Maximizing synchronizability of networks with community structure based on node similarity

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ABSTRACT

In reality, numerous networks have a community structure characterized by dense intra-community connections and sparse inter-community connections. In this article, strategies are proposed to enhance synchronizability of such networks by rewiring a certain number of inter-community links, where the research scope is complete synchronization on undirected and diffusively coupled dynamic networks. First, we explore the effect of adding links between unconnected nodes with different similarity levels on network synchronizability and find that preferentially adding links between nodes with lower similarity can improve network synchronizability more than that with higher similarity, where node similarity is measured by our improved Asymmetric Katz (AKatz) and Asymmetric Leicht–Holme–Newman (ALHNII) methods from the perspective of link prediction. Additional simulations demonstrate that the node similarity-based link-addition strategy is more effective in enhancing network synchronizability than the node centrality-based methods. Furthermore, we apply the node similarity-based link-addition or deletion strategy as the valid criteria to the rewiring process of inter-community links and then propose a Node Similarity-Based Rewiring Optimization (NSBRO) algorithm, where the optimization process is realized by a modified simulated annealing technique. Simulations show that our proposed method performs better in optimizing synchronization of such networks compared with other centrality-based heuristic methods. Finally, simulations on the Rössler system indicate that the network structure optimized by the NSBRO algorithm also leads to better synchronizability of coupled oscillators.

Synchronization is one of the widely studied problems in complex systems and has a wide range of applications in the field of neuroscience, computer science, and so on. For example, synchronized communication, as a kind of data communication, is beneficial in reaching data consensus or achieving efficient communication due to its fast transmission rate and strong real-time performance. It has been found that many real-world networks have a distinct community structure, and such networks often have worse synchronizability due to the low connectivity among communities. Hence, a vital topic in this field is to find the optimal inter-community topology for networks with the community structure, for reaching consensus or achieving efficient communication, etc. Usually adding links between communities will improve network synchronizability, but this is achieved by increasing the communication cost of the network. Therefore, it is worth exploring to find the optimal link-addition method when the number of inter-community links is limited. In this article, we propose a node similarity-based rewiring algorithm and optimize it with a modified simulated annealing technique to choose the optimal links between communities. Numerical simulations show that the proposed method outperforms some well-known heuristic methods while having lower computation complexity. Our proposed algorithm can provide the design ideas for optimal interconnections between communities for real-world networks such as power networks, in terms of enhancing synchronizability and reducing control costs.

I. INTRODUCTION

Network science has made tremendous progress in the past two decades due to advances in computational tools and the high availability of data. As a general way of describing complex systems and complexity science, the methods and theories proposed by network science are widely used in disciplines such as physics, computer science, biology and so on.[1,2] In many systems, synchronization is always a very important dynamic process. Understanding and controlling the collective dynamics has significant theoretical and practical implications.[3–8] In recent years, complex networks with highly synchronized properties have had an increasingly wide range of applications in science and engineering, and different kinds of methods to enhance network synchronizability are constantly being proposed.[9–12] One common approach is to add new links directly to the network,[13–15] where the main issue is how to maximize synchronizability of the network by adding links as few as possible. In addition, network synchronizability can also be enhanced by means of link weighting[16–19] or rewiring.[20–23]

Apart from synchronization, link prediction, as an important part of studying network structural information and evolutionary trends, is also a heated topic in network science that can be used to extract missing information, identify spurious interactions, and evaluate network evolutionary mechanisms.[24–26] Most mainstream methods for link prediction are based on similarity,[27] which can be classified into three categories: local indices, global indices, and quasi-local indices. In addition, there is a class of complex methods in link prediction that are based on the likelihood analysis, such as the hierarchical structure model,[28,29] stochastic block model,[30] and so on. However, the above methods are proposed only from the perspective of network topology, where the dynamic characteristics of each node are not considered. In fact, the evolution of the network structure will cause the change of node dynamics, and in reverse, the node dynamics can influence the evolution mechanism of the network structure.[31] It is worth mentioning that network synchronization is exactly one of the areas where the relationship between the network structure and node dynamics is studied simultaneously.

To explore the special relationship between network structure evolution and node dynamics behaviors, Pan *et al.*[31] introduced the node dynamics model into static networks that are considered in link prediction and then analyzed the impact of the link prediction-based linking strategy on the synchronization behavior of network dynamics.[31] Through numerical simulations and theoretical analysis, it is found that networks that evolve based on link prediction are characterized by the stability of synchronizability, that is to say, the synchronizability of networks will not be greatly improved due to the newly added links. In addition, the real evolution of networks concludes both the link prediction part and the random evolution part, which reflects the slight difference in the dynamic mechanism between link prediction evolution and real network evolution. Inspired by this interesting phenomenon, a practical and significant question naturally comes to our mind whether synchronizability of the network can be improved by the operation of "anti-link prediction." Specifically, the "anti-link prediction" refers to artificially adding links between nodes with low similarity in the network rather than those with high similarity. This link-addition strategy may help to break the stability of synchronizability for normally evolving networks, thereby improving the synchronizability of networks.

Many real-world networks exhibit distinct community structure such as social networks and functional brain networks,[32–35] which are characterized by dense intra-community links and sparse inter-community links. Networks with a community structure generally have a worse synchronizability owing to the much lower proportion of inter-community links compared with that of intra-community links.[36] Previous studies found that in a network with a community structure, the inter-community links play a major role in determining network synchronizability.[37,38] To build a network with maximum synchronizability, the inter-community links should be placed optimally. A natural idea is to add connections between important nodes from different communities, such as high-degree nodes or high-betweenness nodes. Aguirre *et al.*[39] have discussed that connecting high-degree nodes from each community is the best strategy to improve synchronizability,[39] but they did not indicate which high-degree nodes should be connected. Therefore, to find the best inter-community links among several high-degree nodes requires trying all possible inter-community link combinations, which is generally impractical due to the high computational complexity. Zhao *et al.*[40] considered the cortical neural network and further explored the role of intra-community and inter-community links in balancing synchronization of individual communities and global synchronization of the whole network.[40] It is found in their study that randomly selecting end-nodes (i.e., the nodes at both ends of one link) to create inter-community links is more effective than connecting high-degree ones. Furthermore, Jalili *et al.* (2016) point out that another strategy is to limit the search space by restricting end-nodes to important nodes (e.g., high degree or high betweenness) from each community.[41] However, there may exist many possible connections between these important nodes from different communities, and finding the best combination among them still needs expensive cost.

In this article, we investigate how to set up inter-community connections (without changing the internal structure of the original community) for achieving optimal synchronizability of the entire network. We first define the similarity between nodes from the perspective of link prediction, wherein the network topology and node importance are both considered. It is found through simulations that adding links between nodes with lower similarity can improve synchronizability of networks more, compared to adding links between nodes with higher similarity. This finding to some extent confirms the rationality of our anti-link prediction conjecture. Based on previous statements, the node similarity-based strategy is chosen as the criterion to guide the selection of the best topology of inter-community connections. Then, a modified simulated annealing technique is adopted to optimize this rewiring process for maximizing synchronizability of the networks with a community structure. Numerical simulations on synthetic networks show that the proposed algorithm can find the near-best inter-community links, much better than several heuristic methods such as finding the best combinations of inter-community links between high-degree or high-betweenness nodes. Finally, simulations on the Rössler system further indicate that our algorithm also performs well for synchronization of coupled oscillators.

The rest of our work is organized as follows. Section II introduces the definition of network synchronization, the improved indicators that characterize the similarity of nodes, and two synthetic network models. Section III explains our proposed NSBRO algorithm. Section IV displays the experimental design and simulation results. Finally, Sec. V closes the article with conclusions and discussions.

II. SYNCHRONIZABILITY OF DIFFUSIVELY COUPLED NETWORKS

A. Definition of synchronizability

We consider a complex dynamic network consisting of N identical nonlinear coupled nodes, where each node is an n-dimensional dynamical system. The dynamics of the network can be described as

$$\dot{x}_i(t) = f(x_i(t)) - c\sum_{j=1}^{N} l_{ij} H\left(x_j(t)\right), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n$ is the state variable of node i, $f(\cdot): \mathbb{R}^n \to \mathbb{R}^n$ is a well-defined vector function representing the dynamic equation of nodes in the system, the constant c is the coupling strength, and $H(\cdot): \mathbb{R}^n \to \mathbb{R}^n$ is the coupling function. $L = (l_{ij})_{N\times N}$ is the Laplacian matrix corresponding to the adjacency matrix $A = (a_{ij})_{N\times N}$ of the network, defined as $l_{ij} = -a_{ij}$ when $i \neq j$ and $l_{ii} = \sum_{j\neq i} a_{ij}$ otherwise. Since $\sum_{j=1}^{N} l_{ij} = 0$, the coupling part on the right side of Eq. (1) disappears when the node states are all the same.

According to the method of the master stability function,[42] synchronizability of a network can be judged by observing the eigenvalues of its Laplacian matrix. In our work, we assume that the network is undirected and unweighted, and then its corresponding Laplacian matrix is symmetric and positive semi-definite. Thus, the eigenvalues of the Laplacian matrix L are all non-negative and can be arranged as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{\max}$. Typically, we use λ_2 and $R = \lambda_{\max}/\lambda_2$ to evaluate synchronizability of a network with unbounded and bounded synchronized regions, respectively. The larger the λ_2 is, the better synchronizability the network has. And the smaller the eigenratio R is, the better synchronizability the network has. Given that the measure of synchronizability is based on the master stability function, synchronization that our method aims to characterize in this article refers to complete synchronization.

B. Characterization of node similarity

To characterize the similarity between nodes in a network, many link prediction indicators are proposed based on the similarity of the network structure. Previous studies related to path similarity-based metrics, such as local path index,[43,44] Katz index,[45] and Leicht–Holme–Newman (LHNII) index,[46] have indicated that characterizing node similarity by considering the path length and the corresponding path number can exactly improve the link prediction accuracy, but most of them ignore the importance of nodes on the path. In fact, nodes with different importance tend to have different contributions to the formation of new links. Inspired by the dynamics of resource allocation in complex networks,[47] we think that important nodes will contribute less to the formation of direct links between unconnected nodes compared to the less important nodes. For example, as shown in Fig. 1, consider a pair of nodes i and j, which are indirectly connected and nodes k_1 and k_2 as their only common neighbors. Here, the only difference between nodes k_1 and k_2 is that node k_1 has more neighbors, which means that node k_1 is more important than node k_2 to some extent. In the simplest case, we assume that these two common neighbor nodes are not adjacent and each node in the diagram has a certain unit of resources that will

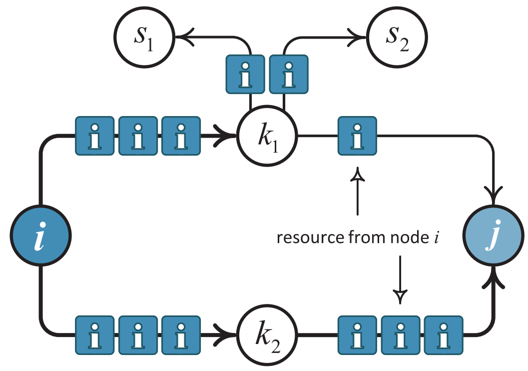

FIG. 1. Resources in node i are divided into two equal pieces and then flow to its neighbors. The thicker lines imply there are more resources flowing. It is worth pointing out that, to give a clearer illustration, the resources flow into node i or out of other nodes are not presented. Specifically, we only show the resource flows from the source node i here.

be equally distributed to all its neighbors. In this way, it is clear that node j will receive more resources from node i with the common neighbor node k_2 as the transmitter. Consequently, the probability of forming a direct connection via node k_2 between end-nodes i and j will be higher. Based on above illustrations, we hold that the impact of node importance on forming new links cannot be ignored. In detail, less important nodes on one path are more likely to promote indirectly connected nodes to form new connections than important nodes.

According to this analysis, we take the Katz index and LHNII index that are well-known in link prediction as examples, consider penalizing important nodes according to their centrality, and then the Asymmetric Katz (AKatz) index and the Asymmetric LHNII (ALHNII) index are proposed to measure the similarity between nodes. The asymmetry here refers to the unequal contribution between nodes to each other, which is reflected in the asymmetry of the importance contribution matrix. The specific form and other details will be introduced later. First, the AKatz algorithm defines

$$s_{xy}^{\mathrm{AKatz}} = \sum_{l=1}^{\infty} \beta^l \cdot |\mathrm{wpaths}_{xy}^{\langle l\rangle}| = \beta\tilde{A}_{xy} + \beta^2(\tilde{A}^2)_{xy} + \beta^3(\tilde{A}^3)_{xy} + \cdots \tag{2}$$

as the similarity between node x and node y in a network, where $\mathrm{wpaths}_{xy}^{\langle l\rangle}$ is the set of all weighted paths with l-hops connecting x and y, $|X|$ denotes the cardinality of set X, and β is a free parameter controlling the weight of paths with different hops. Smaller β means that long-hop paths contribute less to the similarity of nodes. Here, $\tilde{A}$ is the weight configuration matrix reflecting the average contribution between nodes, which can be constructed as follows.

First $\vec{h} = [\log^{-1}(\mathrm{e}+x_1), \log^{-1}(\mathrm{e}+x_2), \ldots, \log^{-1}(\mathrm{e}+x_N)]^T$ is defined to describe the penalty scale of each node in the network, where x_i represents the normalized centrality value of node i, $(\cdot)^T$ represents the transpose of $(\cdot)$, and the form $\log^{-1}(\mathrm{e}+x_i)$ is taken here to avoid negative numbers. Let $H = [\vec{h}, \vec{h}, \ldots, \vec{h}]_{N\times N}$. The importance contribution matrix $\hat{A}$ can be expressed as $\hat{A} = A \circ H^T$,

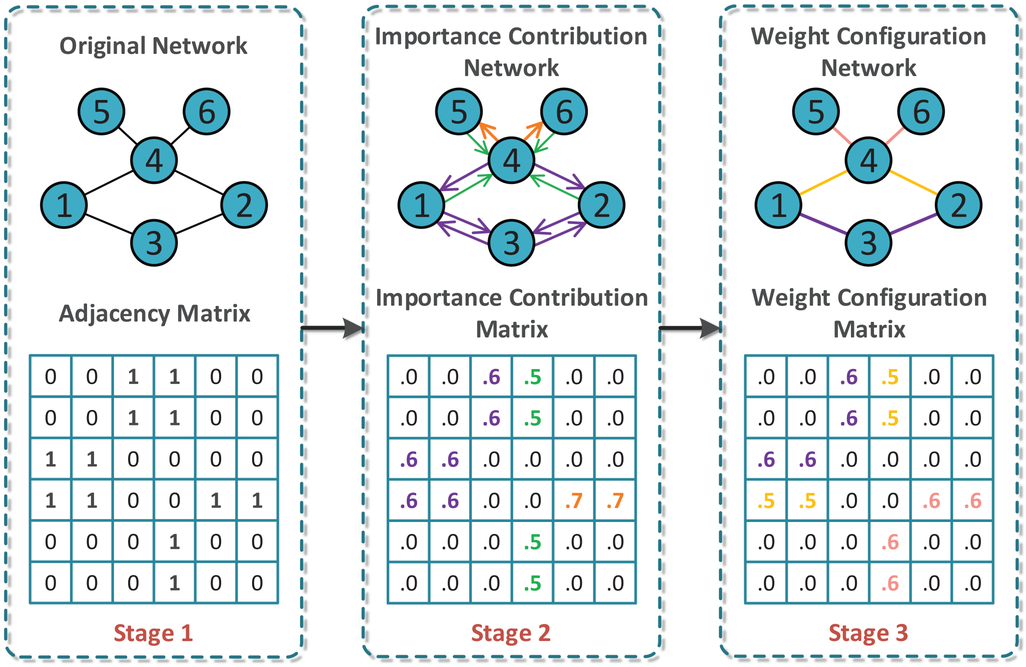

FIG. 2. The construction framework of the weight configuration matrix $\tilde{A}$. In stage 1, the original network is demonstrated, where the adjacency matrix is denoted as A. In stage 2, the importance contribution matrix $\hat{A}$ is obtained by calculating $\hat{A} = A \circ H^T$, where we take the simplest degree centrality as an example to measure the node resource in this diagram. In stage 3, the average link weight is characterized and the weight configuration matrix $\tilde{A}$ can be obtained according to $\tilde{A} = (\hat{A} + \hat{A}^T)/2$. The numerical results in stage 2 and stage 3 are rounded to one decimal place.

where A is the adjacency matrix of the original undirected and unweighted network, and the Hadamard product of $A = [a_{ij}]$ and $B = [b_{ij}]$ with the same dimensions is the matrix $A \circ B = [a_{ij} b_{ij}]$. Under the framework of the master stability function, the synchronizability of an undirected network can be quantified by the eigenratio R of the corresponding Laplacian matrix. The smaller R, the stronger the network synchronizability will generally be. Since the contribution between nodes is unequal, that is, the importance contribution matrix used to reflect the network topology is asymmetric, this spectral method cannot be directly applied to this kind of directed network because complex values will appear in the eigenvalues when the Laplacian matrix is asymmetric. Therefore, we use the average value of the importance contribution between two nodes as the weight of this connection, and then the symmetric weight configuration matrix can be obtained as $\tilde{A} = (\hat{A} + \hat{A}^T)/2$. An illustration of this construction process is provided in Fig. 2.

If the sequence represented by Eq. (2) converges, that is, parameter β is less than the reciprocal of the largest eigenvalue of the adjacency matrix, then the similarity matrix can be expressed as

$$S^{\mathrm{AKatz}} = (I - \beta \tilde{A})^{-1} - I, \tag{3}$$

where X^{-1} represents the inverse of the matrix X and I represents the identity matrix. By means of this matrix expression, the average similarity between nodes in a network can be easily calculated.

In the same way, a compact self-consistent matrix formula can be obtained based on the well-known LHNII method (the LHNII index is a variant of the Katz index, which holds that two nodes are similar if their immediate neighbors are themselves similar), i.e., the ALHNII algorithm, which can be expressed as

$$S^{\mathrm{ALHNII}} = 2m\lambda_1 \tilde{D}^{-1}\left(I - \frac{\phi \tilde{A}}{\lambda_1}\right)\tilde{D}^{-1}, \tag{4}$$

where m is the total number of edges in the network and λ_1 is the largest eigenvalue of the original adjacency matrix A. Here, $\tilde{D}$ is the node centrality matrix with $\tilde{D}_{ij} = \delta_{ij} x_j$, and $\phi (0 < \phi < 1)$ is a free parameter. The choice of ϕ depends on the investigated network, and smaller ϕ means assigning more weights on short-hop paths.

In the process of calculating the similarity of each pair of nodes according to Eqs. (3) and (4), we use the semi-local iterative algorithm (semi-IA) proposed by Luan *et al.*[48] to characterize the importance of nodes in the network.[48] The semi-IA index exhibits a better performance than many well-known indices due to the fact that the impact of the shortest path length and the shortest paths' number on node centrality is simultaneously considered, which is defined as

$$X_i(t+1) = \sum_{j=1, j\neq i, j\in\varphi_i}^{N} \frac{n_{ij}^{\gamma}}{d_{ij}} X_i'(t), \; i = 1, 2, \ldots, N, \tag{5}$$

where the initial value $X_i(t) = 0$ and n_{ij} and d_{ij} represent the shortest path number and the shortest path length between node i and node j, respectively. φ_i is the set of neighbors whose distance from node i is less than or equal to a given value r, $0 \leq \gamma \leq 1$ is the weighting factor of the shortest path number, and $X_i'(t)$ is the normalized importance value of node i at time t. When the iterative process of Eq. (5) reaches a steady state, we use these scores as the centrality values of nodes in the network.

C. Network models

Previous research has indicated that many real-world networks have a community structure. In this article, a simple approach is considered to build network models with the community structure.

In the first step, suppose the network consists of M communities with each community containing N nodes. Since real-world networks have the scale-free or small-world characteristic, each community here is constructed based on the Barabási–Albert scale-free model[49] and the Watts–Strogatz small-world model.[50] Specifically, we use the original preferential attachment mechanism proposed by Barabási and Albert for generating scale-free networks. Starting with m_0 randomly connected nodes, at each proceeding step, a new node with m links is added to the network. In this way, the degree distribution of these constructed networks follows a power law and the average degree is about $2m$. In order to construct Watts–Strogatz networks, first, a regular ring graph with N nodes each connected to its K-nearest neighbors is constructed, where K is an even number. Then, the edges are rewired one by one with a probability P, provided that self-loops and duplication of edges are prohibited. Obviously, the resulting networks have a high clustering coefficient, and the average degree is K.

In the second step, a few links are added between these communities, thus forming a network with the community structure. In this article, the goal is to optimize synchronizability of such networks, specifically to find a certain number of inter-community links that can maximize synchronizability of the whole network. For simplicity, we consider the case where the network only contains two communities, i.e., $M = 2$, and set $m = 2, 3, \ldots, 10$, $K = 4, 6, \ldots, 20$, and $P = 0.1$ in the following parts.

III. MAXIMIZING SYNCHRONIZABILITY THROUGH EFFICIENT REWIRINGS

Networks with high synchronization properties are required in many practical applications, such as designing efficient sensor networks and optimizing computer performance based on the synchronization of processes. There are many methods designed to enhance synchronizability of networks, such as assigning appropriate coupling strength to certain links, giving directionality to the links, rewiring or adding appropriate links in the network, etc. In this work, we use the rewiring strategy to minimize the eigenratio R of networks with the community structure, whereby the average degree properties of the network are preserved.

Previously, Pan *et al.*[31] concluded that networks that evolve based on link prediction are characterized by the stability of synchronizability.[31] Combined with the definition of similarity indices in link prediction, it shows that the stability is determined by the principle of link prediction. Inspired by this result, we decide to further explore the effect of the similarity-based evolution process on network synchronizability. First, we take the scale-free network and the small-world network with each network consisting of 100 nodes as examples. According to Eqs. (3) and (4), the similarity of each pair of nodes is calculated, respectively. Among all unconnected nodes, only one link is added at a time and then deleted after the effect of this added edge on network synchronizability is investigated. This operation is terminated until every pair of unconnected nodes has been considered. Define the Similarity Ranking (SR) as the similarity level order among all pairs of unconnected nodes in a network and then implement the above experimental procedure over 500 independent experiments. It can be seen from Fig. 3 that adding links between nodes with lower similarity level is generally more conducive to the improvement of network synchronizability, compared with adding links between those with higher similarity level. Therefore, we initially believe that adding links between nodes with the lowest similarity is the realistic choice to improve synchronizability of one network.

In order to compare the performance between the above node similarity-based link-addition strategy and the best link addition strategy, we observe the network synchronizability by adding one link between two nodes with the lowest similarity level and adding the globally optimal one. Furthermore, the cases of adding one link between two nodes with the highest degree centrality or betweenness centrality are also compared. The above procedure is performed over 500 independent realizations. As shown in Fig. 4, under two node similarity-based link-addition strategies, synchronizability for both BA networks and WS networks is improved with the increase in the average degree. For each average degree of the BA network or the WS network, the proposed link-addition strategy is all much closer to the global optimal link-addition strategy, which embodies in enabling the network to reach almost the global highest eigenvalue λ_2 and the global lowest eigenratio R. In addition, our proposed strategies are always better than heuristic methods such as preferentially adding links between nodes with the largest degree or largest betweenness, especially for cases where the synchronized regions are bounded. It is worth mentioning that when the synchronization domain is bounded, the node similarity-based link-addition strategies are evidently more advantageous than the node centrality-based strategies for smaller m and smaller K, and all methods appear to have the same performance for $m > 7$ and $K > 16$. A possible explanation is that an increase in the network average degree means an increase in the total number of edges. Therefore, no matter which edge is added to a network with a large average degree, it will not significantly impact the network synchronizability. In addition, finding the global optimum requires trying all possible combinations between unconnected nodes, which is not practical in large-scale networks. However, our proposed similarity-based strategy provides a clear link-addition scheme, which can be effectively applied to large-scale networks and can serve as an efficient approach to find optimal solutions in terms of maximizing network synchronizability. For simplicity, we assume that the synchronized regions of networks are bounded, and the eigenratio R is adopted as the indicator to measure the network synchronizability in the following work.

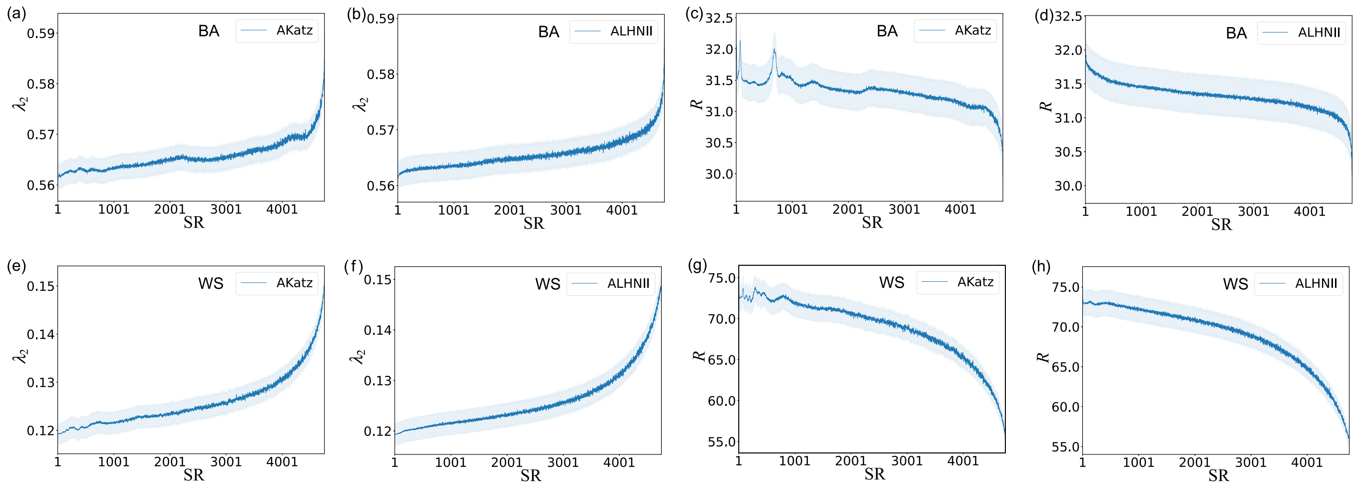

FIG. 3. The effect of adding links between unconnected nodes with different similarity levels on network synchronizability. The abscissa is the similarity ranking (SR), the larger the SR, the lower the similarity between the corresponding nodes. The ordinate represents synchronizability of the network after adding one edge between a pair of unconnected nodes. The upper panels [(a)–(d)] and the lower panels [(e)–(h)] are simulation results under BA scale-free networks and WS small-world networks, respectively. Here, the network size $N = 100$, the average degree is about 4 ($m = 2$, $K = 4$) and the rewiring probability of WS networks is $P = 0.1$. The results averaged over 500 independent experiments are reported as blue lines and shaded areas, representing mean values and values within the 95% confidence interval.

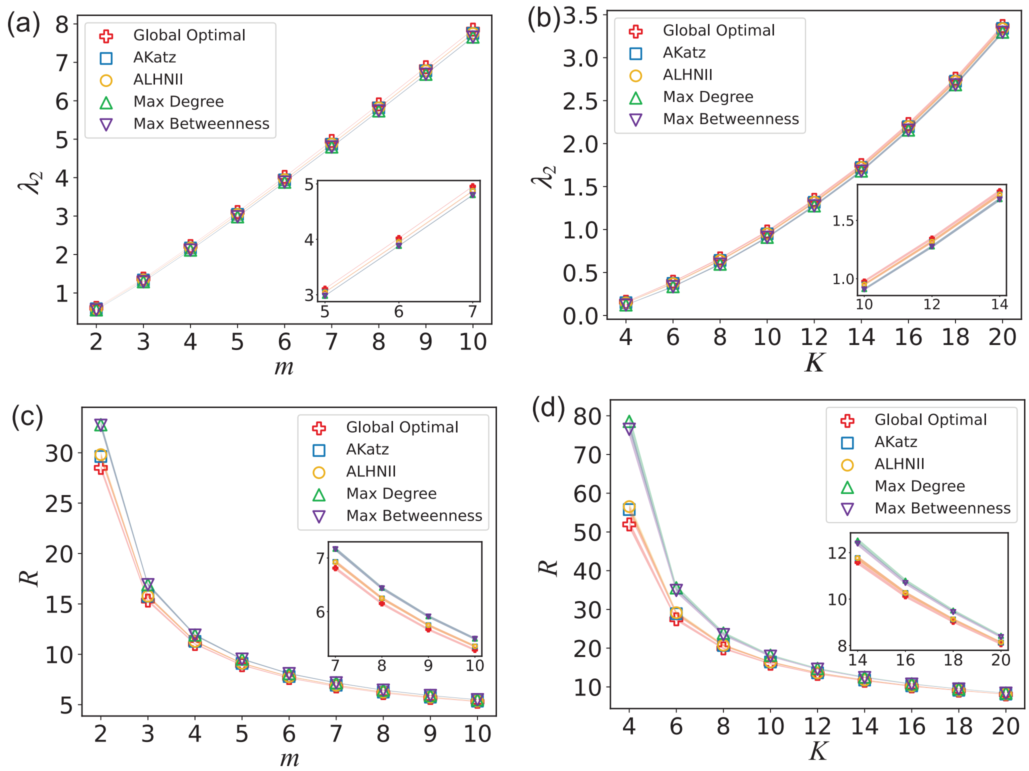

FIG. 4. The smallest nonzero eigenvalues λ_2 and the eigenratio R varying with m for the BA network [(a) and (c)] and varying with K for the WS network [(b) and (d)]. Here, λ_2 and R are computed by adding only one link to the original network based on different link-addition strategies. All results are averaged over 500 independent experiments. Other parameter settings are the same as Fig. 3.

Early work in the field of optimizing network synchronizability was based on stochastic rewiring and optimization algorithms, such as simulated annealing. But this kind of random rewiring strategy typically requires millions of iterations until the algorithm converges because valid criteria for link-addition and link-deletion are not provided.[41] However, the previously introduced Eqs. (3) and (4) for characterizing node similarity can be used to guide the rewiring operations. Specifically, after calculating the similarity between all nodes in the network, it is considered that the probability of disconnecting one link between any pair of connected nodes is proportional to $\exp(S)$, and the probability of creating one link between any pair of unconnected nodes is proportional to $\exp(-S)$, where $\exp(\cdot)$ represents the exponential function. Here, the rewiring operations are performed one after the other, that is, one link is disconnected and another is created at each iteration step. In fact, the greedy rewiring operation will often cause the solution to get stuck in a local optimum. To avoid this situation and find the global optimum as much as possible, we employ a modified simulated annealing optimization technique.[51] In this way, the network structure with better synchronizability can be obtained.

Based on the above analysis, our proposed Node Similarity-Based Rewiring Optimization (NSBRO) algorithm is summarized as follows:

(1) Two disjoint communities Q_1 and Q_2 are constructed using the Barabási–Albert scale-free model or the Watts–Strogatz small-world model. Then, a certain number of inter-community links are created between randomly selected nodes from these two communities, whereby a network with the community structure is constructed.
(2) Calculate the eigenratio of the initial network $R_0 = \lambda_N/\lambda_2$. In the first iteration step, $R_{\min} = R_0$.
(3) The similarity matrix S of the network is calculated according to Eqs. (3) and (4). Since the rewiring operations are limited to inter-community links, all entries corresponding to intra-community links are forced to be 0. Then, existing inter-community links are selected and disconnected with a probability proportional to $\exp(S)$ [this can be achieved by normalizing each element of $\exp(S)$ with the maximum value and taking it as a probability value], and nonexisting inter-community links are selected with a probability proportional to $\exp(-S)$ and create a new inter-community link between these non-adjacent nodes.
(4) After rewiring the inter-community links, eigenratio R of the new network is recalculated. If $R < R_{\min}$, it means that the rewiring operation improves the network synchronizability, and thus, we accept this rewiring operation and update $R_{\min} = R$. Otherwise, it means that this rewiring operation fails to promote or even inhibits the improvement of network synchronizability, and then this rewiring operation will be accepted with probability p, which is defined as $p = \exp[10(R_{\min} - R)]$.
(5) The rewiring algorithm is terminated when a predetermined number of iterations is reached or when the network eigenratio keeps unchanged. Otherwise, return to step (3) to continue the rewiring optimization process.

In this work, we restrict rewiring operations to the links between communities, but the operations generally can be extended to all links across the whole network. Owing to the number of inter-community links remaining the same during the rewiring process, the degree sequence of the network remains unchanged. In addition, since the rewiring process does not involve links within each community, the heterogeneity or homogeneity of communities in the network remains almost unchanged, whereby the eigenratio of the whole network can be optimized to some minimum value.

IV. SIMULATION RESULTS

A. Enhancement of the eigenratio

We build two networks Q_1 and Q_2 with node number $N = 50$ using the BA scale-free and WS small-world models. The parameters are set as $m_0 = m = 2$ for the BA network and $K = 4$ and $P = 0.1$ for the WS network. Then, how to place a certain number of links between Q_1 and Q_2 that can enable the entire network to achieve maximum synchronizability is explored. In this article, we consider the cases with 5 and 10 links between the two communities, respectively.

Figures 5(a) and 5(b) represent the eigenratio R as a function of iteration time t under the two proposed rewiring optimization algorithms NSBRO-AKatz and NSBRO-ALHNII, respectively. The experimental results here are calculated over 50 independent realizations. It is clear that the rewiring process can improve synchronizability of all networks. Specifically, when only five inter-community links are added, the rewiring optimization algorithms can improve synchronizability of networks with BA and WS models as individual community structures by more than 29% and 27% on average, respectively. In addition, the improvement is more than 21% and 16% on average, respectively, when there are 10 inter-community links. It is worth mentioning that synchronizability of the optimized BA network is worse than that of the optimized WS network when the number of inter-community links is fixed. This is mainly due to the higher heterogeneity of the BA network compared to the WS network. Since we did not rewire the intra-community links in the process of optimization, the original heterogeneity or homogeneity of the model is almost preserved. As we all know, synchronizability of a heterogeneous network is generally worse than that of a corresponding homogeneous network.

Next, we investigate the topological characteristics of the nodes that are selected to be connected by NSBRO-AKatz and NSBRO-ALHNII algorithms. In this respect, some well-known centrality indices, such as degree,[52] betweenness,[53] vulnerability,[54] eigenvector,[55] closeness,[56] and clustering,[57] are first adopted. Then, the average end-node rankings of inter-community links for NSBRO-based optimized networks are calculated over 50 independent experiments. The results in Tables I and II indicate that the degree centrality of nodes is more influential in the NSBRO-based optimization process, followed by the betweenness centrality and others. However, the NSBRO-based algorithm does not select all highest degree nodes from each community to form inter-community links. In other words, if the end-nodes are all the highest-degree nodes, the average ranking for them would be 3 (5.5) when 5 (10) inter-community links exist. However, when there are 10 links between these two communities, the average ranking for the 10 chosen nodes is 7.6 for scale-free topologies and 10.4 for small-world topologies. In addition, the proposed rewiring algorithm will

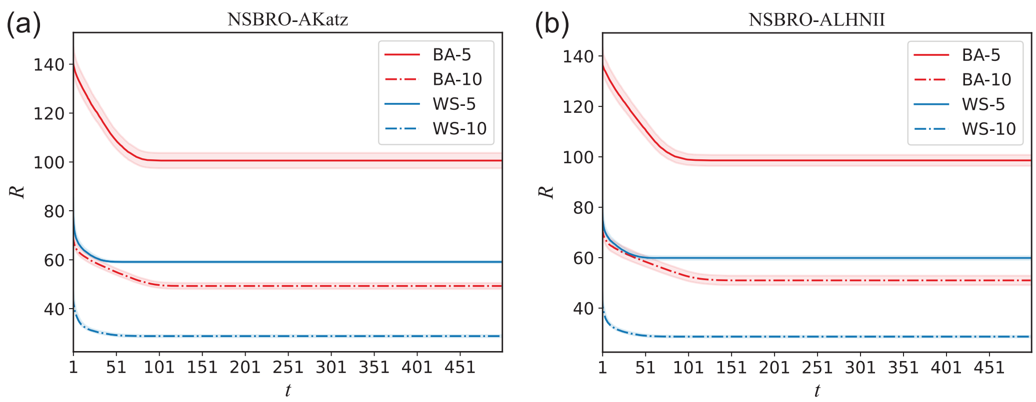

FIG. 5. The eigenratio R as a function of iteration time t in networks with the BA and WS models as individual community structures. Panels (a) and (b) represent the rewiring optimization process under NSBRO-AKatz and NSBRO-ALHNII algorithms, respectively. BA-5 stands for the case where there are five links between communities with the BA network as the topology of individual communities. Here, there are two communities each with $N = 50$ nodes and an average degree 4. Data show mean values and standard errors over 50 independent experiments.

be more inclined to select those less important nodes to form inter-community links, with the increase of inter-community link numbers. This phenomenon can be explained from the perspective of information dissemination that the high-influence nodes generally have more similar neighbors and the overlap of influence area is large, and thus adding links only between these hub nodes is often not the best way for information to spread quickly in the entire network.

To validate the effectiveness of our proposed NSBRO algorithm, we also compare its performance with several heuristic methods for selecting inter-community links. In this part, we adopt the method of restricting search space to find the best possible topology optimized based on heuristic methods. Specifically, an exhaustive search is performed among the top-H hub nodes in each community, so that the linear search method becomes feasible, where H is the number of inter-community links. Table III shows the eigenratio R based on our proposed NSBRO algorithms, the best top-5 nodes in terms of several centrality indices, and the random selection of 5 nodes in each community. The results are obtained from 50 independent average experiments, which exhibit that the proposed algorithms can significantly enhance network synchronizability compared with centrality-based heuristic algorithms, especially for networks with the WS model as individual community structures. By observing the values of eigenratio R before and after optimization in Table III, it is surprising to find that adopting

TABLE I. The centrality ranking of nodes that are selected to be connected from each community by the NSBRO-AKatz algorithm.[a]

Centrality metric	BA-5	BA-10	WS-5	WS-10
Degree	3.8 ± 0.5	7.5 ± 0.6	6.1 ± 0.9	10.4 ± 1.5
	4.0 ± 0.5	8.2 ± 0.7	6.3 ± 1.1	9.7 ± 1.9
Betweenness	4.7 ± 0.7	9.4 ± 0.9	20.4 ± 2.4	26.4 ± 1.8
	4.0 ± 0.5	8.2 ± 0.7	6.3 ± 1.1	9.8 ± 1.9
Vulnerability	4.6 ± 0.7	9.8 ± 1.0	20.0 ± 2.2	25.1 ± 1.6
	4.9 ± 0.8	10.7 ± 1.0	20.2 ± 2.3	25.5 ± 1.6
Eigenvector	6.9 ± 1.2	13.1 ± 1.2	20.5 ± 1.1	24.4 ± 0.8
	7.6 ± 1.3	13.8 ± 1.4	20.8 ± 1.4	24.1 ± 0.9
Closeness	5.9 ± 1.1	12.3 ± 1.3	21.5 ± 1.7	25.3 ± 1.4
	6.4 ± 1.1	13.0 ± 1.3	21.7 ± 1.6	25.4 ± 1.1
Clustering	13.9 ± 1.2	14.6 ± 1.1	15.7 ± 3.2	13.8 ± 1.9
	13.3 ± 1.4	14.8 ± 1.2	15.9 ± 3.1	13.1 ± 2.2

[a]The ranking in this table is in the form of $m \pm n$, where m and n denote the mean and the standard deviation of data, respectively. The experimental results are obtained from 50 independent average experiments.

TABLE II. The centrality ranking of nodes that are selected to be connected from each community by the NSBRO-ALHNII algorithm.[a]

Centrality metric	BA-5	BA-10	WS-5	WS-10
Degree	4.0 ± 0.4	7.7 ± 0.7	6.2 ± 1.3	10.4 ± 1.8
	4.0 ± 0.5	8.2 ± 0.6	5.9 ± 0.9	10.5 ± 1.8
Betweenness	4.8 ± 0.7	9.9 ± 1.0	20.8 ± 2.7	25.7 ± 1.4
	4.0 ± 0.5	8.2 ± 0.6	5.9 ± 0.9	10.5 ± 1.8
Vulnerability	4.8 ± 0.6	10.0 ± 1.0	20.3 ± 2.5	25.6 ± 1.4
	4.7 ± 0.6	10.6 ± 0.9	19.5 ± 2.2	25.6 ± 1.6
Eigenvector	6.8 ± 1.1	13.3 ± 1.1	20.6 ± 1.3	24.4 ± 0.9
	7.1 ± 1.2	14.1 ± 1.1	20.1 ± 1.2	24.2 ± 1.0
Closeness	6.2 ± 0.9	12.6 ± 1.2	21.8 ± 1.7	25.4 ± 1.2
	6.2 ± 0.9	13.3 ± 1.1	20.9 ± 1.7	25.2 ± 1.2
Clustering	13.5 ± 1.0	14.6 ± 1.1	15.7 ± 3.3	13.9 ± 1.8
	13.0 ± 1.4	14.6 ± 1.2	16.8 ± 3.0	14.1 ± 1.9

[a]The explanations are the same as Table I.

the centrality-based methods will reduce synchronizability of networks with the WS model as individual community structures. Since it is impractical to search all possible connections, the end-nodes of inter-community links are limited to those with the highest centrality. On the contrary, our proposed NSBRO algorithm can search all the space and find the (close to) optimal solution. In addition, it is worth mentioning that finding the optimal inter-community topology between the top-H nodes becomes more complicated and often impractical with the number of inter-community links increasing. To find the optimal topology in such a case, one has to perform the order of H^H eigenvalue computation. However, our proposed algorithm can usually reach convergence in cH steps with c being a relatively small number. In Appendix A, we numerically analyze the complexity of the similarity- and centrality-based algorithms. It is clear that the proposed NSBRO algorithm is more efficient than the methods that restrict end-nodes to important nodes, especially for large values of H. In fact, the proposed NSBRO method can be applied to optimize synchronizability of networks with any network structure and any number of inter-community links, as long as the eigenvalues of the Laplacian are computable.

It should be noticed that the parameters chosen for the experimental design that led to results in Table III are $m = 2$ and $K = 4$, respectively. These values correspond to the values in Fig. 4 for which our proposed methods outperform the other methods. Therefore, it is necessary to explore the parameter space (m_1, m_2) and (K_1, K_2) to guarantee a broader understanding of our methods' strengths and limitations, where m_i and K_i represent the network structure parameters of community $Q_i, i = 1, 2$ here. In Fig. 6, we take the NSBRO-AKatz method and the Max-Degree method as examples. The values in the grid represent the gap in maximizing network synchronizability in terms of the above two methods, and the larger the value, the more effective our proposed method is. It can be observed that the advantage of our proposed similarity-based method over the centrality-based method would decrease when the average degree of both adjacent communities is relatively large. On the contrary, when there exist communities with a small average degree, our method would have a significant outperformance in optimizing the network synchronizability. This phenomenon is especially obvious when the network structures of two adjacent communities are both constructed by the WS model. In Appendix B, we also investigate the performance gap of the proposed method relative to the centrality-based method when the network structures of two adjacent communities are inconsistent. It can be seen from Fig. 8 that when the average degree of the community with the WS model as the network structure is small, regardless of the average degree of the other community with the BA model as the network structure, the proposed method will have a more significant advantage compared to the centrality-based method.

TABLE III. The Eigenratio R of the networks when the inter-community topologies are optimized based on the NSBRO algorithm, the best option between nodes with the highest centrality and random selection. The experimental results are obtained from 50 independent average experiments.[a]

Algorithm	BA	WS
NSBRO-Akatz	99.9 ± 3.3	59.1 ± 0.7
NSBRO-ALHNII	98.5 ± 3.1	59.8 ± 1.2
Max-Degree	107.4 ± 3.8	94.6 ± 3.8
Max-Betweenness	108.4 ± 4.4	89.3 ± 2.7
Max-Vulnerability	107.9 ± 4.3	89.5 ± 3.8
Max-Eigenvector	106.5 ± 4.5	142.3 ± 7.8
Max-Closeness	108.8 ± 4.6	96.7 ± 5.0
Max-Clustering	154.2 ± 8.3	112.0 ± 9.8
Random	148.8 ± 9.9	95.4 ± 9.2

[a]The values of eigenratio R before optimization are averaged at about 140.3 and 81.4 for networks with the BA model and the WS model as individual community structures, respectively.

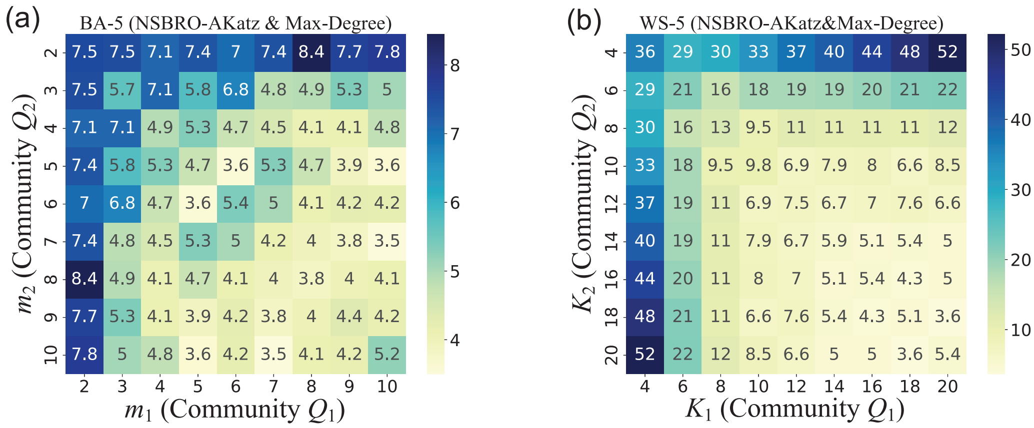

FIG. 6. The performance gap in maximizing network synchronizability in terms of the NSBRO-AKatz method and the Max-Degree method for all combinations of parameter pairs (m_1, m_2) and (K_1, K_2) that listed in Fig. 4. Here, the left panel (a) and the right panel (b) correspond to the cases where both community structures are constructed by the BA network model and the WS network model, respectively. Each data in the grids are averaged over 40 independent realizations.

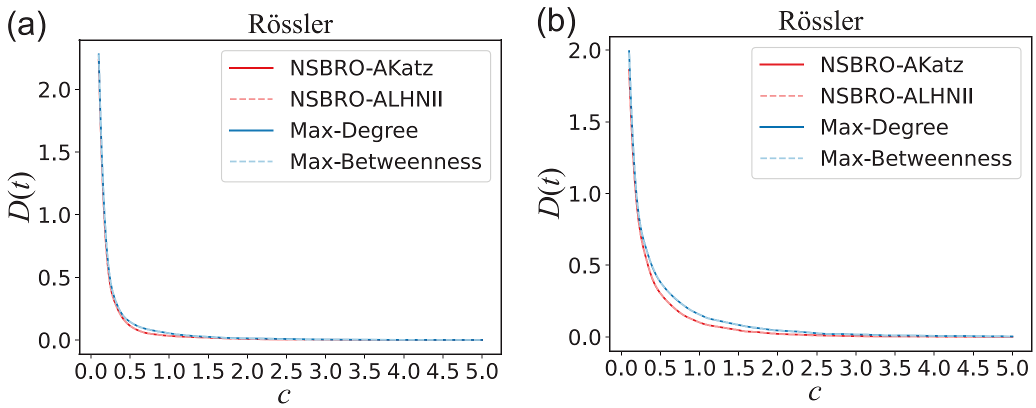

FIG. 7. Synchronization error $D(t)$ at $t = 5$ as a function of the coupling strength c for networks optimized by NSBRO algorithms and centrality-based heuristic methods that find the best five inter-community links between the five highest-degree (Max-Degree) or highest-betweenness (Max-Betweenness) nodes from each community. Here, $D(t)$ is obtained from 40 independent average experiments of randomly set initial values.

B. Enhancement of phase synchronization

For the purpose of evaluating whether the network structures optimized for eigenratio R by our proposed NSBRO algorithms still have good synchronization properties of coupled oscillators, we employ the Rössler system to analyze the dynamic behavior on these optimized networks.[58] Here, we consider an optimized system formed by two disjoint communities Q_1 and Q_2, each consisting of N Rössler systems interconnected by a certain number of links, which is described by

$$\dot{x}_i(t) = f_i\,(t, x_i(t)) - c\sum_{j=1}^{2N} l_{ij} H(x_j(t)), \quad i = 1, 2, \ldots, 2N, \tag{6}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T$ is the state variable of the ith node and $f_i\,(t, x_i(t))$ is the three-dimensional dynamic equations of node i. In detail, $f_{i1}\,(t, x_i(t)) = -x_{i2} - x_{i3}$, $f_{i2}\,(t, x_i(t)) = x_{i1} + 0.15x_{i2}$, and $f_{i3}\,(t, x_i(t)) = 0.2 + x_{i3}(x_{i1} - 4.7)$. $H = [1, 0, 0; 0, 1, 0; 0, 0, 1]$ is the inner coupling matrix.

To quantify the synchronization degree of the Rössler systems in optimized networks, the complete synchronization error is introduced as

$$D(t) = \frac{1}{2}\sum_{i=1}^{2N}\sum_{j=1}^{3}\left(x_{ij} - \bar{x}_j\right)^2, \tag{7}$$

where $\bar{x}_j = [1/(2N)]\sum_{i=1}^{2N} x_{ij}$ is the average value of the jth $(j = 1, 2, 3)$ state variable of all systems in the optimal networks. Obviously, the closer the $D(t)$ value is to 0, the better the synchronizability of the Rössler systems. In numerical simulations, we adopt the fourth-order Runge–Kutta method to solve the differential equations shown by Eq. (6), where the initial values are arbitrarily selected in the interval $[0, 1]$ and the time step is set to 0.1.

Figure 7 displays the synchronization error varying with the coupling strength for networks optimized by NSBRO algorithms and centrality-based heuristic methods, wherein the optimized networks considered here are the networks with the lowest eigenratio reported in Table III. In this part, we explore the synchronization degree of coupled oscillators at time $t = 5$ and the complete synchronization error $D(t)$ is calculated over 40 independent realizations. It is seen that synchronizability of the systems is enhanced with increasing c for the two types of network structures optimized by all optimization methods. Furthermore, we can also find that $D(t)$ of NSBRO-AKatz and NSBRO-ALHNII is evidently lower than that of Max-Degree and Max-Betweenness when the coupling strength c is about 0.7 for both two types of networks. In addition, our method exhibits a prominent advantage over a larger range of coupling strengths c for networks with the WS model as individual community topology. It is again shown that the network structure optimized based on the NSBRO algorithm exhibits better synchronizability than those centrality-based heuristic methods, such as finding the best combinations of inter-community links between hub nodes.

V. CONCLUSIONS AND DISCUSSIONS

In this article, we proposed a node similarity-based strategy to enhance synchronizability of networks with a community structure by rewiring a certain number of inter-community links, where the node similarity is measured by the link prediction method. First, we proposed two improved link prediction algorithms called the Asymmetric Katz (AKatz) method and the Asymmetric LHNII (ALHNII) method by considering the fact that the contribution between nodes is unequal. These two methods can be used to measure the similarity between each pair of nodes in a network, and it is found through our simulations that adding links between nodes with a low similarity level is more conducive to the improvement of network synchronizability than adding links between nodes with a high similarity level. In addition, the proposed link-addition strategy based on node similarity has a close effectiveness as the global optimal link-addition strategy which enables the network to achieve maximal synchronizability and exhibits better performance than heuristic methods such as preferentially adding links between nodes with the largest degree or largest betweenness, especially for networks with a smaller average degree. Furthermore, the node similarity-based methods are chosen as a standard rule to guide the selection of the optimal inter-community topology for networks with the community structure, in terms of enhancing synchronizability of the entire network. Then, a modified simulated annealing technique is adopted to optimize this rewiring process for finding the global optimal solutions. Our research found that NSBRO-based algorithms can significantly improve synchronizability of networks with the BA model and the WS model as each community topology compared with several centrality-based heuristic methods, especially when the average degree of each community is small. In addition, we investigate the properties of end-nodes connected to the final inter-community links optimized by the NSBRO algorithm and discover that the centrality values of these nodes are much smaller than the hub nodes. Finally, we apply the Rössler system to evaluate the properties of synchronization for optimal networks. It is concluded that the network structure optimized based on the NSBRO algorithm shows better synchronizability than the ones based on heuristic methods, such as finding the best inter-community links between the high-degree or high-betweenness nodes from each community.

It is important to delimit the scope of our findings. The major limitation of our results is that they are obtained in the context of complete synchronization, where the proposed method is limited to undirected and diffusively coupled dynamic networks with only two communities of the same size. However, it has potential for a much wide range of real-world networks. A possible direction for future work is to develop this method by considering the complexity of the real world, such as unbalanced networks with more than two communities or even giving the edge directionality. The second limitation is that our method has not fully accounted for all the types of synchronized regions in the latter simulations. However, according to the experimental results about the nonzero smallest eigenvalue λ_2 in the article, we can find that it is not difficult to extend our method to the case where the network synchronized regions are unbounded. Future work may further enrich this aspect by designing new controlled experiments to assess our proposed method's effectiveness more comprehensively.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Yangyang Luan: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Resources (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review and editing (equal). **Xiaoqun Wu:** Formal analysis (equal); Funding acquisition (equal); Supervision (equal); Validation (equal); Writing – review and editing (equal). **Binghong Liu:** Supervision (equal); Validation (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: ALGORITHM COMPLEXITY ANALYSIS

In Sec. IV A, we described the NSBRO algorithms that can be used to maximize synchronizability of networks with the community structure. In Table IV, we record the simulation time of the NSBRO-based algorithms in the case of 5, 10, and 15 inter-community links, where the results are obtained from 50 independent average experiments. In addition, the simulation time of the centrality-based methods in the case of 5 inter-community links is also obtained, which takes about 40 s on average for one independent experiment. It is worth mentioning that computing the simulation time for centrality-based heuristic methods in the presence of 10 and 15 inter-community links is significantly expensive, as the order of eigenvalue computations would be 3.2×10^6 times and 1.4013×10^{14} times than that in the case of 5 inter-community links.

APPENDIX B: PARAMETER SPACE INVESTIGATION FOR COMMUNITIES WITH DIFFERENT STRUCTURES

The case where the two adjacent communities are constructed by different network models is considered in this part. Specifically, one community of the given network is a scale-free topology and the other is a small-world topology. For all combinations of parameter pairs (m, K), the performance gap in maximizing network synchronizability in terms of the NSBRO-AKatz method and the Max-Degree method is investigated in Fig. 8.

TABLE IV. Simulation time (s) for 50 simulations on networks with the BA model and the WS model as each community topology. Other parameter settings about the network structure are the same as in Table III. Simulations were conducted in MATLAB version 2018b on a Quad-Core Intel Core i5 processor.

NSBRO-?	BA-5	WS-5	BA-10	WS-10	BA-15	WS-15
Akatz	160.8	208.3	366.4	443.9	599.5	595.7
ALHNII	379.9	252.3	764.7	416.8	1012.5	504.5

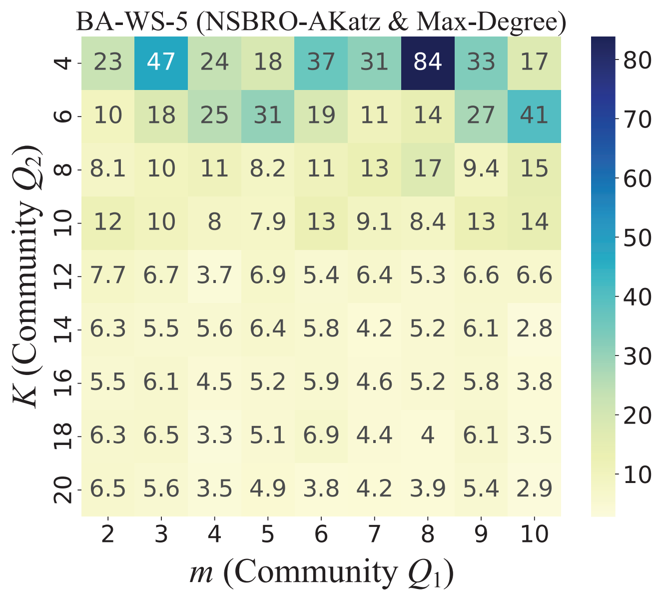

FIG. 8. The performance gap in maximizing network synchronizability in terms of the NSBRO-AKatz method and the Max-Degree method for all combinations of parameter pairs (m, K) listed in Fig. 4. BA-WS-5 stands for the case where there are five inter-community links, with one community Q_1 being the scale-free topology and another community Q_2 being the small-world topology. Here, the size of each communities is $N = 50$. Each data in the grid are averaged over 40 independent realizations.

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