Geometry of Arithmetic Expressions: I. Basic Concepts and Unsolved Problems (Draft)

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Abstract

This paper introduces a novel geometric framework for studying arithmetic expressions, establishing a rigorous connection between algebraic operations and hyperbolic geometry. We formalize arithmetic expressions as syntactic structures and demonstrate how they can be embedded into continuous geometric spaces where addition and multiplication correspond to movements along orthogonal directions. Central to our approach is a flow equation that governs how expression values propagate through this geometric space. We construct the first kind arithmetic expression space $\mathfrak{E}_1$ on the upper half-plane with a hyperbolic metric, where the assignment function satisfies the flow equation and serves as an eigenfunction of the Laplacian. This construction reveals that arithmetic torsion—the non-commutativity of addition and multiplication—directly corresponds to geometric area, analogous to how curvature measures deviation from flatness. The paper establishes arithmetic expressions as geometric objects with intrinsic invariants, opening new avenues for exploring the interplay between computation and geometry.

***K*eywords** arithmetic expressions, hyperbolic geometry

Contents

1 Introduction

From the syntactic trees of arithmetic expressions to the hyperbolic geometry of modular forms, the interplay between discrete algebra and continuous geometry has long fascinated mathematicians. This work addresses a fundamental question: Can the evaluation dynamics of arithmetic expressions themselves form a geometric space? We demonstrate that threadlike arithmetic expressions naturally embed into hyperbolic surfaces through a novel flow equation.

1.1 Key Definitions

We formalize arithmetic expressions $a \in \mathbb{E}[\mathbb{Q}]$ using production rules that generate terms through addition, subtraction, multiplication, and division operations. The evaluation $\nu(a)$ of these expressions can be viewed from multiple perspectives:

- *Syntactically* as tree structures with branch nodes (operators) and leaf nodes (constants)
- *Algebraically* as compositions of elementary operations
- *Geometrically* as paths through a continuous space

Of particular importance are threadlike expressions, where all left nodes are leaf nodes. These expressions, analogous to paths in homotopy theory, provide a natural bridge between algebraic and geometric perspectives.

Through currying and path notation, we establish a formal framework for representing threadlike expressions as sequences of elementary operations:

$$xa_1a_2\cdots a_n \coloneqq a_n(a_{n-1}(\cdots a_2(a_1(x))\cdots)) \tag{1}$$

This representation reveals that the commutator of addition and multiplication operations exhibits a non-trivial torsion:

$$\tau = x \oplus_\mu \otimes_\lambda - x \otimes_\lambda \oplus_\mu = \mu(e^\lambda - 1) \tag{2}$$

1.2 Foundational Results

The central insight of our approach is that arithmetic operations—specifically addition and multiplication—can be interpreted as movements along orthogonal directions in a properly constructed geometric space. This interpretation transforms arithmetic evaluation into geometric propagation, with expression values corresponding to points in a hyperbolic manifold.

The embedding of arithmetic expressions into geometry is governed by a flow equation:

$$\frac{da}{ds} = \mu\cos\theta + a\lambda\sin\theta \tag{3}$$

This partial differential equation describes how assignment values propagate through space along directions with angle θ. In its coordinate-free form:

$$\|\nabla a\| = \sqrt{\mu^2 + a^2\lambda^2} \tag{4}$$

This is an Eikonal equation equivalent to a special Hamilton-Jacobi equation, connecting our construction to fundamental concepts in analytical mechanics and differential geometry.

We establish a specific realization of this framework in the first kind arithmetic expression space $\mathfrak{E}_1$, defined on the upper half-plane $\mathcal{B} = \{(x, y) \mid y > 0\}$ with a hyperbolic metric:

$$ds^2 = \frac{1}{y^2}\left(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2}\right) \tag{5}$$

In this space, the assignment function $a = -\frac{x}{y}$ satisfies the flow equation with parameters μ and λ. Moreover, a is an eigenfunction of the Laplacian with eigenvalue 2, reinforcing the intrinsic geometric nature of this construction.

1.3 Implications

1.4 Roadmap

Our main results establish: (1) A flow equation governing arithmetic propagation in curved spaces; (2) The $\mathfrak{E}_1$ space as a universal geometric framework for expression evaluation;

Through this work, we demonstrate that arithmetic expressions do indeed form a geometric space—one with rich structure and deep connections to hyperbolic geometry, differential equations, and algebraic systems.

2 Basic concepts

Historically, researchers such as Post[10] and Chomsky[1] linked formal grammars to automata and rewriting systems, providing the classic recipe for generating well-formed strings via production rules. Meanwhile, foundational work by Russell[13], Church[2][3], and later Martin-Löf[7][8] introduced type-theoretic frameworks to avoid paradoxes and to systematically handle partial operations, especially critical when dealing with real numbers and division by zero. These two trajectories—rewriting rules versus type discipline—ultimately converge on the need for both syntactic generativity and semantic rigor[4][14].

In this paper, we begin by defining arithmetic expressions through basic production rules for addition, subtraction, multiplication, and division over $\mathbb{Q}$. We keep this first step lean, acknowledging that purely rewriting-based definitions can clash with hidden subtleties when extended to $\mathbb{R}$. The result is a fertile middle ground: a tree-structured syntax and partial evaluation that highlight geometric properties (e.g., "threadlike" paths), but also carry unresolved issues around singularities, infinite expansions, and semantic gaps. We postpone those deeper type-theoretic and topological remedies to the final sections, where we address how these complexities reveal new facets of "arithmetic torsion" and other structural phenomena in the geometry of expressions.

2.1 Arithmetic expression

In order to define arithmetic expressions involving real numbers $\mathbb{R}$ in a rigorous way, we need to use a sophisticated type theory. However, in order to keep things simple and maintain clarity, we will start by using only production rules, but with certain semantic restrictions. We will also begin with rational numbers $\mathbb{Q}$ to avoid the difficulties inside real numbers $\mathbb{R}$.

Definition 2.1. *An arithmetic expression a over $\mathbb{Q}$ is a structure given by the following production rules:*

$$\begin{aligned} a &\longleftarrow x \\ a &\longleftarrow (a+a) \\ a &\longleftarrow (a-a) \\ a &\longleftarrow (a\times a) \\ a &\longleftarrow (a\div a) \end{aligned} \tag{6}$$

where $x \in \mathbb{Q}$, and we denote this as $a \in \mathbb{E}[\mathbb{Q}]$.

During the production process, we can obtain both a string representation and a tree representation of arithmetic expression a, where the two representations are equivalent. For instance, the string representation of a might be:

$$(((((1\times 2)\times 2)-1)\times(2+1))-6) \tag{7}$$

and the parsed syntax tree is depicted in Figure 1.

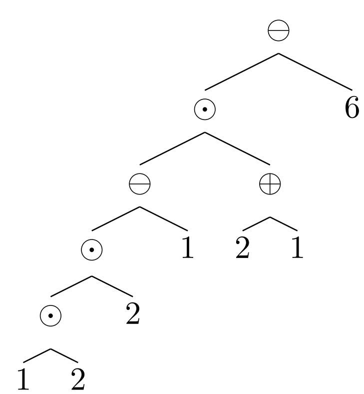

Figure 1: a tree representation of an arithmetic expression

If we interpret the target as a string and the building processes in production rule (6) as string building, we get the *string representation*. On the other hand, if the target is a tree, tree building leads to the *tree representation*. We can easily obtain the string representation of a from its tree representation by performing a pre-order traversal.

The concept of a *sub-expression* can also be derived from the concept of a subtree. The branch nodes are all labeled with operators: $+$, $-$, $\times$, $\div$. The leaf nodes are all labeled with numbers.

Evaluation ν is a partial function that operates on arithmetic expression $a \in \mathbb{E}[\mathbb{Q}]$. It is undefined only if division by zero occurs during the recursive evaluation process.

We can define evaluation $\nu(a)$ of a recursively as follows:

- Constant leaf: for any $x \in \mathbb{Q}$, $\nu(x) = x$.
- Compositional node by $+$: For any $(a+b)$, $\nu((a+b)) = \nu(a) + \nu(b)$.
- Compositional node by $-$: For any $(a-b)$, $\nu((a-b)) = \nu(a) - \nu(b)$.
- Compositional node by $\times$: For any $(a \times b)$, $\nu((a \times b)) = \nu(a)\nu(b)$.
- Compositional node by $\div$: For any $(a \div b)$, if $\nu(b) \neq 0$, then $\nu((a \div b)) = \nu(a)/\nu(b)$.

We say that an arithmetic expression a is *evaluable* if $\nu(a)$ is defined. In the rest of this article, we will only consider evaluable arithmetic expressions unless stated otherwise.

Given an arithmetic expression a, whatever evaluable or not, we can obtain its tree representation. If a node l is a leaf node, its corresponding subexpression s is a number, so we consider it to be already "evaluated". If a node b is a branch node, its corresponding subexpression s is an expression, and we can apply ν to it to obtain a number $\nu(s)$. During the recursive evaluation process, starting from the leaves and moving towards the root, the subexpressions are evaluated one after another. However, the order of evaluations is generally not unique.

Definition 2.2. *The evaluation order of an arithmetic expression a is an ordering of branch nodes in the tree representation of a such that every node (sub-expression) is evaluated before its parent.*

For example, the possible evaluation orders of the arithmetic expression in Figure 1 are:

- $1 \times 2 \to \underline{2}; \underline{2} \times 2 \to \underline{4}; \underline{4} - 1 \to \underline{3}; 2 + 1 \to \underline{3}; \underline{3} \times \underline{3} \to \underline{9}; \underline{9} - 6 \to 3$
- $1 \times 2 \to \underline{2}; \underline{2} \times 2 \to \underline{4}; 2 + 1 \to \underline{3}; \underline{4} - 1 \to \underline{3}; \underline{3} \times \underline{3} \to \underline{9}; \underline{9} - 6 \to 3$
- $1 \times 2 \to \underline{2}; 2 + 1 \to \underline{3}; \underline{2} \times 2 \to \underline{4}; \underline{4} - 1 \to \underline{3}; \underline{3} \times \underline{3} \to \underline{9}; \underline{9} - 6 \to 3$
- $2 + 1 \to \underline{3}; 1 \times 2 \to \underline{2}; \underline{2} \times 2 \to \underline{4}; \underline{4} - 1 \to \underline{3}; \underline{3} \times \underline{3} \to \underline{9}; \underline{9} - 6 \to 3$

The underlined numbers are the numbers that are evaluated during the evaluation process.

Below are examples of expressions that have a unique evaluation order. These include right-expanded, left-expanded, and combinations of them, as shown in Figure 2 and Figure 3.

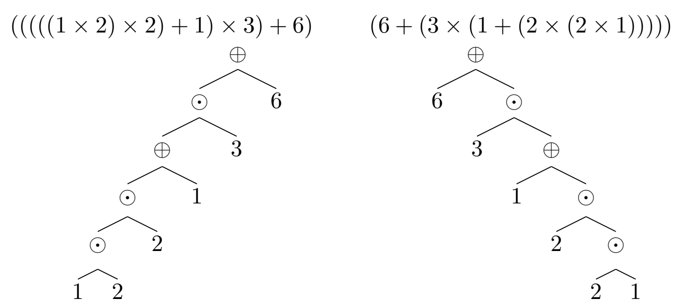

Figure 2: right-expanded and left-expanded expressions

The evaluation order of an arithmetic expression is related to the topological order of its tree representation, but they are not the same. The topological order of a tree is an ordering of nodes such that every node is visited before its parent[5]. However, we are only interested in the ordering of branch nodes, as leaf nodes have already been evaluated and can be ignored. Additionally, the topological order goes from parent to children, while the evaluation order goes from children to parent.

Definition 2.3. *A threadlike expression is an arithmetic expression that all the left nodes in its tree representation are leaf nodes.*

$((3 \times ((2 \times (1 \times 2)) + 1)) + 6)$

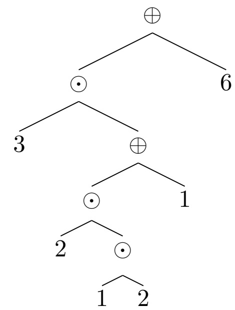

Figure 3: combinations of right-expanded and left-expanded expressions

So a threadlike expression is right-expanded and its evaluation order is unique. One example of threadlike expressions is shown on the left side of Figure 2.

Threadlike expressions are significant here because they are analogous to the concept of paths in homotopy theory in geometry. In a more general context, certain special types of threadlike expressions are also interesting: for example, *alternating threadlike expressions* are expressions in which the additional and multiplicative operators appear in an alternating manner. In the field of computing, a hardware component called *multiplier-accumulator* (MAC) unit has been implemented [11], which is a special case of an alternating threadlike expression. As a result, some numerical algorithms based on MAC units have been studied [6].

2.2 A scalar field and a mesh grid

Consider the upper half plane $\{\mathcal{H} : (x, y)|y > 0\}$ equipped with an inner product and metrics defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = [a_x \quad a_y] \begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2 \ln^2 2} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

and

$$ds^2 = \frac{1}{y^2}(dx^2 + \frac{dy^2}{\ln^2 2})$$

We consider a scalar field satisfying

$$a = -\frac{x}{y} \tag{8}$$

We call this field an *assignment*.

Proper assignments allow us to establish a connection between paths in homotopy and threadlike arithmetic expressions, and to incorporate function theory into the study of arithmetic expression geometry.

We can draw a grid on the scalar field A and underlying upper half plane $\mathcal{H}$ as shown in Figure 4. The blue lines encode a $+1$ relationship, the green lines encode a $\times 2$ relationship, and they are line families that are perpendicular to each other. The length of the line segments between two neighboring crossing points are unit length(calculations in lemma **??**). The red value at the crossing points is the value of the scalar field A at that point. Based on the relationships encoded by the lines, we can encode threadlike arithmetic expressions, which will be introduced in the subsection 2.3.

The addition-multiplication grid is also scale-invariant under the transformation

$$\begin{cases} x' = \alpha x \\ y' = \alpha y \end{cases}$$

where $\alpha = 2^k, k \in \mathbb{Z}$.

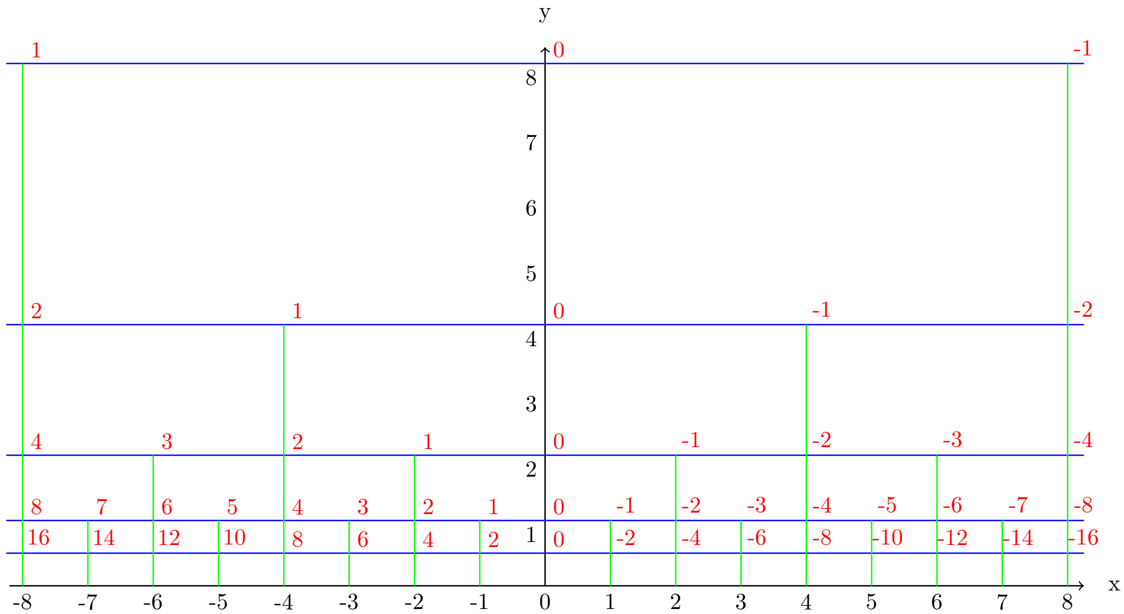

Figure 4: An addition-multiplication grid by generators with $\mu = 1$ and $\lambda = \ln 2$

We can imagine if we make the grid finer and finer, the grid will become a continuous space. This leads to a rigorous treatment of arithmetic expressions as a geometric space in section **??**.

2.3 Encoding threadlike expressions on the addition-multiplication grid

If we interpret the horizontal blue lines as $+1$ and the vertical green lines as $\times 2$ in Figure 4, we can encode threadlike expressions on the addition-multiplication grid. For example, in Figure 5 we encode $((((1 \times 4) - 1) \times 2) - 3)$ as the bold black lines.

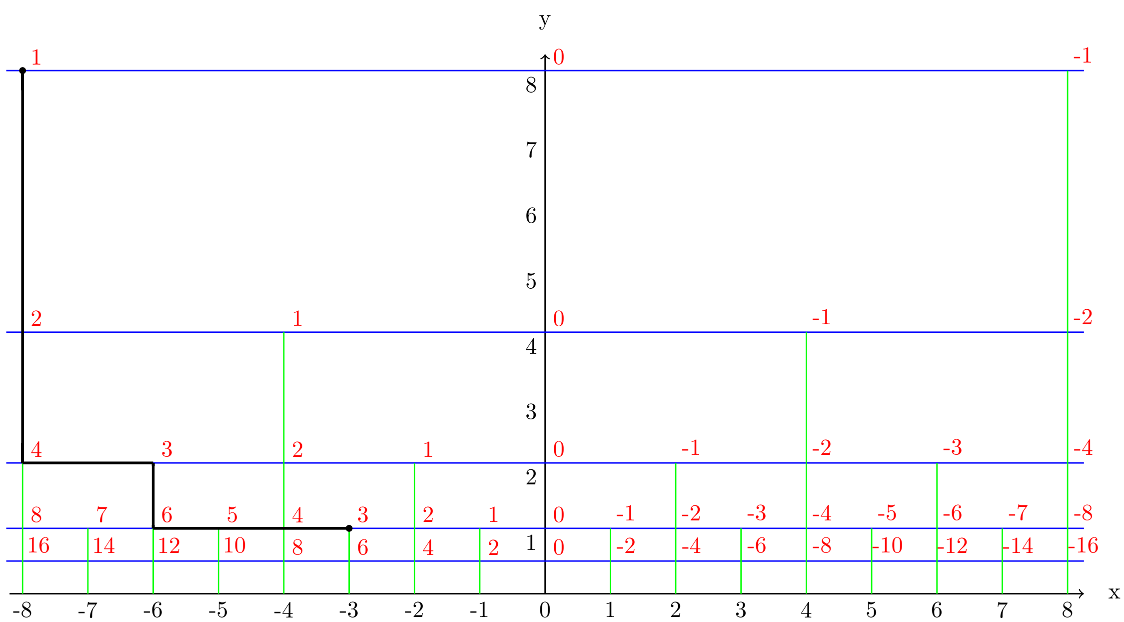

Figure 5: encoding threadlike expression

The zigzag lines in Figure 5 can be divided into four parts:

- the vertical line from 1 to 4: encoded as multiplication by 4
- the horizontal line from 4 to 3: encoded as subtraction by 1
- the vertical line from 3 to 6: encoded as multiplication by 2
- the horizontal line from 6 to 3: encoded as subtraction by 3

2.4 From a scalar field to a space of threadlike expressions

As shown in Figure 6, we have the following paths and expressions:

- the black path: $((1 \times 8) - 5) = 3$
- the purple path: $((1 - \frac{5}{8}) \times 8) = 3$
- the brown path: $((((((1 - \frac{1}{8}) \times 2) - \frac{1}{2}) \times 2) - 1) \times 2) = 3$
- the orange path: infinite many addition-multiplication terms accumulated together, a special kind of integration

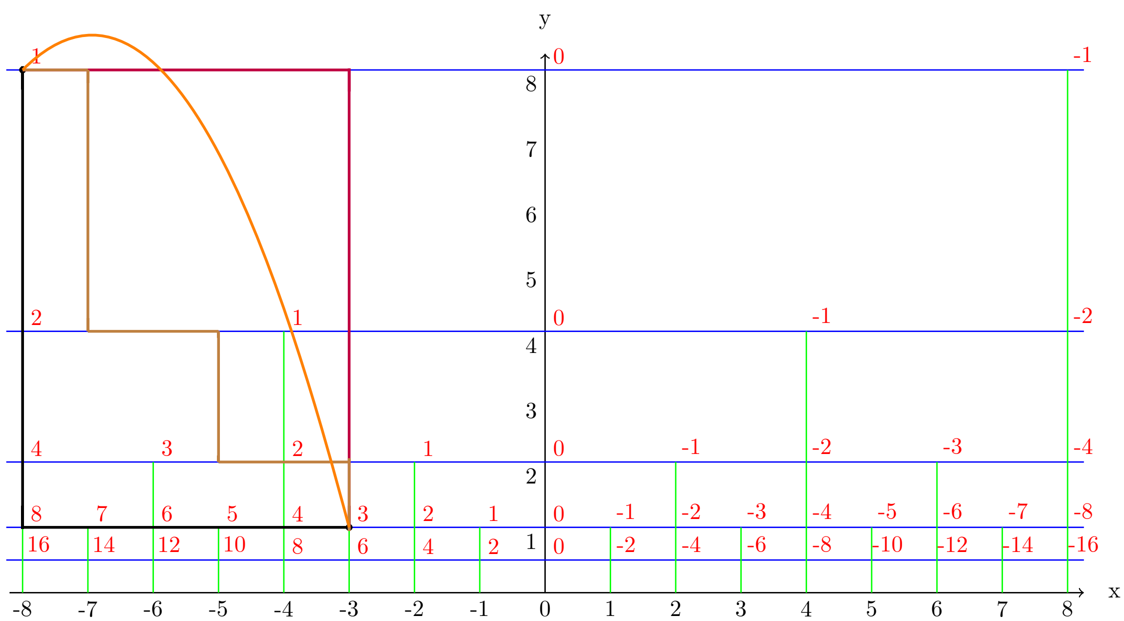

Figure 6: different encodings and their canonical form

All of the paths in Figure 6 have the same source 1 and same target 3. We will discuss a canonical form for these paths.

It is easy to see that the expressions can be transformed into each other by using the multiplication distributive law and by combining and decomposing terms.

Conversion form brown path to black path

$$3 = ((((((1 - \frac{1}{8}) \times 2) - \frac{1}{2}) \times 2) - 1) \times 2) \tag{9}$$

$$= 1 \times 8 - \frac{1}{8} \times 8 - \frac{1}{2} \times 4 - 1 \times 2 \tag{10}$$

$$= ((1 \times 8) - 5) \tag{11}$$

Conversion form brown path to purple path

$$3 = ((((((1 - \frac{1}{8}) \times 2) - \frac{1}{2}) \times 2) - 1) \times 2) \tag{12}$$
$$= (1 - \frac{1}{8}) \times 8 - \frac{1}{2} \times 4 - 1 \times 2 \tag{13}$$
$$= (1 - \frac{1}{8}) \times 8 - \frac{1}{4} \times 8 - \frac{1}{4} \times 8 \tag{14}$$
$$= (1 - \frac{1}{8} - \frac{1}{4} - \frac{1}{4}) \times 8 \tag{15}$$
$$= ((1 - \frac{5}{8}) \times 8) \tag{16}$$

Therefore, we can define the black and purple paths in Figure 6 as a pair of canonical paths, which represent all threadlike expressions connecting the source 1 and the target 3.

Once we have such canonical paths, we can determine the canonical form of the whole space relative to an arbitrary source point O and any other target point P. This allows us to define the space as a space of threadlike expressions.

2.5 Currying and path notation

Currying is a basic technique in functional programming[12], which is used to transform a function with multiple arguments into a sequence of functions with one argument. By currying a threadlike arithmetic expression, we can obtain a sequence of functions that operate on an operand, which is the leftmost leaf node.

We introduce the following notation for currying a threadlike arithmetic expression:

- initial operand: the leftmost leaf node
- operator: $\oplus_y : x \mapsto x + y$
- operator: $\ominus_y : x \mapsto x - y$
- operator: $\otimes_y : x \mapsto x \cdot e^y$
- operator: $\oslash_y : x \mapsto x \cdot e^{-y}$

For example, the threadlike arithmetic expression $(((((1 \times 2) \times 2) + 1) \times 3) + 6)$ can be curried as

$$\oplus_6(\otimes_{\ln 3}(\oplus_1(\otimes_{\ln 2}(\otimes_{\ln 2}(1)))))$$

Suppose we have a series of operators $a_1, a_2, \cdots a_{n-1}, a_n$, we introduce a *path notation*.

$$xa_1a_2 \cdots a_{n-1}a_n \coloneqq a\_n(a_{n-1}(\cdots a_2(a_1(x)) \cdots))$$

So, the above example can be written as

$$1 \otimes_{\ln 2} \otimes_{\ln 2} \oplus_1 \otimes_{\ln 3} \oplus_6$$

If a path begins with a number, we refer to it as a *bounded path*. If it does not, we refer to it as a *free path*, similar to the concept of vectors from the origin versus vectors at arbitrary points. a bounded path results in a number, while a free path results in a function.

Now we will verify that the operators within a path are associative.

Lemma 2.1. *The operators within a path are associative, i.e. we have*

$$a[bc] = [ab]c$$

Proof. We use normal typeface to express the path notation, and bold typeface to express the function notation.

For a free path, follow the definition, we have

$$a[bc] = [bc](\mathbf{a}) = \mathbf{c}(\mathbf{b}(\mathbf{a}))$$

$$[ab]c = \mathbf{c}([ab]) = \mathbf{c}(\mathbf{b}(\mathbf{a}))$$

hence, we have

$$a[bc] = [ab]c$$

is hold for a free path.

For a bounded path, we have

$$xa[bc] = [bc](\mathbf{a}(x)) = \mathbf{c}(\mathbf{b}(\mathbf{a}(x)))$$
$$x[ab]c = \mathbf{c}([ab](x)) = \mathbf{c}(\mathbf{b}(\mathbf{a}(x)))$$

hence, we have

$$a[bc] = [ab]c$$

is hold for a bounded path.

□

Definition 2.4. *The concatenation of paths $p_1 \cdot p_2$ is defined as the composite of functions:*

$$p_1 \cdot p_2 \coloneqq p_2 \circ p_1$$

When a sequence of paths is concatenated, and only the first path can be bounded. If the first path is bounded, the concatenated result is a bounded path. Otherwise, the concatenated result is a free path.

2.6 Alternating threadlike expressions

Now we can define alternating threadlike expressions, which were mentioned in Section **??**, using the path notion.

$$\alpha = a_1 b_1 a_2 b_2 \cdots a_l b_l, a_i = \otimes_{\lambda_i}, b_i = \oplus_{\mu_i}, \lambda_i, \mu_i \in \mathbb{R} \tag{17}$$

where $\bigoplus$ and $\bigotimes$ denote addition and multiplication, respectively, and the expression is a zigzag of alternating addition and multiplication operations. α is a free path, and we can bind a number to it.

Since 0 is the identity element for addition and 1 is the identity element for multiplication, it is straightforward to see that any arithmetic expression can be converted into an alternating threadlike expression by introducing more 0 and 1 into the original expression. So alternating threadlike expression is a kind of canonical form.

We can derive a formula for perturbations in alternating threadlike expressions.

Let us define the left-to-right accumulated sum of λ_i as $\check{\lambda}_i$, such that:

$$\check{\lambda}_i = \sum_{j=1}^{i} \lambda_j, \check{\lambda}_0 = 0 \tag{18}$$

Then we also have right-to-left accumulated sum of λ_i

$$\hat{\lambda}_i = \check{\lambda}_l - \check{\lambda}_{l-i}, \hat{\lambda}_0 = 0 \tag{19}$$

Expanding equation (17) using the distributive law and the above notion at point μ_0, we obtain:

$$\alpha(\mu_0) = e^{\lambda_l}(\cdots(e^{\lambda_2}(e^{\lambda_1}\mu_0 + \mu_1) + \mu_2)\cdots) + \mu_l \tag{20}$$
$$= e^{\hat{\lambda}_l}\mu_0 + e^{\hat{\lambda}_{l-1}}\mu_1 + e^{\hat{\lambda}_{l-2}}\mu_2 + \cdots + e^{\hat{\lambda}_1}\mu_{l-1} + e^{\hat{\lambda}_0}\mu_l \tag{21}$$

Next, at the starting point μ_0, we introduce a perturbation $\tilde{\mu}_0 = e^{\eta_0}\mu_0 + \epsilon_0$, where η_0 and ϵ_0 are the disturbance terms added by the summation and multiplication operations, respectively. Then, we have:

$$\alpha(\tilde{\mu}_0) = e^{\hat{\lambda}_l}(\tilde{\mu}_0) + e^{\hat{\lambda}_{l-1}}\mu_1 + e^{\hat{\lambda}_{l-2}}\mu_2 + \cdots + e^{\hat{\lambda}_1}\mu_{l-1} + e^{\hat{\lambda}_0}\mu_l \tag{22}$$
$$= \alpha(\mu_0) + e^{\hat{\lambda}_l}(\tilde{\mu}_0 - \mu_0) \tag{23}$$

As a result, purely from an arithmetic perspective, without the need for limits, we can derive the following meaningful ratio:

$$\frac{\alpha(\tilde{\mu}_0) - \alpha(\mu_0)}{\tilde{\mu}_0 - \mu_0} = e^{\hat{\lambda}_l} = e^{\check{\lambda}_l} \tag{24}$$

Now we extend this relationship from the starting point μ_0 to the entire process, we define the recursive formula

$$w_i = e^{\lambda_i} w_{i-1} + \mu_i, w_0 = 0$$

and then we have

$$\frac{\tilde{w}_i - w_i}{\tilde{\mu}_0 - \mu_0} = e^{\check{\lambda}_i}, i \in \{1, ..., l\} \tag{25}$$

So, we have

$$\tilde{w}_i - w_i = e^{\check{\lambda}_i} (\tilde{\mu}_0 - \mu_0)$$

and hence

$$\tilde{w}_i - w_i = e^{\lambda_i} (\tilde{w}_{i-1} - w_{i-1}) \tag{26}$$

That means the perturbation along the path is controlled by the multiplication terms of e^{λ_i}.

2.7 Generated structure, commutator and arithmetic torsion

In order to study mesh grids like the one described in subsection 2.2, we need to investigate the algebraic structure of the threadlike arithmetic expressions that are generated.

For real number $\mathbb{R}$ and elements $\mu, \lambda \in \mathbb{R}$, we consider all the arithmetical expressions that are freely generated from

- initial operand: 0
- operator: $\oplus_\mu : x \mapsto x + \mu$
- operator: $\ominus_\mu : x \mapsto x - \mu$
- operator: $\otimes_\lambda : x \mapsto x \cdot e^\lambda$
- operator: $\oslash_\lambda : x \mapsto x \cdot e^{-\lambda}$

We denote these expressions as $E(\mu, \lambda)$, where μ is the additional generator and e^λ is the multiplicative generator. In cases where the context is clear, we may omit μ and λ from the index. Our goal is not to study only a single $E(\mu, \lambda)$, but rather to use a family of $E(\mu, \lambda)$ to approach a continuous space.

Since $\oplus_\mu$ and $\ominus_\mu$ are mutually inverse operations, it follows that $\otimes_\lambda$ and $\oslash_\lambda$ are also mutually inverse. This means that $E(\mu, \lambda)$ forms a group. An observation is that the commutator of this group is not equal to identity generally, especially the commutator of the generators.

$$x \oplus_\mu \otimes_\lambda \ominus_\mu \oslash_\lambda - x = \mu(1 - e^{-\lambda}) \tag{27}$$
$$x \otimes_\lambda \oplus_\mu \oslash_\lambda \ominus_\mu - x = -\mu(1 - e^{-\lambda}) \tag{28}$$

Formula 27 obey the right-hand rule, and formula 28 obey the left-hand rule.

Or equivlently[1], we define below difference τ obey the right-hand rule:

$$\tau = x \oplus_\mu \otimes_\lambda - x \otimes_\lambda \oplus_\mu = \mu(e^\lambda - 1) \tag{29}$$

These differences are constant, indicating a type of torsion in the generated group. And torsion τ is specifically referred to as the arithmetic torsion.

We will reveal that τ is related to the curvature of the surface in later sections.

[1]Please reference section 3.5, the equivlence here is refered to the same order of the infinitesimal

2.8 Levels of Equality, Singularity, and Symmetry Problems

From theoretical and computational perspectives, it is useful to consider different levels of equality within structures generated freely from arithmetic operations. These levels reveal distinct stages of abstraction, from the most basic sequential representation to the final numerical or geometric realization. We can distinguish at least the following key levels:

1. **Literal Equality**:
This is the finest level of equality, judged directly by the sequence of operations (or its string representation). Two threadlike expressions are literally equal only if their sequences are identical. While this level might be too strict to reflect computational outcomes, it provides the richest *base textures* without any imposed algebraic constraints, forming the foundation upon which more complex structures and spaces can be woven.
2. **Operational/Syntactic Equality**:
This level is based on fundamental rules governing the composition of operations, chief among them being **associativity** (as proven in Lemma 2.1). Associativity ensures that the result of composing a sequence of operations is well-defined, independent of the grouping (parenthesization). This endows our arithmetic expressions with at least a foundational **monoid** or **groupoid** structure, enabling the consistent definition of "paths" and their composition, which is essential for geometrization. It is crucial to note that the sequential path operations central to this framework (e.g., compositions of addition $\oplus_\mu$ and multiplication $\otimes_\lambda$) generally do **not** satisfy the standard **distributive law** at this level. This is evidenced by the non-zero arithmetic torsion τ, a key feature indicating the non-commutative and non-distributive nature of these combined operations.
3. **Relational/Algebraic Equality**:
Building upon the operational structure above, specific contexts often introduce **additional, concrete algebraic relations**. These relations might stem from:
   - The existence of inverses for the operations, leading to a group structure.
   - Geometric or topological constraints, such as identities that must be satisfied by relators in a knot group.
   - Arithmetic identities satisfied by parameters involved in the operations, such as a minimal polynomial $P(\lambda) = 0$.

   This level of equality is defined by the **specific presentation (generators and relations)** of the resulting algebraic structure, shaping the particular algebraic object of interest (e.g., a specific group). Formal tools from combinatorial group theory, such as **HNN extensions** and **amalgamated free products**, provide a precise framework for describing these algebraic structures arising from geometric, topological, or arithmetic constraints. Results like Britton's Lemma ensure that these relationally-defined structures are typically non-trivial.
4. **Semantic Equality**:
This is the coarsest level of equality, determined by the final **numerical evaluation** of the expression or by the **value of the assignment function** a in a geometric model. Two expressions might be distinct at the Relational/Algebraic level (representing different group elements) but still yield the same semantic value. This level directly corresponds to the computational output or the state within the geometric realization.

Hierarchy and Challenges:

We can view these levels of equality as forming a lattice or hierarchy based on refinement, ordered by implication from finer to coarser levels:

$$E_{\text{Literal}} \implies E_{\text{Op/Syn}} \implies E_{\text{Rel/Alg}} \implies E_{\text{Semantic}}$$

where E_X denotes "equality at level X", and $\implies$ means "implies". A finer equality necessarily satisfies any coarser equality.

It is important, however, to distinguish this lattice of *equality relations* (ordered by refinement) from the relationship between the corresponding *algebraic structures*. For instance, in the context of HNN extensions relevant to fibered knots (corresponding to the Relational/Algebraic level, e.g., the knot group $\pi_1(K)$), the structure typically contains the base group (corresponding to the finer Operational/Syntactic level, e.g., $\pi_1(F)$) as a (normal) **subgroup**. That is, the algebraic structure defined by the coarser ("higher" in the lattice) equality relation embeds the structure defined by the finer ("lower") relation. This is not a contradiction; it highlights that a coarser equivalence relation (identifying more pairs) can define a larger, more complex algebraic structure which contains the structure defined by the finer relation as a component.

Significant theoretical challenges arise from the "distance" between these different levels, particularly between the Relational/Algebraic structure and its Semantic/Geometric realization. For example:

- **Role of Background Algebra**: While the AEG path operations themselves might be non-distributive, the discussion of Relational/Algebraic equality (e.g., involving $P(\lambda) = 0$) relies on the parameters (like λ) residing within standard algebraic systems (fields, rings) where distributivity holds, allowing polynomials and minimal polynomials to be defined. These relations from standard algebra are then imposed as constraints on the non-distributive AEG operational structure.
- **Singularity Problem**: Certain literally or syntactically valid expressions might be semantically invalid (e.g., involving division by zero), especially when extending definitions from $\mathbb{Q}$ to $\mathbb{R}$ or $\mathbb{C}$. How do these semantic singularities manifest in the geometric model? Do they correspond to singularities in the space or specific behaviors of the assignment function a?
- **Symmetry Problem**: What is the relationship between the symmetries of the algebraic structure (e.g., automorphisms of the HNN group) and the symmetries of the final geometric space (e.g., isometries of the $\mathfrak{E}_1$ space)? How does the process of constructing the geometric space ("weaving the textures into a space") selectively realize or break the symmetries present at the algebraic level? Different levels of equality correspond to different "symmetry resources".

Exploring these problems is central to our research, aiming to develop a unified theoretical framework capable of accommodating these different structural levels and revealing the deep connections between arithmetic, algebra, geometry, and topology.

Foundational problem: A careful reader may have noticed that the definition 2.1 is based on rational numbers $\mathbb{Q}$. Why can't we use real numbers $\mathbb{R}$ instead? The answer is that syntactically valid expressions may not be semantically valid. Dividing by zero can lead to invalid expressions, and the evaluation of the expression cannot be defined in this situation. Therefore, in real numbers, an expression may be syntactically valid but semantically not valid, and there is no algorithm that can decide whether an expression is semantically valid or not. How can we bridge this gap and provide a continuous geometry space? We will attempt to partially solve this problem in some special cases in section **??**.

Singular point problem: We have a very strong intuition that semantically invalid expressions lead to singular points. The way we discussed in complex analysis may be borrowed here: essential singularities and poles.

Symmetry and classification problem: We conjecture that the equality lattice may not only play a role in the construction of a space, but also determine the symmetry of that space. We can imagine that, at certain levels of the lattice, we weave syntactically generated substructures into points to form a space, and the weaving process uses up some symmetrical resources, leaving the rest to form a symmetry on the space. The structure within the total symmetry may provide us with a systematic way of constructing spaces, and allow us to classify spaces based on their symmetries.

3 Flow equation

This section derives the central flow equation governing how assignment values propagate through geometric space: $\frac{da}{ds} = \mu \cos\theta + a\lambda \sin\theta$, where θ represents the angle of movement. The flow equation can be reformulated in various ways, including its coordinate-free form ($||\nabla a|| = \sqrt{\mu^2 + a^2\lambda^2}$), which is an Eikonal equation equivalent to a special Hamilton-Jacobi equation.

The section demonstrates that the flow equation is consistent with discrete generating processes and can be expressed in contour-gradient form. It establishes a relationship between arithmetic torsion and geometric area ($d\tau = \mu\lambda dudv$), linking non-commutativity in arithmetic to measurable geometric properties. The flow equation is shown to have a geometric propagation interpretation, where assignment values correspond to wavefront evolution in hyperbolic space. The section concludes with a discussion of the existence of metrics that make functions satisfy the flow equation, presenting a local morphing process while noting that the global case remains unsettled.

3.1 Derivation of the flow equation

Consider an infinitesimal generating process on a Riemannian surface M using two generators: one for an additional action μ and the other for a multiplicative action e^{λ}. These two generators are perpendicular. This generation process produces an assignment $A : M \to R$ over the surface.

For any point with an assignment a_0, if we consider a movement of distance ϵ in a direction with angle θ over a time period of δ, we can establish the following:

$$a_\delta = (a_0 + \mu\epsilon\cos\theta)e^{\lambda\epsilon\sin\theta} + O(\epsilon^2)$$

or

$$a_\delta = a_0 e^{\lambda\epsilon\sin\theta} + \mu\epsilon\cos\theta + O(\epsilon^2)$$

Both formula can be simplified to the same result:

$$a_\delta = a_0 + \epsilon(a_0\lambda\sin\theta + \mu\cos\theta) + O(\epsilon^2)$$

Then, we have the following equation:

$$\frac{1}{\delta}(a_\delta - a_0) = \frac{\epsilon}{\delta}(\mu\cos\theta + a_0\lambda\sin\theta)$$

When both δ and ϵ are towards zero, we get da/dt, and hence

$$\frac{da}{dt} = u(\mu\cos\theta + a\lambda\sin\theta)$$

Or, we can change it to another form

$$\frac{da}{ds} = \mu\cos\theta + a\lambda\sin\theta \tag{30}$$

We name this equation (30) as the flow equation.

The left side of this equation is governed by the distance structure, while the right side is governed by the angle structure. So that the isometries of the surface keep the flow equation (30).

We can also get a direct formal solution of the flow equation (30)(details in Appendix A).

$$a = (a_0 + \frac{\mu}{\lambda}\cot\theta)e^{\lambda s\sin\theta} - \frac{\mu}{\lambda}\cot\theta \tag{31}$$

3.2 The matrix Lie algebra perspective

To place the flow equation (30) on a firm Lie–theoretic footing, we model each evaluation state as an element of the affine group $G = \mathrm{Aff}(1)$ acting on the line by $x \mapsto \Phi x + a$. In matrix form, this action is represented by the faithful representation $\rho : G \to \mathrm{GL}(2, \mathbb{R})$:

$$\rho(g) = \begin{pmatrix} \Phi & a \\ 0 & 1 \end{pmatrix}, \tag{32}$$

where $\Phi \in \mathbb{R}_{>0}$ is the accumulated multiplicative scaling (corresponding to the flow of λ) and $a \in \mathbb{R}$ is the current translation (corresponding to the flow of μ). Group multiplication in G is realized as ordinary matrix multiplication.

The Lie algebra $\mathfrak{aff}(1)$ consists of all matrices of the form

$$X = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{R},$$

with a basis given, for instance, by the dilation generator E_{11} and the translation generator E_{12}. For a path $s \mapsto g(s)$ in G with matrix representation $\rho(g(s))$, we prescribe an infinitesimal generator $\Omega \in \mathfrak{aff}(1)$ that combines the local multiplicative and additive intensities λ and μ along a direction θ:

$$\Omega = \lambda \sin\theta \, E_{11} + \mu \cos\theta \, E_{12} = \begin{pmatrix} \lambda \sin\theta & \mu \cos\theta \\ 0 & 0 \end{pmatrix}. \tag{33}$$

The evolution of the arithmetic state is governed by the (right–invariant) differential equation:

$$\frac{d}{ds}\rho(g(s)) = \Omega \, \rho(g(s)). \tag{34}$$

Substituting the matrix forms, we have:

$$\begin{pmatrix} d\Phi/ds & da/ds \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda \sin\theta & \mu \cos\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi & a \\ 0 & 1 \end{pmatrix}. \tag{35}$$

Performing the matrix multiplication on the right-hand side, the upper-right entry yields:

$$\frac{da}{ds} = (\lambda \sin\theta)\, a + (\mu \cos\theta) \cdot 1 = \mu \cos\theta + a\lambda \sin\theta, \tag{36}$$

which is exactly the flow equation (30) derived initially from the geometric propagation picture.

Mathematical Note. In this formulation, the generator takes the form $\Omega = \lambda \sin\theta \, E_{11} + \mu \cos\theta \, E_{12}$, so the flow equation is structurally identical to the action of the Lie algebra $\mathfrak{aff}(1)$ on the affine line. The term $\mu \cos\theta$ is the contribution of the translation generator E_{12}, while the term $a\lambda \sin\theta$ arises from the interaction between the dilation generator E_{11} and the current state a.

Crucially, at the level of the affine-group representation, the translation component $D(g)$ (which corresponds to a) satisfies a twisted Leibniz rule:

$$D(uv) = D(u) + \Phi(u)D(v). \tag{37}$$

This is precisely the defining relation of the Fox differential calculus on the group ring. The "Leibniz-like" form of the flow equation is thus the continuous incarnation of this rule, with the off–diagonal entry in the product $\Omega \, \rho(g)$ playing the role of the Fox derivative of the current state. (See Section **??** for further discussion on the connection between $Aff(1)$ representations and global arithmetic torsion).

3.3 Discrete generating

In section 2.7, we have discussed a discrete generating process. Since flow equation governs an infinitesimal generating process, we will show the above discrete generating process can be emerged from the solution of the flow equation (31) naturally. We expand the formula by the Taylor series:

$$a = a_0 e^{\lambda s \sin\theta} + \frac{\mu}{\lambda}[1 + \lambda s \sin\theta + \frac{1}{2!}(\lambda s \sin\theta)^2 + \frac{1}{3!}(\lambda s \sin\theta)^3 + \cdots - 1] \cot\theta$$

Change the formula slightly:

$$a = a_0 e^{\lambda s \sin\theta} + \mu s \cos\theta + \frac{\mu}{\lambda} \sin\theta \cos\theta (\frac{\lambda^2 s^2}{2!} + \frac{\lambda^3 s^3}{3!} \sin\theta + \frac{\lambda^4 s^4}{4!} \sin^2\theta + \cdots)$$

By the formula of double angle, we have

$$a = a_0 e^{\lambda s \sin\theta} + \mu s \cos\theta + \frac{\mu}{2\lambda} \sin 2\theta (\frac{\lambda^2 s^2}{2!} + \frac{\lambda^3 s^3}{3!} \sin\theta + \frac{\lambda^4 s^4}{4!} \sin^2\theta + \cdots)$$

We denote

$$\Psi(s) = \frac{1}{2!} + \frac{\lambda s}{3!} \sin\theta + \frac{\lambda^2 s^2}{4!} \sin^2\theta + \cdots \tag{38}$$

Then we have

$$a = a_0 e^{\lambda s \sin\theta} + \mu s \cos\theta + \frac{\mu\lambda}{2} s^2 \Psi(s) \sin 2\theta \tag{39}$$

This formula gives the discrete generating process, when $\theta = \frac{k\pi}{2}, k = 0, 1, 2, 3 \cdots, s = 0, 1, 2, 3 \cdots$, we have

$$a = a_0 e^{\lambda s \sin\theta} + \mu s \cos\theta \tag{40}$$

Especially, we have the following four cases:

- $\theta = 0$: $a_s = a_0 + \mu s$
- $\theta = \frac{\pi}{2}$: $a_s = a_0 e^{\lambda s}$
- $\theta = \pi$: $a_s = a_0 - \mu s$
- $\theta = \frac{3\pi}{2}$: $a_s = a_0 e^{-\lambda s}$

This result is straightforward, but it demonstrates that the infinitesimal generating process is consistent with the discrete generating process. And this expands our toolset, enabling us to explore the interplay between discrete and infinitesimal generating processes.

3.4 The contour-gradient form of flow equation

It is easy to derive the contour equation in the local coordinate

$$\mu \cos\theta_c + a\lambda \sin\theta_c = 0 \tag{41}$$

then we have

$$\theta_c = -\arctan\frac{\mu}{a\lambda} \tag{42}$$

the contour and the gradient are perpendicular to each other

$$\theta_g = \pm\frac{\pi}{2} - \arctan\frac{\mu}{a\lambda} \tag{43}$$

then along θ_g we have

$$\frac{da}{ds} = \mu \cos(\pm\frac{\pi}{2} - \arctan\frac{\mu}{a\lambda}) + a\lambda \sin(\pm\frac{\pi}{2} - \arctan\frac{\mu}{a\lambda}) \tag{44}$$

$$\frac{da}{ds} = \pm\sqrt{\mu^2 + \lambda^2 a^2} \tag{45}$$

By introducing the right-hand rotation angle ϕ along the gradient direction, we can establish a local polar coordinate system based on the gradient and contour lines. Then the growth rate of a along the angle ϕ is

$$\frac{da}{ds} = \mu \cos(\frac{\pi}{2} - \arctan\frac{\mu}{a\lambda} + \phi) + a\lambda \sin(\frac{\pi}{2} - \arctan\frac{\mu}{a\lambda} + \phi) \tag{46}$$

And the simplified equation is

$$\frac{da}{ds} = \sqrt{\mu^2 + a^2\lambda^2}\cos\phi \tag{47}$$

or

$$\frac{da_\phi}{ds_\phi} = \sqrt{\mu^2 + a^2\lambda^2}\cos\phi \tag{48}$$

if we want to emphasize the path is along the angle ϕ.

The equation (47) is the flow equation in the contour-gradient coordinate system.

Equation (47) is solvable, and we get the relation between a and s:

$$\tanh(\lambda s\cos\phi - c) = \frac{\lambda a}{\sqrt{\mu^2 + \lambda^2 a^2}} \tag{49}$$

we can further simplify the equation to

$$a = \pm\frac{\mu}{\lambda}\sinh(\lambda s\cos\phi - c) \tag{50}$$

Under the initial condition $a = a_0$ when $s = 0$, we can get the following equation:

$$a = \frac{\mu}{\lambda}\sinh(\lambda s\cos\phi + \operatorname{arcsinh}\frac{a_0\lambda}{\mu}) \tag{51}$$

or

$$a = -\frac{\mu}{\lambda}\sinh(\lambda s\cos\phi - \operatorname{arcsinh}\frac{a_0\lambda}{\mu}) \tag{52}$$

In this coordinate system, the additional line and the multiplicative line are:

$$\phi = \arccos\frac{\mu}{\sqrt{\mu^2 + a^2\lambda^2}} \tag{53}$$

$$\phi = \arcsin\frac{\mu}{\sqrt{\mu^2 + a^2\lambda^2}} \tag{54}$$

3.5 Arithmetic coordinate and area formula

We begin our exploration by examining the flow equation (30) within the framework of a local polar coordinate system:

$$\frac{da}{ds} = \mu\cos\theta + a\lambda\sin\theta \tag{55}$$

In an effort to re-contextualize this equation, we set $du = \cos\theta ds$ and $dv = \sin\theta ds$, where du and dv are perpendicular infinitesimal movements. We can use these movements to construct a local Descartes coordinate system, and the first fundamental form of this system is:

$$ds^2 = du^2 + dv^2 \tag{56}$$

Thereby this enables us to express the flow equation in a different light:

$$da = \mu du + a\lambda dv \tag{57}$$

Our attention now turns to the concept of arithmetic torsion, particularly at an infinitesimal level. Delving into the interplay between two infinitesimal generating processes, we observe that:

$$d\tau = (a_0 + \mu du)e^{\lambda dv} - (a_0 e^{\lambda dv} + \mu du) \tag{58}$$

From this relationship, we deduce:

$$d\tau = \mu du(e^{\lambda dv} - 1) \tag{59}$$

This leads us to an area formula, capturing the essence of this interaction:

$$d\tau = \mu\lambda du dv \tag{60}$$

and because the area element have a form

$$dS = du dv \tag{61}$$

Then we have

$$\frac{d\tau}{\mu\lambda} = dS \tag{62}$$

This formula is compelling as it establishes a link between area elements and arithmetic torsion. Such formulations find parallels in the realms of classic analysis and differential geometry. For instance, they resonate with concepts akin to Stokes' theorem or the Gauss-Bonnet theorem. We intend to expand upon this formula in the ensuing section**??**, aiming to forge a connection with curvature and delve into the intricacies of the Gauss-Bonnet theorem.

It's noteworthy to emphasize the distinctiveness of the local Descartes coordinate system. This system, by integrating the assignment, lays the foundation for a theoretical framework. We refer to this as the *arithmetic coordinate system*, given its unique properties and alignment with arithmetic principles.

3.6 The coordinate-free form of flow equation

From the contour-gradient form of the flow equation (47), we can derive a coordinate-free form of the flow equation. Let's consider the direction of $\phi = 0$ in the contour-gradient coordinate system, and we have

$$\frac{da}{ds}|_{\phi=0} = \sqrt{\mu^2 + a^2\lambda^2}\cos 0$$

Notice the gradient of a is not dependent on the coordinate system, and we have the coordinate-free form of the flow equation:

$$||\nabla a|| = \sqrt{\mu^2 + a^2\lambda^2} \tag{63}$$

It should be noted that the coordinate-free form of the flow equation (63) is an Eikonal equation, and can be viewed as a special Hamilton–Jacobi equation

$$H(x, a, \nabla a) = 0$$

where the Hamiltonian is

$$H(x, a, p) = ||p|| - \sqrt{\mu^2 + a^2\lambda^2} \tag{64}$$

3.7 Propagation and Rectification

While Equation (47) can be solved directly, its structure suggests a simplifying change of variables that "rectifies" the flow. Let us define a new variable f by

$$f = \operatorname{arcsinh}\left(\frac{\lambda a}{\mu}\right).$$

This choice is motivated by the desire to find a variable whose gradient has a constant magnitude. Using the chain rule,

$$\nabla f = \frac{df}{da}\nabla a = \frac{\lambda/\mu}{\sqrt{1+(\lambda a/\mu)^2}}\nabla a = \frac{\lambda}{\sqrt{\mu^2+\lambda^2 a^2}}\nabla a.$$

Taking the norm of both sides and using the Eikonal form (63), we find

$$\|\nabla f\| = \frac{\lambda}{\sqrt{\mu^2+\lambda^2 a^2}}\|\nabla a\| = \lambda.$$

The rectified variable f has a gradient of constant magnitude. Consequently, its rate of change along a direction at an angle ϕ to the gradient is simply

$$\frac{df}{ds} = \langle \nabla f, T\rangle = \|\nabla f\|\cos\phi = \lambda\cos\phi.$$

This linearized equation is trivial to integrate: $f(s) = f_0 + \lambda s\cos\phi$. Substituting back for a gives $a = (\mu/\lambda)\sinh(f)$, which yields the general solution for propagation from an initial value f_0:

$$a = \frac{\mu}{\lambda}\sinh\Big(\lambda s\cos\phi + \operatorname{arcsinh}\frac{\lambda a_0}{\mu}\Big). \tag{65}$$

In the case of propagation along the gradient ($\phi = 0$) from a zero-value contour ($a_0 = 0$), this simplifies to

$$a = \pm\frac{\mu}{\lambda}\sinh(\lambda s).$$

This form is reminiscent of the circumference of a hyperbolic circle of radius s, $C \propto \sinh(\sqrt{-k}s)$, suggesting that a can be interpreted as a propagating wavefront in a space of constant negative curvature. A detailed geometric interpretation will be given in Section 4.

3.8 Flow and function

In this section, we aim to present novel insight into functions. Namely, the treatment of functions as flows will be discussed.

Definition 3.1. *Given a function k on the real domain R, we can introduce a mapping l on the arithmetic expression space H such that the following diagram commutes.*

$$\begin{array}{ccc} H & \xrightarrow{l} & H \\ \nu\downarrow & & \downarrow\nu \\ R & \xrightarrow[k]{} & R \end{array}$$

where ν is the evaluation function of the expression. Then we call the mapping l is the promotion of the function k, or function k is the projection of the mapping l.

Giving an arithmetic expression space as definition at the beginning of the section2.2, we will show examples of flows as functions in the following Section 4.

3.9 The existence theorem

There are two existence problems related to the flow equation (30). The first existence problem concerns the existence of a function a on a Riemannian surface S that satisfies the flow equation (30). The second existence problem concerns the existence of a metric g on a Riemannian surface S that makes a function a satisfy the flow equation (30).

We can proof there is a local morphing process over metric g to make a function a satisfy the flow equation (30) locally. But the global morphing process is more complicated, and we need to consider the global structure of the surface S, which is still not settled.

Lemma 3.1. *(By Le Zhang) Given an oriented compact Riemannian surface S, and a smooth function a over S, there exists a metric g on S that makes a satisfying the flow equation* (30)*.*

Proof. Local perspective:

Consider a point p on the surface S, and there is a neighborhood U around p. In this area, we can find a local isothermal coordinate system in which the metric takes the form:

$$ds^2 = e^{2\rho}(du^2 + dv^2),$$

where u and v are the coordinates of U, and ρ is a function of u, v in U. The gradient of a in this local isothermal coordinate system is expressed as:

$$\nabla a = \frac{\partial a}{\partial u}du + \frac{\partial a}{\partial v}dv.$$

Using the definition of the directional derivative, we obtain:

$$\frac{da_\psi}{ds_\psi} = ||\nabla a|| \cos\psi,$$

where $||\nabla a||$ is the norm of ∇a, and ψ is the angle between ∇a and the direction of movement.

Now, considering the flow equation 48 in the gradient-contour coordinate system, we have:

$$\frac{da_\phi}{ds_\phi} = \sqrt{\mu^2 + a^2\lambda^2}\cos\phi.$$

Note that $||\nabla a||$ is fixed for the given function a and the local coordinate system, and $\sqrt{\mu^2 + a^2\lambda^2}$ is also fixed for the given function a. We can scale $e^{2\rho}$ with a linear factor α to make $||\nabla a||$ match the fixed value of $\sqrt{\mu^2 + a^2\lambda^2}$, thus we have a morphing process controlled by α that

$$\begin{aligned} ds^2 &= \alpha e^{2\rho}(du^2 + dv^2) && (66)\\ &= e^{2\rho + \ln\alpha}(du^2 + dv^2). && (67) \end{aligned}$$

Under the morphing ratio α, we have:

$$||\nabla_\alpha a|| = \alpha^{-1}||\nabla a||, \tag{68}$$

and when α is set to the value of:

$$||\nabla_\alpha a|| = \sqrt{\mu^2 + a^2\lambda^2},$$

the flow equation (30) is satisfied in the local coordinate system.

The morphing ratio α is calculated as follows:

$$\alpha = \frac{||\nabla a||}{\sqrt{\mu^2 + a^2\lambda^2}}. \tag{69}$$

□

When we consider the broader scope of the surface S, it's possible to extend the morphing process to every point, ensuring that the flow equation (30) is satisfied on a global scale. However, this expansion necessitates a harmonious integration of the morphing process across neighboring locales. Specifically, this means that the morphing should not only preserve the circles centered at point p within its immediate local chart but also maintain the integrity of these circles within the adjacent charts of point p. In essence, the morphing process must be seamlessly coordinated across the various local regions to achieve a unified global transformation. How to achieve this harmonious integration remains an open question, and further exploration is needed to address this challenge.

3.10 Defining the Arithmetic Expression Space

The preceding sections have established the flow equation as the fundamental differential relation governing the interplay between arithmetic operations and a continuous geometric background. We have derived its various forms and established a local existence theorem (Lemma 3.1), showing that a metric can be morphed to accommodate a given scalar field as a valid assignment. This provides the necessary motivation to formally define the central object of our study.

Definition 3.2 (Arithmetic Expression Space). *An* Arithmetic Expression Space *(AES) is a pair* (M, a)*, where:*

- M *is a two-dimensional Riemannian manifold, equipped with a metric* g*.*
- $a : M \to \mathbb{R}$ *is a smooth scalar field on* M*.*

such that for given constant parameters $\mu, \lambda \in \mathbb{R}$*, the pair* (M, a) *satisfies the flow equation:*

$$\frac{da}{ds} = \mu \cos\theta + a\lambda \sin\theta$$

or, equivalently, its coordinate-free Eikonal form:

$$||\nabla a||_g = \sqrt{\mu^2 + a^2\lambda^2}$$

The manifold M *provides the geometric stage, defining distance and angles, while the scalar field* a *embodies the arithmetic assignment. The flow equation enforces the core consistency condition that links the geometry of* M *to the arithmetic propagation defined by* a*. In the subsequent sections, we will construct and analyze concrete examples of such spaces, beginning with the first kind space* $\mathfrak{E}_1$*.*

4 The first kind space $\mathfrak{E}_1$

This section introduces the first kind arithmetic expression space ($\mathfrak{E}_1$), providing a geometric framework for analyzing arithmetic expressions. The space is constructed on the upper half-plane with a hyperbolic metric: $ds^2 = \frac{1}{y^2}(dx^2+dy^2)$, where the assignment function $a = -\frac{x}{y}$ satisfies the flow equation and serves as an eigenfunction of the Laplacian with eigenvalue 2.

Two equivalent examples of $\mathfrak{E}_1$ are presented: the upper half-plane model and a horocycle-based coordinate system, connected through Möbius transformation. The section explores geometric propagation mechanisms, showing how the assignment value propagates like expanding concentric circles in hyperbolic space. It examines grid structures in $\mathfrak{E}_1$, revealing dual grids reflecting the geometric structure of the Baumslag-Solitar group, and demonstrates how arithmetic torsion corresponds precisely to hyperbolic areas enclosed between evaluation paths. The section concludes by introducing tube structures, which extend $\mathfrak{E}_1$ to parameterized families, enabling analysis of how expressions evolve across parameter variations.

4.1 Foundational exemplars

We present two analytically equivalent examples that belong to the class of spaces designated as the first kind arithmetic expression space $\mathfrak{E}_1$.

4.1.1 Example 1: Upper Half Plane Model

Consider the upper half plane $\mathcal{H} : (x, y) \mid y > 0$ equipped with the following inner product and metric tensor:

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_x & a_y \end{bmatrix} \begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

On this manifold, we define an assignment field a as follows:

$$a = -\frac{x}{y} \tag{70}$$

Theorem 4.1. *The assignment a defined by formula* (70) *satisfies the flow equation* (30).

Proof. We initiate with the differential of the assignment:

$$da = d\left(-\frac{x}{y}\right) = \frac{xdy - ydx}{y^2} = -\frac{dx + ady}{y}$$

The differential of arc length is given by:

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

Therefore:

$$\frac{da}{ds} = -\frac{dx + ady}{y} \cdot \frac{y}{\sqrt{dx^2 + dy^2}} = -\frac{dx + ady}{\sqrt{dx^2 + dy^2}}$$

In the local coordinate system determined by $(-1, 0)$ and $(0, -1)$ under the right-hand rule, we have:

$$\cos\theta = \frac{-dx}{\sqrt{dx^2 + dy^2}} \quad \text{and} \quad \sin\theta = \frac{-dy}{\sqrt{dx^2 + dy^2}}$$

Substituting these values:

$$\frac{da}{ds} = \cos\theta + a\sin\theta$$

This precisely corresponds to the flow equation (30) with $\mu = 1$ and $\lambda = 1$. □

We can verify that a constitutes an eigenfunction of the Laplacian operator:

$$\Delta a = -y^2 \left(\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \right) = y^2 \left(\frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \frac{x}{y} \right) \right) = 2a$$

4.1.2 Example 2: Horocycle-Based Coordinate System

For our second exemplar, we introduce a horocycle-based coordinate system for hyperbolic surfaces. This global coordinate system comprises two orthogonal families of curves: horocycles sharing the same ideal point, and geodesics perpendicular to these horocycles.

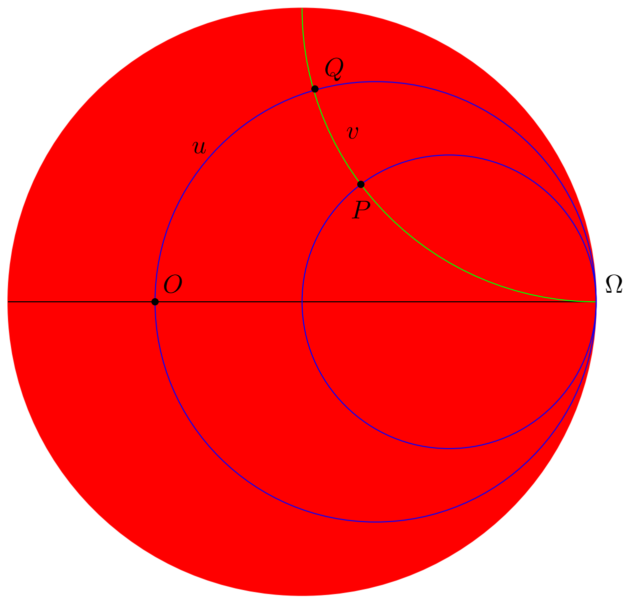

Figure 7: A horocycle-based coordinate system on the Poincaré disc. Blue curves represent horocycles tangent at ideal point Ω, green lines depict perpendicular geodesics.

On the Poincaré disc $\mathcal{P}$, the coordinates of a point P are denoted by (u, v), where:

- u represents the signed length of OQ
- v represents the signed length of QP
- The sign conventions adhere to the right-hand rule and orientation relative to the ideal point Ω

We equip this coordinate system with the inner product:

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_u & a_v \end{bmatrix} \begin{bmatrix} e^{-2v} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_u \\ b_v \end{bmatrix}$$

And the corresponding metric tensor:

$$ds^2 = e^{-2v} du^2 + dv^2$$

The Laplacian operator in this coordinate system is expressed as:

$$\Delta = e^{2v} \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - \frac{\partial}{\partial v}$$

In this coordinate framework, we define an assignment:

$$a = ue^{-v} \tag{71}$$

Theorem 4.2. *The assignment a defined by formula* (71) *satisfies the flow equation* (30).

Proof. We establish this result by demonstrating that examples 1 and 2 are equivalent through a Möbius transformation. Consider the complex representation of the upper half plane:

$$z = x + yi$$

The Möbius transformation mapping the upper half plane to the Poincaré disc is given by:

$$z \mapsto \frac{z-i}{z+i}$$

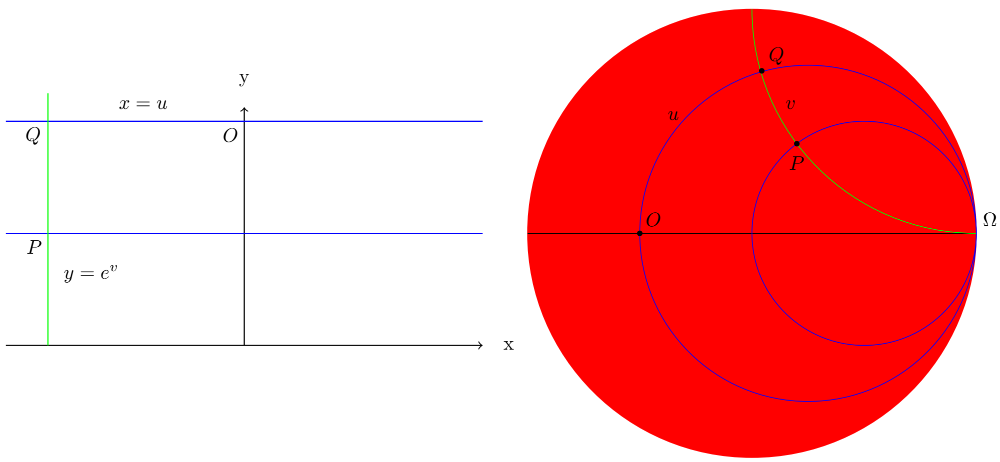

Figure 8: Mapping between the upper half plane and Poincaré disc models

This conformal transformation maps horizontal lines in $\mathcal{H}$ to horocycles sharing the ideal point $\Omega = 1$ in $\mathcal{P}$, and vertical geodesics in $\mathcal{H}$ to perpendicular geodesics in $\mathcal{P}$.

Expressed in the target coordinate system, this transformation yields:

$$\begin{cases} x = u \\ y = e^v \end{cases}$$

Substituting into the assignment from Example 1:

$$a = -\frac{x}{y} = -\frac{u}{e^v} = -ue^{-v}$$

Since the Möbius transformation is conformal and preserves the flow equation, and accounting for the orientation change, we obtain $a = ue^{-v}$ satisfying the flow equation. □

As in Example 1, we can verify that a constitutes an eigenfunction of the Laplacian:

$$\Delta a = e^{2v}\frac{\partial^2(ue^{-v})}{\partial u^2} + \frac{\partial^2(ue^{-v})}{\partial v^2} - \frac{\partial(ue^{-v})}{\partial v} = 2a$$

These two examples, emerging from the same geometric foundation but expressed in different coordinate systems, demonstrate the fundamental properties of the first kind arithmetic expression space.

4.2 Theoretical framework of $\mathfrak{E}_1$ space

Building upon the foundational exemplars, we now establish a comprehensive theoretical framework for the first kind arithmetic expression space $\mathfrak{E}_1$.

Consider the upper half plane $\mathcal{B}$:

$$\{\mathcal{B} : (x, y) | y > 0\}$$

equipped with an inner product and metric tensor parameterized by constants μ and λ:

$$\mathbf{a} \cdot \mathbf{b} = [a_x \quad a_y] \begin{bmatrix} \frac{1}{\mu^2 y^2} & 0 \\ 0 & \frac{1}{\lambda^2 y^2} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$ds^2 = \frac{1}{y^2} \left(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2} \right)$$

The assignment function in this generalized framework maintains the form:

$$a = -\frac{x}{y} \tag{72}$$

This defines the first kind arithmetic expression space $\mathfrak{E}_1$, characterized by the following theorem:

Theorem 4.3. *The assignment a given by* (72) *satisfies the flow equation* (30) *with parameters μ and λ, independent of the specific values of these generators.*

Proof. The differential of the assignment is given by:

$$da = d\left(-\frac{x}{y}\right) = \frac{xdy - ydx}{y^2} = -\frac{dx + ady}{y}$$

The differential of arc length is expressed as:

$$ds = \frac{1}{y} \sqrt{\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2}}$$

Therefore:

$$\frac{da}{ds} = -\frac{dx + ady}{y} \cdot \frac{y}{\sqrt{\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2}}} = -\frac{dx + ady}{\sqrt{\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2}}}$$

In the local coordinate system determined by $(-1, 0)$ and $(0, -1)$ according to the right-hand rule:

$$\cos\theta = \frac{-\frac{dx}{\mu}}{\sqrt{\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2}}} \quad \text{and} \quad \sin\theta = \frac{-\frac{dy}{\lambda}}{\sqrt{\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2}}}$$

Substituting these values:

$$\frac{da}{ds} = \mu \cos\theta + a\lambda \sin\theta$$

This precisely corresponds to the flow equation (30) with the given parameters μ and λ. □

The $\mathfrak{E}_1$ space is distinguished by its intrinsic connection to hyperbolic geometry and the property that the assignment function $a = -x/y$ constitutes an eigenfunction of the Laplacian operator with eigenvalue 2. This space provides a natural geometric framework for analyzing arithmetic expressions, particularly those involving addition and multiplication operations.

4.3 Geometric propagation mechanisms

The flow equation (30) in the $\mathfrak{E}_1$ space, $da/ds = \mu \cos\theta + a\lambda \sin\theta$, provides a dynamic interpretation of arithmetic expression evaluation as a propagation process. This perspective extends the propagation method discussed in Section **??**.

We can visualize this process using the concept of wavefronts. Considering the locus where the assignment $a = 0$ (the y-axis, $x = 0$) as the initial source or baseline, the "information" or "value" propagates outwards into the upper half-plane $\mathcal{H}$. The flow equation governs how the assignment value a changes as this propagation occurs.

Points with the same assignment value form equipotential lines, or contours, defined by $a = -x/y = a_0$ (constant). In the (x, y) coordinate chart of $\mathcal{H}$, these contours are rays emanating from the origin, described by the equation $x = -a_0 y$.

A key insight arises when observing how these equipotential lines behave as the propagation proceeds. Propagation, fundamentally occurring along directions related to the gradient of a (which are orthogonal to the contours), leads to an increase in the magnitude $|a|$ as points move further from the initial $a = 0$ line.

Now, consider the orientation of the contour ray $x = -a_0 y$. Its slope in the (x, y) plane is $dy/dx = -1/a_0$.

- When $|a_0|$ is very small (i.e., close to the initial $a = 0$ state on the y-axis), the slope $-1/a_0$ is very large in magnitude, meaning the ray is nearly vertical, aligned closely with the y-axis.
- As the wavefront propagates outwards, the magnitude $|a_0|$ increases. Consequently, the magnitude of the slope $|-1/a_0|$ decreases, approaching zero as $|a_0| \to \infty$. This means the ray becomes increasingly horizontal, aligning more closely with the x-axis.

Therefore, the geometric propagation driven by the flow equation manifests as a dynamic **sweeping or change in orientation of the equipotential rays** $a = a_0$. As the magnitude $|a|$ increases due to propagation away from the zero line, the corresponding ray appears to **rotate** from a near-vertical orientation (along the y-axis for $a = 0$) towards a near-horizontal orientation (along the positive x-axis if $a \to -\infty$, or the negative x-axis if $a \to +\infty$).

This increase in magnitude $|a|$ with the propagation distance s along the gradient is quantified by the relationship $|a| = (\mu/\lambda)\sinh(\lambda s)$ (derived from (65) for propagation from $a_0 = 0$), confirming that further propagation indeed leads to larger $|a|$ values and thus more horizontally oriented contour lines.

This dynamic picture of equipotential rays sweeping from vertical towards horizontal alignment provides a core geometric interpretation of the propagation mechanism inherent in the flow equation within the $\mathfrak{E}_1$ space.

4.4 Grid structures, chirality, and their interrelation via conformal mapping

A significant geometric characteristic of the first kind arithmetic expression space ($\mathfrak{E}_1$) is the presence of two distinct yet interrelated grid structures. These structures provide a geometric realization for arithmetic operations and reveal a deep connection to the Baumslag–Solitar groups $BS(2, 1)$ and $BS(1, 2)$. Both grid types are constructed within the upper half-plane model $\mathcal{H} = \{(x, y) \mid y > 0\}$, utilizing the assignment function $a = -x/y$ as the underlying scalar field that defines the meaning of arithmetic operations.

4.4.1 The Rectilinear Grid: Geometric Realization of $BS(2, 1)$ and its Chirality

The first grid structure, the **rectilinear grid** (Figure 9), is formed by lines parallel to the Cartesian axes.

- **Addition Loci (Horizontal Lines, $y = c_0$):** Movement along these blue lines, changing x, corresponds to an **addition** operation on the assignment value a. Specifically, an operation $a_{new} = a_{old} + s$ (where s is the amount added) is realized by changing x from x_{old} to $x_{new} = x_{old} - s \cdot y_0$. The red numerals on these lines in the original Figure 9 (which this refers to) indicate the value of $a = -x/y$ at those points.
- **Multiplication Loci (Vertical Lines, $x = c_0$):** Movement along these green lines, changing y, corresponds to a **multiplication** operation on a. An operation $a_{new} = a_{old} \cdot k$ (where k is the multiplication factor) is realized by changing y from y_{old} to $y_{new} = y_{old}/k$. The blue numerals on these lines in the original Figure 9 indicate the value of $a = -x/y$ at those points. For this rectilinear grid, we consider a specific multiplication factor $k = 2$.

The assignment $a = -x/y$ at grid intersections reflects the outcome of these operations. This rectilinear grid, with an addition step s and multiplication factor $k = 2$, serves as a geometric realization of the **Baumslag–Solitar group** $BS(2, 1)$, defined by $\langle x_g, y_g \mid y_g^{-1} x_g^2 y_g = x_g \rangle$. Using Currying-style operators $\oplus_s$ for x_g (representing adding s) and $\otimes_k$ for y_g (representing multiplying by k), the group relation $v + 2s/k = v + s$ is satisfied if $k = 2$ (for $s \neq 0$). This confirms the consistency of the rectilinear grid with $BS(2, 1)$ when the multiplication factor is 2.

4.4.2 Chirality of the Rectilinear Grid

We define the **chirality** (handedness) of this operational system at any point as the sense of rotation from the local direction of "increasing a via addition" to the local direction of "increasing a via multiplication (with factor $k > 1$)."

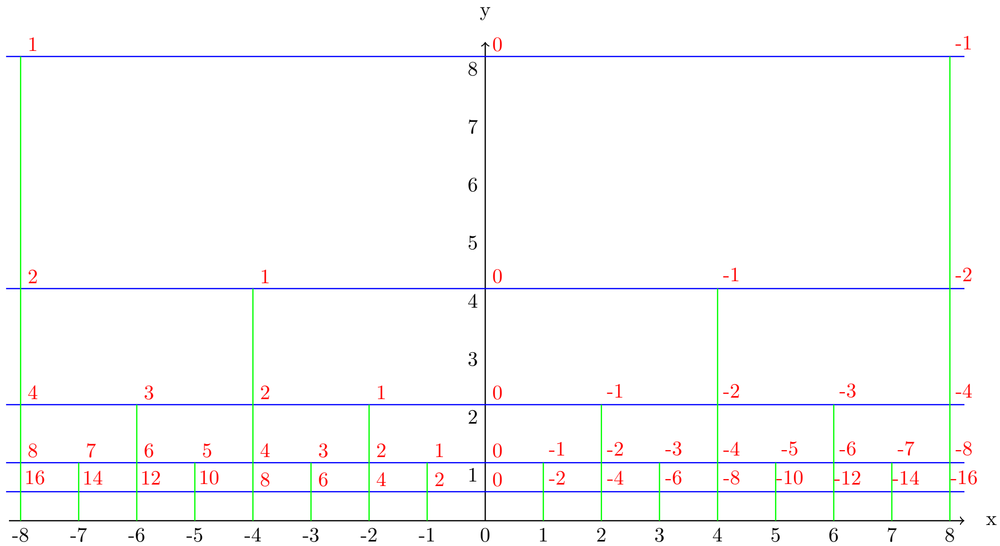

Figure 9: Rectilinear grid structure in $\mathfrak{E}_1$, a geometric realization of $BS(2,1)$ (with multiplication factor 2), exhibiting a clockwise (right-handed) operational chirality.

- **Direction of Increasing Addition (V_{add}):** To increase a by $s > 0$, x must decrease (since $y > 0$). V_{add} is locally in the $-x$ direction (left).
- **Direction of Increasing Multiplication (V_{mul}):** To increase a by a factor $k = 2$ (assuming $a \neq 0$), y must decrease to $y/2$. V_{mul} is locally in the $-y$ direction (down).

The rotation from V_{add} (left) to V_{mul} (down) in the (x, y) plane is **clockwise**. We designate this H_0, and for this discussion, associate it with a "right-handed system."

4.4.3 The Transformed Grid via $w = -1/z$: Emergence of $BS(1,2)$ through Chirality Preservation

The second grid structure, the **transformed (or curved) grid** (Figure 10), results from applying the conformal transformation $w = -1/z$ to the rectilinear grid, where $z = x + iy$ and $w = u + iv$. This specific Möbius transformation maps $\mathcal{H}$ to itself, transforming horizontal lines ($y = c_0$) in the z-plane (addition lines) to semicircles $u^2 + (v - 1/(2c_0))^2 = (1/(2c_0))^2$ and vertical li*nes ($x = c_0$) in the z-plane (multiplication lines) to semicircles $(u + 1/(2c_0))^2 + v^2 = (1/(2c_0))^2$.

The assignment function transforms as $a_{orig} = -x/y \to a_{new} = -u/v = x/y = -a_{orig}$. This sign inversion has a crucial impact on the interpretation of arithmetic operations and the system's chirality:

1. **Effect on Addition and Chirality (Intermediate State):** The conformal map $w = -1/z$ is orientation-preserving. Let J be its Jacobian. An operation that increases a_{orig} by s (direction V_{add}) now changes a_{new} by $-s$. Thus, to achieve an increase in a_{new} by s' (i.e., $a'_{new} = a'_{old} + s'$), the underlying operation corresponds to changing a_{orig} by $-s'$. The local direction for this, V'_{add}, is $J(-V_{add})$. If we provisionally keep the multiplication factor as $k = 2$, the direction for increasing a_{new} by this factor, V'_{mul}, is $J(V_{mul})$. The rotation from $V'_{add} \sim J(-V_{add})$ to $V'_{mul} \sim J(V_{mul})$ is now **counter-clockwise** (an H_1 or "left-handed" chirality).
2. **Chirality Preservation and Adjustment of Multiplication:** We posit that for the transformed system to maintain a consistent operational framework analogous to the original, the original chirality H_0 (clockwise) must be preserved. The observed flip to H_1 (counter-clockwise) in the intermediate state (due to the $a \to -a$ induced inversion of the effective addition direction) necessitates a compensatory "logical inversion" in the multiplication operation. This logical inversion is interpreted as changing the multiplication factor from k to $1/k$. Thus, the multiplication factor for the transformed grid becomes $1/2$ (the reciprocal of the original $k = 2$).
3. **Final State of the Transformed Grid and $BS(1,2)$:** The arithmetic operations for the transformed grid, principled by chirality preservation, are:

- Effective addition step for a_{new}: $-s$ (where s was the step for a_{orig}).
- Effective multiplication factor for a_{new}: $1/2$.

Let $V''_{add} \sim J(-V_{add})$ be the direction of increasing a_{new} via an addition of (positive) $-s$. Let V''_{mul} be the direction of increasing a_{new} via multiplication by $1/2$. Since $1/2 < 1$, for $a_{new} > 0$, this operation decreases a_{new}. The direction corresponding to the operator "multiply by $1/2$" is thus opposite to that of "multiply by 2". Hence, $V''_{mul} \sim J(-V_{mul})$. The rotation $V''_{add} \to V''_{mul}$ (i.e., $J(-V_{add}) \to J(-V_{mul})$) is now **clockwise**, restoring the original chirality H_0. A system with addition $\oplus_{-s}$ and multiplication $\otimes_{\ln(1/2)}$ (factor $1/2$) satisfies the $BS(1,2)$ relation $(y')^{-1}x'y' = (x')^2$, where x' is $\oplus_{-s}$ and y' is $\otimes_{\ln(1/2)}$. This is because $(-s)/(1/2) = 2(-s)$. Therefore, the transformed grid, under this chirality-preserving reinterpretation of its operations, serves as a geometric realization of $BS(1,2)$.

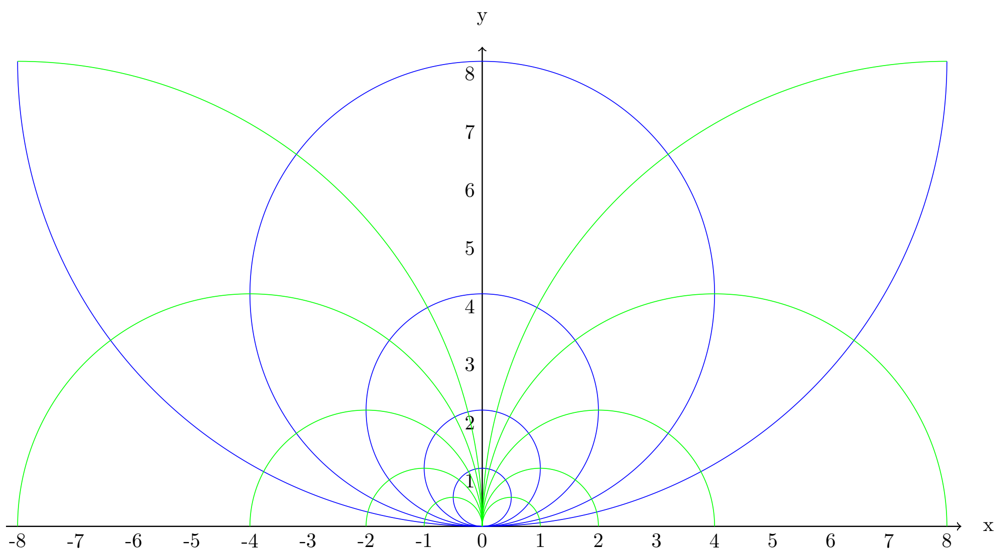

Figure 10: Transformed (curved) grid structure in $\mathfrak{E}_1$ via $w = -1/z$. Through chirality preservation, its operations (effective addition step $-s$, effective multiplication factor $1/2$) correspond to $BS(1,2)$.

4.4.4 Interrelation and Symmetry

The rectilinear grid (realizing $BS(2,1)$ with $k = 2$) and the transformed curved grid (realizing $BS(1,2)$ with $k = 1/2$) are intrinsically linked by the conformal map $w = -1/z$. This geometric transformation, when augmented by the principle of preserving a defined operational chirality, reveals a profound connection between these two distinct Baumslag–Solitar groups. It is not merely a visual transformation of grid lines but a process that systematically reinterprets the arithmetic operations (addition step $s \to -s$, multiplication factor $k \to 1/k$) to maintain a fundamental handedness. This framework elucidates how different algebraic structures can manifest from the same underlying geometric space $\mathfrak{E}_1$ and assignment field $a = -x/y$, connected by conformal symmetry and consistent operational principles.

4.5 Torsion under scale transformation

The addition-multiplication grid introduced in Section 2.2 has a natural embedding in the arithmetic expression space $\mathfrak{E}_1$. This grid consists of two orthogonal families of curves:

1. **Addition curves** (blue lines): horizontal geodesics along which y remains constant, representing iterated additions.
2. **Multiplication curves** (green lines): vertical or logarithmically scaled geodesics where the ratio x/y remains constant, representing multiplicative transformations.

This grid structure facilitates the geometric analysis of *arithmetic torsion*—a quantity arising from the non-commutativity of certain additive and multiplicative expression sequences. Specifically, torsion quantifies the discrepancy between two seemingly equivalent but differently ordered expressions.

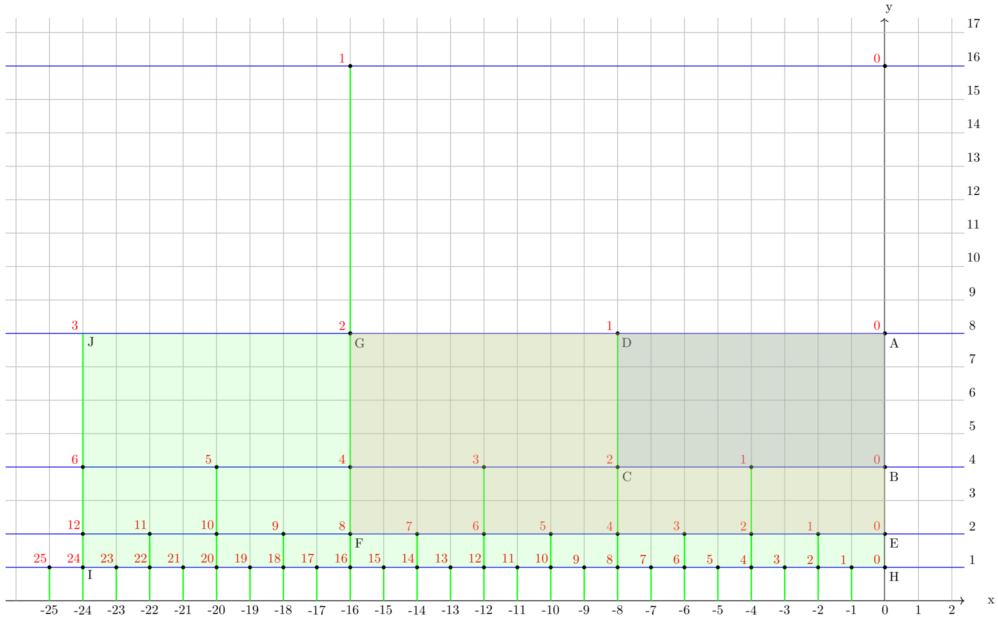

Figure 11: Illustration of the correspondence between hyperbolic area and arithmetic torsion

Figure 11 illustrates the relationship between the area enclosed by expression paths in the grid and the resulting torsion. Consider the following expression identity comparisons:

- One-step case:
$$x \times 2 + 1 - (x + 1) \times 2 = -1 \tag{73}$$
- Two-step case:
$$x \times 4 + 2 - (x + 2) \times 4 = -6 \tag{74}$$
- Three-step case:
$$x \times 8 + 3 - (x + 3) \times 8 = -21 \tag{75}$$

These differences correspond precisely to the hyperbolic areas enclosed between alternative evaluation paths within these specific grid examples:

- The region $ABCD$ encompasses 1 unit cell.
- The region $AEFG$ encompasses 6 unit cells.
- The region $AHIJ$ encompasses 21 unit cells.

These examples compellingly suggest that arithmetic torsion accumulates in proportion to the area enclosed by the grid paths, indicating a potential connection between algebraic non-commutativity and geometric surface area.

Further supporting this connection, we established a differential formulation:

$$d\tau = \mu\lambda \, du \, dv \tag{76}$$

where $d\tau$ represents the infinitesimal arithmetic torsion, and $du\,dv$ denotes the area element in the (u, v) coordinate system adapted to the grid. This equation provides the precise microscopic law linking infinitesimal torsion to the infinitesimal area element.

However, while we possess both macroscopic observations suggesting a torsion-area relationship (from the scaled grid examples) and the exact microscopic differential law (76), the explicit integral theorem rigorously bridging these scales is currently underdeveloped. Formulating how the microscopic torsion density $d\tau$ integrates over a finite region to yield the total accumulated torsion—potentially involving boundary terms in analogy with Stokes' theorem, or relating the total torsion to curvature and topology similarly to the Gauss-Bonnet theorem—remains a key objective for future work.

The analogy with curvature in differential geometry therefore becomes particularly pertinent: just as Gaussian curvature encodes deviation from flatness, arithmetic torsion quantifies deviation from commutativity in arithmetic flow. In this sense, torsion constitutes a measure of the operational significance of evaluation order.

The $\mathfrak{E}_1$ space thus provides a mathematical framework where algebraic non-commutativity manifests as measurable geometric distortion—establishing a novel interpretation of arithmetic structure as a form of discrete curvature. This opens avenues for investigating further geometric invariants such as torsion density, torsion-induced flow bifurcation, and **realizing** a Gauss–Bonnet-type integral identity for arithmetic surfaces.

4.6 Tube structure

In preceding sections, we introduced the first kind arithmetic expression space $\mathfrak{E}_1$ as a geometric realization of arithmetic flow under **fixed** generator parameters μ and λ. However, more complex structures emerge when we consider the entire family of spaces indexed by the parameter λ (or potentially both μ and λ) and analyze how expression behavior evolves across this family. This naturally leads to the concept of a *tube structure*.

4.6.1 From Slices to Parameterized Families

Each individual $\mathfrak{E}_1$ space, denoted $\mathfrak{E}_1^{(\lambda)}$, can be conceptualized as a single *slice* or *fiber* (in the sense of fiber bundles) within the family of expression spaces indexed by the parameter λ. Within each slice, the evaluation of arithmetic expressions is realized through traversal along (geodesic) paths, the result is governed by the scalar field a, and the flow is determined by the metric tensor corresponding to that slice.

Consider a fixed algebraic structure—for instance, an alternating path (with a fixed internal multiplier) corresponding to a polynomial $P(x)$—and examine how its evaluation result $P(e^\lambda)$ evolves as the tube structure parameter λ varies. For each value of λ, the evaluation $P(e^\lambda)$ corresponds to a point (or more accurately, the assignment value a at that point) within the λ-slice $\mathfrak{E}_1^{(\lambda)}$. As λ varies continuously, these points trace out a continuous trajectory through the family of spaces. We refer to such a trajectory generated by P as a *section* or a *λ-trajectory*. The collection of all slices corresponding to the allowed λ values, along with these structures upon them, together form a new, higher-dimensional entity: the tube structure.

4.6.2 Tube Structure as Total Space

We define a **tube structure** $\mathcal{T}$ as the *total space* formed by the family of $\mathfrak{E}_1$ spaces indexed by a continuous parameter λ (typically $\lambda > 0$), which can be formally written as the disjoint union:

$$\mathcal{T} = \bigsqcup_{\lambda>0} \mathfrak{E}_1^{(\lambda)} \tag{77}$$

This total space needs to be endowed with an appropriate topology (and possibly a differential or fiber bundle structure) to support coherent analysis along the λ-direction.

In this structure:

- The *base space* is the parameter domain Λ for λ (e.g., $\mathbb{R}^+$).
- The *fiber* over each point λ in the base space is the geometric expression space $\mathfrak{E}_1^{(\lambda)}$.
- Fixed algebraic expression structures (especially those corresponding to polynomials $P(x)$, via the evaluation $P(e^\lambda)$) trace *canonical sections* or λ-trajectories through $\mathcal{T}$. These sections connect the fibers for different λ.

4.6.3 Zero Loci and Nodal Evolution

A primary motivation for studying tube structures is to investigate how *zero loci*—the sets of points where an expression evaluates to zero ($a = 0$)—evolve with the parameter λ.

- **In the Tube Structure $\mathcal{T}_1$ based on $\mathfrak{E}_1$**: For the $\mathfrak{E}_1$ space ($a = -x/y$) that we have discussed in detail, the zero locus within **each slice** $\mathfrak{E}_1^{(\lambda)}$ is always the **same simple** line: the y-axis ($x = 0$). Consequently, in the tube structure $\mathcal{T}_1 = \bigsqcup \mathfrak{E}_1^{(\lambda)}$, the overall zero locus is the trivial hyperplane $x = 0$.
- **Outlook for Non-Trivial Spaces**: However, as our research suggests, the simplicity of the zero locus in $\mathfrak{E}_1$ might limit its capacity to explain more complex phenomena (like those observed in knot theory examples). Therefore, there is strong motivation to seek and construct **"non-trivial" arithmetic expression spaces** $\mathfrak{E}_{NT}$,

where a single slice $\mathfrak{E}_{NT}^{(\lambda)}$ might possess **multiple or morphologically more complex zero lines**. In the tube structures $\mathcal{T}_{NT}$ built from such non-trivial spaces, the zero locus itself could evolve with λ, potentially exhibiting various interesting phenomena, such as:

- **Bifurcation**: New zero lines might emerge or merge with existing ones as λ varies.
- **Branching**: The zero locus of certain expressions might exhibit multi-valued behavior along the λ-direction.
- **Topology change**: The overall zero surface might develop handles (genus), singularities, or undergo other changes in its topological structure.

The analysis of such complex zero loci and their evolution (potentially within $\mathcal{T}_{NT}$) constitutes a core direction for studying expression dynamics, particularly when considering families of expressions or differential equations involving λ.

4.6.4 Investigative Approaches and Outlook

The formalism of tube structures opens up multiple avenues for research in arithmetic expression geometry:

- Examining the global properties of zero surfaces (in the general case where they might be non-trivial), such as their genus, regions of curvature concentration, and dependence on the parameter λ.
- Studying the geometric properties of sections γ_P corresponding to polynomials $P(e^\lambda)$ within the tube structure, and investigating whether imposing geometric continuity conditions leads to algebraic rigidity.
- Exploring the possibility of establishing flow equations across the λ-family, perhaps defining a notion of *connection* or *parallel transport* between different λ-slices.
- Defining *moduli spaces* of expression geometries as structured fiber bundles over parameter spaces.

Ultimately, tube structures provide a mathematical framework wherein the dynamics of arithmetic expressions can be analyzed analogously to field theory. In this analogy, expressions (or their underlying algebraic structures) act as structured sections, while quantities like arithmetic torsion, curvature, and zero loci serve as local or global invariants.

5 The accumulative commutative space

5.1 Global arithmetic torsion and the accumulative commutative space

The non-commutative nature of fundamental arithmetic operations, such as addition and multiplication, is the very source of arithmetic torsion. Consider the local effect: evaluating an expression like $x \oplus_\mu \otimes_\lambda$ versus $x \otimes_\lambda \oplus_\mu$. While both "path segments" originate from the same initial value x, their terminal points in the arithmetic expression space will differ if a non-zero local torsion $\tau_{\text{local}} = \nu(x \oplus_\mu \otimes_\lambda) - \nu(x \otimes_\lambda \oplus_\mu)$ exists. These two segments do not naturally form a closed loop; instead, they represent a "gap" or a "tear" caused by the order of operations. This inherent "openness" or "torn state" at the local level signifies a deviation from commutativity and poses a challenge for directly applying integral theorems like Green's or Stokes', which typically rely on closed paths or well-defined bounded regions within the space of action.

The crucial question then arises: how do these elemental "tears" accumulate along an extended, arbitrary arithmetic path γ? How can we quantify the total geometric impact of these accumulated deviations? This is the heart of the "torsion-area problem" which we previously highlighted with specific examples in the $\mathfrak{E}_1$ space (cf. Material 4, Section 4.5) and the local differential formulation $d\tau_{\text{local}} = \mu\lambda du dv$. We seek a general principle to measure this overall "deviation from commutativity" for any path γ.

Let us reflect further on the meaning of torsion. Global arithmetic torsion, which we will define algebraically as $\tau_{\text{alg}}(\gamma) = \nu(\gamma) - \nu(\bar{\gamma})$ (the difference between evaluating a path and its reversed-sequence counterpart), is precisely a measure of this accumulated "torn state" across the entire path. It quantifies how far the sequence of operations, taken as a whole, has strayed from a kind of "effective commutativity" that would see $\nu(\gamma)$ and $\nu(\bar{\gamma})$ be equal. This persistent difference, this "tear" magnified globally, intrinsically suggests the need for a different kind of space—a "commutative space"—wherein these path-dependent discrepancies can be consistently charted and their collective magnitude assessed. We need a framework where the comparison between γ and $\bar{\gamma}$ can be geometrically realized by forming a closed boundary.

Thus, the **accumulative commutative space** (ACS) is born. This space, constructed by the *commutative accumulation* of operational parameters ($A_\gamma = \sum \mu_k$ and $M_\gamma = \sum \lambda_k$) from a path γ, provides exactly such a commutative canvas. Critically, in this ACS, any path γ and its reversed-sequence counterpart $\bar{\gamma}$ share common start $(0, 0)$ and end points (A_γ, M_γ). This allows them to genuinely form the boundary $\partial\Sigma_\gamma$ of a well-defined planar region Σ_γ. It is upon this stage that we can rigorously explore and establish the geometric nature of global arithmetic torsion.

Algebraic definition of global arithmetic torsion $\tau(\gamma)$: comparison via path reversal

Let γ be an arbitrary arithmetic path, representing an ordered sequence of operations $op_n \circ \cdots \circ op_1$ acting on an initial value x. The local non-commutative effects accumulate along this path. To quantify the total, accumulated arithmetic torsion for the entire path γ, we compare its standard evaluation $\nu(\gamma)$ with the evaluation of its **reversed path,** $\bar{\gamma}$. The reversed path $\bar{\gamma}$ is defined by applying the *literal sequence of operations* from γ in the exact reverse order ($op_1 \circ op_2 \circ \cdots \circ op_n$) to the same initial value x.

The **global arithmetic torsion,** $\tau(\gamma)$, is then defined through its **algebraic evaluation form**:

$$\tau_{\text{alg}}(\gamma) = \nu(\gamma) - \nu(\bar{\gamma}) \tag{78}$$

A crucial property, verified for the operations $\oplus_\mu$ and $\otimes_\lambda$, is that $\tau_{\text{alg}}(\gamma)$ is independent of the initial value x, making it an intrinsic characteristic of the operational structure of path γ. The evaluations $\nu(\gamma)$ and $\nu(\bar{\gamma})$ occur within the **arithmetic expression space** (which may be modeled by, e.g., the hyperbolic $\mathfrak{E}_1$ space or more generally over $\mathbb{R}$ or $\mathbb{C}$).

Geometric representation of paths in the accumulative commutative space and the region Σ_γ

The accumulative commutative space (ACS), whose conceptual motivation was just outlined, is a Euclidean plane. For any given arithmetic path γ, its representation in this space is constructed by accumulating the parameters of its constituent operations:

- $A_\gamma = \sum \mu_k$: The total accumulated additive charge from all $\oplus_{\mu_k}$ operations in γ.
- $M_\gamma = \sum \lambda_k$: The total accumulated logarithmic multiplicative charge from all $\otimes_{\lambda_k}$ operations in γ.

When the sequence of operation parameters defining path γ is plotted as a trajectory in this ACS, it charts a path starting from the origin $(0, 0)$ to a final point (A_γ, M_γ). Critically, the reversed path $\bar{\gamma}$, having the same set of operations, also maps from $(0, 0)$ to the *same* endpoint (A_γ, M_γ), though typically via a different trajectory in the (A, M) plane (which are the coordinates of the ACS). These two trajectories—one for γ and one for $\bar{\gamma}$—naturally enclose a two-dimensional region, which we denote as Σ_γ.

Core conclusion: the triple identity of global arithmetic torsion

Within this framework, our central finding, verified through numerous examples, is that the global arithmetic torsion $\tau(\gamma)$ possesses three equivalent formulations. This **triple identity** bridges its algebraic definition with concrete geometric measures in the accumulative commutative space:

(I) Algebraic evaluation form (as defined in Eq. 78):

$$\tau(\gamma) = \nu(\gamma) - \nu(\bar{\gamma})$$

(II) Geometric interior integral form in the accumulative commutative space: The torsion is precisely equal to the integral of the 2-form $e^M dM \wedge dA$ over the region Σ_γ:

$$\tau(\gamma) = \iint\limits_{\Sigma_\gamma} e^M dM \wedge dA$$

(The order $dM \wedge dA$ versus $dA \wedge dM$ depends on the chosen orientation of Σ_γ; the kernel e^M reflects the exponential scaling inherent in multiplicative operations).

(III) Geometric boundary integral form in the accumulative commutative space: By Green's (or Stokes') Theorem, this area integral is equivalent to a line integral of the 1-form $\omega = e^M dA$ over the oriented boundary $\partial\Sigma_\gamma$ of the region Σ_γ. If $\partial\Sigma_\gamma$ is oriented as traversing $\bar{\gamma}$ and then $-\gamma$ (the reverse of path γ):

$$\tau(\gamma) = \oint_{\partial\Sigma_\gamma} e^M dA$$

In its discrete summation form, for paths composed of $\oplus_{\mu_k}$ and $\otimes_{\lambda_k}$ operations, this becomes:

$$\tau(\gamma) = \left(\sum_{(\mu_k, M_k) \in \bar{\gamma}} \mu_k e^{M_k} \right) - \left(\sum_{(\mu_j, M_j) \in \gamma} \mu_j e^{M_j} \right)$$

where the sum is over the additive steps μ in each path, and M is the accumulated logarithmic multiplicative charge at the point of that addition.

In summary, our main theorem states the equality of these three forms:

$$\tau_{\text{alg}}(\gamma) = \tau_{\text{int}}(\gamma) = \tau_{\text{bound}}(\gamma) \tag{79}$$

The establishment of this triple identity, particularly the equivalence $\tau_{\text{alg}}(\gamma) = \iint_{\Sigma_\gamma} e^M dM \wedge dA$, provides a definitive affirmative answer to our initial "torsion-area problem": global arithmetic torsion for an arbitrary path γ can indeed be precisely quantified as an e^M-weighted area within the accumulative commutative space. This lends a concrete geometric meaning to an otherwise abstract algebraic difference, distinct from, yet inspired by, the initial observations in $\mathfrak{E}_1$.

While the accumulative commutative space (ACS), with its specific accumulation rules for A and M, has proven effective for this geometric quantification, its introduction might initially appear as a purpose-built construct. However, the significance of this parameter space is greatly amplified by an independent algebraic perspective. As our broader research reveals (and as will be discussed further), this very (A, M) coordinate system (which defines the ACS) also emerges naturally as the target space for group homomorphisms from F_2 (the free group of arithmetic operations) and related groups such as $BS(m, n)$. These homomorphisms map operational sequences to their net (A, M) "charges." This dual role of the ACS—both as a geometric stage for quantifying torsion and as an algebraic target for characterizing process structure—makes our theoretical narrative more cohesive and interesting, highlighting the ACS as a truly pivotal element in the emerging geometry of arithmetic expressions.

Significance and interpretation: a Stokes-like theorem for arithmetic expressions

This **triple identity** (Eq. 79) is a cornerstone of our current understanding of global arithmetic torsion. It establishes a precise and verified equivalence between:

- A purely algebraic quantity ($\tau_{\text{alg}}(\gamma)$) derived from evaluation differences within the (potentially non-Euclidean) arithmetic expression space (such as $\mathfrak{E}_1$).
- A geometric interior integral ($\tau_{\text{int}}(\gamma)$) representing an e^M-weighted area in the flat Euclidean accumulative commutative space.

- An equivalent geometric boundary integral ($\tau_{\text{bound}}(\gamma)$) also in the accumulative commutative space.

The dual-space model—distinguishing the arithmetic expression space (where operations are realized and $\nu(\gamma)$ is determined) from the accumulative commutative space (where torsion is geometrically quantified via parameter accumulation)—is pivotal. The e^M kernel serves as the crucial bridge, translating the multiplicative scaling dynamics inherent in arithmetic operations (and potentially manifest in the arithmetic expression space) into the weighting of geometric elements in the accumulative commutative space.

This framework provides a concrete "Stokes-like theorem for global arithmetic torsion," relating a global algebraic effect (the total torsion) to the integral of a local geometric density (in the accumulative commutative space) or a boundary sum. It forms a solid foundation for further investigations, including relating this accumulative commutative space integral back to potential curvature integrals in spaces like $\mathfrak{E}_1$, thereby advancing towards a more comprehensive arithmetic Gauss-Bonnet type theorem.

5.2 F_2 homomorphisms and ideal-like structures

In the preceding section, the accumulative commutative space (ACS) was introduced as a crucial geometric stage. Its commutative nature—where coordinates (A_γ, M_γ) are formed by the simple summation of operational parameters, irrespective of their order in a path γ—allowed any path γ and its reversed-sequence counterpart $\bar{\gamma}$ to share common start and end points. This was instrumental in defining a bounded region Σ_γ, thereby enabling the geometric quantification of global arithmetic torsion $\tau(\gamma)$ as an e^M-weighted area. Having established its utility as a "commutative canvas" for geometric measurement, we now turn to explore the deeper algebraic structures associated with this ACS and its connection to the fundamental generative structure of arithmetic paths.

Arithmetic paths, in their most abstract and unconstrained form, are generated by a sequence of elementary operations. Let us consider two fundamental types of abstract generative processes: an "additive process" X_A, which intends to shift a value by a basic unit μ_0, and a "logarithmic multiplicative process" X_M, which intends to scale a value by a factor e^{λ_0} (thus shifting its logarithm by λ_0). The set of all possible finite sequences composed of these processes and their inverses (X_A^{-1} for subtraction by μ_0, X_M^{-1} for division by e^{λ_0} or scaling by $e^{-\lambda_0}$) forms the free group on two generators, $F_2 = \langle X_A, X_M \rangle$. It is crucial to note that at this stage, we are considering F_2 purely as a group of symbolic operational sequences or "generative structures." We are not yet concerned with the arithmetic evaluation $\nu(\gamma)$ of a path $\gamma \in F_2$ to a specific numerical outcome, but rather with the algebraic properties inherent in these sequences themselves as revealed by their cumulative operational parameters.

This perspective leads us very naturally to define a group homomorphism from the (generally non-commutative) free group F_2 to an abelian group representing the accumulated parameters in the accumulative commutative space. Let $\mathbb{A}_{(A,M)}$ denote this target abelian group (which defines the ACS), typically $\mathbb{Z}\mu_0 \times \mathbb{Z}\lambda_0$ if μ_0, λ_0 are considered indivisible units associated with single applications of X_A and X_M respectively (or $\mathbb{R}^2$ for continuous parameters). We define the homomorphism $\Phi : F_2 \to \mathbb{A}_{(A,M)}$ by its action on the generators:

- $\Phi(X_A) = (\mu_0, 0)$
- $\Phi(X_M) = (0, \lambda_0)$
- $\Phi(X_A^{-1}) = (-\mu_0, 0)$
- $\Phi(X_M^{-1}) = (0, -\lambda_0)$

For any path $\gamma = g_1 g_2 \dots g_k \in F_2$ (where $g_i \in \{X_A, X_M, X_A^{-1}, X_M^{-1}\}$), its image is $\Phi(\gamma) = \sum_{i=1}^{k} \Phi(g_i) = (A_\gamma, M_\gamma)$, where A_γ is the total accumulated "additive charge" and M_γ is the total accumulated "logarithmic multiplicative charge". This map Φ is indeed a group homomorphism because composition of paths in F_2 corresponds to vector addition of their (A, M) images in the abelian group $\mathbb{A}_{(A,M)}$. This homomorphism effectively translates the non-commutative compositional structure of F_2 paths into the simpler, commutative world of their net operational "footprints" (A_γ, M_γ) within the ACS.

We can further consider the component homomorphisms:

- $\phi_A : F_2 \to \mathbb{Z}\mu_0$ mapping $\gamma \mapsto A_\gamma$.
- $\phi_M : F_2 \to \mathbb{Z}\lambda_0$ mapping $\gamma \mapsto M_\gamma$.

These are also group homomorphisms. A key insight arises when we consider the preimages of "ideal-like" structures (specifically, subgroups of the form $k\mathbb{Z}\mu_0$ or $k\mathbb{Z}\lambda_0$) from the target abelian groups back into F_2. For any integer k, let:

- $K_{(k),A} = \phi_A^{-1}(k\mathbb{Z}\mu_0) = \{\gamma \in F_2 \mid A_\gamma \text{ is an integer multiple of } k\mu_0\}$
- $K_{(k),M} = \phi_M^{-1}(k\mathbb{Z}\lambda_0) = \{\gamma \in F_2 \mid M_\gamma \text{ is an integer multiple of } k\lambda_0\}$

Since ϕ_A and ϕ_M are group homomorphisms and their target groups are abelian (making all subgroups normal), $K_{(k),A}$ and $K_{(k),M}$ are normal subgroups of F_2. For instance, $K_{(k),A}$ comprises all operational sequences whose net additive effect (in terms of μ_0 units) is "trivial modulo k." These subgroups thus impose an arithmetic classification upon the abstract generative paths in F_2.

Remarkably, the lattice structure of these normal subgroups in F_2 (ordered by inclusion) is perfectly compatible with the lattice structure of ideals in $\mathbb{Z}$ and, consequently, with the nesting of closed sets in the Zariski topology on Spec $\mathbb{Z}$. For example, considering the A-component (and assuming $\mu_0 = 1$ for simplicity, so $\phi_A : F_2 \to \mathbb{Z}$):

$$K_{(n),A} \subseteq K_{(d),A} \iff n\mathbb{Z} \subseteq d\mathbb{Z} \iff d|n$$

And similarly for the M-component. This corresponds directly to the nesting of Zariski closed sets $V(k\mathbb{Z})$ (where $V(\cdot)$ here denotes the variety of an ideal in algebraic geometry):

$$V(d\mathbb{Z}) \subseteq V(n\mathbb{Z}) \iff d|n$$

Thus, we have the chain:

$$K_{(n),A} \subseteq K_{(d),A} \iff d|n \iff V(d\mathbb{Z}) \subseteq V(n\mathbb{Z})$$

This demonstrates a profound structural correspondence: the way these specific normal subgroups $K_{(k),A}$ (or $K_{(k),M}$) are organized within F_2 faithfully mirrors a fundamental organizing principle of the Zariski topology on Spec $\mathbb{Z}$.

In this manner, the accumulative commutative space (ACS), initially introduced as a geometric tool for understanding global torsion as an area, reveals its deeper algebraic significance. It serves as a "character space" for F_2, where abstract operational sequences are mapped to quantitative "charges" (A_γ, M_γ). These charges, in turn, allow us to define ideal-like normal subgroups within F_2 that resonate with classical algebraic and topological structures found in number theory and algebraic geometry. This uncovers an intrinsic arithmetic order within the purely generative framework of F_2, even before any notion of arithmetic evaluation $\nu(\gamma)$ is applied. This understanding forms a crucial foundation as we later consider how these structures behave when F_2 is "condensed" by imposing relations to form other groups, such as the Baumslag-Solitar groups.

5.3 Conclusion and Transition

In this section, we have demonstrated the necessity of the Accumulative Commutative Space (ACS) to provide a consistent geometric interpretation for global arithmetic torsion. By projecting non-commutative operational sequences onto a commutative parameter plane, we established that torsion is equivalent to a weighted area, captured by the Triple Identity. Furthermore, we revealed the deeper algebraic significance of the ACS as a character space for the free group of operations.

Thus far, the ACS has been treated as an abstract Euclidean plane for global accounting. This raises a further question: can this abstract parameter space be endowed with a richer differential structure that explains the *local* dynamics of arithmetic? In the next section, we will demonstrate that the ACS can be realized as the base space of a three-dimensional *contact manifold*. This new structure will provide the precise differential-geometric engine for the arithmetic flow, unifying the global, commutative picture of torsion with the local, non-commutative nature of operations.

6 Arithmetic expression contact structure

In the preceding section, we introduced the Accumulative Commutative Space (ACS) as an abstract parameter plane essential for quantifying global arithmetic torsion. While powerful for global analysis, the ACS lacks an intrinsic mechanism to describe the *local*, non-commutative dynamics of arithmetic operations.

This section bridges that gap by endowing the ACS with a rich differential structure. We construct a three-dimensional **Arithmetic Contact Manifold**, whose two-dimensional base is precisely a local, differential version of the ACS. This framework provides the geometric engine for the arithmetic flow, unifying the global, commutative picture of torsion with the local, non-commutative nature of operations. It is here that the flow equation from Section **??** finds its most natural home, and we develop the differential calculus necessary for its study.

6.1 Core definitions and the contact structure

We begin by establishing our geometric space. Consider a 3-dimensional manifold with coordinates $(u, v, a) \in \mathbb{R}^3$, where the base plane spanned by (u, v) is understood as the differential realization of the ACS. The geometry is built upon two fundamental 1-forms, defined using constant real parameters μ and λ:

$$\omega := \mu\, du + \lambda a\, dv, \qquad \alpha := da - \omega. \tag{80}$$

The 1-form α is central to our construction. Its primary role is to define a "horizontal" plane at each point, known as the contact distribution. To see how it works, let's consider its action on an arbitrary vector field $X = x_u\partial_u + x_v\partial_v + x_a\partial_a$. Recalling that $da(X) = x_a$, $du(X) = x_u$, and $dv(X) = x_v$, the action is given by:

$$\alpha(X) = da(X) - \mu\, du(X) - \lambda a\, dv(X) = x_a - \mu x_u - \lambda a x_v.$$

The set of all vectors X for which $\alpha(X) = 0$ constitutes this horizontal plane.

Throughout this work, we use two types of differentials in parallel: (i) the standard de Rham exterior derivative d, which is always nilpotent ($d^2 = 0$); and (ii) the **expression differential** δ, which is axiomatically defined in Section 7.

The contact property. A key property of α is that it defines a contact structure on our 3D space. This is verified by checking if the 3-form $\alpha \wedge d\alpha$ is a volume form (i.e., non-zero everywhere). First, we compute the exterior derivative of ω:

$$d\omega = d(\mu\, du + \lambda a\, dv) = \lambda\, da \wedge dv.$$

Then, the exterior derivative of α is simply $d\alpha = d(da - \omega) = -d\omega = -\lambda\, da \wedge dv$. Now, we can compute the wedge product:

$$\begin{aligned}
\alpha \wedge d\alpha &= (da - \omega) \wedge (-\lambda\, da \wedge dv) \\
&= -da \wedge (\lambda\, da \wedge dv) + \omega \wedge (\lambda\, da \wedge dv) \\
&= 0 + (\mu\, du + \lambda a\, dv) \wedge (\lambda\, da \wedge dv) \\
&= \mu\lambda\, du \wedge da \wedge dv + \lambda^2 a\, \underbrace{dv \wedge da \wedge dv}_{=0} \\
&= \mu\lambda\, du \wedge da \wedge dv.
\end{aligned}$$

Provided $\mu\lambda \neq 0$, this is a volume form, confirming that α is a **contact form**.

Normal form, Reeb field, and contact distribution. By introducing natural units $\tilde{u} = \mu u$ and $\tilde{v} = \lambda v$, the form α is reduced to its canonical form $\alpha_0 = da - d\tilde{u} - a\, d\tilde{v}$, which facilitates comparison with standard literature.

The Reeb vector field R, defined by $i_R d\alpha = 0$ and $\alpha(R) = 1$, is $R = -(1/\mu)\partial_u$. The contact distribution $\mathcal{H}$, as introduced earlier, is formally the kernel of α:

$$\mathcal{H} := \ker \alpha = \{X \in TM : \alpha(X) = 0\}.$$

Viewing the tangent bundle as a composition of the (u, v)-base and the a-fiber, the form $\alpha = da - \omega$ intrinsically links the "vertical" change da to the "horizontal" displacement defined by ω.

6.2 The geometry on the contact distribution $\ker \alpha$

We now define two special vector fields that form a basis for the horizontal plane $\mathcal{H}$ at every point. These are the horizontal lifts of the base coordinate vectors, which we term the **expression directional derivatives**:

$$D_u := \partial_u + \mu\, \partial_a, \qquad D_v := \partial_v + \lambda a\, \partial_a. \tag{81}$$

These fields are constructed specifically to be horizontal, a fact we can verify directly. For D_u, its coordinate components are $(x_u, x_v, x_a) = (1, 0, \mu)$. Applying the formula for $\alpha(X)$:

$$\alpha(D_u) = x_a - \mu x_u - \lambda a x_v = \mu - \mu(1) - \lambda a(0) = 0.$$

For D_v, its components are $(x_u, x_v, x_a) = (0, 1, \lambda a)$. Applying α:

$$\alpha(D_v) = x_a - \mu x_u - \lambda a x_v = \lambda a - \mu(0) - \lambda a(1) = 0.$$

Since both D_u and D_v are annihilated by α, and they are clearly linearly independent, they form a basis for the 2-dimensional contact distribution:

$$\ker \alpha = \operatorname{span}\{D_u, D_v\}.$$

For any smooth scalar field $F(u, v, a)$, we define its expression differential δF and the directional derivative D_θ as:

$$\delta F := (D_u F)\, du + (D_v F)\, dv, \qquad D_\theta := \cos\theta\, D_u + \sin\theta\, D_v. \tag{3}$$

This construction effectively reduces the geometry from the three-dimensional space (u, v, a) to the two-dimensional contact distribution $\mathcal{H}$.

Structural relation between d and δ. For any scalar field F, the two differentials are related by a fundamental identity:

$$\boxed{\delta F = dF - (\partial_a F)\, \alpha}$$

This identity can be interpreted as projecting dF onto the horizontal distribution $\mathcal{H}$ by subtracting its vertical component along α. Expanding this definition yields:

$$\delta F = (F_u + \mu F_a)\, du + (F_v + \lambda a F_a)\, dv,$$

which is consistent with Eq. (15). In particular, we recover $\delta a = \omega$, $\delta u = du$, and $\delta v = dv$.

6.3 Legendrian flow and rectification

A curve $\gamma(s) = (u(s), v(s), a(s))$ is **Legendrian** if its tangent vector lies in the contact distribution, i.e., $\dot\gamma(s) \in \ker\alpha$. For such a curve, the evolution of a is governed by the **flow equation**:

$$\frac{da}{ds} = D_\theta a = \mu\cos\theta + \lambda a \sin\theta, \tag{4}$$

where θ parameterizes the angle of the tangent vector in the basis $\{D_u, D_v\}$. With respect to a given metric on the base manifold, this flow equation can be written in its Eikonal form:

$$\|\nabla a\| = \sqrt{\mu^2 + \lambda^2 a^2}. \tag{5}$$

We introduce a **rectifying variable** y:

$$y = \operatorname{arcsinh}\left(\frac{\lambda a}{\mu}\right) \quad \Rightarrow \quad \|\nabla y\| = \lambda. \tag{6}$$

This rectification transforms the non-linear velocity field for a into a constant-speed flow for y, which is advantageous for geometric constructions and enhances numerical stability.

Non-commutativity and curvature. The commutator of the horizontal vector fields yields a purely vertical vector, reflecting the "curvature" of the contact distribution. This phenomenon can be described as a "vertical return":

$$[D_u, D_v] = \mu\lambda\, \partial_a, \qquad \delta^2 F = \mu\lambda(\partial_a F)\, du \wedge dv, \qquad \delta^2 a = \mu\lambda\, du \wedge dv. \tag{7}$$

The circulation-area formula provides a tool for quantifying mesh singularities and global topological constraints:

$$\oint_{\partial\Sigma} \omega = \iint_{\Sigma} d\omega = \mu\lambda \iint_{\Sigma} du \wedge dv. \tag{8}$$

Compatibility with de Rham cohomology. The exterior derivative of ω is $d\omega = \lambda\, da \wedge dv$. By pulling this 2-form back to a section where $\alpha = 0$ (i.e., substituting $da = \omega$), we obtain:

$$(d\omega)^* = \lambda\, \omega \wedge dv = \mu\lambda\, du \wedge dv,$$

which is identical to $\delta^2 a$. This demonstrates the precise collaborative relationship between d and δ.

Example: Basis Flows. Flowing along D_u implies $\dot{a} = \mu \Rightarrow a(s) = a_0 + \mu s$. Flowing along D_v implies $\dot{a} = \lambda a \Rightarrow a(s) = a_0 e^{\lambda s}$. Linear combinations of these generate the general flow given by Eq. (4).

6.4 Algebraic Structure and Classification of the AEG Lie Algebra

The contact geometry framework developed previously not only provides a dynamical system for the arithmetic flow but also reveals a deep underlying algebraic structure. The horizontal vector fields D_u, D_v, and the emergent vertical field ∂_a form a real Lie algebra under the Lie bracket. Classifying this algebra is essential for understanding the fundamental nature of the interplay between addition and multiplication.

6.4.1 Commutation Relations and Lie Algebra Verification

Let us define the basis for our Lie algebra, $\mathfrak{g}$, as:

- $e_1 = D_u = \partial_u + \mu \partial_a$
- $e_2 = D_v = \partial_v + \lambda a \partial_a$
- $e_3 = \partial_a$

Their commutation relations are directly computed as:

$$[e_1, e_2] = \mu \lambda e_3 \tag{82}$$
$$[e_1, e_3] = [\partial_u + \mu \partial_a, \partial_a] = 0 \tag{83}$$
$$[e_2, e_3] = [\partial_v + \lambda a \partial_a, \partial_a] = -\lambda e_3 \tag{84}$$

As vector fields on a smooth manifold, they are guaranteed to satisfy the Jacobi identity, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, thus confirming that $\mathfrak{g}$ is a bona fide Lie algebra.

6.4.2 Classification: Solvable but not Nilpotent

To classify $\mathfrak{g}$, we examine its derived and lower central series.

- **Solvability:** The derived series is $\mathcal{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_3\}$ and $\mathcal{D}^2\mathfrak{g} = [\mathcal{D}^1\mathfrak{g}, \mathcal{D}^1\mathfrak{g}] = \{0\}$. Since the series terminates at zero, the algebra is **solvable**.
- **Nilpotency:** The lower central series is $\mathcal{L}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_3\}$. However, the next term, $\mathcal{L}^2\mathfrak{g} = [\mathfrak{g}, \mathcal{L}^1\mathfrak{g}]$, contains the non-zero bracket $[e_2, e_3] = -\lambda e_3$, which means $\mathcal{L}^2\mathfrak{g} = \text{span}\{e_3\}$. The series stabilizes and never reaches zero, hence the algebra is **non-nilpotent**.

Thus, the AEG Lie algebra is a three-dimensional, solvable, non-nilpotent real Lie algebra.

6.4.3 Deeper Structure: A Central Extension of $\mathfrak{aff}(1)$

A more precise characterization comes from analyzing the algebra's internal structure.

- **Center:** The center $Z(\mathfrak{g})$ consists of elements that commute with all elements of $\mathfrak{g}$. A direct calculation shows that $Z(\mathfrak{g}) = \text{span}\{e_1 - \mu e_3\} = \text{span}\{\partial_u\}$.
- **Quotient Algebra:** The quotient algebra $\mathfrak{g}/Z(\mathfrak{g})$ is a two-dimensional Lie algebra spanned by the cosets $\bar{e}_2$ and $\bar{e}_3$. Their bracket is $[\bar{e}_2, \bar{e}_3] = -\lambda \bar{e}_3$. This is precisely the Lie algebra of the 1D affine group, $\mathfrak{aff}(1)$.

This reveals a crucial insight: **the AEG Lie algebra $\mathfrak{g}$ is a central extension of the affine Lie algebra $\mathfrak{aff}(1)$ by a one-dimensional center**.

Furthermore, this structure is "almost-abelian," as it contains both the 2D non-abelian subalgebra $\text{span}\{e_2, e_3\} \cong \mathfrak{aff}(1)$ and the 2D abelian ideal $\text{span}\{e_1, e_3\}$.

6.4.4 Isomorphism Class and Significance

For non-zero parameters μ, λ, we can normalize the basis via the linear transformation $e_2' = e_2/\lambda$ and $e_3' = \mu e_3$. The commutation relations become:

$$[e_1, e_2'] = e_3', \quad [e_1, e_3'] = 0, \quad [e_2', e_3'] = -e_3' \tag{85}$$

This demonstrates that for any $\mu\lambda \neq 0$, the resulting Lie algebra belongs to a **single isomorphism class** over $\mathbb{R}$. The parameters μ and λ only scale the basis, they do not change the intrinsic algebraic structure.

In conclusion, the contact form $\alpha = da - \mu\, du - \lambda a\, dv$, which in natural units ($\tilde{u} = \mu u, \tilde{v} = \lambda v$) takes the canonical shape $da - d\tilde{u} - a\, d\tilde{v}$, provides the precise geometric stage for this "geometrized affine algebra." The fact that this geometry realizes a central extension of $\mathfrak{aff}(1)$ confirms that the AEG framework captures the fundamental algebraic signature of arithmetic—a structure born from the interaction of a translation-like operation (D_u) with a value-dependent scaling operation (D_v).

7 Differential calculus on the contact structure

7.1 Axiomatic Definition and Basic Formulas

A1 (Definition of the Expression Differential) We define the action of δ on the coordinate functions as:

$$\delta a = \omega, \quad \delta u = du, \quad \delta v = dv, \tag{9}$$

and extend its action to all scalar fields F by linearity and the Leibniz rule.

A2 (Directional Derivatives) The operators D_u, D_v are defined as in Eq. (**??**), with directional synthesis given by Eq. (3).

A3–A4 (Chain Rules) For univariate and bivariate compositions, $\Phi(a)$ and $F(E_1, E_2)$, the following chain rules hold:

$$\delta\Phi(a) = \Phi'(a)\,\omega, \qquad \delta F = \partial_1 F\,\delta E_1 + \partial_2 F\,\delta E_2. \tag{10-11}$$

These are consistent with the equivalent definition $\delta F = dF - (\partial_a F)\alpha$.

T1–T5 (Fundamental Theorems) The axiomatic system yields five fundamental theorems:

- T1 **Flow Equation:** Setting $F = a$ in the definition of δ immediately gives Eq. (4).
- T2 **Non-Commutativity:** $[D_u, D_v] = \mu\lambda\,\partial_a$.
- T3 **Curvature / Covariant Second Differential:** $\delta^2 F = \mu\lambda(\partial_a F)\,du \wedge dv$.
- T4 **Compatibility with de Rham:** $(d\omega)^* = \mu\lambda\,du \wedge dv$ on the section $\alpha = 0$.
- T5 **Circulation-Area Theorem:** As stated in Eq. (8).

7.2 Quick Reference and Computation Rules

Let $\omega = \mu\,du + \lambda a\,dv$. The following rules apply:

$$\delta(a^n) = na^{n-1}\omega\ (n \in \mathbb{Z}), \quad \delta(\ln a) = \frac{\omega}{a}\ (a > 0), \quad \delta(e^a) = e^a\,\omega, \tag{13}$$

$$\delta(\sin a) = \cos a\,\omega, \quad \delta(\cos a) = -\sin a\,\omega, \quad \delta(\arcsin a) = \frac{\omega}{\sqrt{1+a^2}}. \tag{14}$$

If $E = E(u, v, a)$, its expression derivatives and differential are:

$$D_u E = E_u + \mu E_a, \quad D_v E = E_v + \lambda a E_a, \quad \delta E = (D_u E)\,du + (D_v E)\,dv. \tag{15}$$

These identities provide a unified interface for automatic differentiation, optimization, and geometric modeling.

7.3 Geometric Interpretation: An Ehresmann Connection

If we view the projection $\pi : (u, v, a) \mapsto (u, v)$ as a fiber bundle, the condition $\alpha = 0$ defines a horizontal distribution $\mathcal{H}$. The operators D_u, D_v are the horizontal lifts of the base vector fields ∂_u, ∂_v. The curvature 2-form of this connection corresponds to $\delta^2 a$. This perspective rigorously explains why applying δ is equivalent to restricting d to $\ker\alpha$ and reveals the geometric meaning of the non-commutativity factor $\mu\lambda$.

7.4 Numerics and Scaling: Natural Units and Preconditioning

In natural units where $\tilde{u} = \mu u$ and $\tilde{v} = \lambda v$, all formulas retain their form. For numerical optimization, working with the rectified variable y (or, equivalently, preconditioning the gradients for a with the pointwise factor $(\mu^2 + \lambda^2 a^2)^{-1}$) can significantly smooth the loss landscape and improve convergence stability.

7.5 Extending polynomials: the affine–Appell basis

Natural units: $\tilde{u} = \mu u,\ \tilde{v} = \lambda v$. Define scaled powers

$$B_n(a, v) := e^{-n\tilde{v}} a^n \quad (n \in \mathbb{N}).$$

Let

$$\mathcal{B} := \Big\{ \sum_{n=0}^{N} P_n(u, v, e^{\tilde{v}})\, B_n(a, v)\ \text{(finite)} \Big\}.$$

Closure Theorem. $\mathcal{B}$ is closed under the mixed calculus generated by

$$D_u = \partial_u + \mu\, \partial_a, \qquad D_v = \partial_v + \lambda a\, \partial_a.$$

Rules:

$$D_u(PB_n) = (\partial_u P)B_n + \mu n\, e^{-\tilde{v}} P\, B_{n-1}, \quad D_v(PB_n) = (\partial_v P)B_n.$$

7.6 Antiderivatives and a finite upward sweep

If P is independent of u,

$$D_u^{-1}\big(P(v, e^{\tilde{v}})\, B_n\big) = \frac{e^{\tilde{v}} P(v, e^{\tilde{v}})}{\mu(n+1)}\, B_{n+1}.$$

In general, define $Q_n^{(0)} = \dfrac{e^{\tilde{v}}}{\mu(n+1)} P_n$ and set $G^{(0)} = \sum_n Q_n^{(0)} B_{n+1}$. Then $F - D_u G^{(0)} = -\sum_n (\partial_u Q_n^{(0)}) B_{n+1}$. Repeat for finitely many steps.

Example. For $F = a^3 e^{\lambda v} = e^{4\tilde{v}} B_3$,

$$D_u^{-1} F = \frac{e^{\tilde{v}}}{4\mu} B_4 = \frac{e^{\lambda v}}{4\mu} a^4, \quad D_u\left(\frac{e^{\lambda v}}{4\mu} a^4\right) = a^3 e^{\lambda v}.$$

8 Arithmetic holomorphic function

8.1 Definition and Intuition

A pair of real-valued functions $(f, g) = (f(u, v, a), g(u, v, a))$ is said to be **arithmetic holomorphic** if it satisfies the AEG-Cauchy-Riemann equations:

$$D_u f = D_v g, \qquad D_v f = -D_u g. \tag{16}$$

This definition lifts the classic Cauchy-Riemann conditions from the complex plane $\mathbb{C} \cong \mathbb{R}^2(u, v)$ to the horizontal geometry of the arithmetic expression contact structure.

8.2 Basic Properties and Proof Sketches

1. **Conformality and Modulus Equality**: The vectors $(D_u f, D_u g)$ and $(D_v f, D_v g)$ are orthogonal and have equal magnitude:
$$(D_u f)^2 + (D_u g)^2 = (D_v f)^2 + (D_v g)^2.$$
The proof follows from direct substitution using Eq. (16).
2. **Closure Properties**: The set of arithmetic holomorphic pairs is closed under addition and an "AEG complex multiplication" defined as $(f, g) \odot (\tilde{f}, \tilde{g}) = (f\tilde{f} - g\tilde{g},\ f\tilde{g} + \tilde{f}g)$. Furthermore, composition with a classic holomorphic function of (u, v) preserves arithmetic holomorphicity.
3. **Classical Limit**: If f and g are independent of a, then $D_u = \partial_u$ and $D_v = \partial_v$, and the AEG-CR equations reduce to the standard Cauchy-Riemann equations.
4. **Rigidity**: If $f = f(a)$ and $g = g(a)$, the only solutions are constants. (*Hint*: Eq. (16) forces the vectors $(f'(a), g'(a))$ and $(\mu, \lambda a)$ to be simultaneously orthogonal and equal in magnitude, which is impossible unless they are zero.)
5. **Scaling of the Curvature Density**: In arithmetic holomorphic coordinates (f, g), the curvature 2-form transforms as:
$$d\omega = \frac{\mu\lambda}{|F'|^2}\, df \wedge dg, \qquad \text{where } |F'|^2 := (D_u f)^2 + (D_u g)^2. \tag{17}$$
The area density scales by the Jacobian factor, but the total integral remains invariant. This provides an area formula for "expression conformal coordinates".

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A Solution of the flow equation

We can also get a direct formal solution of the flow equation ((30)) step by step:

$$\frac{da}{\mu\cos\theta + a\lambda\sin\theta} = ds$$

$$\frac{1}{\lambda\sin\theta}\frac{d(\mu\cos\theta + a\lambda\sin\theta)}{\mu\cos\theta + a\lambda\sin\theta} = ds$$

$$\frac{1}{\lambda\sin\theta} ln(\mu\cos\theta + a\lambda\sin\theta) = s + C$$

$$\mu\cos\theta + a\lambda\sin\theta = e^{\lambda s\sin\theta}e^{C\lambda\sin\theta}$$

Considering the initial condition

$$\mu\cos\theta + a_0\lambda\sin\theta = e^{C\lambda\sin\theta}$$

We have

$$\mu\cos\theta + a\lambda\sin\theta = e^{\lambda s\sin\theta}(\mu\cos\theta + a_0\lambda\sin\theta)$$

$$a = \frac{\mu\cos\theta + a_0\lambda\sin\theta}{\lambda\sin\theta}e^{\lambda s\sin\theta} - \frac{\mu}{\lambda}\cot\theta$$

$$a = (a_0 + \frac{\mu}{\lambda}\cot\theta)e^{\lambda s\sin\theta} - \frac{\mu}{\lambda}\cot\theta$$

$$a = a_0 e^{\lambda s\sin\theta} + \frac{\mu}{\lambda}(e^{\lambda s\sin\theta} - 1)\cot\theta \tag{86}$$

$$a = a_0 e^{\lambda s\sin\theta} + \frac{\mu}{\lambda}(e^{\lambda s\sin\theta} - 1)\cot\theta \tag{87}$$

B Geometry calculation

Giving

$$ds^2 = \frac{1}{y^2}(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2})$$

we calculate the geometric quantities. We follow the notion in the text book[9], and the first fundamental form is given by

$$ds^2 = A^2dx^2 + B^2dy^2$$

where

$$A = \frac{1}{\mu y}, \quad B = \frac{1}{\lambda y}$$

B.1 Line element

The line element is already given by above equation.

B.2 Area element

The area element is given by

$$dS = ABdxdy$$

hence we have

$$dS = \frac{1}{\mu\lambda y^2}dxdy$$

B.3 Gauss curvature

Gauss curvature K is given by

$$K = -\frac{1}{AB}\left(\partial_y\left(\frac{\partial_y A}{B}\right) + \partial_x\left(\frac{\partial_x B}{A}\right)\right)$$

so we have

$$K = -\mu\lambda y^2\left(\partial_y\left(\lambda y\partial_y\left(\frac{1}{\mu y}\right)\right) + \partial_x\left(\mu y\partial_x\left(\frac{1}{\lambda y}\right)\right)\right)$$

$$K = -\lambda^2 y^2\left(\partial_y\left(y\partial_y\left(\frac{1}{y}\right)\right)\right)$$

$$K = -\lambda^2 y^2\frac{1}{y^2}$$

$$K = -\lambda^2$$

B.4 Laplacian

Given a metric tensor

$$g = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix},$$

where A and B are functions of the coordinates (typically x and y), the Laplacian of a function $f(x, y)$ can be derived from the general expression of the Laplace-Beltrami operator for a Riemannian manifold. The formula for the Laplacian Δf in such a setting, using the metric components g_{ij}, is given by:

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right),$$

where $|g|$ is the determinant of the metric tensor g_{ij}, g^{ij} are the components of the inverse metric tensor, and ∂_i denotes partial differentiation with respect to the ith coordinate.

Given the metric tensor, the determinant $|g|$ is $A^2 B^2$. The inverse metric tensor g^{ij} is simply:

$$g^{ij} = \begin{bmatrix} \frac{1}{A^2} & 0 \\ 0 & \frac{1}{B^2} \end{bmatrix}.$$

Plugging these into the formula for the Laplacian, we get:

$$\Delta f = \frac{1}{AB} \left[\partial_x \left(B A^{-1} \partial_x f \right) + \partial_y \left(A B^{-1} \partial_y f \right) \right],$$

In our setting, $A = \frac{1}{\mu y}$ and $B = \frac{1}{\lambda y}$:

$$\Delta f = y^2 \left(\mu^2 \frac{\partial^2 f}{\partial x^2} + \lambda^2 \frac{\partial^2 f}{\partial y^2} \right)$$

And for the function $f = -\frac{x}{y}$, we have

$$\Delta f = -\frac{2\lambda^2 x}{y} = 2\lambda^2 f$$

So, we reach the conclusion that the function $f = -\frac{x}{y}$ is a eigenfunction of the Laplacian with eigenvalue $2\lambda^2$.