Module 7: Reliability

The reliability of a product (or system) can be defined as the probability that a product will perform a required function under specified conditions for a certain period of time.

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The manufacturing organisations, the government and civilian communities <u>are trying to purchase products (or systems) with higher reliability and lower maintenance costs.</u>

<u>As consumers, we are mainly concerned with buying products that last longer and are cheaper to maintain, i.e., have higher reliability.</u>

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The reliability (or) reliability function of the component at time t, denoted by $R(t)$, is defined as the probability that the component works properly at time t.

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If a component is put into operation at some specified time, say $t = 0$, and if it fails to function properly at the time T, T is called the life length or time to failure of the component.

Obviously, T is a continuous random variable with some probability density function $f(t)$.

If T is the time till the failure of the unit (a random variable) occurs, then the probability that it will not fail in a given environment before time t (or its reliability) is

$$R(t) = P(T > t) = 1 - P(T \le t) = 1 - F(t) \qquad (1)$$

Probability that system will survive at least until time t

where $F(t)$ is the cumulative distribution function of T, given by $F(t) = \int_0^t f(t)dt$.

Now, from (1), we have

$$R(t) = 1 - \int_0^t f(t)dt = \int_t^\infty f(t)dt \qquad (2)$$

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From (1) or (2) and since $F(0) = 0$ and $F(\infty) = 1$

by the property of cumulative distribution function,

we have $R(\infty) = 0$ and $R(0) = 1$.

There fore, $0 \le R(t) \le 1$.

Now, since $\dfrac{dF(t)}{dt} = f(t)$ and from (1), we get

$$f(t) = -\frac{dR(t)}{dt} = -R'(t) \qquad (3)$$

At time t = 0, the number of survivors is equal to number of items put on test. Therefore, the reliability at t = 0 is R(0) = 1 = 100% and no component will survive till the end. So R(inf)=0.

Consider the conditional probability that the system will fail in the interval (t, t + Δt), given that it has survived up to time *t*.

$$P(t < T < t + \Delta\mathrm{t} \mid T > t)$$

$$= \left[\frac{P(t < T < t + \Delta\mathrm{t})}{P(T > t)} \right]$$

$$= \left[\frac{F(t + \Delta\mathrm{t}) - F(t)}{R(t)} \right]$$

$$\boxed{\mathrm{P}(a < X \leq b) = F_X(b) - F_X(a)}$$

Hazard (or Failure) Rate Function

Hazard rate is an instantaneous failure rate at time t.

The failure rate (or, hazard function) is defined as:

$$\lambda(t) = \lim_{\Delta t \to 0} \left[\frac{P(t < T < t + \Delta \mathrm{t} \mid T > t)}{\Delta t} \right]$$

$$= \lim_{\Delta t \to 0} \left[\frac{F(t + \Delta \mathrm{t}) - F(t)}{\Delta t . R(t)} \right] = \lim_{\Delta t \to 0} \left[\frac{R(t) - R(t + \Delta \mathrm{t})}{\Delta t . R(t)} \right]$$

$$= -\frac{R^{'}(t)}{R(t)} = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)} \qquad (4)$$

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The hazard rate refers to the rate of death for an item of a given age (t), and it is also known as the failure rate.

From (4), we have followings:

$$\lambda(t) = -\frac{R'(t)}{R(t)} \qquad (5)$$

$$\lambda(t) = \frac{f(t)}{R(t)} \quad \text{(From (3))} \qquad (6)$$

and

$$f(t) = \lambda(t)R(t) \qquad (7)$$

Now, using (5) and integrating both sides with respect to t between 0 and t, we have

$$-\int_0^t \lambda(t)dt = \int_0^t \frac{R^{'}(t)}{R(t)}dt \; = \left[\log R(t)\right]_0^t$$

$$= \log R(t) \quad \{\text{since } \log R(0) = 1\}$$

This implies that, $R(t) = e^{-\int_0^t \lambda(t)dt}$

Now, from (7), we have $f(t) = \lambda(t)e^{-\int_0^t \lambda(t)dt}$ (8)

Note that from (8), $f(t)$ is determined uniquely if hazard rate $\lambda(t)$ is given.

Conditional Reliability

Conditional reliability is another concept useful to describe the reliability of a component or system following a wear-in period (burn-in period) or after a warranty period.
It is defined as

$$R(t/t_0) = P(\text{item survives further time } t \mid \text{survives to } t_0)$$

$$= P(T > t + t_0 / T > t_0)$$

$$= \frac{P(T > t + t_0)}{P(T > t_0)} = \frac{R(t_0 + t)}{R(t_0)}$$

Mean Time to Failure (MTTF)

The mean time to failure is the average or mean of the life distribution.

Mean time to failure is usually used for products that have only one life , that is, non-repairable.

Now, MTTF

$$= E(T) = \int_0^\infty t f(t) dt = \int_0^\infty t \left(-\frac{dR}{dt} \right) dt$$

$$= -\int_0^\infty t dR = -\left[(tR(t))_0^\infty - \int_0^\infty R dt \right] = \int_0^\infty R(t) dt$$

Then, MTTF $= E(T) = \int_0^\infty R(t) dt$ and

$$\text{Var}(T) = E(T^2) - [E(T)]^2 = \int_0^\infty t^2 f(t) dt - (MTTF)^2 .$$

The **design life** of a component or product is the period of time during which the item is expected by its designers to work within its specified parameters; in other words, the life expectancy of the item. It is the length of time between placement into service of a single item and that item's onset of wear out.

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Problem1: The STAR company manufactures motors. The time to failure in years of these motors has the density function $f(t) = \dfrac{200}{(t+10)^3}, \ t \geq 0.$

(i) Find the reliability function and determine the reliability for the first year of operation.

(ii) Compute the MTTF.

(iii) What is the design life for reliability 0.95?

(iv)What is the reliability function after one year of operation ?

Solution: Given that $f(t)=\dfrac{200}{(t+10)^3},\ t\geq 0$

(i) Reliability $R(t)=\int_t^\infty f(t)dt=\int_t^\infty \dfrac{200}{(t+10)^3}dt=\left[\dfrac{-100}{(t+10)^2}\right]_t^\infty=\dfrac{100}{(t+10)^2}$

Reliability for the first year of operation is $R(1)$

Therefore, $R(1)=\dfrac{100}{(1+10)^2}=0.8264.$

(ii) MTTF = $\int_0^\infty R(t)dt=\int_0^\infty \dfrac{100}{(t+10)^2}dt=\left[\dfrac{-100}{t+10}\right]_0^\infty=10$ years.

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(iii) **Design life is the time to failure** (t_D) that corresponds to a specified reliability.

Now it is required to find t_D corresponding to $R = 0.95$.

Therefore, $$\frac{100}{(t_D + 10)^2} = 0.95$$

This implies that, $$(t_D + 10)^2 = \frac{100}{0.95}$$

Therefore, $t_D = 0.2598$ years ≈ 95 days.

(iv) By definition, we know that $$R(t/t_0) = \frac{R(t_0 + t)}{R(t_0)}$$

Therefore, $$R(t/1) = \frac{R(t+1)}{R(1)} = \frac{100}{(t+11)^2} \times \frac{11^2}{10^2} = \frac{121}{(t+11)^2}$$

Example 2 *The time to failure in operating hours of a critical solid-state power unit has the hazard rate function* $\lambda l(t) = 0.003\left(\frac{t}{500}\right)^{0.5}$*, for* $t \geq 0$.

(a) *What is the reliability if the power unit must operate continuously for 50 hours?*
(b) *Determine the design ofe if a reliability of 0.90 is desired.*
(c) *Compute the MTTF.*
(d) *Given that the unit has operated for 50 hours, what is the probability that it will survive a second 50 hours of operation?*

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Solution

$$\text{(a)}\quad R(t) = \exp\left[-\int_0^t \lambda(t)\,dt\right]$$

$$\therefore \qquad R(50) = \exp\left[-\int_0^{50} 0.003\left(\frac{t}{500}\right)^{0.5} dt\right]$$

$$= \exp\left[-\frac{0.003}{\sqrt{500}}\cdot\frac{2}{3}t^{3/2}\right]_0^{50}$$

$$= \exp\left[-\frac{0.003}{\sqrt{500}}\times\frac{2}{3}\times 50\sqrt{50}\right]$$

$$= \exp[-0.03162]$$

$$= 0.9689$$

(b) $R(t_D) = 0.90$

i.e., $$\exp\left[-\int_0^{t_D} 0.003\left(\frac{t}{500}\right)^{0.5} dt\right] = 0.90$$

i.e., $$-\int_0^{t_D} \frac{0.003}{\sqrt{500}} t^{1/2} dt = -0.10536$$

i.e., $$\frac{0.003}{\sqrt{500}} \times \frac{2}{3} t_D^{3/2} = 0.10536$$

$\therefore$ $$t_D = \left\{\frac{3 \times \sqrt{500} \times 0.10536}{2 \times 0.003}\right\}^{2/3} = 111.54 \text{ hours.}$$

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(c) $\text{MTTF} = \int_0^\infty R(t)\,dt$

$$= \int_0^\infty e^{-\left(\frac{0.003}{\sqrt{500}}\times\frac{2}{3}\times t^{3/2}\right)} dt$$

$$= \int_0^\infty e^{-at^{3/2}}\,dt, \text{ where } a = \frac{0.003\times 2}{3\times\sqrt{500}}$$

$$= \int_0^\infty e^{-x}.\frac{2}{3a^{2/3}}\,x^{-1/3}\;dx, \text{ on putting } x = at^{3/2}$$

$$= \frac{2}{3a^{2/3}}\,\Gamma(2/3) = \frac{2}{3a^{2/3}}\,\frac{3}{2}\,\Gamma(5/3)$$

$$= \frac{0.9033}{a^{2/3}}, \text{ from the table of values of Gamma function.}$$

$$= 45.65 \text{ hours.}$$

(d) $P(T \geq 100/T \geq 50) = \dfrac{P(T \geq 100)}{P(T \geq 50)} = \dfrac{R(100)}{R(50)}$

$$= \exp\left[\geq \int_{50}^{100} \lambda(t)dt \right]$$

$$= \exp\left[\left\{ -\frac{0.002}{\sqrt{500}} \times 100^{3/2} \right\} - \left\{ \frac{-0.002}{\sqrt{500}} \times 50^{3/2} \right\} \right]$$

$$= \exp[\{- 0.08944\} - \{- 0.03162\}]$$

$$= 0.9438$$

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Special Failure Distributions

If the time to failure T follows an exponential distribution with parameter λ

then

$$MTTF = E(T) = \frac{1}{\lambda}$$

$$R(t) = e^{-\lambda t}$$

$$\mathrm{Var}(T) = \sigma_T^2 = \frac{1}{\lambda^2}$$

$$R(t/T_0) = \frac{R(T_0 + t)}{R(T_0)} = \frac{e^{-\lambda(T_0+t)}}{e^{-\lambda T_0}}$$
$$= e^{-\lambda t}$$

A manufacturer determines that, on the average, a television set is used 1.8 hours per day. A one-year warranty is offered on the picture tube having a MTTF of 2000 hours. If the distribution is exponential, what percentage of the tubes will fail during the warranty period?

Solution

Since the distribution of the time to failure of the picture tube is exponential,

$$R(t) = e^{-\lambda t} \text{ where } \lambda \text{ is the failure rate}$$

Given that

$$\text{MTTF} = 2000 \text{ hours}$$

i.e.,

$$\int_0^\infty e^{-\lambda t}\, dt = 2000$$

i.e.,

$$\frac{1}{\lambda} = 2000 \text{ or } \lambda = 0.0005/\text{hour}$$

$$P(T \leq 1 \text{ year}) = P(T \leq 365 \times 1.8 \text{ hours}) \; [\because \text{ the T.V. is operated for 1.8 hours/day}]$$

$$= 1 - P\{T > 657\}$$

$$= 1 - R(657)$$

$$= 1 - e^{-0.0005 \times 657}$$

$$= 0.28$$

i.e., 28% of the tubes will fail during the warranty period.

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Example 3 *The reliability of a turbine blade is given by* $R(t) = \left(1 - \frac{t}{t_0}\right)^2, 0 \le t \le t_0$, *where* t_0 *is the maximum life of the blade.*

(a) *Show that the blades are experiencing wear out.*

(b) *Compute MTTF as a function of the maximum life.*

(c) *If the maximum life is 2000 operating hours, what is the design life for a reliability of 0.90?*

Solution

(a) $R(t) = \left(1 - \frac{t}{t_0}\right)^2, 0 \le t \le t_0$

Now $$\lambda(t) = -\frac{R'(t)}{R(t)}$$

$$= -\left(1 - \frac{t}{t_0}\right)^{-2}\left\{-\frac{2}{t_0}\left(1 - \frac{t}{t_0}\right)\right\}$$

$$= \frac{2}{t_0 - t}$$

$$\lambda'(t) = \frac{2}{(t_0 - t)^2} > 0 \text{ and so } \lambda(t) \text{ is an increasing function of } t.$$

When the failure rate increases with time, it indicates that the blades are experiencing wear out.

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(b) $$\text{MTTF} = \int_0^{\infty} R(t)dt = \int_0^{t_0} \left(1 - \frac{t}{t_0}\right)^2 dt$$

$$= \left[-\frac{t_0}{3}\left(1 - \frac{t}{t_0}\right)^3\right]_0^{t_0} = \frac{t_0}{3}$$

(c) When $t_0 = 2000$, $R(t_D) = 0.90$

i.e., $$\left(1 - \frac{t_D}{2000}\right)^2 = 0.90$$

$\therefore$ $$1 - \frac{t_D}{2000} = 0.9487$$

$\therefore$ $t_D = 102.63$ hours.

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Weibull Distribution

$$R(t) = \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

$$\text{MTTF} = E(T) = \theta\Gamma\left(1+\frac{1}{\beta}\right)$$

$$\text{Var}(T) = \sigma_T^2 = \theta^2\left\{\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}$$

$$R(t/T_0) = \frac{R(t+T_0)}{R(T_0)}$$

$$= \frac{\exp\left[-\left(\frac{t+T_0}{\theta}\right)^{\beta}\right]}{\exp\left[-\left(\frac{T_0}{\theta}\right)^{\beta}\right]}$$

$$= \exp\left[-\left(\frac{t+T_0}{\theta}\right)^{\beta}+\left(\frac{T_0}{\theta}\right)^{\beta}\right]$$

Example 9 *For a system having a Weibull failure distribution with a shape parameter of 1.4 and a scale parameter of 550 days, find*

(a) R(100 days); (b) the B1 life; (c) MTTF; (d) the standard deviation, (e) the design life for a reliability of 0.90.

Solution

The *pdf* of the Weibull distribution is given by $f(t) = \frac{\beta}{\theta} \cdot \left(\frac{t}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\theta}\right)^{\beta}\right\}$, $t \geq 0$. Now $\beta = 1.4$ and $\theta = 550$ days

(a)
$$R(t) = \exp\left\{-\left(\frac{t}{\theta}\right)^{\beta}\right\}$$

$$\therefore \quad R(100) = \exp\left\{-\left(\frac{100}{550}\right)^{1.4}\right\} = 0.9122$$

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(b)

Note: The life time corresponding to a reliability of 0.99 is called the $B1$ life. Similarly that corresponding to $R = 0.999$ is called the $B.1$ life.

Let t_R be the $B1$ life of the system

Then $$R(t_R) = 0.99$$

i.e., $$\exp\left\{-\left(\frac{t_R}{550}\right)^{1.4}\right\} = 0.99$$

$\therefore$ $$\left(\frac{t_R}{550}\right)^{1.4} = 0.01005$$

$$\therefore \quad t_R = 550 \times (0.01005)^{\frac{1}{1.4}} = 20.6 \text{ days}$$

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(c) $\text{MTTF} = \theta \cdot \Gamma\left(1+\frac{1}{\beta}\right) = 550 \times \Gamma\left(1+\frac{1}{1.4}\right) = 550 \times \Gamma(1.71)$

$$= 550 \times 0.91057, \text{ from the Gamma tables}$$

$$= 500.8 \text{ days}$$

(d) $$(\text{S.D.})^2 = \theta^2 \left\{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}$$

$$= 550^2 \left[\Gamma\left(1+\frac{2}{1.4}\right) - \left\{\Gamma\left(1+\frac{1}{1.4}\right)\right\}^2\right]$$

$$= 550^2 \left[\Gamma(2.43) - \left\{\Gamma(1.71)\right\}^2\right]$$

$$= 550^2 \left[1.43 \times \Gamma(1.43) - \left\{\Gamma(1.71)\right\}^2\right]$$

$$= 550^2 \left[1.43 \times 0.88604 \times (0.91057)^2\right]$$

$\therefore$ $$\text{S.D} = 550 \times 0.66174 = 363.96 \text{ days}$$

(e) Let t_D be the required design life for $R = 0.90$

$$\therefore \qquad R(t_D) = \exp\left\{-\left(\frac{t_D}{550}\right)^{1.4}\right\} = 0.90$$

$$\therefore \qquad \left(\frac{t_D}{550}\right)^{1.4} = 0.10536$$

$$\therefore \qquad t_D = 550 \times (0.10536)^{\frac{1}{1.4}} = 110.2 \text{ days}$$

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Example 10 *A device has a decreasing failure rate characterised by a two-parameter Wibull distribution with $\theta = 180$ years and $\beta = 0.5$. The device is required to have a design life reliability of 0.90.*

(a) What is the design life, if there is no wear-in period?

(b) What is the design life, if there is a wear-in period of 1 month in the beginning?

Answer

(a) Let t_D be the required design life for $R = 0.90$

$$\therefore \qquad R(t_D) = \exp\left\{-\left(\frac{t_D}{180}\right)^{0.5}\right\} = 0.90$$

$$\therefore \qquad \left(\frac{t_D}{180}\right)^{0.5} = 0.10536$$

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$$\therefore\ t_D = 180 \times (0.10536)^2 = 1.998 \simeq 2 \text{ years.}$$

(b) If T_0 is the wear-in period, then

$$R(t/T_0) = \exp\left\{-\left(\frac{t+T_0}{\theta}\right)^{\beta} + \left(\frac{T_0}{\theta}\right)^{\beta}\right\}$$

Let t_W be the required design life with wear-in.

Then $$\exp\left\{-\left(\frac{t_W + \frac{1}{12}}{180}\right)^{0.5} + \left(\frac{\frac{1}{12}}{180}\right)^{0.5}\right\} = 0.90$$

i.e., $$-\left(\frac{12t_W + 1}{12 \times 180}\right)^{0.5} + \left(\frac{1}{12 \times 180}\right)^{0.5} = -\,0.10536$$

i.e., $$\left(\frac{12t_W + 1}{12 \times 180}\right)^{0.5} = 0.12688$$

$\therefore$ $$12t_W + 1 = 12 \times 180 \times 0.01610 = 34776$$

i.e., $$t_W = 2.18 \text{ years}$$

Thus a wear-in period of 1 month increases the design life by nearly 0.81 year or 10 months.

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