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IS WAVE PROPAGATION COMPUTABLE OR CAN WAVE COMPUTERS BEAT THE TURING MACHINE?

KLAUS WEIHRAUCH AND NING ZHONG

1. *Introduction*

By the Church–Turing Thesis a numerical function is computable by a physical device if and only if it is computable by a Turing machine. The 'if'-part is plausible since every (sufficiently small) Turing machine can be simulated by a computer program which operates correctly as long as sufficient time and storage are available and no errors occur. On the other hand, every program for a modern digital computer can be simulated by a Turing machine. What is more, most physicists believe that for processes which can be described by well-established theories (finitely many point masses interacting gravitationally, electromagnetic waves, quantum systems etc.) the future behavior can be computed with arbitrary precision, at least in principle, from sufficiently precisely given initial conditions, where the computations can be performed on digital computers, and hence on Turing machines. Nevertheless, there might exist physical processes which are not Turing computable in this way. For discussions and further references see [**4**, **13**].

Is Turing's computability concept sufficiently powerful to model all kinds of physical processes? If the answer is 'yes', then perhaps the Church–Turing Thesis could be derivable from the laws of physics, or perhaps it should even be considered as a fundamental law of physics itself. Otherwise, the Church–Turing Thesis should possibly be corrected. In this paper we shall concentrate on a special type of physical process, namely one on scalar waves in Euclidean space. We start from remarkable results by Pour-El and Richards [**11**] and Pour-El and Zhong [**9**], who constructed computable initial conditions f for the three-dimensional wave equation

$$\begin{cases} u_{tt} = \Delta u, \\ u(0,x) = f(x), \quad u_t(0,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

such that the unique solutions are not computable. In particular, there is a three-dimensional wave u such that $x \mapsto u(0,x)$ is computable and $x \mapsto u(1,x)$ is not computable. These examples have considerably disconcerted logicians and computer scientists, as well as physicists, most of whom accept the Church–Turing Thesis or at least believe that wave propagation can be predicted (arbitrarily precisely) by means of digital computers. Various interpretations of the examples have been proposed and the discussions are still going on. Do these examples mean that the behavior of some physical devices cannot be predicted by Turing machines? If so, there might be a numerical function which is computable on a physical machine but not on a Turing machine and the Church–Turing Thesis might be false.

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While for discrete sets there is a generally accepted computability theory, for real functions and other functions from analysis several non-equivalent computability concepts have been proposed [**12**; **16**, Chapter 9]. In this paper we use 'Type-2 Theory of Effectivity', TTE for short [**7**, **16**], a computability concept for analysis, which is based on Turing's definition of computable real numbers [**15**] and Grzegorczyk's definition of computable real functions [**6**]. This model of computation is realistic, since the TTE-computable functions can be realized by Turing machines. TTE is consistent with the approach used by Pour-El and Richards [**10**] but more expressive and particularly suitable for our purposes.

In § 2 we summarize some definitions from TTE (for details see [**16**]) and introduce the canonical computability concepts on the spaces we shall use later, in particular on $\mathbb{R}^n$, $C^k(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

In § 3 we study wave propagation on spaces of continuously differentiable real functions $C^k(\mathbb{R}^3)$. While so far only the behavior of solution operators on computable initial functions has been investigated [**11**, **10**, **9**], the framework of TTE allows the study of computability of the operators themselves. We prove that for the canonical computability concepts on these function spaces the solution operator is computable (Theorem 3.2) and hence (as is well known) continuous, but one order of differentiability may be lost (Examples 3.5, 3.6 and 3.7). However, the solution operator has computational irregularities for instances in which it is not continuous. This includes the Pour-El–Richards–Zhong counter-example (Theorems 3.10 and 3.11), which can be proved by Pour-El and Richards' more general 'First Main Theorem' [**10**].

In § 4 we consider waves from Sobolev spaces $H^s(\mathbb{R}^3)$ (with $s \in \mathbb{R}$), where $H^s(\mathbb{R}^3)$ is the space of all generalized complex-valued functions u on $\mathbb{R}^3$ such that the weighted Fourier transform $(1 + |\xi|^2)^{s/2}\widehat{u}(\xi)$ is in $L^2(\mathbb{R}^3)$. We introduce a canonical representation of this space and prove that in this very natural setting also the solution operator for the Cauchy problem of the wave equation is computable without any 'loss of regularity' (Theorem 4.5).

Finally, in § 5 we discuss whether the Pour-El–Richards paradox can be used to design a 'wave computer' which beats the Turing machine. It turns out that at least there is no obvious way to construct such a wave computer. Note that [**17**, **18**] are preliminary versions of this paper.

2. *Computability on second countable spaces*

This section summarizes some basic concepts from Type-2 Theory of Effectivity (TTE) and introduces the computability concepts on the spaces we need for studying wave propagation. Details about TTE can be found in the textbook [**16**]. In TTE, Turing machines which accept and produce finite strings from Σ^* (where Σ is a finite alphabet) are generalized to *Type*-2 *machines* which can read and write from left to right infinite sequences from Σ^ω. For a machine M with a single infinite input and infinite output, for $p, q \in \Sigma^\omega$, $f_M(p) = q$ if and only if the machine M applied to p computes forever and prints q on its output tape. The generalization to machines with finitely many input tapes and an output tape, some of which may be finite, is straightforward. Every computable function is continuous with respect to the discrete topology on Σ^* and Cantor topology on Σ^ω.

Finite or infinite sequences of symbols can be used as 'codes' or 'names' of objects from 'abstract' sets M. This way Turing machines can be used to compute functions on sets like the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$. A notation (representation) of a set M is a partial or total surjective function $\nu :\subseteq \Sigma^* \to M$ ($\delta :\subseteq \Sigma^\omega \to M$). Let (M, δ) and (M', δ') be represented sets. An element $x \in M$ is δ-computable if and only if it has a computable δ-name $p \in \Sigma^\omega$ (that is, $\delta(p) = x$). A function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$ is a (δ, δ')-realization of a function $f :\subseteq M \to M'$ if and only if $f \circ \delta(p) = \delta' \circ g(p)$ for all $p \in \mathrm{dom}(f \circ \delta)$. The function f is called (δ, δ')-computable (-continuous) if and only if it has a computable (continuous) (δ, δ')-realization. An extension to functions with several arguments is straightforward. The representation δ is reducible (t-reducible) to δ', where $\delta \leqslant \delta'$ ($\delta \leqslant_t \delta'$), if and only if the identical embedding is (δ, δ')-computable (-continuous). Equivalence (t-equivalence) is defined by $\delta \equiv \delta' :\iff \delta \leqslant \delta'$ and $\delta' \leqslant \delta$ ($\delta \equiv_t \delta' :\iff \delta \leqslant_t \delta'$ and $\delta' \leqslant_t \delta$). Two representations of a set M induce the same computability (continuity) on M if and only if they are equivalent (t-equivalent). An extension of the above concepts to notations is straightforward. For details see [**16**, § 3.1].

There is an 'effective' representation $\eta : \Sigma^\omega \to F^{\omega\omega}$ (where $F^{\omega\omega}$ is the set of continuous functions $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ with G_δ-domain; it contains an extension of every continuous partial function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$) which satisfies the utm-theorem (universal Turing machine theorem) and the smn-theorem [**16**, § 2.3]. A natural representation $[\delta \to \delta']$ of the (δ, δ')-continuous functions $f : M \to M'$ is defined by $[\delta \to \delta'](p) = f$ if and only if $\eta(p)$ is a (δ, δ')-realization of f. It has the remarkable property that for any representation γ of a set of (δ, δ')-continuous functions $f : M \to M'$, the evaluation function $(f, x) \mapsto f(x)$ is $(\gamma, \delta, \delta')$-computable if and only if $\gamma \leqslant [\delta \to \delta']$ [**16**, § 3.3]. We will repeatedly apply the following lemma on type conversion (see [**16**, § 3.3]).

LEMMA 2.1. *Let* $\delta_i :\subseteq \Sigma^\omega \to M_i$ $(i = 0, \dots, k)$ *be representations and let* $f : M_1 \times \dots \times M_k \to M_0$. *For*

$$T(f)(x_1, \dots, x_{k-1})(y) := f(x_1, \dots, y),$$

f *is* $(\delta_1, \dots, \delta_k, \delta_0)$*-computable if and only if* $T(f)$ *is* $(\delta_1, \dots, \delta_{k-1}, [\delta_k \to \delta_0])$*-computable.*

Among the numerous representations of a set, *admissible representations* are of particular interest [**16**, § 3.2]. Roughly speaking, for an admissible representation, a name of an element $x \in M$ is an infinite sequence of properties $A \subseteq M$ of x from a countable supply σ, where x has the property A if and only if $x \in A$. For defining a computability concept on M, the user specifies a countable supply $\sigma \subseteq 2^M$ of 'atomic properties' and a notation $\nu :\subseteq \Sigma^* \to \sigma$ of this set σ. In the following, let Σ and Γ be sufficiently large finite alphabets with $\# \notin \Gamma$ and $\Sigma = \Gamma \cup \{\#\}$ and let $\Gamma^+ := \Gamma^* \setminus \{\lambda\}$.

DEFINITION 2.2. An *effective topological space* is a triple $\mathbf{S} = (M, \sigma, \nu)$, where M is a set, $\sigma \subseteq 2^M$ is a countable set such that $\{A \in \sigma \mid x \in A\} = \{A \in \sigma \mid y \in A\}$ implies $x = y$ for all $x, y \in M$, and $\nu :\subseteq \Sigma^* \to \sigma$ is a notation of the set σ (with $\mathrm{dom}(\nu) \subseteq \Gamma^+$). Let $\tau_{\mathbf{S}}$ be the topology generated by σ as a subbase. We say that $\mathbf{S}$ is computable if and only if $\{(u, v) \mid \nu(u) = \nu(v)\}$ is recursively enumerable.

We shall call the elements of σ 'atomic properties'. Each element of M can be identified by its atomic properties, that is, $\tau_{\mathbf{S}}$ is a T_0-space with countable base, and ν has a recursively enumerable domain, if $\mathbf{S}$ is computable. The induced representation is as follows.

DEFINITION 2.3. Let $\mathbf{S} = (M, \sigma, \nu)$ be an effective topological space. Define the standard representation $\delta_{\mathbf{S}} :\subseteq \Sigma^\omega \to M$ by $\delta_{\mathbf{S}}(p) = x$ if and only if

(1) $w \in \mathrm{dom}(\nu)$ if $w \neq \lambda$ and $\#w\#$ is a subword of p, and

(2) $\{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \#w\# \text{ is a subword of } p\}$.

Therefore, a $\delta_{\mathbf{S}}$-name of x is a list of all atomic properties of x in arbitrary order. For effective topological spaces $\mathbf{S} = (M, \sigma, \nu)$ and $\mathbf{S}' = (M', \sigma', \nu')$, a function $f :\subseteq M \to M'$ is $(\delta_{\mathbf{S}}, \delta_{\mathbf{S}'})$-computable if and only if there is some Type-2 machine which for any $x \in \mathrm{dom}(f)$ maps any list of all atomic properties of x to a list of all atomic properties of $f(x)$.

For functions between effective topological spaces $\mathbf{S} = (M, \sigma, \nu)$ and $\mathbf{S}' = (M', \sigma', \nu')$ continuity has a very elementary interpretation in terms of atomic properties. By definition, a function $f :\subseteq M \to M'$ is continuous at $x \in \mathrm{dom}(f)$ if and only if for all $A' \in \sigma'$ with $f(x) \in A'$ there are $A_1, A_2, \ldots, A_k \in \sigma$ such that $x \in A_1 \cap A_2 \cap \ldots \cap A_k$ and $f[A_1 \cap A_2 \cap \ldots \cap A_k] \subseteq A'$. Informally, this means that every atomic property A' of $f(x)$ already follows from *finitely many* atomic properties $A_1, A_2, \ldots, A_k$ of x. Since for a Type-2 machine every finite portion of an output depends only on a finite portion of the input, the following fundamental continuity theorem can be proved straightforwardly [**16**, § 3.2].

THEOREM 2.4. *A function is $(\delta_{\mathbf{S}}, \delta_{\mathbf{S}'})$-continuous if and only if it is $(\tau_{\mathbf{S}}, \tau_{\mathbf{S}'})$-continuous. In particular, every $(\delta_{\mathbf{S}}, \delta_{\mathbf{S}'})$-computable function is $(\tau_{\mathbf{S}}, \tau_{\mathbf{S}'})$-continuous.*

The above computability definition is applicable to most of the spaces considered in analysis and, in particular, in physics. For fixing a computability concept on M choose a countable set σ of 'atomic properties' (which selects a concept of 'approximation' on M) and a notation ν of σ (which selects a concept of computation). Although, for a set M, the set σ and a notation ν of σ can be chosen almost arbitrarily, in most applications the set σ of atomic properties (and therefore the topology $\tau_{\mathbf{S}}$), as well as the notation ν of it, are determined by the available properties such as results of better and better measurements, of a preceding computation, or of throwing the dice. In § 5 we will use effective topological spaces as mathematical models for physical measurement devices.

For separable metric spaces, Cauchy representations are convenient [**16**, § 8.1].

DEFINITION 2.5. An *effective metric space* is a tuple $\mathbf{M} = (M, d, A, \alpha)$ such that α is a notation of the set A dense in the metric space (M, d). The Cauchy representation δ_C for $\mathbf{M}$ is defined by $\delta_C(p) = x$ if and only if $p = u_0\#u_1\#\ldots$ such that $d(\alpha(u_i), x) \leqslant 2^{-i}$ for all $i \in \mathbb{N}$. We say that $\mathbf{M}$ is *computable* if and only if $\nu_{\mathbb{Q}}(t) < d(\alpha(u), \alpha(v)) < \nu_{\mathbb{Q}}(w)$ is recursively enumerable in $t, u, v, w \in \Sigma^*$ (where $\nu_{\mathbb{Q}}$ is a standard notation of $\mathbb{Q}$).

In the following we introduce some concrete effective topological spaces which we will use later [**16**, §§4.1, 6.1, 6.4, 8.1].

DEFINITION 2.6. Define representations via effective topological spaces and effective metric spaces, respectively, as follows.

(1) Euclidean space $\mathbb{R}^n$. Let $\sigma := \{B(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}, r > 0\}$ be the set of open balls with rational center and radius, and let ν be some standard notation of σ. Let $\rho^n := \delta_{\mathbf{S}} :\subseteq \Sigma^\omega \to \mathbb{R}^n$ be the associated representation ($\rho := \rho^1$).

(2) $C(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ is continuous}\}$. Let

$$\sigma_n := \{R_{arcd} \mid a \in \mathbb{Q}^n, r, c, d \in \mathbb{Q}, r > 0, c < d\}$$

be the set of atomic properties where $R_{arcd} := \{ f \in C(\mathbb{R}^n) \mid f[\overline{B}(a, r)] \subseteq (c; d)\}$, and let ν_n be some standard notation of σ_n. Let $\delta_n := \delta_{\mathbf{S}}$ be the associated representation of $C(\mathbb{R}^n)$.

(3) $C^1(\mathbb{R}^n) := \{ f \in C(\mathbb{R}^n) \mid \partial_{x_i} f \in C(\mathbb{R}^n)\ (1 \leqslant i \leqslant n)\}$. Let

$$\sigma_n^1 := \{(i, R_{arcd}^i) \mid 0 \leqslant i \leqslant n, a \in \mathbb{Q}^n, r, c, d \in \mathbb{Q}, r > 0, c < d\}$$

with $R_{arcd}^0 := R_{arcd} \cap C^1(\mathbb{R}^n)$ and $R_{arcd}^i := \{ f \in C^1(\mathbb{R}^n) \mid \partial_{x_i} f \in R_{arcd}\}$ be the set of atomic properties, and let ν_n^1 be a standard notation of σ_n^1. Let $\tau_n^1 := \tau_{\mathbf{S}}$ and $\delta_n^1 := \delta_{\mathbf{S}}$.

(4) $C^k(\mathbb{R}^n)$ of k-times continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$. Generalize the case of $C^1(\mathbb{R}^n)$ straightforwardly.

(5) Computable metric spaces. For an effective metric space $\mathbf{M} = (M, d, A, \alpha)$ define an effective topological space $\mathbf{S} = (M, \sigma, \nu)$ as follows: σ is the set of all open balls $B(a, r)$ with $a \in A$ and $r \in \mathbb{Q}$ and ν is some standard notation of σ. If $\mathbf{M}$ and $\mathbf{S}$ are computable, then $\delta_{\mathbf{S}} \equiv \delta_C$ (the Cauchy representation is equivalent to $\delta_{\mathbf{S}}$).

(6) $L^2(\mathbb{R}^n)$. For $x_i, y_i \in \mathbb{R}$ define $(x_1, \dots, x_n) < (y_1, \dots, y_n)$ if and only if $x_i < y_i$ for $i = 1, \dots, n$. For $a, b \in \mathbb{R}^n$ with $a < b$ define $\mathbf{1}_{ab}(x) := 1$ if $a < x < b$ and $\mathbf{1}_{ab}(x) := 0$ otherwise. Let

$$A := \left\{ \sum_{i=1}^{k} c_i \cdot \mathbf{1}_{a_i b_i} \;\middle|\; k \in \mathbb{N},\ c_i \text{ rational complex},\ a_i, b_i \in \mathbb{Q}^n,\ a_i < b_i \right\}$$

be the countable set of rational complex-valued finite step functions and let μ be some standard notation of A. Then $(L^2(\mathbb{R}^n), d, A, \mu)$, where

$$d(f, g) := \left(\int_{\mathbb{R}^n} |f(x) - g(x)|^2 \, dx \right)^{1/2},$$

is a computable metric space. Let δ_L be the Cauchy representation.

Obviously in (1), $\tau_{\mathbf{S}}$ is the standard topology $\tau_{\mathbb{R}^n}$ on $\mathbb{R}^n$. A function $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ is (ρ^n, ρ)-computable if and only if it is computable in the sense of Grzegorczyk [**6**, **10**], G-computable for short.

In (2) the associated topology $\tau_n := \tau_{\mathbf{S}}$ is known as the compact open or strong topology on $C(\mathbb{R}^n)$. One can show that $\delta_n \equiv [\rho^n \mapsto \rho]$ [**16**, §6.1]. Consequently, $f, x \mapsto f(x)$ is (δ, ρ^n, ρ)-computable if and only if $\delta \leqslant \delta_n$ for all representations δ of $C(\mathbb{R}^n)$, that is, δ_n is $\leqslant$-complete in the set of all representations δ of $C(\mathbb{R}^n)$ for which the evaluation function is computable. Therefore, the representation δ_n is tailor-made for computing function values.

In (3), τ_n^1 is the smallest topology τ on $C^1(\mathbb{R}^n)$ such that the functions $f \mapsto f$ and $f \mapsto \partial_{x_i} f$ (where $1 \leqslant i \leqslant n$) are (τ, τ_n)-continuous. Furthermore, the representation δ_n^1 is $\leqslant$-complete in the set of all representations δ of $C^1(\mathbb{R}^n)$ for which the functions $f \mapsto f$ and $f \mapsto \partial_{x_i} f$ $(1 \leqslant i \leqslant n)$ are (δ, δ_n)-computable. Roughly speaking, a δ_n^1-name of f is a combination of δ_n-names of f, $\partial_{x_1} f$, $\partial_{x_2} f$, ... and $\partial_{x_n} f$. The representation δ_n^1 is tailor-made for computing function values as well as first partial derivatives.

In (4) a δ_n^k-name of a function $f \in C^k(\mathbb{R}^n)$ is a combination of δ_n-names of all partial derivatives $\partial^\alpha f$ with $|\alpha| \leqslant k$.

By definition, for an effective topological space $\mathbf{S} = (M, \sigma, \nu)$, a standard name of $x \in M$ is a list of names of *all* atomic properties $A \in \sigma$ with $x \in A$. In all practical cases there are equivalent representations (defining the same computability concept on M) which are much simpler and therefore more useful for concrete computations.

3. *Continuously differentiable initial conditions*

Pour-El and Richards [**10**] and Pour-El and Zhong [**9**] show that for the Cauchy problem of the three-dimensional wave equation (1.1) there exists a C^1 initial function f which is G-computable (computable in the classical Grzegorczyk sense) in $C(\mathbb{R}^3)$ such that the corresponding unique solution u of (1.1) is not G-computable.

These results bother physicists, in particular, who are convinced that wave propagation is computable and in fact write computer programs which predict the future behavior of waves from initial conditions. What causes the non-computability? In the following we discuss this question in detail.

Consider the Cauchy problem of the three-dimensional wave equation

$$\begin{cases} u_{tt} = \Delta u, \\ u(0, x) = f(x), \quad u_t(0, x) = g(x), \quad t \in \mathbb{R},\, x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

If the Cauchy data satisfy $f \in C^k(\mathbb{R}^3)$ and $g \in C^{k-1}(\mathbb{R}^3)$ (with $k \geqslant 1$), then by classical analysis the initial value problem (3.1) has a unique solution given by

$$\begin{aligned} u(t, x) &= \frac{1}{4\pi t} \int_{|y-x|=t} g(y)\, d\sigma(y) + \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{|y-x|=t} f(y)\, d\sigma(y) \right) \\ &= \int_{S^2} [tg(x + tn) + f(x + tn) + t \nabla f(x + tn) \cdot n] d\sigma(n), \end{aligned} \tag{3.2}$$

where $u \in C^{k-1}(\mathbb{R}^4)$. First we prove that the solution operator mapping f and g to u is computable. We prepare for the proof with a lemma. For ρ-computable $a, b \in \mathbb{R}$, the function $f \mapsto \int_a^b f(x)\, dx$ is (δ_1, ρ)-computable on $C(\mathbb{R})$ [**16**, § 6.4]. We use this result to show the following.

LEMMA 3.1. *The function $f \mapsto \int_{S^2} f(n)\, d\sigma(n)$ is (δ_3, ρ)-computable.*

Proof. We have

$$\int_{S^2} (n)\, d\sigma(n) = \int_0^{2\pi} \int_0^{\pi} f(\Phi(\theta, \varphi)) \sin\theta \, d\theta \, d\varphi,$$

where $\Phi(\theta, \varphi) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$. Since $(f, n) \mapsto f(n)$ is (δ_3, ρ^3, ρ)-computable and Φ, sin and multiplication on $\mathbb{R}$ are computable with respect to ρ, $(f, \varphi, \theta) \mapsto f(\Phi(\theta, \varphi)) \sin\theta$ is $(\delta_3, \rho, \rho, \rho)$-computable. By Lemma 2.1 the function H with $H(f, \varphi)(\theta) := f(\Phi(\theta, \varphi)) \sin\theta$ is $(\delta_3, \rho, [\rho \to \rho])$-computable, and hence $(\delta_3, \rho, \delta_1)$-computable. Therefore, $F : (f, \varphi) \mapsto \int_0^\pi f(\Phi(\theta, \varphi)) \sin\theta \, d\theta$ is (δ_3, ρ, ρ)-computable. By Lemma 2.1, the function G with $G(f)(\varphi) := F(f, \varphi)$ is $(\delta_3, [\rho \to \rho])$-computable, and hence (δ_3, δ_1)-computable. Then

$$f \mapsto \int_0^{2\pi} G(f)(\varphi) = \int_0^{2\pi} \int_0^{\pi} f(\Phi(\theta, \varphi)) \sin\theta \, d\theta \, d\varphi$$

is (δ_3, ρ)-computable. □

THEOREM 3.2. *For $k \geqslant 1$ the solution operator $S : (f, g) \mapsto u$ of the Cauchy problem* (3.1) *mapping $f \in C^k(\mathbb{R}^3)$ and $g \in C^{k-1}(\mathbb{R}^3)$ to the solution $u \in C^{k-1}(\mathbb{R}^4)$ is $(\delta_3^k, \delta_3^{k-1}, \delta_4^{k-1})$-computable.*

Proof. First we consider the case $k = 1$. Since $h \mapsto \partial_{x_i} h$ is (δ_3^1, δ_3)-computable, $(h, x) \mapsto h(x)$ is (δ_3, ρ^3, ρ)-computable, and addition and multiplication are computable with respect to ρ, the integrand of (3.2),

$$(f, g, (t, x), n) \mapsto [tg(x + tn) + f(x + tn) + t\nabla f(x + tn) \cdot n],$$

is $(\delta_3^1, \delta_3, \rho^4, \rho^3, \rho)$-computable. Therefore, by Lemma 2.1 the function I, defined by

$$I(f, g, (t, x))(n) := tg(x + tn) + f(x + tn) + t\nabla f(x + tn) \cdot n,$$

is $(\delta_3^1, \delta_3, \rho^4, [\rho^3 \to \rho])$-computable, and hence $(\delta_3^1, \delta_3, \rho^4, \delta_3)$-computable. Then by Lemma 3.1,

$$(f, g, (t, x)) \mapsto \int_{S^2} [tg(x + tn) + f(x + tn) + t\nabla f(x + tn) \cdot n] d\sigma(n)$$

is $(\delta_3^1, \delta_3, \rho^4, \rho)$-computable. Again with Lemma 2.1 we see that $(f, g) \mapsto u$, where u is the solution of (3.1), is $(\delta_3^1, \delta_3, \delta_4)$-computable.

Suppose that $k \geqslant 2$. It suffices to show that $(f, g) \mapsto \partial^\alpha u$, mapping the initial condition to the partial derivative of order $|\alpha|$ of the solution $u : \mathbb{R}^4 \to \mathbb{R}$, is $(\delta_3^k, \delta_3^{k-1}, \delta_4)$-computable for all $|\alpha| < k$. We have

$$\partial^\alpha u(t, x) = \partial^\alpha \int_{S^2} h_{fg}(t, x, n) \, d\sigma(n) = \int_{S^2} \partial^\alpha h_{fg}(t, x, n) \, d\sigma(n),$$

where $h_{fg}(t, x, n) := tg(x + tn) + f(x + tn) + t\nabla f(x + tn) \cdot n$ and partial derivatives are with respect to the variables t, x_1, x_2 and x_3. By shifting partial derivatives to the interior, the expression $\partial^\alpha h_{fg}(t, x, n)$ can be transformed to a term T containing partial derivatives of f of order at most k and of g of order less than k. Since $h \mapsto \partial^\beta h$ is (δ_3^m, δ_3)-computable for $h \in C^m(\mathbb{R}^3)$ and $|\beta| \leqslant m$, $(f, g, (t, x), n) \mapsto T$ is $(\delta_3^k, \delta_3^{k-1}, \rho^4, \rho^3, \rho)$-computable. From here we can continue as in the case $k = 1$. □

COROLLARY 3.3. *Consider $k \geqslant 1$.*

(1) *The solution operator*

$$S' : (f, g, t) \mapsto u(t, \cdot)$$

of the Cauchy problem (3.1) mapping $f \in C^k(\mathbb{R}^3)$, $g \in C^{k-1}(\mathbb{R}^3)$ *and $t \in \mathbb{R}$ to the solution $u(t, \cdot) \in C^{k-1}(\mathbb{R}^3)$ is $(\delta_3^k, \delta_3^{k-1}, \rho, \delta_3^{k-1})$-computable.*

(2) *For each* $t \in \mathbb{R}$ (*computable* $t \in \mathbb{R}$) *the solution operator*

$$S''(t)(f, g) \mapsto u(t, \cdot)$$

of the Cauchy problem (3.1) *mapping* $f \in C^k(\mathbb{R}^3)$ *and* $g \in C^{k-1}(\mathbb{R}^3)$ *to the solution* $u(t, \cdot) \in C^{k-1}(\mathbb{R}^3)$ *at time* t *is* $(\delta_3^k, \delta_3^{k-1}, \delta_3^{k-1})$*-continuous* (*-computable*).

Remember that 'δ-computable' implies 'δ-continuous' and by Theorem 2.4 for standard representations of effective topological spaces 'δ-continuous' is equivalent to 'continuous'. Remember also that computable operators map computable elements to computable ones. A loss of regularity cannot be observed for initial conditions $f, g \in C^\infty(\mathbb{R}^3)$. For $p_0, p_1, \ldots \in \Sigma^\omega$ define the infinite tupling by $\langle p_0, p_1, p_2, \ldots \rangle \langle i, j \rangle := p_i(j)$, where $\langle i, j \rangle$ is Cantor's pairing function.

THEOREM 3.4. *For* $m \geqslant 1$ *define a representation* $\delta_m^\infty :\subseteq \Sigma^\omega \to C^\infty(\mathbb{R}^m)$ *by* $\delta_m^\infty \langle p_0, p_1, p_2, \ldots \rangle = f$ *if and only if* $f = \delta_m^i(p_i)$ *for all i. Then the solution operator* $S : (f, g) \mapsto u$ *of the Cauchy problem* (3.1) *mapping* $f \in C^\infty(\mathbb{R}^3)$ *and* $g \in C^\infty(\mathbb{R}^3)$ *to the solution* $u \in C^\infty(\mathbb{R}^4)$ *is* $(\delta_3^\infty, \delta_3^\infty, \delta_4^\infty)$*-computable. Corollary* 3.3 *holds accordingly.*

For a proof observe that the procedure described in the proof of Theorem 3.2 is uniformly computable in k.

Theorem 3.2 and Corollary 3.3 indicate that the solution u of the wave equation can be less regular than the initial data. There is a possible loss of one order of differentiability: $u(0, \cdot) = f \in C^k(\mathbb{R}^3)$ and $u_t(0, \cdot) = g \in C^{k-1}(\mathbb{R}^3)$ guarantee only $u \in C^{k-1}(\mathbb{R}^4)$, $u(t, \cdot) \in C^{k-1}(\mathbb{R}^3)$ and $u_t(t, \cdot) \in C^{k-2}(\mathbb{R}^3)$. By the following examples this loss is proper in general.

EXAMPLE 3.5. Again we consider the Cauchy problem of the three-dimensional wave equation with zero initial velocity (1.1). Assume that $k \geqslant 1$ and $F \in C^k(\mathbb{R}) \setminus C^{k+1}(\mathbb{R})$. Without loss of generality, we can assume that the support of F lies outside of the interval $(-\frac{1}{2}, \frac{1}{2})$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x) = F(|x|)$. Then f is spherically symmetric about the origin of $\mathbb{R}^3$ and has the same regularity as F. Since the initial velocity g is zero, for any future time $t > 0$ at place $0 \in \mathbb{R}^3$ the state of the wave is given by

$$u(t, 0) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{|y|=t} F(|y|)\, d\sigma(y) \right) = \frac{\partial}{\partial t} [F(t) \cdot t].$$

Obviously, $u(\cdot, 0) \in C^{k-1}(\mathbb{R}) \setminus C^k(\mathbb{R})$, and hence $u \notin C^{k-1}(\mathbb{R}^4)$ ('one derivative is lost').

We show that the loss of regularity can also be proper in space coordinates (Corollary 3.3(2)). Recall that if $f \in C^k(\mathbb{R}^3)$, with $k \geqslant 0$, the Cauchy problem (1.1) has a unique solution given by

$$u(t, x) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{|y-x|=t} f(y)\, d\sigma(y) \right).$$

Thus $f \in C^k(\mathbb{R}^3)$ guarantees that $u \in C^{k-1}(\mathbb{R}^4)$ for any $k \geqslant 1$. Let ρ, θ and φ be the spherical coordinates

$$x_1 = \rho \cos\theta \sin\varphi, \quad x_2 = \rho \sin\theta \sin\varphi, \quad x_3 = \rho \cos\varphi,$$

where $\rho \geqslant 0$, $0 \leqslant \theta \leqslant 2\pi$, and $0 \leqslant \varphi \leqslant \pi$. The Jacobian $\partial(x_1, x_2, x_3)/\partial(\rho, \theta, \varphi)$ of the coordinate transformation is $\rho^2 \sin\varphi$. For any function f of x, denote

$$\widetilde{f}(\rho, \theta, \varphi) = f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi) = f(x_1, x_2, x_3).$$

Then the regularity of f with respect to x is the same as that of $\widetilde{f}$ with respect to (ρ, θ, φ) whenever $\rho \neq 0$, $\varphi \neq 0$, and $\varphi \neq \pi$.

EXAMPLE 3.6. There exists a continuous initial function f such that the corresponding unique solution u of (1.1) is no longer continuous at a later time.

Let

$$f(x) = \begin{cases} (\rho - 1)(2-\rho)(\varphi - \frac{1}{6}\pi)(\frac{1}{4}\pi - \varphi) & \text{if } 1 \leqslant \rho \leqslant 2 \text{ and } \frac{1}{6}\pi \leqslant \varphi \leqslant \frac{1}{4}\pi, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $\widetilde{f}(\rho, \theta, \varphi) = f(x)$ is continuous with respect to (ρ, θ, φ). Since $f \equiv 0$ in some neighborhoods of $\rho = 0$, $\varphi = 0$, and $\varphi = \pi$, the regularity of f with respect to x is the same as that of $\widetilde{f}$ with respect to (ρ, θ, φ). Thus $f \in C(\mathbb{R}^3)$. (Actually f is Lipschitz continuous on $\mathbb{R}^3$.) Let u denote the solution of (1.1) with the initial condition f. Then $u(1, \cdot)$ describes the state of the wave propagation at time $t = 1$. We prove in the following that $u(1, \cdot)$ is no longer continuous. Thus a wave propagation by beginning with a continuous initial state can become discontinuous at a later time. The proof consists of two observations.

OBSERVATION 1. $u(1, 0) \neq 0$.

Proof. The solution at the location $x = 0$ is

$$u(t, 0) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y|=t} f(y)\, d\sigma(y)\right).$$

Thus for any $1 \leqslant t \leqslant 2$ we have

$$\begin{aligned} u(t,0) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y|=t} f(y)\, d\sigma(y)\right) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_0^{2\pi}\int_{\pi/6}^{\pi/4} (t-1)(2-t)(\varphi - \tfrac{1}{6}\pi)(\tfrac{1}{4}\pi - \varphi)t^2 \sin\varphi\, d\varphi\, d\theta\right) \\ &= \frac{\partial}{\partial t}\left(\frac{3t^2 - t^3 - 2t}{2}\int_{\pi/6}^{\pi/4} (\varphi - \tfrac{1}{6}\pi)(\tfrac{1}{4}\pi - \varphi)\sin\varphi\, d\varphi\right) \\ &= \frac{\partial}{\partial t}\left(\frac{3t^2 - t^3 - 2t}{2}\cdot(-\sqrt{2} + \sqrt{3} - \tfrac{1}{24}\pi - \tfrac{1}{24}\sqrt{2}\,\pi)\right) \\ &= \tfrac{1}{2}(6t - 3t^2 - 2)\cdot(-\sqrt{2} + \sqrt{3} - \tfrac{1}{24}\pi - \tfrac{1}{24}\sqrt{2}\,\pi). \end{aligned}$$

In particular, $u(1, 0) = \frac{1}{2}(-\sqrt{2} + \sqrt{3} - \frac{1}{24}\pi - \frac{1}{24}\sqrt{2}\,\pi) \neq 0.$ □

OBSERVATION 2. *For any* $x = (0, 0, -r)$ *with* $0 < r < 1$, $u(1, x) = 0$.

Proof. Consider the sphere $|y-x|=t$ for any t with $0 \leqslant t < \sqrt{1+r^2}$. Let $y=(y_1, y_2, y_3)$. The portion of the sphere with $y_3<0$ lies outside of the cone $\varphi = \frac{1}{4}\pi$. If $y_3 \geqslant 0$, then from $|y-x|=t$ and $r>0$ we get

$$y_1^2+y_2^2+y_3^2 = t^2-2y_3r-r^2 < 1-2y_3r < 1.$$

Thus the portion of the sphere $|y-x|=t$ when $y_3 \geqslant 0$ (if there is any) is inside the unit sphere $\rho=1$. Then, by the definition of f, for any $x=(0,0,-r)$ with $0<r<1$, any t with $0 \leqslant t < \sqrt{1+r^2}$, and any y satisfying $|y-x|=t$, we have $f(y)\equiv 0$. Hence

$$u(t,x) = \frac{\partial}{\partial t}\left(\int_{|y-x|=t} f(y)\,d\sigma(y)\right) = 0.$$

In particular, $u(1,x)=0$ for all $x=(0,0,-r)$ with $0<r<1$. □

Combining Observations 1 and 2 we see that $u(1,\cdot)$ has a jump at $x=0$, and thus is not continuous. The continuity of the initial state is lost.

By a similar argument we can construct a C^1 initial function such that the state of the wave at time $t=1$ is continuous but not of class C^1. We outline the construction. Let

$$g(x) = \begin{cases} (\rho-1)^2(2-\rho)^2(\varphi-\frac{1}{6}\pi)^2(\frac{1}{4}\pi-\varphi)^2 & \text{if } 1 \leqslant \rho \leqslant 2 \text{ and } \frac{1}{6}\pi \leqslant \varphi \leqslant \frac{1}{4}\pi, \\ 0 & \text{otherwise.} \end{cases}$$

The function g is of class C^1. The partial derivative, g_{x_3}, of g with respect to x_3 is equal to

$$2((\rho-1)(2-\rho)^2-(\rho-1)^2(2-\rho))(\varphi-\tfrac{1}{6}\pi)^2(\tfrac{1}{4}\pi-\varphi)^2\cdot\cos\varphi$$
$$+\,2(\rho-1)^2(2-\rho)^2((\varphi-\tfrac{1}{6}\pi)(\tfrac{1}{4}\pi-\varphi)^2-(\varphi-\tfrac{1}{6}\pi)^2(\tfrac{1}{4}\pi-\varphi))\cdot\rho^{-1}(-\sin\varphi)$$

if $1 \leqslant \rho \leqslant 2$ and $\frac{1}{6}\pi \leqslant \varphi \leqslant \frac{1}{4}\pi$ and $g_{x_3}=0$ otherwise. We also note that the first order partial derivatives of g are Lipschitz continuous on $\mathbb{R}^3$. Now we consider the Cauchy problem

$$\begin{cases} v_{tt} = \Delta v, \\ v(0,x) = g_{x_3}(x), \quad v_t(0,x)=0, \quad t\in\mathbb{R},\ x\in\mathbb{R}^3. \end{cases} \tag{3.3}$$

The same computation as was used in Observations 1 and 2 leads to the following results:

(1) $v(1,0) = \int_{\pi/6}^{\pi/4}(\varphi-\frac{1}{6}\pi)^2(\frac{1}{4}\pi-\varphi)^2\cos\varphi\sin\varphi\,d\varphi > 0$;

(2) $v(1,x)=0$ for all $x=(0,0,-r)$ with $0<r<1$.

Thus $v(1,\cdot)$ is not continuous. Next we consider another Cauchy problem (3.4) with C^1 initial state g:

$$\begin{cases} w_{tt} = \Delta w, \\ w(0,x) = g(x), \quad w_t(0,x)=0, \quad t\in\mathbb{R},\ x\in\mathbb{R}^3. \end{cases} \tag{3.4}$$

Since $g\in C^1(\mathbb{R}^3)$, the solution $w(t,x)$ of (3.4) belongs to $C^0(\mathbb{R}^4)$. In particular, $w(1,\cdot)\in C^0(\mathbb{R}^3)$. Differentiating each equation in (3.4) with respect to x_3 yields

$$\begin{cases} (w_{x_3})_{tt} = \Delta(w_{x_3}), \\ w_{x_3}(0,x) = g_{x_3}(x), \quad (w_{x_3})_t(0,x)=0, \quad t\in\mathbb{R},\ x\in\mathbb{R}^3. \end{cases}$$

This Cauchy problem is the same as (3.3). Since the solution of a Cauchy problem is unique, $w_{x_3} \equiv u$. Thus $w_{x_3}(1, \cdot)$ is not continuous; in other words, $w(1, \cdot) \notin C^1(\mathbb{R}^3)$. $\square$

Example 3.6 shows that a wave that begins by being continuous or C^1 smooth can lose regularity (in space coordinates) later on. In the next example we show that the loss of regularity is proper no matter how smooth the initial state is. In other words, for any integer $k \geqslant 0$, there exists a wave propagation which begins with a C^k smooth state but becomes less smooth at a later time.

EXAMPLE 3.7. For any integer $k \geqslant 0$, there exists an initial function $F \in C^k(\mathbb{R}^3)$ such that the future state of the wave at time $t = 1$, $U(1, \cdot)$, of the Cauchy problem

$$\begin{cases} U_{tt} = \Delta U, \\ U(0, x) = F(x), \quad U_t(0, x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^3. \end{cases} \tag{3.5}$$

is in $C^{k-1}(\mathbb{R}^3) \backslash C^k(\mathbb{R}^3)$.

The example is the consequence of Example 3.6 and the lemma below, which says that if the solution of (3.5) loses one derivative for some $C^{k,\alpha}$ smooth initial condition at a certain future time, then there is a $C^{k+2,\alpha}$ smooth initial function such that its corresponding solution of (3.5) also loses one derivative. Thus $C^{k,\alpha}(\mathbb{R}^3)$, with $0 < \alpha \leqslant 1$, is the set of all $f \in C^k(\mathbb{R}^3)$ such that the kth order partial derivatives of f are Hölder continuous. Example 3.6 provides the cases where $k = 0$ and $k = 1$. The lemma allows us to use induction for $k \geqslant 2$.

LEMMA. *Let $k \geqslant 0$ be an integer. Suppose there exists an $f \in C^{k,\alpha}(\mathbb{R}^3)$, with $0 < \alpha \leqslant 1$, such that the solution, $u(1, \cdot)$, of the Cauchy problem*

$$\begin{cases} u_{tt} = \Delta u, \\ u(0, x) = f(x), \quad u_t(0, x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^3, \end{cases} \tag{3.6}$$

at time $t = 1$ is in $C^{k-1}(\mathbb{R}^3) \backslash C^k(\mathbb{R}^3)$. Then there exists an $F \in C^{k+2,\alpha}(\mathbb{R}^3)$ such that the solution, $U(1, \cdot)$, of the Cauchy problem (3.5) *at time $t = 1$ belongs to $C^{k+1}(\mathbb{R}^3) \backslash C^{k+2}(\mathbb{R}^3)$.*

Proof. By the classical theory of elliptic equations (see, for example, the book by Gilbarg and Trudinger [**5**]), the Poisson equation

$$\Delta u(x) = f(x), \quad x \in \mathbb{R}^3,$$

with the forcing term $f \in C^{k,\alpha}(\mathbb{R}^3)$, has a solution in $C^{k+2,\alpha}(\mathbb{R}^3)$ (elliptic gain of 2). Let $F \in C^{k+2,\alpha}(\mathbb{R}^3)$ be such a solution. Then the solution $U(t, x)$ of (3.5) with F being the initial state is in $C^{k+1}(\mathbb{R}^4)$; in particular, $U(1, \cdot) \in C^{k+1}(\mathbb{R}^3)$. Applying the Laplace operator to each equation in (3.5) yields

$$\begin{cases} (\Delta U)_{tt} = \Delta(\Delta U), \\ \Delta U(0, x) = \Delta F(x) = f(x), \quad (\Delta U)_t(0, x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^3. \end{cases}$$

The resulting Cauchy problem is the same as (3.6). Thus $\Delta U \equiv u$. Since $u(1, \cdot) \notin C^k(\mathbb{R}^3)$, $U(1, \cdot)$ cannot be in $C^{k+2}(\mathbb{R}^3)$. Thus at time $t = 1$ the wave is only C^{k+1} smooth. One derivative is lost. $\square$

By Corollary 3.3, on $C^1(\mathbb{R}^3)$ the special solution operator $f \mapsto S'(f, 0, 1)$ is

(δ_3^1, δ_3)-computable, and hence (τ_3^1, τ_3)-continuous by Theorem 2.4. We show, however, that it is not (τ_3, τ_3)-continuous, and hence not (δ_3, δ_3)-computable, on $C^1(\mathbb{R}^3)$. (Remember that a linear operator is continuous everywhere or nowhere.)

THEOREM 3.8. *For any $t \in \mathbb{R}\setminus\{0\}$, the wave propagator $S_t : f \mapsto S'(f, 0, t)$, which sends the initial Cauchy data $f \in C^1(\mathbb{R}^3)$ and $g \equiv 0$ to the solution at time t, is not (τ_3, τ_3)-continuous.*

Proof. Let $f_n(x) = n^{-1} \sin n|x|^2$ and $f(x) := 0$ for any $x \in \mathbb{R}^3$ and $n > 0$. Then $f_n \in C^1(\mathbb{R}^3)$ for all $n \in \mathbb{N}$. We obtain $S_t(f) = 0$ and

$$\begin{aligned}
S_t(f_n)(0) &= u_n(t, 0) \\
&= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y|=t} f_n(y)\, d\sigma(y)\right) \\
&= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y|=t} \frac{1}{n}\sin n|y|^2\, d\sigma(y)\right) \\
&= \frac{\partial}{\partial t}\left(\frac{1}{n}\sin(nt^2)\cdot t\right) \\
&= 2t^2\cos(nt^2) + n^{-1}\sin(nt^2).
\end{aligned}$$

In particular, $|S_t(f_n)(0)| > d$ infinitely often for some $d > 0$. Consider an open subbase element $R_{0,1,-e,e} \in \sigma$ from Definition 2.6(2) with $0 < e < d$. We have $S_t(f) = 0 \in R_{0,1,-e,e}$. If S_t is continuous in f, then $f \in U := A_1 \cap \ldots \cap A_k$ and $S_t(U) \subseteq R_{0,1,-e,e}$ for some $A_1, \ldots, A_k \in \sigma$. We have $f_n \in U$ for any sufficiently large n. For some n we obtain $f_n \in U$ and $|S_t(f_n)(0)| > d$; hence $S_t(f_n) \notin R_{0,1,-e,e}$. Therefore, S_t is not (τ_3, τ_3)-continuous. □

As we have shown, no finite set of atomic properties listed by a δ_3-name of $f \in C^1(\mathbb{R}^3)$ suffices to guarantee a 'narrow' atomic property for a δ_3-name of the solution. In other words, the topology τ_3 associated with the representation δ_3 on the domain $C^1(\mathbb{R}^3)$ of the solution operator is not sufficiently fine for determining a δ_3-name of the continuous solution.

In summary, for $t \neq 0$ the solution operator S_t does not necessarily map $C(\mathbb{R}^3)$ into itself, does not necessarily map $C^1(\mathbb{R}^3)$ into itself, maps $C^1(\mathbb{R}^3)$ to $C(\mathbb{R}^3)$ and is (τ_3^1, τ_3)-continuous but not (τ_3, τ_3)-continuous on $C^1(\mathbb{R}^3)$.

Parallel to these topological irregularities of the solution operator there are computational ones. Myhill gave an example [**8**] of a δ_1-computable differentiable function $f \in C^1(\mathbb{R})$ such that the derivative $f' \in C(\mathbb{R})$ is not δ_1-computable. (Notice that the differentiation operator D on $C^1(\mathbb{R})$ is *not* (τ_1, τ_1)-continuous and that τ_1 is the topology associated with the representation δ_1 of the domain of D.) We apply Myhill's result to show that the solution operator S from Theorem 3.2 maps some computable initial condition to a non-computable solution.

EXAMPLE 3.9. Consider Example 3.5. By Myhill's theorem there is a δ_1-computable function $F \in C^1(\mathbb{R})$ such that $F'(1)$ is not ρ-computable. We have

$$u(t, 0) = \frac{\partial}{\partial t}[F(t)\cdot t] = F(t) + F'(t)\cdot t.$$

We obtain $u(1,0) = F(1) + F'(1)$, a number which is not ρ-computable. Since the function u maps the computable argument $(1,0)$ to a non-computable number, it cannot be (ρ^4, ρ)-computable.

A more general situation where non-continuity causes non-computability is the 'First Main Theorem' by Pour-El and Richards [**10**]. As an important application they show the following result.

THEOREM 3.10 (Pour-El and Richards). *There is a δ_3-computable function $f \in C^1(\mathbb{R}^3)$ (that is, a continuously differentiable function) such that the solution, $u(1,\cdot)$, of the Cauchy problem* (1.1) *at time $t = 1$ is not δ_3-computable.*

This result has been strengthened by Pour-El and Zhong [**9**] as follows.

THEOREM 3.11 (Pour-El and Zhong). *For any compact set $D \subset \mathbb{R}^3 \times [0, +\infty)$, there is a δ_3-computable function $f \in C^1(\mathbb{R}^3)$ such that the solution u of the Cauchy problem* (1.1) *is not δ_3-computable in any neighborhood of any point of the set D.*

According to a general theorem by Brattka [**2**] any function of sufficiently high degree of discontinuity with some weak computability properties maps some computable elements to non-computable ones. For the instances we have considered, the solution operator of the Cauchy problem (3.1) is computable if it is continuous, and has computational irregularities if it is not continuous. Remember that for each of the considered spaces the topology and the representation defining the computability concept are simultaneously generated by an effective topological space, where the topology is the final topology of the representation (Definition 2.3 and Theorem 2.4).

4. *The wave propagator on Sobolev spaces*

While the wave propagator does not preserve $C^k \times C^{k-1}$ regularity, for any real number s it preserves $H^s \times H^{s-1}$ regularity, where H^s is the Sobolev space of order s. In this section we show that the wave propagator is even computable when operating on initial data from Sobolev spaces.

Sobolev spaces $H^s(\mathbb{R}^d)$ are defined in an analogous fashion to C^k, but with L^2 taking over the role of the continuous functions. Let k be a non-negative integer. The Sobolev space $H^k(\mathbb{R}^d)$ is the set of all functions $u \in L^2(\mathbb{R}^d)$ such that $\partial^\alpha u \in L^2(\mathbb{R}^d)$ for all distributional derivatives with $|\alpha| \leqslant k$. Thus $H^k(\mathbb{R}^d)$ is a separable Hilbert space with the norm

$$\|u\|_{H^k(\mathbb{R}^d)} := \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^d)} \right)^{1/2}.$$

In terms of Fourier transform, there is an equivalent definition for $H^k(\mathbb{R}^d)$:

$$H^k(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) \mid (1 + |\xi|^2)^{k/2}\, \widehat{u}(\xi) \in L^2(\mathbb{R}^d)\}$$

and

$$\|u\|_{H^k(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^k\, |\widehat{u}(\xi)|^2\, d\xi \right)^{1/2},$$

where $\widehat{u}(\xi)$ is the Fourier transform of u. In this alternative definition of $H^k(\mathbb{R}^d)$, k does not have to be a non-negative integer. This observation leads to the following definition of the Sobolev space $H^s(\mathbb{R}^d)$ for any $s \in \mathbb{R}$.

DEFINITION 4.1. For any $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^d)$ is the set of all generalized functions u such that $(1+|\xi|^2)^{s/2}\widehat{u}(\xi) \in L^2(\mathbb{R}^d)$. Thus $H^s(\mathbb{R}^d)$ is a separable Hilbert space with the norm

$$\|u\|_s = \|u\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi \right)^{1/2}.$$

Note that L^2-computability induces a natural computability concept on $H^s(\mathbb{R}^d)$. In the following let δ_L be the representation of $L^2(\mathbb{R}^3)$ from Definition 2.6(6).

DEFINITION 4.2. For any $s \in \mathbb{R}$ define a representation δ^s of $H^s(\mathbb{R}^3)$ by

$$\delta^s(p) = f :\iff \delta_L(p) = (1+|\xi|^2)^{s/2}\widehat{f}.$$

Consider the Cauchy problem (3.1) of the three-dimensional wave equation

$$\begin{cases} u_{tt} = \Delta u, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), \quad t \in \mathbb{R}, x \in \mathbb{R}^3. \end{cases}$$

For any fixed $t \in \mathbb{R}$, let

$$S_W(t)(u(0,x), u_t(0,x)) := (u(t,x), u_t(t,x))$$

be the wave propagator which sends the initial Cauchy data to the Cauchy data at time t.

The Sobolev spaces $H^s(\mathbb{R}^3)$ are naturally associated with the wave equation. We have seen in the previous section that the wave operator S_W does not propagate $C^k \times C^{k-1}$ regularity. Littman's theorem [**11**] asserts that S_W does not propagate norms based on L^p for $p \neq 2$ in dimensions greater than 1. However, if one measures regularity in the H^s sense, there is no loss of derivatives in any dimension. According to the classical theory for the wave equation, for any $s \in \mathbb{R}$ and $t \in \mathbb{R}$, the wave propagator $S_W(t)$ maps $H^s \times H^{s-1}$ to itself and the first component of $S_W(t)$ is the unique solution of the Cauchy problem (3.1).

Sobolev spaces are suitable for analyzing the wave equation computationally as well: the solution operator of (3.1) propagates Sobolev computability. As our main theorem of this section we prove that the operator $(f,g,t) \mapsto S_W(t)(f,g)$ is a computable operator which for any t sends every initial Cauchy data $(f,g) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ to the Cauchy data $(f_t, g_t) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ at time t. In other words, there is a Type-2 Turing machine which computes a name for the corresponding solution when fed with a name of the time t and names of the initial data.

We prepare the proof with computational versions of the facts that $f + f' \in L^2$ and $f(x)g(x) \in L^2$ if $f, f' \in L^2$ and g is continuous and bounded.

LEMMA 4.3. (1) *Addition* $(f,f') \mapsto f+f'$ *on* L^2 *is* $(\delta_L, \delta_L, \delta_L)$*-computable.*

(2) *Multiplication*

$$\mathrm{Mul} : (f,g,N) \mapsto fg \text{ where } fg(x) := f(x)g(x)$$

for $f \in L^2(\mathbb{R}^3)$, $g \in C(\mathbb{R}^3)$ and $N \in \mathbb{R}$, with $|g(x)| < N$ for all x, is $(\delta_L, \delta_3, \rho, \delta_L)$-computable.

Proof. Let μ be the notation of the finite rational step functions introduced in Definition 2.6(6) for three dimensions and let δ_L be the associated Cauchy representation of $L^2(\mathbb{R}^3)$.

First consider addition. There is a Type-2 machine which transforms any two infinite input sequences $u_0\#u_1\#\dots$ and $v_0\#v_1\#\dots$ in $\mathrm{dom}(\delta_L)$ to a sequence $w_0\#w_1\#\dots$ where $\mu(w_n) = \mu(u_{n+1}) + \mu(v_{n+1})$. Then $w_0\#w_1\#\dots \in \mathrm{dom}(\delta_L)$ and $\delta_L(w_0\#w_1\#\dots) = \delta_L(u_0\#u_1\#\dots) + \delta_L(v_0\#v_1\#\dots)$. Therefore, addition is $(\delta_L, \delta_L, \delta_L)$-computable.

For proving the second statement we introduce another representation δ of $C(\mathbb{R}^3)$. Define a metric on $C(\mathbb{R}^3)$ by

$$d(f,g) := \sum_{k=1}^{\infty} 2^{-k} \frac{|f-g|_k}{1+|f-g|_k} \quad \text{where } |f-g|_k := \sup_{|x|\leqslant k} |f(x)-g(x)|.$$

Let D be the set of polynomials on $\mathbb{R}^3$ with rational coefficients. Then D is dense in the metric space $(C(\mathbb{R}^3), d)$. Let ν be a canonical notation of the set of these functions. Then the associated Cauchy representation $\delta :\subseteq \Sigma^\omega \to C(\mathbb{R}^3)$ is equivalent to the representation δ_3. We prove that Mul is $(\delta_L, \delta_3, \rho, \delta_L)$-computable.

We describe an algorithm for a machine M which computes a δ_L-name of the multiplication fg from a δ_L-name of f, a δ_3-name of g, and a ρ-name of a bound N of g. Assume that $\{\mu(a_i)\}$ is a sequence of rational step functions which converges fast to f in L^2-norm, and that $\{\nu(b_j)\}$ is a sequence of elements from the dense set D converging fast to g with respect to the metric d. We will produce a sequence $\{\mu(u_n)\}_{n\in\omega}$ which converges fast to fg in L^2-norm. Consider $n \in \omega$. Choose $i_n > n + 2 + \log_2 N$. Since $f = \lim \mu(a_i)$ in L^2-norm,

$$\|f - \mu(a_{i_n})\|_{L^2} < 2^{-(n+2)}/N.$$

Find natural numbers $\alpha(n)$ and $\beta(n)$ such that $\mathrm{supp}(\mu(a_{i_n})) \subset B(0, \alpha(n))$ and $\|\mu(a_{i_n})\|_{L^2} < 2^{\beta(n)}$.

Choose $j_n := n + \alpha(n) + \beta(n) + 4$. Then

$$d(g, \nu(b_{j_n})) \leqslant 2^{-j_n} < 2^{-(n+\alpha(n)+\beta(n)+3)}.$$

Selecting the $\alpha(n)$-term from the infinite sum $d(g, \nu(b_{j_n}))$ we obtain

$$2^{-\alpha(n)} \frac{|g-\nu(b_{j_n})|_{\alpha(n)}}{1+|g-\nu(b_{j_n})|_{\alpha(n)}} < 2^{-(n+\alpha(n)+\beta(n)+3)};$$

hence

$$|g - \nu(b_{j_n})|_{\alpha(n)} < 2^{-(n+\beta(n)+2)}.$$

Therefore,

$$
\begin{aligned}
\|fg - \mu(a_{i_n})\nu(b_{j_n})\|_{L^2} &\leqslant \|fg - \mu(a_{i_n})g\|_{L^2} + \|\mu(a_{i_n})g - \mu(a_{i_n})\nu(b_{j_n})\|_{L^2} \\
&\leqslant N\|f - \mu(a_{i_n})\|_{L^2} + \left(\int_{\mathbb{R}^3} |\mu(a_{i_n})(g - \nu(b_{j_n}))|^2 \, dx \right)^{1/2} \\
&< 2^{-(n+2)} + \left(\int_{|x| < \alpha(n)} |\mu(a_{i_n})(g - \nu(b_{j_n}))|^2 \, dx \right)^{1/2} \\
&= 2^{-(n+2)} + \left(\int_{|x| < \alpha(n)} |\mu(a_{i_n})|^2 |g - \nu(b_{j_n})|^2 \, dx \right)^{1/2} \\
&\leqslant 2^{-(n+2)} + 2^{-(n+\beta(n)+2)} \left(\int_{|x| < \alpha(n)} |\mu(a_{i_n})|^2 \, dx \right)^{1/2} \\
&\leqslant 2^{-(n+2)} + 2^{-(n+\beta(n)+2)} \|\mu(a_{i_n})\|_{L^2} \\
&\leqslant 2^{-(n+2)} + 2^{-(n+2)} \\
&\leqslant 2^{-(n+1)}.
\end{aligned}
$$

(Recall that $\|\mu(a_{i_n})\|_{L^2} < 2^{\beta(n)}$.)

Finally, Machine M searches for u_n such that

$$\|\mu(u_n) - \mu(a_{i_n})\nu(b_{j_n})\|_{L^2} < 2^{-(n+1)}.$$

We note that by definition $\|\mu(u) - \mu(v)\nu(w)\|_{L^2} < 2^{-n}$ is decidable in u, v, w and n. Machine M gives u_n to the output at stage n. We observe that

$$
\begin{aligned}
\|\mu(u_n) - fg\|_{L^2} &\leqslant \|\mu(u_n) - \mu(a_{i_n})\nu(b_{j_n})\|_{L^2} + \|\mu(a_{i_n})\nu(b_{j_n}) - fg\|_{L^2} \\
&< 2^{-(n+1)} + 2^{-(n+1)} < 2^{-n}.
\end{aligned}
$$

Obviously, M produces an infinite sequence $\mu(u_0), \mu(u_1), \ldots$ of rational step functions which converges fast to fg. The proof of the lemma is complete. □

COROLLARY 4.4. *Let $h : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be (ρ, ρ^3, ρ)-computable and $b : \mathbb{R} \to \mathbb{R}$ be (ρ, ρ)-computable such that $|h(t, x)| \leqslant b(t)$ for all $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Then $(f, t) \mapsto f h(t, \cdot) \in L^2$, for $f \in L^2$ and $t \in \mathbb{R}$, is $(\delta_L, \rho, \delta_L)$-computable.*

Proof. Define $h'(t)(x) := h(t, x)$. By Lemma 2.1, the function h' is $(\rho, [\rho^3 \to \rho])$-computable, and hence (ρ, δ_3)-computable. For all t, the number $b(t)$ is a uniform bound of the function $h'(t) \in C(\mathbb{R}^3)$. By Lemma 4.3, $(f, t) \mapsto f h'(t) = f h(t, \cdot)$ is $(\delta_L, \rho, \delta_L)$-computable. □

For the precise statement of the theorem we use the product $[\delta, \delta']$ of two representations δ and δ' canonically defined by $[\delta, \delta']\langle p, p' \rangle := (\delta(p), \delta'(p'))$.

THEOREM 4.5. *Let $s \in \mathbb{R}$ be an arbitrary real number. Then the wave propagator*

$$S'_W : H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \times \mathbb{R} \to H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$$

mapping initial data in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ to the solution in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ at time t is $([\delta^s, \delta^{s-1}], \rho, [\delta^s, \delta^{s-1}])$-computable.

Proof. Applying the Fourier transform to (3.1) with respect to x yields

$$\widehat{u}_{tt}(t,\xi)-\widehat{u}(t,\xi)=0$$

with

$$\widehat{u}(0,\xi)=\widehat{f}(\xi),\quad \widehat{u}_t(0,\xi)=\widehat{g}(\xi),$$

which is a second order linear ordinary differential equation. The equation can be solved explicitly and the solution is given by

$$\widehat{u}(t,\xi)_{f,g}=\widehat{f}(\xi)\cos(|\xi|t)+\widehat{g}(\xi)\sin(|\xi|t)/|\xi|.$$

From a δ^s-name of f, a δ^{s-1}-name of g and a ρ-name of t we want to determine

(i) a δ^s-name of $u(t,\xi)_{f,g}$ and

(ii) a δ^{s-1}-name of $u_t(t,\xi)_{f,g}$.

First we consider Case (i). Define $h(t,\xi):=\cos(|\xi|t)$ and $b(t):=1$. By Corollary 4.4, $(F(\xi),t)\mapsto F(\xi)\cos(|\xi|t)$ is (δ_L,ρ,δ_L)-computable.

Define

$$h(t,\xi):=(1+|\xi|^2)^{1/2}\sin(|\xi|t)/|\xi|$$

and

$$b(t):=2+2t.$$

Then $h:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ and b are computable and $|h(t,\xi)|\leqslant b(t)$ for $|\xi|\leqslant 1$ as well as for $|\xi|\geqslant 1$. By Corollary 4.4, $(G(\xi),t)\mapsto G(\xi)(1+|\xi|^2)^{1/2}\sin(|\xi|t)/|\xi|$ is (δ_L,ρ,δ_L)-computable. Then by Lemma 4.3(1),

$$H:(F(\xi),G(\xi),t)\mapsto F(\xi)\cos(|\xi|t)+G(\xi)(1+|\xi|^2)^{1/2}\sin(|\xi|t)/|\xi|$$

is $(\delta_L,\delta_L,\rho,\delta_L)$-computable. Therefore, there is a computable function h with $H(\delta_L(p),\delta_L(q),\rho(r))=\delta_L h(p,q,r)$ (whenever the left-hand side is defined).

Suppose $f=\delta^s(p)$, $g=\delta^{s-1}(q)$ and $t=\rho(r)$. We obtain

$$\begin{aligned}
&(1+|\xi|^2)^{s/2}\widehat{u}(t,\xi)_{f,g}\\
&\quad=(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\cos(|\xi|t)+(1+|\xi|^2)^{(s-1)/2}\widehat{g}(\xi)(1+|\xi|^2)^{1/2}\sin(|\xi|t)/|\xi|\\
&\quad=\delta_L(p)\cos(|\xi|\rho(r))+\delta_L(q)(1+|\xi|^2)^{1/2}\sin(|\xi|\rho(r))/|\xi|\\
&\quad=H(\delta_L(p),\delta_L(q),\rho(r))\\
&\quad=\delta_L h(p,q,r).
\end{aligned}$$

This means that $u(t,\xi)_{f,g}=\delta^s h(p,q,r)$. Therefore, $(f,g,t)\mapsto u(t,\xi)_{f,g}$ is $(\delta^s,\delta^{s-1},\rho,\delta^s)$-computable. Observe that the computable function h does not depend on the parameter s!

We consider Case (ii). From the explicit solution we obtain

$$\widehat{u}_t(t,\xi)_{f,g}=-\widehat{f}(\xi)|\xi|\sin(|\xi|t)+\widehat{g}(\xi)\cos(|\xi|t)$$

and

$$\begin{aligned}
(1+|\xi|^2)^{(s-1)/2}\,\widehat{u}_t(t,\xi)_{f,g}&=-(1+|\xi|^2)^{s/2}\widehat{f}(\xi)|\xi|\sin(|\xi|t)/(1+|\xi|^2)^{1/2}\\
&\quad+(1+|\xi|^2)^{(s-1)/2}\,\widehat{g}(\xi)\cos(|\xi|t).
\end{aligned}$$

The functions $|\xi|\sin(|\xi|t)/(1+|\xi|^2)^{1/2}$ and $\cos(|\xi|t)$ of t and ξ are bounded by 1. With the arguments from Case (i) we obtain the fact that $(f, g, t) \mapsto u_t(t, \xi)_{f,g}$ is $(\delta^s, \delta^{s-1}, \rho, \delta^{s-1})$-computable. □

COROLLARY 4.6. *For each real number s and each computable real number t the wave propagator*

$$S_W(t) : H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \to H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$$

mapping initial data in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ to a solution in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ at time t is $([\delta^s, \delta^{s-1}], [\delta^s, \delta^{s-1}])$-computable.

As we have observed, the Type-2 machine computes the solution of (3.1) from the initial condition and the time does not depend on s and does not need a name of s as an input. Therefore, the real number s determining the Sobolev space $H^s(\mathbb{R}^3)$ does not need to be computable. It is merely part of the definition of the representation δ^s (Definition 4.2).

5. *Relation to physical reality*

In the preceding sections we have studied computability properties of the solution operator of the wave equation. From the variety of computability models for analysis which have been proposed [**16**, Chapter 9] we have chosen 'Type 2 Theory of Effectivity' (TTE), which is conceptually simple, very flexible and very realistic. In contrast to 'real number models' [**14**, **1**] where programs handle real numbers as entities, in TTE programs operate on 'finite properties' of 'infinite objects' like real numbers or continuous real functions. This is a much more realistic model, since no digital computer can read, store or write arbitrary real numbers. For every set under consideration, we have introduced computability via a notation of a set of 'atomic' properties (effective topological space, Definition 2.2). A name of a point is an infinite list of all of its atomic properties (Definition 2.3).

Given two effective topological spaces $S_i = (M_i, \sigma_i, \nu_i)$ with standard representations δ_i where $i = 1, 2$ (Definitions 2.2 and 2.3), a function $f :\subseteq M_1 \to M_2$ is computable if some Turing machine (or any equivalent device) transforms any δ_1-name of an argument x to a δ_2-name of $f(x)$. Since for such a computation every finite portion of the output depends only on a finite portion of the input, every property $A_2 \in \sigma_2$ of $f(x)$ can already be determined from finitely many properties $A_{11}, \ldots, A_{1m} \in \sigma_1$ of x. This is equivalent to the following condition: f is continuous with respect to the topologies generated by σ_1 and σ_2, respectively, as subbases.

Effective topological spaces $\mathbf{S} = (M, \sigma, \nu)$ cannot only be used for defining topology (or approximation) and computability on sets M but also offer themselves as models of observation in physics. In this case M is the set of states of a physical object and σ is the set of possible observations. (The notation ν of the set σ is given canonically in all applications.) If we accept that the result of an observation must be stored in a memory, then, since every memory (available today) has only finite capacity, the set of possible results is countable. Therefore, restriction to countable sets σ is realistic. As an example, consider $M = \mathbb{R}$ as the set of positions of a particle and σ as the set of all real intervals with rational endpoints (Definition 2.6(1)).

We will need some assumptions about physical measurements or observations which we formulate in our mathematical model. We assume that there is a measuring procedure P which can be applied to any state $x \in M$ and any finite set $\{A_1, \ldots, A_m\} \subseteq \sigma$ (but no infinite set) of properties such that:

(P1) P applied to x and $\{A_1, \ldots, A_m\}$ answers 'yes' if x has the properties $A_1, \ldots, A_m$, and gives no answer otherwise;

(P2) a given answer is absolutely correct (not only with high probability);

(P3) observation does not disturb the state.

Notice that this includes arbitrarily 'exact' observations, such as $x \in (1; 1 + 10^{-1000})$ in the above example.

It is a matter of physics to investigate how realistic the various models of measurement are. In this paper, we have considered several models, that is, effective topological spaces, for physical waves: $C(\mathbb{R}^3)$, $C^k(\mathbb{R}^3)$, $C^\infty(\mathbb{R}^3)$ where atomic properties concern bounds of amplitudes of the wave and its derivatives, and the Sobolev spaces $H^s(\mathbb{R}^3)$ where atomic properties concern L^2-properties of weighted Fourier transforms of waves.

In particular, for the space $C(\mathbb{R}^3)$ we have introduced continuity and computability by the set σ_3 of 'amplitude boxes' generating the topology τ_3 and the representation δ_3 (Definition 2.6(2)). For the space $C^1(\mathbb{R}^3)$ we have introduced continuity and computability by the set σ_3^1 consisting of amplitude boxes of functions and of amplitude boxes of partial derivatives generating the topology τ_3^1 and the representation δ_3^1 (Definition 2.6(3)).

By Corollary 3.3, the solution operator S_1 mapping any initial condition $f \in C^1(\mathbb{R}^3)$ of the special Cauchy problem (1.1) to $u(1, x)$, the amplitude at time 1, is (δ_3^1, δ_3)-computable and so (τ_3^1, τ_3)-continuous. In particular, S_1 maps every δ_3^1-computable function $f \in C^1(\mathbb{R}^3)$ to a δ_3-computable function $S_1(f) \in C(\mathbb{R}^3)$. By Theorem 3.8 however, S_1 is not (τ_3, τ_3)-continuous. By the Pour-El–Richards result, S_1 maps some δ_3-computable function $f_{\mathrm{PR}} \in C^1(\mathbb{R}^3)$ to a function $f \in C(\mathbb{R}^3)$ which is not δ_3-computable. Obviously, f_{PR} cannot be δ_3^1-computable, that is, the amplitude boxes of its partial derivatives are not recursively enumerable. Thus, non-computability of the gradient of the initial condition f_{PR} causes non-computability of the amplitude of the wave at time 1 (in each case computability with respect to δ_3).

We return to the question for a wave computer which beats the Turing machine and ask whether the Pour-El–Richards theorem, Theorem 3.10, can be used to construct a physical device computing a number function $h : \mathbb{N} \to \mathbb{N}$ which is not Turing computable. We consider three-dimensional waves from $C(\mathbb{R}^3)$ or $C^1(\mathbb{R}^3)$ as states which propagate according to the wave equation (1.1) and assume that rational amplitude boxes as well as rational amplitude boxes of the partial derivatives (for differentiable states) can be observed. We discuss three attempts.

(1) We would like to start propagation with the δ_3-computable wave $f_{\mathrm{PR}} \in C^1(\mathbb{R}^3)$ of the Pour-El–Richards counterexample. This function is defined by an infinite recursively enumerable set of atomic properties $A \in \sigma_3$. Since, in accordance with our model, only finitely many of them can be guaranteed by measurement, it is impossible to prepare f_{PR} at time 0 exactly.

(2) As an alternative we perform experiments $i = 1, 2, \ldots$ with initial conditions $f_i \in C^1(\mathbb{R}^3)$ satisfying the first i properties $A \in \sigma_3$ of f_{PR} in some fixed computable

enumeration. Although the sequence $i \mapsto f_i$ converges to f_{PR} with respect to the topology τ_3, the sequence $i \mapsto S_1(f_i)$ in general does not converge (with respect to the topology τ_3), since S_1 is not (τ_3, τ_3)-continuous. Therefore performing measurements on the $S_1(f_i)$ at time 1 is useless.

(3) As a further alternative we would like to perform experiments $i = 1, 2, \dots$ with initial conditions $g_i \in C^1(\mathbb{R}^3)$ satisfying the first i properties $A \in \sigma_3^1$ of f_{PR} in some fixed enumeration. Since the sequence $i \mapsto g_i$ converges to f_{PR} with respect to the topology τ_3^1 and S_1 is (τ_3^1, τ_3)-continuous, the sequence $i \mapsto S_1(g_i)$ converges (with respect to τ_3) to $S_1(f_{\mathrm{PR}})$, which is not δ_3-computable. So by repeated measurements at time 1 it might be possible to find a δ_3-name of $S_1(f_{\mathrm{PR}})$, which is a non-computable discrete function. Unfortunately, the function f_{PR} is not δ_3^1-computable, and so we are not able to enumerate the properties $A \in \sigma_3^1$ of f_{PR} effectively.

In summary, even under very idealizing assumptions about measurements and wave propagation in reality, it seems to be very unlikely that the Pour-El–Richards counterexample can be used to build a physical machine with a 'wave subroutine' computing a function which is not Turing computable. We may still believe that the Church–Turing Thesis holds.

Notice that in (3.1) and (1.1) we have considered the wave equation for velocity $c = 1$. The general wave equation

$$\begin{cases} v_{tt} = c^2 \cdot \Delta v, \\ v(0, y) = f(y), \quad v_t(0, y) = g(y), \quad t \in \mathbb{R}, y \in \mathbb{R}^3. \end{cases} \tag{5.1}$$

can be reduced to the special case by setting $u(t, x) := v(t, cx)$. Therefore, whenever the wave propagator for (3.1) is computable, the wave propagator for (5.1) is also computable in c as an additional variable, and computable as before if c is a computable real number.

Apart from wave propagation, two other partial differential equations with applications in physics have been analysed similarly, the linear and a non-linear Schrödinger equation [**19**, **20**]. In both cases the propagator is computable. Furthermore, for the Korteweg–de Vries equation the periodic solution is computable if the initial condition is computable [**3**]. However, computability of many other physical processes, for example, boundary value problems, has not yet been investigated. Therefore, there still might be a physical device which beats the Turing machine.

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Klaus Weihrauch
Department of Computer Science
FernUniversität Hagen
58084 *Hagen*
Germany
Klaus.Weihrauch@FernUni-Hagen.de

Ning Zhong
Department of Mathematics
Clermont College
University of Cincinnati
Batavia
OH 45103-1785
USA
Ning.Zhong@uc.edu