Experimental Calibration of AEG Parameter t via 4_1 Knot Pseudo-Anosov Dynamics

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1 Introduction

This document details the third proposed research direction: the experimental calibration of the Arithmetic Expression Geometry (AEG) parameter t by leveraging the pseudo-Anosov dynamics associated with the figure-eight knot (4_1). The core idea is to connect the stretching factor λ_{4_1} of the 4_1 knot's monodromy to the parameter t in AEG and observe the consequences for arithmetic torsion and AEG flow dynamics.

The figure-eight knot is a fibered knot, and its complement admits a pseudo-Anosov map as its monodromy. The stretching factor of this map is a key topological invariant.

2 Pseudo-Anosov Dynamics and the Parameter t

2.1 The Stretching Factor λ_{4_1}

For the figure-eight knot (4_1), the monodromy of its fibration is a pseudo-Anosov diffeomorphism of a punctured torus. The stretching factor (or dilatation) of this map is denoted λ_{4_1}. This value is the largest root of the Alexander polynomial $\Delta_{4_1}(t) = t^2 - 3t + 1 = 0$. Specifically, $\lambda_{4_1} = \frac{3+\sqrt{5}}{2}$, which is the square of the golden ratio, ϕ^2, where $\phi = \frac{1+\sqrt{5}}{2}$. Its inverse, $\lambda_{4_1}^{-1} = \frac{3-\sqrt{5}}{2}$, is the other root of the Alexander polynomial.

2.2 Calibrating AEG Parameter t

The proposal is to set the multiplicative parameter t within the AEG framework (e.g., in an arithmetic interpretation like $a \mapsto \otimes_t, b \mapsto \oplus_1$ as suggested in Notes 2) to one of these dynamically significant values, i.e., $t = \lambda_{4_1}$ or $t = \lambda_{4_1}^{-1}$.

3 Experimental Investigation

3.1 Effect on Arithmetic Torsion

With t set to a root of $\Delta_{4_1}(t)$, the Alexander polynomial term in the global arithmetic torsion formula $\mathcal{T}(S) = \Delta_{4_1}(t)(t^K - 1)$ becomes zero. Therefore, for any path S and any integer K, the arithmetic torsion $\mathcal{T}(S)$ should vanish: $\mathcal{T}(S) = 0 \cdot (t^K - 1) = 0$.

This provides a direct experimental test:

1. Implement the arithmetic interpretation for $G(4_1)$ with $t = \lambda_{4_1}$ (or $t = \lambda_{4_1}^{-1}$).
2. Compute the value of various relator paths S in $G(4_1)$ under this interpretation.
3. Verify that the resulting values, which correspond to $\mathcal{T}(S)$, are indeed zero (or numerically very close to zero, accounting for potential floating-point inaccuracies if numerical methods are used).

This experiment would confirm a fundamental consistency between the AEG framework and the known properties of the 4_1 knot's Alexander polynomial and its relation to pseudo-Anosov dynamics.

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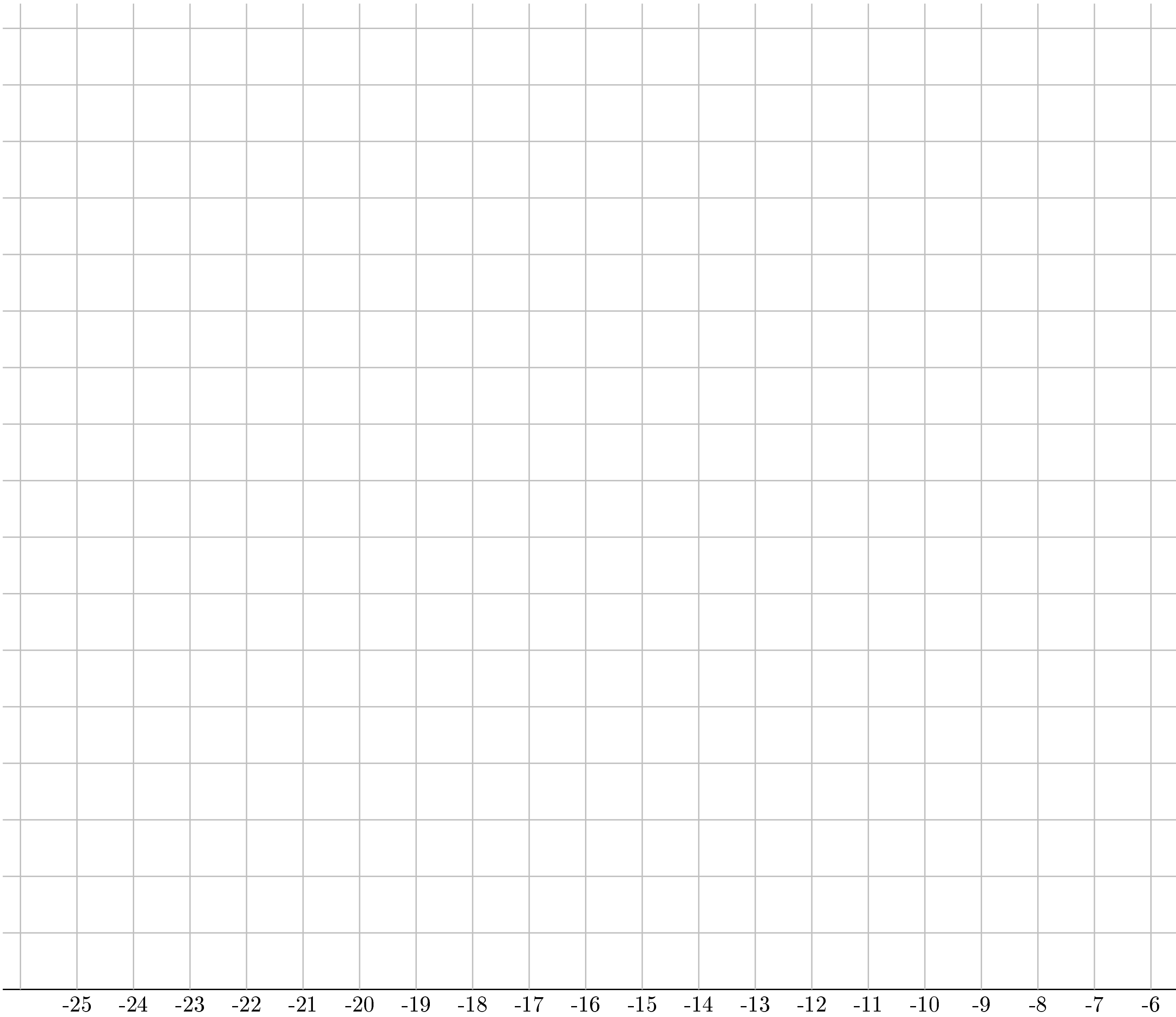

Figure 1: The Figure-Eight Knot (4_1).

3.2 Exploring AEG Flow Dynamics

A more speculative but potentially fruitful direction is to explore the AEG flow equation:

$$\frac{da}{ds} = \mu \cos\theta + a\lambda \sin\theta$$

Here, a is the assignment function, s is a path parameter, and μ, θ, λ are parameters of the flow. The question is whether setting the AEG parameter λ (distinct from λ_{4_1} used for t) in relation to $t = \lambda_{4_1}$ could reveal interesting dynamics.

For instance:

- Could specific choices of λ (perhaps $\lambda = \ln \lambda_{4_1}$ or a related value) lead to AEG paths in the (U, V)

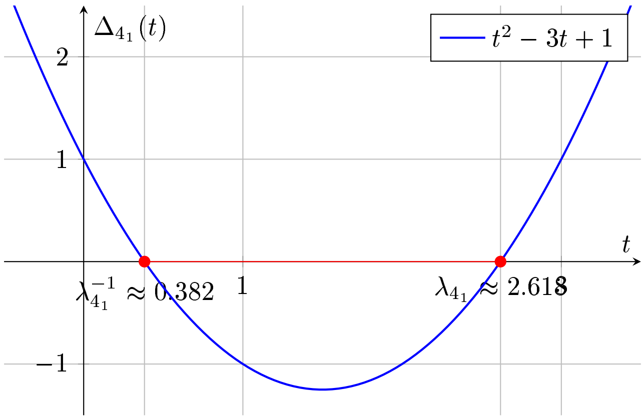

Figure 2: The Alexander Polynomial $\Delta_{4_1}(t) = t^2 - 3t + 1$ for the 4_1 knot, showing its roots $\lambda_{4_1}^{-1}$ and λ_{4_1}.

space that exhibit characteristics reminiscent of the stable or unstable foliations of the pseudo-Anosov map on the fiber surface of the 4_1 knot complement?

- Does the AEG space, when probed with this calibrated t, naturally exhibit structures or symmetries that align with the hyperbolic geometry of $S^3 \setminus 4_1$ or its fundamental group $G(4_1)$?

This part of the investigation is more open-ended and aims to see if the dynamical properties inherent in the 4_1 knot (via its monodromy) can be reflected or modeled within the AEG framework by a careful choice of its parameters.

4 Expected Outcomes

This experimental calibration is expected to:

- Verify the consistency of the AEG framework with known properties of the 4_1 knot, specifically that arithmetic torsion vanishes when t is a root of the Alexander polynomial.
- Provide a concrete, dynamically motivated choice for the parameter t in AEG when studying the 4_1 knot.
- Potentially uncover deeper connections between AEG flow dynamics and the geometric/topological structures of the 4_1 knot complement, such as its fibration and pseudo-Anosov monodromy.

Successfully linking the AEG parameter t to the intrinsic dynamics of the knot itself would be a significant step in establishing AEG as a relevant tool for knot theory.