

January 13, 2018
Heroin Model

S=susceptibles

P=prescribed opioid users, A=addicted to opioids, H=heroin users/addicted, R=treatment/rehabilitation

$S(0) = S_0, P(0) = P_0, A(0) = A_0, H(0) = H_0, R(0) = R_0$

Assume $H_0 > 0, \mu_H > \mu_A$ and $\theta_1 > \theta_2, \theta_3$

$$\frac{dS}{dt} = -\alpha S - \beta(1 - \xi)SA - \beta\xi SP - \theta_1 SH + \epsilon P + \delta R + \mu(P + R) + (\mu + \mu_A)A + (\mu + \mu_H)H$$

$$\frac{dP}{dt} = \alpha S - \epsilon P - \gamma P - \theta_2 PH - \mu P$$

$$\frac{dA}{dt} = \gamma P + \sigma_A R + \beta(1 - \xi)SA + \beta\xi SP - \zeta A - \theta_3 AH - (\mu + \mu_A)A$$

$$\frac{dH}{dt} = \theta_1 SH + \theta_2 PH + \theta_3 AH + \sigma_H R - \nu H - (\mu + \mu_H)H$$

$$\frac{dR}{dt} = \zeta A + \nu H - \delta R - \sigma_A R - \sigma_H R - \mu R$$

The following is a brief description of each parameter in the system:

α : the rate at which people are prescribed opioids

β : total probability of becoming addicted to opioids other than by prescription

$\beta(1 - \xi)$: proportion of which the non-prescribed, susceptible population becomes addicted to opioids by black market drugs and other addicts

$\beta\xi$: proportion of which the non-prescribed, susceptible population obtains extra prescription opioids and becomes addicted

θ_1 : rate at which the non-prescribed, susceptible population becomes addicted to heroin by black market drugs and other addicts

ϵ : rate at which people come back to the susceptible class after being prescribed opioids (i.e. not addicted)

δ : rate at which people come back to the susceptible class after successfully finishing treatment

μ : natural death rate

μ_A : enhanced death rate for opioid addicts (overdose rate which results in death)

μ_H : enhanced death rate for heroin addicts (overdose rate which results in death)

γ : rate at which prescribed opioid users become addicted

θ_2 : rate at which opioid prescribed user population becomes addicted to heroin

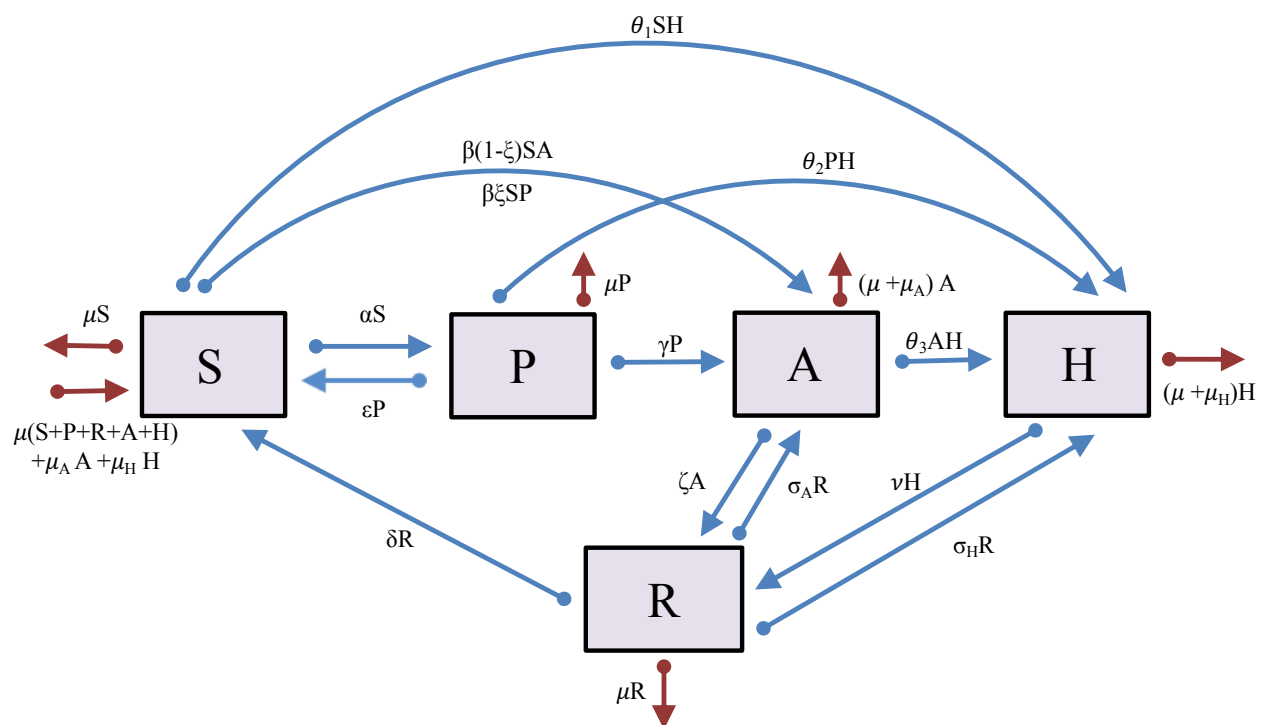
σ_A : rate at which people relapse from treatment into the opioid addicted class

ζ : rate at which addicted opioid users enter treatment/rehabilitation

θ_3 : rate at which the opioid addicted population becomes addicted to heroin

σ_H : rate at which people relapse from treatment into the heroin addicted class

ν : rate at which heroin users enter treatment/rehabilitation



To find the addiction-free equilibrium (AFE), we set Eqs. (FILL IN) equal to zero and require that $A = H = R = 0$. We are left with the system

$$\begin{aligned} 0 &= -\alpha S^* - \beta \xi S^* P^* + \epsilon P^* + \mu P^* \\ 0 &= \alpha S^* - \epsilon P^* - \gamma P^* - \mu P^* \\ 0 &= \gamma P^* + \beta \xi S^* P^* \end{aligned}$$

If $P = 0$, then the only solution is $S^* = P^* = H^* = R^* = 0$. Thus, will assume $P \neq 0$. This forces $\gamma + \beta \xi S^* = 0$ and since all of our parameters and variables are non-negative, then it must be $\gamma = 0$ and either $\beta = 0$ or $\xi = 0$. Under the assumption that $\gamma = 0 = \xi$ to ensure the existence of our AFE and that $1 = S + P + A + H + R$, we calculate the AFE to be

$$\begin{aligned} S^* &= \frac{\epsilon + \mu}{\alpha + \epsilon + \mu} \\ P^* &= \frac{\alpha}{\alpha + \epsilon + \mu} \\ A^* &= 0 \\ H^* &= 0 \\ R^* &= 0 \end{aligned}$$

Calculating the Basic Reproduction Number, R_0

From this point on, we will assume $\gamma = 0$ and $\xi = 0$ (thus, $\beta \neq 0$) in order to ensure the existence of the AFE. This results in the infected compartment Eqns. (FILL IN) reducing to:

$$\begin{aligned} \frac{dA}{dt} &= \sigma_A R + \beta S A - \zeta A - \theta_3 A H - (\mu + \mu_A) A \\ \frac{dH}{dt} &= \theta_1 S H + \theta_2 P H + \theta_3 A H + \sigma_H R - \nu H - (\mu + \mu_H) H \\ \frac{dR}{dt} &= \zeta A + \nu H - \delta R - \sigma_A R - \sigma_H R - \mu R \end{aligned}$$

Thus, under the assumption of A, H and R as the infected compartments and parameter restrictions stated above, the assumptions of the Next Generation Method are satisfied for matrices \mathcal{F} and \mathcal{V} shown below. Note that \mathcal{F}_i represents the rate that secondary infections enter infected compartment i and \mathcal{V}_i represents the difference between the rate of transfer out of compartment i and the rate of transfer into compartment i by means different than a secondary infection. Using this method results in the following matrices:

$$\mathcal{F} = \begin{pmatrix} 0 \\ 0 \\ \beta SA \\ \theta_1 SH + \theta_2 PH \\ 0 \end{pmatrix}$$

$$\mathcal{V} = \begin{pmatrix} \alpha S + \beta SA + \theta_1 SH - \epsilon P - \delta R - \mu(P + R + A + H) - \mu_A A - \mu_H H \\ -\alpha S + \epsilon P + \theta_2 PH + \mu P \\ -\sigma_A R + \zeta A + \theta_3 AH + (\mu + \mu_A) A \\ -\theta_3 AH - \sigma_H R + \nu H + (\mu + \mu_H) H \\ -\zeta A - \nu H + \delta R + \sigma_A R + \sigma_H R + \mu R \end{pmatrix}$$

$$F = \begin{pmatrix} \beta S^* & 0 & 0 \\ 0 & \theta_1 S^* + \theta_2 P^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \zeta + \mu + \mu_A & 0 & -\sigma_A \\ 0 & \nu + \mu + \mu_H & -\sigma_H \\ -\zeta & -\nu & \delta + \sigma_A + \sigma_H + \mu \end{pmatrix}$$

The eigenvalues of FV^{-1} are calculated to be:

$$\sigma(FV^{-1}) = \{0, \frac{(r+s) - \sqrt{(r-s)^2 + 4\beta S^* z \sigma_A \zeta \sigma_H \nu}}{2\det(V)}, \frac{(r+s) + \sqrt{(r-s)^2 + 4\beta S^* z \sigma_A \zeta \sigma_H \nu}}{2\det(V)}\}$$

\mathcal{R}_0 may then be determined as the spectral radius of FV^{-1} :

$$\mathcal{R}_0 = \frac{(r+s) + \sqrt{(r-s)^2 + 4\beta S^* z \sigma_A \zeta \sigma_H \nu}}{2\det(V)}$$

where $a = \zeta + \mu + \mu_A$, $b = \nu + \mu + \mu_H$, $c = \delta + \sigma_A + \sigma_H + \mu$, $z = \theta_1 S^* + \theta_2 P^*$, $r = \beta S^*(bc - \sigma_H \nu)$, $s = z(ac - \sigma_A \zeta)$, and $\det(V) = a(bc - \sigma_H \nu) - \sigma_A \zeta b$.

We note that the radicand $(r-s)^2 + 4\beta S^* z \sigma_A \zeta \sigma_H \nu$ is positive, since all parameters are positive. In addition, r is positive since bc contains the term that cancels with $-\sigma_H \nu$, s is positive since ac contains the term that cancels with $-\sigma_A \zeta$ and finally, $\det(V)$ is positive since abc contains terms that cancel with $-\sigma_A \zeta(\nu + \mu + \mu_H) - \sigma_H \nu$.