Visualization and Analysis of 4_1 Knot Paths in (U,V) Space and e^V Weighted Area

Manus

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1 Introduction

This document explores the second research direction: the visualization and preliminary analysis of geometric properties of paths associated with the figure-eight knot (4_1) within the (U, V) reference space. A key focus is understanding the e^V weighted area, which arises in the geometric interpretation of global arithmetic torsion, $\mathcal{T}(S) = \iint_{\Sigma_S} e^V dV \wedge dU$.

The 4_1 knot (Figure-Eight Knot) is the simplest hyperbolic knot and serves as a fundamental example for exploring connections between knot theory and Arithmetic Expression Geometry (AEG).

2 Geometric Interpretation of Arithmetic Torsion

Material 5 of the provided research background introduces a geometric interpretation of the global arithmetic torsion for a path S (typically a relator in the knot group $G(K)$) as an integral over a surface Σ_S in a (U, V) reference space:

$$\mathcal{T}(S) = \iint_{\Sigma_S} e^V dV \wedge dU$$

Here, U and V are coordinates in a 2-dimensional space where arithmetic operations of addition and multiplication are represented as translations. For an arithmetic expression P, $U(P)$ can be thought of as related to the additive structure and $V(P)$ to the multiplicative structure. Specifically, if an expression involves m multiplications by a factor t_k at each step, V can accumulate as $V = \sum \ln t_k$. If t is a constant parameter for m multiplicative operations, $V = m \ln t$.

3 Visualizing Paths in (U,V) Space

3.1 Conceptual Path Representation

As described in the user's documentation (e.g., 'knots$_0$1.tex'), $pathscorrespondingtorelatorsof$ G(4_1) can be visualized in the (U, V) plane. The path traces a trajectory based on arithmetic operations, and a closed loop (formed by S and potentially S_{rev} or other segments) encloses the region Σ_S.

Figure ?? shows the user's conceptual schematic for such a path.

3.2 Worked Example: Path Construction for $S = $ aBAABabb with $t = 2$

To make the concept concrete, let us consider the specific path $S = $ aBAABabb and map its operations to movements in the (U, V) plane. Following user feedback and common context in AEG discussions, we will use $t = 2$. The mapping is:

- Operation 'a' (multiplication by $t = 2$) corresponds to a change $(0, \ln 2)$ in (U, V).
- Operation 'A' (multiplication by $t^{-1} = 1/2$) corresponds to a change $(0, -\ln 2)$ in (U, V).
- Operation 'b' (addition of 1) corresponds to a change $(1, 0)$ in (U, V).
- Operation 'B' (addition of -1) corresponds to a change $(-1, 0)$ in (U, V).

Starting at $(U_0, V_0) = (0, 0)$:

standalone amsthm amssymb amsfonts amsmath mathtools
pgf pgfplots tikz
tkz-euclide tkz-graph graphicx

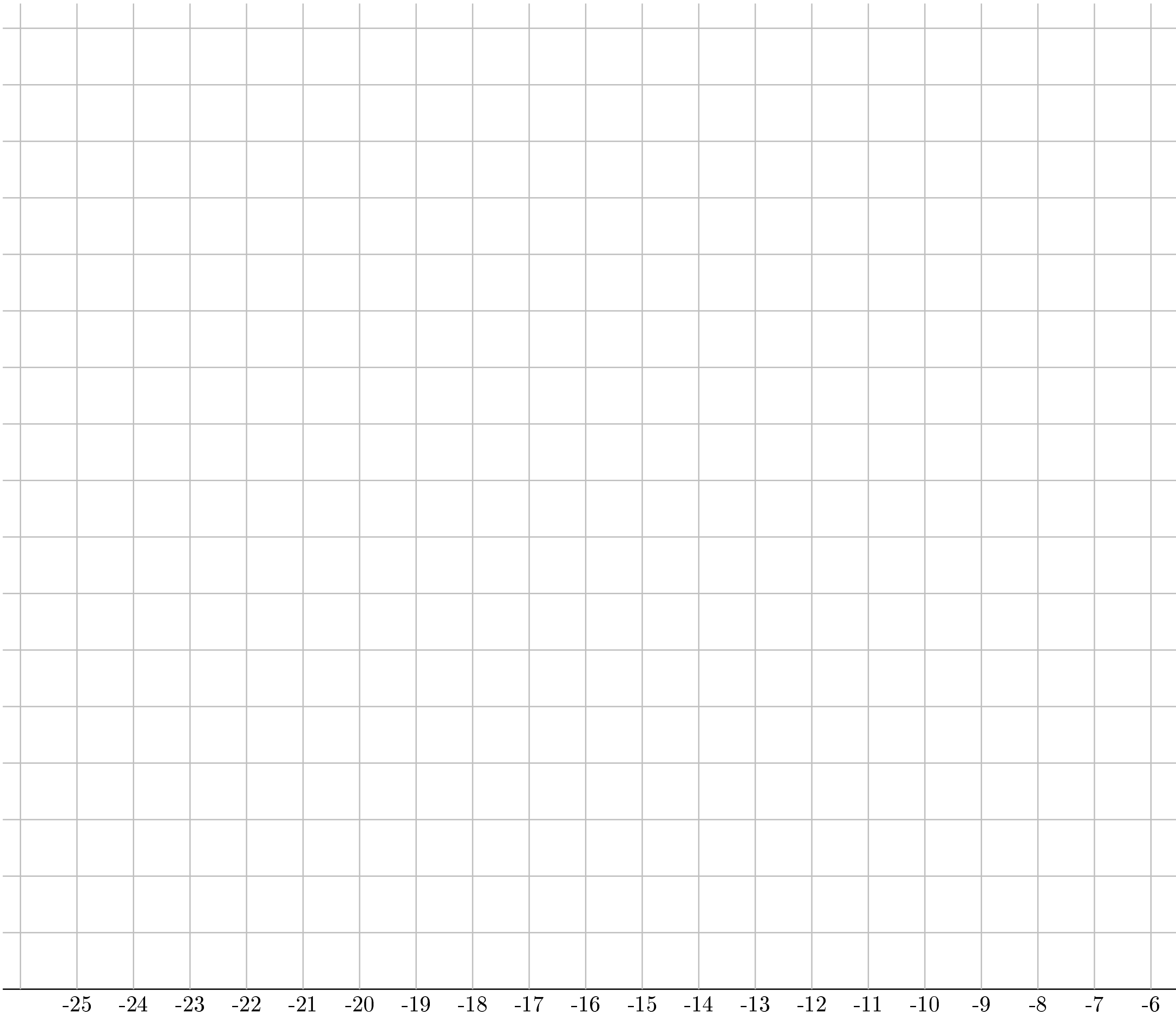

Figure 1: The Figure-Eight Knot (4_1).

1. $P_0 = (0, 0)$
2. ‘a‘: $(U_1, V_1) = (0, 0) + (0, \ln 2) = (0, \ln 2)$
3. ‘B‘: $(U_2, V_2) = (0, \ln 2) + (-1, 0) = (-1, \ln 2)$
4. ‘A‘: $(U_3, V_3) = (-1, \ln 2) + (0, -\ln 2) = (-1, 0)$
5. ‘A‘: $(U_4, V_4) = (-1, 0) + (0, -\ln 2) = (-1, -\ln 2)$
6. ‘B‘: $(U_5, V_5) = (-1, -\ln 2) + (-1, 0) = (-2, -\ln 2)$
7. ‘a‘: $(U_6, V_6) = (-2, -\ln 2) + (0, \ln 2) = (-2, 0)$

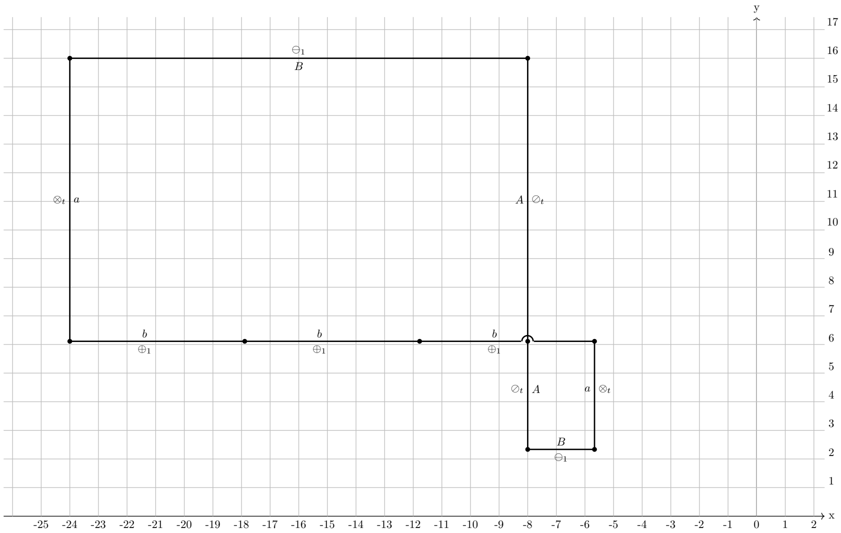

Figure 2: Schematic representation of a relator path in the (U,V) arithmetic expression space (conceptual, from repository).

8. ‘b‘: $(U_7, V_7) = (-2, 0) + (1, 0) = (-1, 0)$ (This point is P_3)
9. ‘b‘: $(U_8, V_8) = (-1, 0) + (1, 0) = (0, 0)$ (This point is P_0, the path closes)

This path $S =$ aBAABabb forms a closed loop in the (U, V) plane. The surface Σ_S is the region enclosed by this path.

Visualization of the path $S =$ aBAABabb in the (U, V) reference space with $t = 2$. The V-axis is scaled by $\ln 2$. The shaded region represents Σ_S.

Figure **??** shows this specific path and the enclosed region Σ_S for $t = 2$.

3.3 Calculating the Weighted Area $\iint_{\Sigma_S} e^V dV \wedge dU$ for $t = 2$

The integral is $\mathcal{T}(S) = \iint_{\Sigma_S} e^V dV \wedge dU$. Using Green's Theorem, this can be written as a line integral $\mathcal{T}(S) = \oint_S -e^V dU$. We sum the contributions from each segment of the path $P_0 \to P_1 \to \cdots \to P_8$. Note that $e^V = e^{k \ln 2} = (e^{\ln 2})^k = 2^k$ when $V = k \ln 2$.

- $P_0(0, 0) \to P_1(0, \ln 2)$: $U = 0$, $dU = 0$. Integral contribution $= 0$.
- $P_1(0, \ln 2) \to P_2(-1, \ln 2)$: $V = \ln 2$, so $e^V = 2^1 = 2$. U goes from 0 to -1, so dU over the segment is -1. $\int_0^{-1} -2dU = -2[-U]_0^{-1} = -2(1 - 0) = -2$.
- $P_2(-1, \ln 2) \to P_3(-1, 0)$: $U = -1$, $dU = 0$. Integral contribution $= 0$.
- $P_3(-1, 0) \to P_4(-1, -\ln 2)$: $U = -1$, $dU = 0$. Integral contribution $= 0$.
- $P_4(-1, -\ln 2) \to P_5(-2, -\ln 2)$: $V = -\ln 2$, so $e^V = 2^{-1} = 1/2$. U goes from -1 to -2, so $dU = -1$. $\int_{-1}^{-2} -(1/2)dU = -(1/2)[-U]_{-1}^{-2} = -(1/2)(2 - 1) = -1/2$.
- $P_5(-2, -\ln 2) \to P_6(-2, 0)$: $U = -2$, $dU = 0$. Integral contribution $= 0$.
- $P_6(-2, 0) \to P_7(-1, 0)$: $V = 0$, so $e^V = 2^0 = 1$. U goes from -2 to -1, so $dU = 1$. $\int_{-2}^{-1} -1dU = -1[-U]_{-2}^{-1} = -1(1 - 2) = 1$.
- $P_7(-1, 0) \to P_8(0, 0)$: $V = 0$, so $e^V = 2^0 = 1$. U goes from -1 to 0, so $dU = 1$. $\int_{-1}^{0} -1dU = -1[-U]_{-1}^{0} = -1(0 - 1) = 1$.

Summing these contributions: $\mathcal{T}(S) = -2 - 1/2 + 1 + 1 = -2 - 0.5 + 2 = -0.5$.

For $t = 2$, the Alexander polynomial is $\Delta_{4_1}(2) = 2^2 - 3(2) + 1 = 4 - 6 + 1 = -1$. If this path $S =$ aBAABabb corresponded to $K = 0$, then $\mathcal{T}(S)$ should be 0. Since $\mathcal{T}(S) = -0.5 \neq 0$, this path does not yield $K = 0$ for $t = 2$. If $\mathcal{T}(S) = \Delta(t)(t^K - 1)$, then $-0.5 = (-1)(2^K - 1)$, so $0.5 = 2^K - 1$, which means $2^K = 1.5$. Thus $K = \log_2(1.5) = \frac{\ln 1.5}{\ln 2} \approx \frac{0.405}{0.693} \approx 0.585$. This is not an integer, indicating that the chosen path S = aBAABabb might not be a true relator for 4_1 or the formula/interpretation needs further refinement for arbitrary paths versus true relators that define K as an integer.

This calculation for $t = 2$ illustrates the method. For actual 4_1 relators known to have specific integer K values (e.g., $K = 0$ or $K = -2$ from Question 1 analysis), a similar calculation should yield results consistent with $\Delta(t)(t^K - 1)$.

3.4 The Weighting Factor e^V

The term e^V acts as a weighting factor for the area element $dV \wedge dU$. Since $V = \sum \ln t_k$, $e^V = e^{\sum \ln t_k} = \prod t_k$. This means the contribution of an area element to the torsion integral is scaled by the product of the multiplicative factors encountered along the path to reach that region. When $t_k = t$ for all multiplicative steps, $V = m \ln t$, and $e^V = t^m$.

4 Analysis and Verification of the Triple Identity

The calculated geometric torsion can then be compared with the algebraically defined torsion $\Delta(t)(t^K - 1)$. This comparison serves as a direct verification of the proposed triple identity for specific instances related to the 4_1 knot.

4.1 Special Case: K=0

When $K = 0$, the algebraic torsion $\mathcal{T}(S)$ becomes zero (assuming $\Delta(t) \neq 0$). This implies that the geometric integral $\iint_{\Sigma_S} e^V dV \wedge dU$ must also be zero. Visualizing paths for which $K = 0$ (identified from the study in Question 1, such as `aBAABabbb`) in the (U, V) space and analyzing their Σ_S and e^V distributions would be particularly insightful. For such paths, the calculation of $\oint_S -e^V dU$ should yield zero.

5 Expected Outcomes

This investigation is expected to:

- Provide concrete visualizations of how paths from $G(4_1)$ map into the (U, V) reference space, using user-provided schematics and detailed worked examples (e.g., S = aBAABabb with $t = 2$).
- Offer a clearer understanding of how the e^V weighting (dependent on the base t for $\ln t$ accumulation in V) affects the geometric interpretation of arithmetic torsion.
- Allow for preliminary verification of the geometric torsion formula for specific 4_1 knot relators by comparing with algebraic results.
- Reveal geometric characteristics of paths or regions Σ_S that correspond to $K = 0$ or other significant torsion values.

This direction aims to solidify the geometric underpinnings of AEG by directly applying its concepts to the 4_1 knot and visualizing the consequences of its arithmetic interpretation in a geometric space, using contextually appropriate parameters like $t = 2$.