Systematic Study of K Values in Global Arithmetic Torsion for the 4_1 Knot

Manus

May 12, 2025

1 Introduction

This document presents a systematic study of the integer K that appears in the global arithmetic torsion formula, $\mathcal{T}(S) = \Delta(t)(t^K - 1)$, specifically for the figure-eight knot (4_1). This investigation aims to understand the conditions under which K takes certain values by analyzing various paths (relators) in the knot group $G(4_1)$, using data provided in the 'aeg-paper' repository.

The figure-eight knot, denoted as 4_1, is a prime, alternating knot with a crossing number of four. It is the simplest hyperbolic knot.

2 Background: Global Arithmetic Torsion

The concept of global arithmetic torsion is given by the expression $\mathcal{T}(S) = \Delta(t)(t^K - 1)$. Here, $\Delta(t)$ is the Alexander polynomial of the knot ($t^2 - 3t + 1$ for the 4_1 knot), t is a parameter, and K is an integer that depends on the chosen path S (a relator) within the knot group $G(4_1)$.

3 Analysis of K Values from Provided Data

Data for the knot 4_1 is available in the 'knots/results.tex' file within the 'aeg-paper' repository. The relevant section for knot 4_1 is presented and analyzed below.

3.1 Data from `results.tex` for Knot 4_1

The provided data indicates the following for knot 4_1:

- Alexander polynomial: $\Delta(a) = a^2 - 3a + 1$.
- Cyclotomic polynomials: $\Phi_1(a) = a - 1$, $\Phi_2(a) = a + 1$.
- Path-dependent terms: $p(a) = \nu(S_R)(0, a)$ and $q(a) = \nu(S_{R_{\text{rev}}})(0, a)$.
- Torsion: $\tau(a) = p(a) - q(a)$.

Table 1: Summary of Arithmetic Torsion Analysis for Knot 4_1 (Unified Mapping, from `results.tex`).

Relator(s) Used	$p(a)$	$q(a)$	**Torsion** $\tau(a) = p(a) - q(a)$	**Torsion Factors**	**Notes** ($k_p, k_q, \sigma_{\text{eff}}$; Cyclot. Factors)
`aaBAbbbAB`, `abbbaBAAB`	$-\Delta(a)$	$-\frac{\Delta(a)}{a^2}$	$\frac{-\Delta(a)(a^2-1)}{a^2}$	$\frac{-\Delta(a)\Phi_1(a)\Phi_2(a)}{a^2}$	$k_p = 0, k_q = 2, \sigma_{\text{eff}} = -1$; Cyclot. $\Phi_1\Phi_2$
`aBAABabbb`, `bbbABaaBA`, `bbbaBAABa`	$-\frac{\Delta(a)}{a}$	$-\frac{\Delta(a)}{a}$	0	0	$k_p = 1, k_q = 1, p(a) = q(a)$; No cyclot. part

3.2 Derivation of K

We use the formula $\mathcal{T}(S) = \Delta(a)(a^K - 1)$. The term $\mathcal{T}(S)$ corresponds to the "Torsion $\tau(a)$" column from Table ?? (with the variable t replaced by a for consistency with the table notation). Thus, $a^K - 1 = \frac{\tau(a)}{\Delta(a)}$.

standalone amsthm amssymb amsfonts amsmath mathtools
pgf pgfplots tikz tkz-euclide tkz-graph graphicx

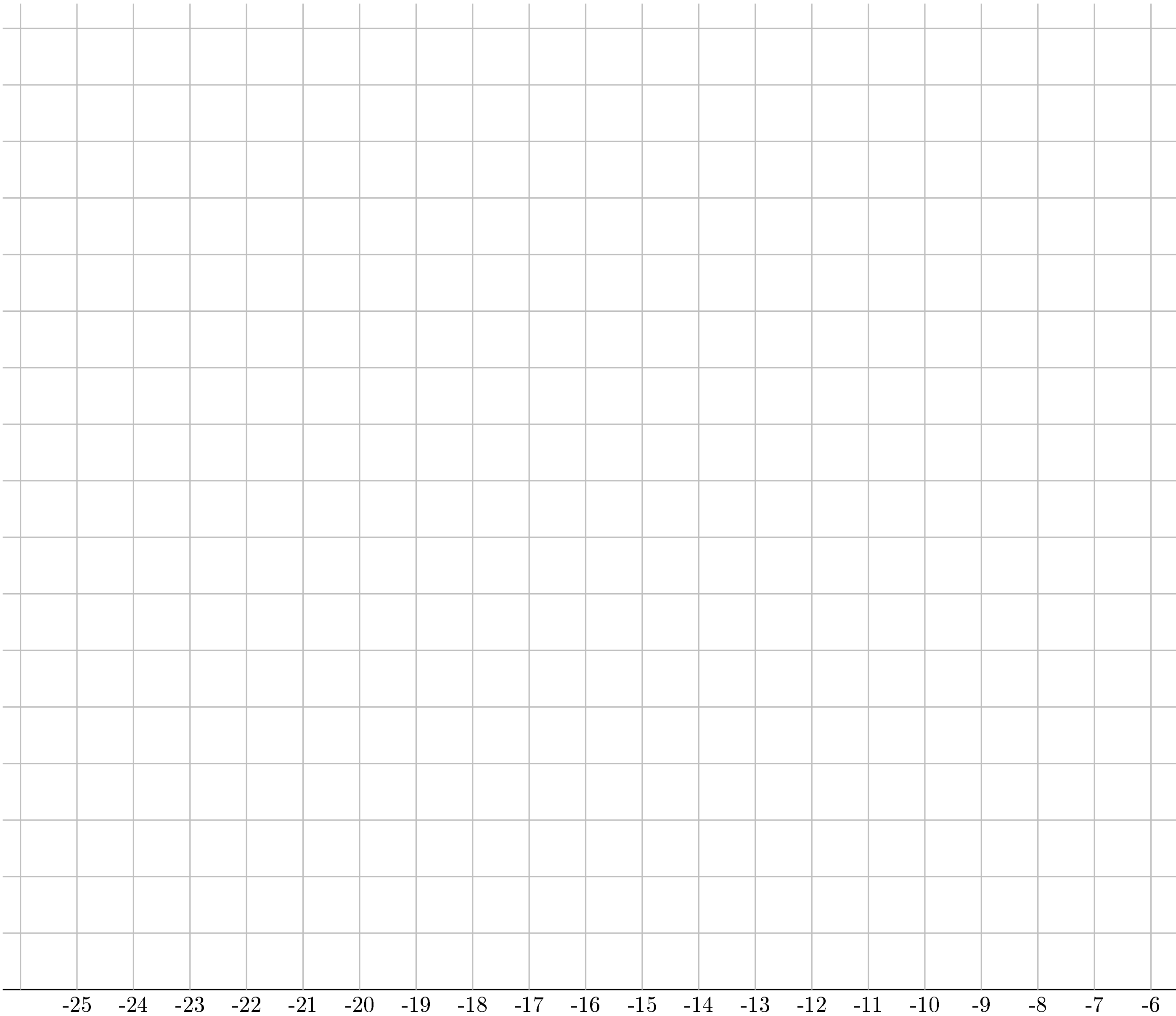

Figure 1: The Figure-Eight Knot (4_1).

3.2.1 Case 1: Relators `aaBAbbbAB`, `abbbaBAAB`

From Table **??**, for these relators:

$$\tau(a) = \frac{-\Delta(a)(a^2-1)}{a^2}$$

Assuming $\Delta(a) \neq 0$, we substitute this into the equation for $a^K - 1$:

$$a^K - 1 = \frac{1}{\Delta(a)}\left(\frac{-\Delta(a)(a^2-1)}{a^2}\right) = \frac{-(a^2-1)}{a^2} = \frac{1-a^2}{a^2}$$

Therefore,

$$a^K = 1 + \frac{1-a^2}{a^2} = \frac{a^2+1-a^2}{a^2} = \frac{1}{a^2} = a^{-2}$$

This implies that for these relators, $K = -2$.

The notes in the table ($k_p = 0, k_q = 2, \sigma_{\text{eff}} = -1$) provide parameters related to the specific computational framework used to derive $p(a)$ and $q(a)$. The derived $K = -2$ is a direct consequence of the torsion formula and the reported $\tau(a)$.

3.2.2 Case 2: Relators `aBAABabbb`, `bbbABaaBA`, `bbbaBAABa`

From Table ??, for these relators:

$$\tau(a) = 0$$

Assuming $\Delta(a) \neq 0$, we have:

$$a^K - 1 = \frac{0}{\Delta(a)} = 0$$

This implies $a^K = 1$. If a is not a root of unity, then $K = 0$. This case is of particular interest as it signifies a vanishing component of the arithmetic torsion. The notes confirm $p(a) = q(a)$ for these paths, leading directly to $\tau(a) = 0$.

4 Discussion of Path Properties and K Values

The analysis reveals two distinct K values for the provided sets of relators for the 4_1 knot:

- $K = -2$ for relators like `aaBAbbbAB`.
- $K = 0$ for relators like `aBAABabbb`.

Further investigation would involve:

1. **Algebraic Structure of Relators:** Analyzing the word structure of these relators in the generators of $G(4_1)$ (e.g., Wirtinger presentation or other common presentations like $\langle x, y \mid xyx^{-1}yxy^{-1}x^{-1}y^{-1}\rangle = 1$) to identify features correlating with $K = -2$ versus $K = 0$. This could involve looking at exponent sums, subword patterns, or relationships to specific group operations.
2. **Geometric Interpretation in (U,V) Space:** Visualizing these paths in the (U, V) reference space, as conceptualized in related research questions. The geometric properties (net displacement, enclosed area, winding numbers) of paths leading to $K = -2$ versus $K = 0$ could reveal underlying geometric reasons for the different K values.

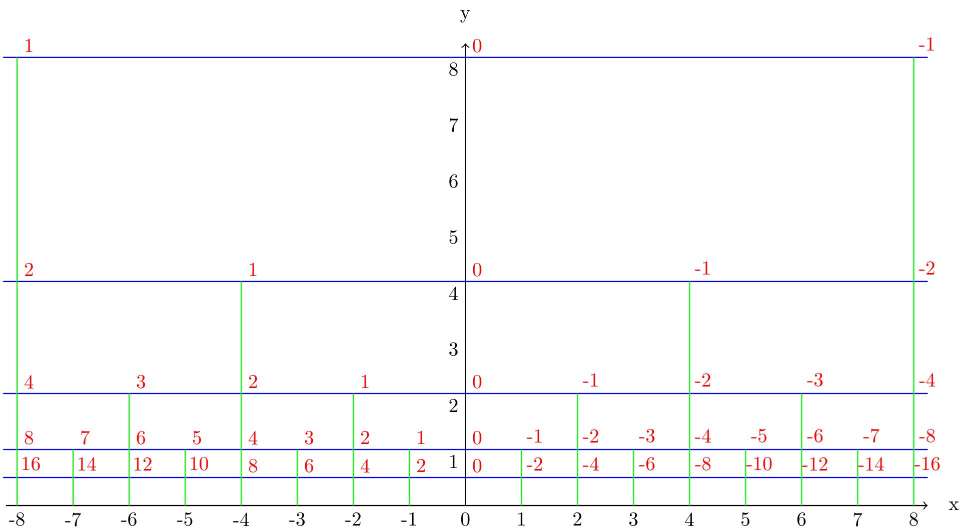

Figure 2: Conceptual example of a path in the (U,V) reference space. Specific paths from Table ?? would be plotted to analyze their geometric characteristics.

3. **Role of $k_p, k_q, \sigma_{\text{eff}}$:** Understanding the precise relationship between the parameters $k_p, k_q, \sigma_{\text{eff}}$ (from the notes in Table ??) and the derived integer K. These parameters likely encode information about how $p(a)$ and $q(a)$ are constructed from $\Delta(a)$ and cyclotomic factors, which in turn determines K.

5 Conclusion

By directly analyzing the provided data for the 4_1 knot, we have determined specific integer K values associated with different relators. For the relator set including `aaBAbbbAB`, $K = -2$. For the set including `aBAABabbb`, $K = 0$ (assuming a is not a root of unity). This detailed analysis, incorporating the explicit data from 'results.tex', provides a concrete foundation for understanding the behavior of K in the arithmetic torsion formula for the 4_1 knot. Future work should focus on correlating these K values with the algebraic and geometric properties of the paths.