1^*+\eta) \leq \Omega_1(t_1^*) < K_1(t_1^*)$, and $K_1(t_1^*+\eta) \leq K_1(t_1^*)$ for all $\eta \geq 0$. Since there always exist directed edges from $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\ldots,h\}}$ to $\mathcal{G}[\Xi_o(t)]$ for $t \geq \tau$, we assume that when $t \geq t_1^*$, there is a directed edge between $v_{i_0} \in \mathcal{V}_{\bar{c}}$ and $v_{i_1} \in \mathcal{V}_{\bar{o}}$. Besides, with Assumption 2, we can define β^* as a positive constant such that none of the nonzero entries of $\mathcal{G}[\tilde{\boldsymbol{B}}_{co}(t)]$ is smaller than it for all $t \geq 0$. Then, one has

$$
\begin{aligned}
\left|\tilde{y}_{co_{i_1}}(t_1^*+1)\right| &= \left|\sum_{j=1}^{n(m_{\bar{c}}+m_o)} \tilde{b}_{co_{i_1 j}}(t_1^*)\tilde{y}_{co_j}(t_1^*)\right| \\
&\leq \left|\tilde{b}_{co_{i_1 i_0}}(t_1^*)\tilde{y}_{co_{i_0}}(t_1^*)\right| + \sum_{j\neq i_0}\left|\tilde{b}_{co_{i_1 j}}(t_1^*)\tilde{y}_{co_j}(t_1^*)\right| \\
&\leq \beta^*\Omega_1(t_1^*) + (1-\beta^*)K_1(t_1^*) \\
&= \Omega_1(t_1^*) + (1-\beta^*)(K_1(t_1^*) - \Omega_1(t_1^*)).
\end{aligned}
$$

Similarly, for $t = t_1^*+2$, further calculation shows that

$$
\begin{aligned}
\left|\tilde{y}_{co_{i_1}}(t_1^*+2)\right| &= \left|\sum_{j=1}^{n(m_{\bar{c}}+m_o)} \tilde{b}_{co_{i_1 j}}(t_1^*+1)\tilde{y}_{co_j}(t_1^*+1)\right| \\
&\leq \left|\tilde{b}_{co_{i_1 i_0}}(t_1^*+1)\tilde{y}_{co_{i_0}}(t_1^*+1)\right| \\
&\quad + \sum_{j\neq i_0}\left|\tilde{b}_{co_{i_1 j}}(t_1^*+1)\tilde{y}_{co_j}(t_1^*+1)\right| \\
&\leq \beta^*\Omega_1(t_1^*) + (1-\beta^*)(\Omega_1(t_1^*) \\
&\quad + (1-\beta^*)(K_1(t_1^*) - \Omega_1(t_1^*))) \\
&= \Omega_1(t_1^*) + (1-\beta^*)^2(K_1(t_1^*) - \Omega_1(t_1^*)).
\end{aligned}
$$

Repeating the above calculation, one derives that, for all $\eta \geq 1$,

$$
\left|\tilde{y}_{co_{i_1}}(t_1^*+\eta)\right| \leq \Omega_1(t_1^*) + (1-\beta^*)^\eta(K_1(t_1^*) - \Omega_1(t_1^*)).
$$

Hence, one can deduce from $\lim_{\eta\to\infty}(1-\beta^*)^\eta = 0$ that $\lim_{\eta\to\infty}\sup|\tilde{y}_{co_{i_1}}(t_1^*+\eta)| \leq \Omega_1(t_1^*)$. Since the above discussion applies to all t_r^*, it holds for all $r = 1, 2, \ldots$, that $\lim_{\eta\to\infty}\sup K_1(t) \leq \Omega_1(t_r^*)$. Considering the fact that $\lim_{t\to\infty}\Omega_1(t) = \Omega_1$, one has $\lim_{t\to\infty}\sup K_1(t) \leq \Omega_1$.

Regarding the other aspect, a similar analysis can be applied. If $K_2(t) < \Omega_2(t)$ holds only for a finite time t, then it is clear that $\lim_{t\to\infty}\inf K_2(t) \geq \lim_{t\to\infty}\Omega_2(t) = \Omega_2$. Besides, when $K_2(t) < \Omega_2(t)$ holds for an infinite time sequence $s = s_1^*, s_2^*, \ldots$, and $s \geq \tau$, we can conclude by picking the specific time s_1^* that $|\tilde{y}_{co_{i_1}}(s_1^*+\eta)| \geq \Omega_2(s_1^*) + (1-\beta^*)^\eta(K_2(s_1^*) - \Omega_2(s_1^*))$ for any $\eta \geq 0$. Further, one has $\lim_{t\to\infty}\inf K_2(t) \geq \Omega_2$.

By combining the above two aspects, we establish that $\lim_{t\to\infty}|\tilde{y}_{co_j}(t)| \in [\Omega_2, \Omega_1]$ for any $v_j \in \mathcal{V}_{\bar{o}}$, indicating that $\lim_{t\to\infty}\tilde{y}_{co_j}(t) \in [-\Omega_1, -\Omega_2] \cup [\Omega_2, \Omega_1]$ for any $v_j \in \mathcal{V}_{\bar{o}}$. This completes the proof of Statement (2). ∎

Theorem 3 is initially limited to the case where an open SCC $\mathcal{G}[\Theta_{o,i}]$ is directly connected to one or multiple closed SCCs $\{\mathcal{G}[\Theta_{c_k,i}]\}_{k\in\{1,\ldots,h\}}$, with $h \geq 1$. However, our analysis can be extended to cases of indirect connections. This implies the existence of a directed path from the closed SCCs $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\ldots,h\}}$ to the open SCC $\mathcal{G}[\Xi_o(t)]$ that traverses other open SCCs of $\mathcal{G}[\tilde{\boldsymbol{B}}_{co}(t)]$. Significantly, in this extended case, the asymptotic results stated in Theorem 3 remain valid, provided that there are no competing logical interdependencies on topics covered by this directed path.

Remark 4. In the case where $h = 1$, indicating that the open SCC $\mathcal{G}[\Theta_{o,i}]$ is solely connected to one closed SCC $\mathcal{G}[\Theta_{c,i}]$, the proof of Theorem 3 states that $\Omega_1 = \Omega_2 = y_{co}^* \in [0, 1]$. Thus, Statement (2) of Theorem 3 includes Statement (1) as a special case.

Further to the conclusions of Theorem 3, we derive the following result for the case where agents' opinions on topics within an open SCC $\mathcal{G}[\Theta_{o,i}]$ take a trivial value.

Corollary 1. *Let the hypotheses in Theorem 3 hold. Suppose that an open SCC $\mathcal{G}[\Theta_{o,i}]$ is solely connected to multiple closed SCCs $\{\mathcal{G}[\Theta_{c_k,i}]\}_{k\in\{1,\ldots,h\}}$, and the maximum value of modulus consensus* *solutions to these closed SCCs is denoted by Ω. If any of the following statements holds, the opinions of agents on the m_o topics will eventually fall within the range of $[-\Omega, \Omega]$:*

(1) At least one of the closed SCCs $\{\mathcal{G}[\Theta_{c_k,i}]\}_{k\in\{1,\dots,h\}}$ is structurally unbalanced.
(2) The open SCC $\mathcal{G}[\Theta_{o,i}]$ is structurally unbalanced.
(3) There are agents with competing logical interdependencies on the $m_c + m_o$ topics.

Proof. In this proof, we follow the notations used in the proof of Theorem 3.

Statement (1): The asymptotic result we need to prove can be immediately obtained by referring to Statement (1) of Theorem 2 and Statement (2) of Theorem 3, given that the minimum value of modulus consensus solutions to these closed SCCs $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\dots,h\}}$ is 0.

Statement (2): According to Theorem 2, it is known that the opinions of agents on topics within $\mathcal{G}[\Theta_{o,i}]$ will eventually achieve a state of trivial consensus, provided that the irreducible logic matrix $\boldsymbol{C}_i$ is structurally unbalanced for all $i \in \mathcal{I}$. Here, for an open structurally unbalanced SCC $\mathcal{G}[\Theta_{o,i}]$, we assume that the maximum value of modulus consensus solutions to $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\dots,h\}}$ is Ω ($\Omega \geq 0$). Let $\Omega(t) = \max_{v_i\in\mathcal{V}_{\bar{c}}} |\tilde{y}_{co_i}(t)|$, and $K(t) = \max_{v_j\in\mathcal{V}_{\bar{o}}} |\tilde{y}_{co_j}(t)|$. Based on the consensus result in Theorem 2, we have $\lim_{t\to\infty} \Omega(t) = \Omega$. By referring to the analytical procedure of Statement (2) in Theorem 3, we find that $|\tilde{y}_{co_{i_1}}(t_1^* + \eta)| \leq \Omega(t_1^*) + (1-\beta^*)^\eta (K(t_1^*) - \Omega(t_1^*))$ holds for all $\eta \geq 1$ and a specific time $t_1^* \geq \tau$. From this, it follows that $\lim_{t\to\infty} \sup K(t) \leq \Omega$. In other words, the opinions of agents on the m_o topics will asymptotically be bounded by Ω. The proof is thus completed.

Statement (3): This statement can be derived by examining two scenarios involving competing logical interdependencies. In the first scenario, assume that there are agents with competing logical interdependencies on topics within the closed SCCs $\{\mathcal{G}[\Theta_{c_k,i}]\}_{k\in\{1,\dots,h\}}$. By applying Statement (2) of Theorem 2, it can be concluded that the corresponding closed SCC $\{\mathcal{G}[\Xi_{c_k}(t)]\}_{k\in\{1,\dots,h\}}$ is structurally unbalanced. Thus, the desired result follows by referring to the proof of Statement (1). In the second scenario, assume that there are agents with competing logical interdependencies on topics within the open SCC $\mathcal{G}[\Theta_{o,i}]$. In this case, the considered network still maintains the same spanning tree structure since competing logical interdependencies only affect the sign of edge weights, not the direction of these edges. By following similar discussions as in Statement (2), the desired result is thus claimed. ■

In summary, we outline the main contributions of this section. For any given set of logic matrices $\boldsymbol{C}_i$ and an initially connected influence network $\mathcal{G}[\boldsymbol{W}_0]$, Lemma 1 verifies the invariance of structural balance in the evolutionary influence network, while Lemma 2 establishes its almost-sure convergence to sign equilibria. Subsequently, Theorem 1 presents the convergence result for the coevolution model with irreducible logic matrices. Lastly, Theorem 2, 3, and Corollary 1 collectively provide sufficient conditions, addressing both irreducible and reducible logic matrices, to systematically determine the limiting distribution of agents' opinions on a given topic.

5. Numerical simulations

This section provides two examples to verify the theoretical results in Section 4. Given the coevolutionary dynamics in Definition 2, we establish an initially connected influence network $\mathcal{G}[\boldsymbol{W}_0]$ consisting of $n = 6$ agents, with

$$\boldsymbol{W}_0 = \begin{pmatrix} 0.2 & 0 & 0 & 0 & 0 & 0.8 \\ -0.7 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.15 & 0 & 0 & 0.45 \\ 0 & 0 & 0 & 0.1 & 0.9 & 0 \\ 0 & 0 & 0 & -0.6 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0 & -0.55 & 0.25 \end{pmatrix}.$$

Note that $\boldsymbol{W}_0$ satisfies Assumption 2. For each simulation example, the initial state $\boldsymbol{x}(0)$ is generated by selecting each $x_i^p(0)$ from a uniform distribution in $[-1, 1]$. Furthermore, we set $\sigma = 1\times 10^{-8}$ for $g_2[x]$ in (6), and instinctively select the step-size parameter $\epsilon = 0.1$ to fulfill Assumption 3. The examples below will help clarify the key role of the set of logic matrices in determining the limiting distribution of multidimensional opinions.

Example 1. For all $i = 1, \dots, 6$, the logic matrix $\boldsymbol{C}_i$ is given by

$$\boldsymbol{C}_i = \begin{pmatrix} \vartheta_i^1 & 0 & 1-\vartheta_i^1 & 0 & 0 \\ \vartheta_i^2 - 1 & \vartheta_i^2 & 0 & 0 & 0 \\ 0 & \vartheta_i^3 - 1 & \vartheta_i^3 & 0 & 0 \\ 0 & 0 & -\vartheta_i^4 & 1-(\vartheta_i^4+\vartheta_i^5) & \vartheta_i^5 \\ 0 & 0 & 0 & 1-\vartheta_i^6 & \vartheta_i^6 \end{pmatrix}, \quad (17)$$

where ϑ_i^k are drawn from a uniform distribution in $(0, 1)$ for $k = 1, \dots, 6$, and appropriate normalization is conducted to ensure the row-sum constraint of $\boldsymbol{C}_i$ in Assumption 1. The trajectories of influence evolution and opinion evolution are depicted in Fig. 1, indicating that the initially connected influence network converges to a fully connected structurally balanced configuration, and the opinions of agents on these five topics converge to the same bipartite consensus state. This result is consistent with Lemma 2, Statement (1) in Theorem 2, and Statement (1) in Theorem 3. To emphasize the key results concerning logic matrices, we omit the detailed evolution of influence weights in the following simulations. For a more comprehensive validation of the network's almost-sure convergence (Lemma 2), see the Monte Carlo simulations in the Appendix, where randomized trials confirm its eventual convergence to a sign equilibrium.

To illustrate the impact of structural imbalances and competing logical interdependencies, we make a simple modification to the simulation setup independently. In the former case, we simply change the sign of the $c_{21,i}$ entry of $\boldsymbol{C}_i$ to obtain a structurally unbalanced logic matrix. As depicted in Fig. 2(a), the opinions of agents on these five topics all reach a trivial consensus. In the latter case, assuming that there exist competing logical interdependencies on Topic 2 between Agent 4 and the other agents, we simply change the $c_{21,4}$ entry of $\boldsymbol{C}_4$ from $\vartheta_4^2 - 1$ to $1 - \vartheta_4^2$. The temporal evolution of agents' opinions, converging to a trivial consensus on all five topics, is illustrated in Fig. 2(b). With the above two modifications, the effectiveness of Statement (2) in Theorem 2, Statement (1) in Theorem 3, and Corollary 1 are verified.

Example 2. For all $i = 1, \dots, 6$, the logic matrix $\boldsymbol{C}_i$ is given by

$$\boldsymbol{C}_i = \begin{pmatrix} \vartheta_i^1 & 0 & 1-\vartheta_i^1 & 0 & 0 & 0 \\ \vartheta_i^2 - 1 & \vartheta_i^2 & 0 & 0 & 0 & 0 \\ 0 & \vartheta_i^3 - 1 & \vartheta_i^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\vartheta_i^4 & 0 & 1-(\vartheta_i^4+\vartheta_i^5) & \vartheta_i^5 \\ 0 & 0 & 0 & -\vartheta_i^6 & 1-(\vartheta_i^6+\vartheta_i^7) & \vartheta_i^7 \end{pmatrix}, \quad (18)$$

where ϑ_i^k, $k = 1, \dots, 7$ are drawn from a uniform distribution in $(0, 1)$ with appropriate normalization. Consistent with Statement

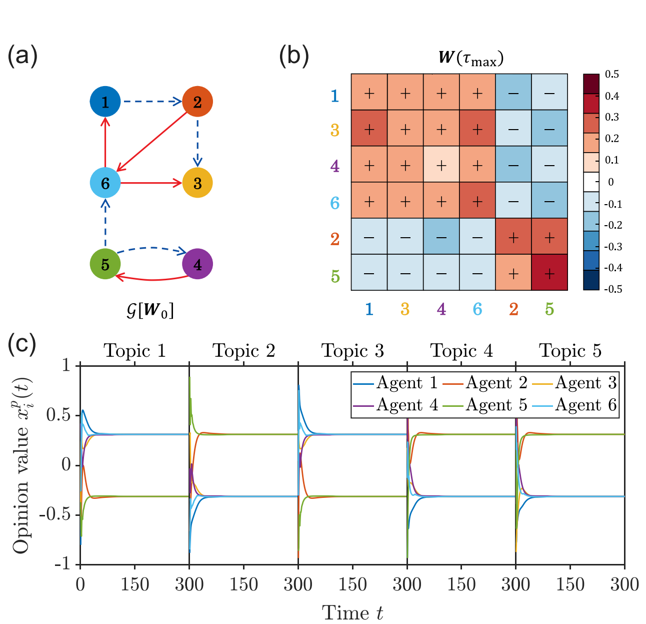

Fig. 1. An illustration of the coevolution of multidimensional opinions and influence weights. (a) The initial influence network $\mathcal{G}[\boldsymbol{W}_0]$ (self-loops and edge weights are hidden for clarity), where the red solid (resp., blue dashed) lines denote positive (resp., negative) edges. (b) The structure of a sign equilibrium $\boldsymbol{W}(\tau_{\max})$ for the initial influence matrix $\boldsymbol{W}_0$, with the symbol $+$ (resp., $-$) representing the positive (resp., negative) influence weight. (c) Evolutionary trajectories of $x_i^p(t)$, with $\boldsymbol{C}_i$ given in (17), $i = 1, \ldots, 6$, $p = 1, \ldots, 5$.

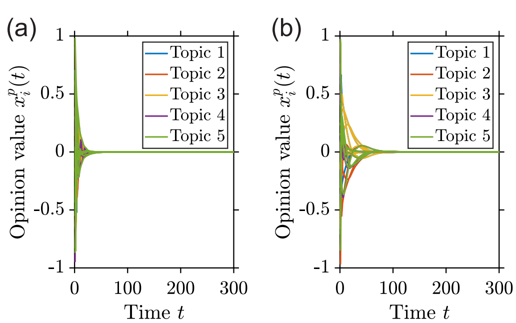

Fig. 2. Evolutionary trajectories of $x_i^p(t)$, with $\boldsymbol{C}_i$ given in (17), except (a) $\boldsymbol{C}_i$ has entry $c_{21,i} = 1 - \vartheta_i^2$, or except (b) $\boldsymbol{C}_4$ has entry $c_{21,4} = 1 - \vartheta_4^2$, $i = 1, \ldots, 6$, $p = 1, \ldots, 5$.

(1) in Theorem 2 and Statement (2) in Theorem 3, the agents' opinions on Topics 1–3 and Topic 4 separately reach a bipartite consensus, while the scope of their opinions on Topics 5 and 6 is determined by those on Topics 1–4. The temporal evolution of agents' opinions on these six topics is depicted in Fig. 3.

6. Concluding remarks

In this article, we have studied a coevolution model that incorporates multidimensional opinions and coopetitive influence networks. Inspired by the Altafini-type update rule and three well-established sociological mechanisms, we formulated the coevolutionary dynamics and analyzed its asymptotic properties. We established the almost-sure convergence of evolutionary influence networks, and provided a set of results to systematically determine the limiting distribution of multidimensional opinions regarding irreducible and reducible logic matrices. It turns out that the structure of logic matrices, including structural balance properties and competing logical interdependencies, plays a key role in determining whether opinions on a given topic will reach a bipartite consensus. Notably, the emergence of such a stable bipartite consensus, through adequate discussions between agents on several logically interdependent topics, reveals some realistic explanations for persistently changing but steadily existing interpersonal conflicts in a social group. Thus, our work offers a fresh perspective to understand the coevolutionary dynamics of multidimensional opinions over coopetitive networks.

The literature review in this article is restricted to linear consensus models, where the bipartite consensus is intricately linked to non-generic assumptions on the coupling topology. For future research, we aim to introduce nonlinear features of real-world opinion evolution, as suggested by Bizyaeva, Franci, and Leonard (2022), Mei, Bullo, Chen, Hendrickx, and Dörfler (2022) and Tian, Wang, and Bullo (2023), to capture a broader range of phenomena within human social groups.

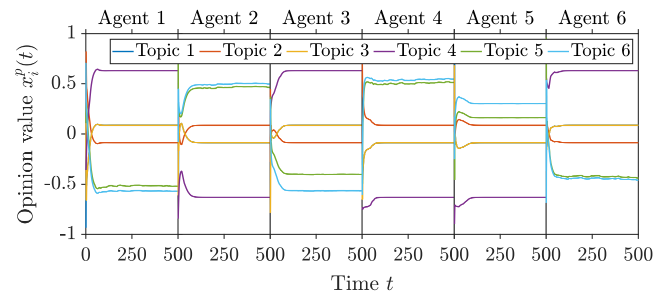

Fig. 3. Evolutionary trajectories of $x_i^p(t)$, with $\boldsymbol{C}_i$ given in (18), $i = 1, \ldots, 6$, $p = 1, \ldots, 5$.

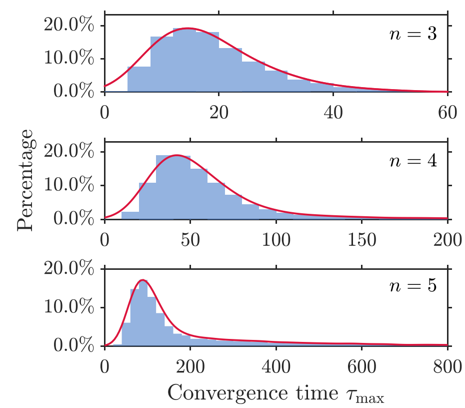

Fig. A.1. Histograms of convergence times, $\tau_{\max}$, across 8,000 independent trials for influence networks of size $n = 3, 4, 5$, with kernel-smoothed density estimates (red curves).

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Appendix. Monte Carlo validation of Lemma 2

To support the theoretical result of almost-sure convergence in the influence network, we conduct Monte Carlo simulations on small-scale networks with $n = 3, 4, 5$ agents, each involving $m = 5$ topics. For each n, we perform 8,000 independent trials, where the initial appraisal matrix $\boldsymbol{W}_0$ is randomly generated under Assumption 2. The initial opinion vector $\boldsymbol{x}(0)$ and logic matrices $\{\boldsymbol{C}_i\}$ are independently sampled, satisfying Assumption 1.

We track the convergence time, $\tau_{\max}$, required for the influence network to reach a sign equilibrium. Fig. A.1 presents the histograms of $\tau_{\max}$, confirming that all trials reach sign equilibria, thereby supporting Lemma 2. As expected, larger networks generally require more iterations due to the asynchronous update process, yet the distributions remain unimodal and well-bounded.

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