SCHOOL OF ADVANCED SCIENCES
DEPARTMENT OF MATHEMATICS
FALL SEMESTER – 2020~2021

MAT2001 – Statistics for Engineers
(Embedded Theory Component)

COURSE MATERIAL

Module 5
Hypothesis Testing – I

Syllabus:
Testing of Hypothesis – Introduction – Types of Errors – Critical Region – Procedure of Testing Hypothesis – Large Sample Tests – Z-Test for Single Proportion, Difference of Proportions, Single Mean and Difference of Means.

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Outline

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Introduction

The method of hypothesis testing uses tests of significance to determine the likelihood that a statement (often related to the mean or variance of a given distribution) is true, and at what likelihood we would, as statisticians, accept the statement as true. While understanding the mathematical concepts that go into the formulation of these tests is important, knowledge of how to appropriately use each test (and when to use which test) is equally important.

- A **Statistical hypothesis** is a conjecture about a population parameter. This conjecture may or may not be true.
- The **null hypothesis**, symbolized by H_0, is a statistical hypothesis that states that there is no difference between a parameter and a specific value or that there is no difference between two parameters.
- The **alternative hypothesis**, symbolized by H_1, is a statistical hypothesis that states a specific difference between a parameter and a specific value or states that there is a difference between two parameters.
  In other words, we can say H_1 is complementary to H_0.

Type of Tests

Two-Tailed Test:

- A medical researcher is interested in finding out whether a new medication will have any undesirable side effects. The researcher is particularly concerned with the pulse rate of the patients who take the medication.
- What are the hypotheses to test whether the pulse rate will be different from the mean pulse rate of 82 beats per minute?
- $H_0 : \mu = 82$ and $H_1 : \mu \neq 82$.

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- $H_0 : \mu = 82$ and $H_1 : \mu \neq 82$.
- This is a **Two-Tailed** test.

Right-Tailed Test:

- A chemist invents an additive to increase the life of an automobile battery. If the mean lifetime of the battery is 36 months, then his hypotheses are
- $H_0 : \mu = 36$ and $H_1 : \mu > 36$

Right-Tailed Test:

- A chemist invents an additive to increase the life of an automobile battery. If the mean lifetime of the battery is 36 months, then his hypotheses are
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- This is a **Right-Tailed** test.

Left-Tailed Test:

- A contractor wishes to lower heating bills by using a special type of insulation in houses. If the average of the monthly heating bills is Rs.78, her hypotheses about heating costs will be
- $H_0 : \mu = Rs.78$ and $H_1 : \mu < Rs.78$

Right-Tailed Test:

- A chemist invents an additive to increase the life of an automobile battery. If the mean lifetime of the battery is 36 months, then his hypotheses are
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Left-Tailed Test:

- A contractor wishes to lower heating bills by using a special type of insulation in houses. If the average of the monthly heating bills is Rs.78, her hypotheses about heating costs will be
- $H_0 : \mu = Rs.78$ and $H_1 : \mu < Rs.78$
- This is a **Left-Tailed** test.

- A **test statistic** uses the data obtained from a sample to make a decision about whether or not the null hypothesis should be rejected.
- The numerical value obtained from a test statistic is called the **calculated value**.

Errors in Hypothesis Testing:

- A **Type I error** occurs if one rejects the null hypothesis when it is true. This is similar to a good product being rejected by the consumer and hence Type I error is also known as *producer's risk*.

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- The probability of committing Type I error is the **level of significance** (LOS) which is denoted by α. Typical significance levels are: 0.10, 0.05, 0.02 and 0.01.
- A **Type II error** occurs if one does not reject the null hypothesis when it is false. As this error is similar to that of accepting a product of inferior quality, it is known as *consumer's risk.* The probability of committing Type II error is denoted by β.

The critical values for some standard LOS's are given in the following table:

Table 1:

Type of Test	$\alpha = 1\%(0.01)$	$\alpha = 2\%(0.02)$	$\alpha = 5\%(0.05)$	$\alpha = 10\%(0.1)$
Two-Tailed	$\|z_\alpha\| = 2.58$	$\|z_\alpha\| = 2.33$	$\|z_\alpha\| = 1.96$	$\|z_\alpha\| = 1.645$
Right-Tailed	$z_\alpha = 2.33$	$z_\alpha = 2.055$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left-Tailed	$z_\alpha = -2.33$	$z_\alpha = -2.055$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

Critical/Rejection Region:

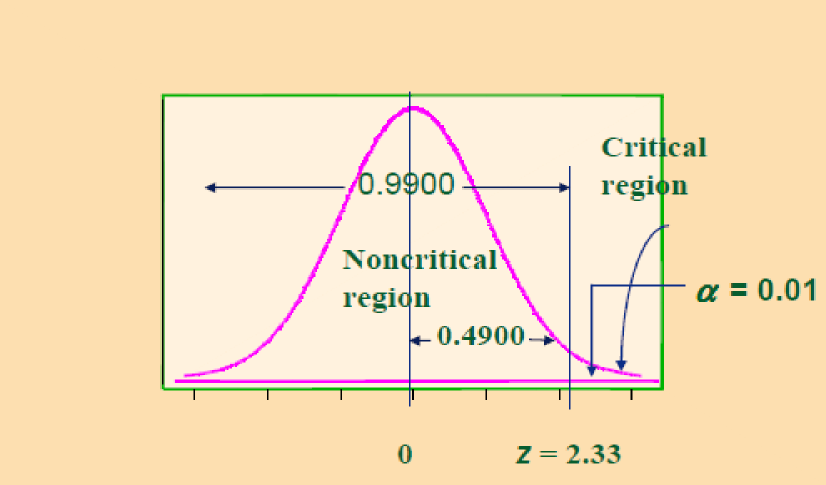
The **critical** or **rejection region** is the range of values of the calculated value that indicates that there is a significant difference and that the null hypothesis should be rejected.

Noncritical/Nonrejection Region:

- The region complementary to the critical region is called the **noncritical** or **nonrejection region**.
- The critical value separates the critical region from the noncritical region.

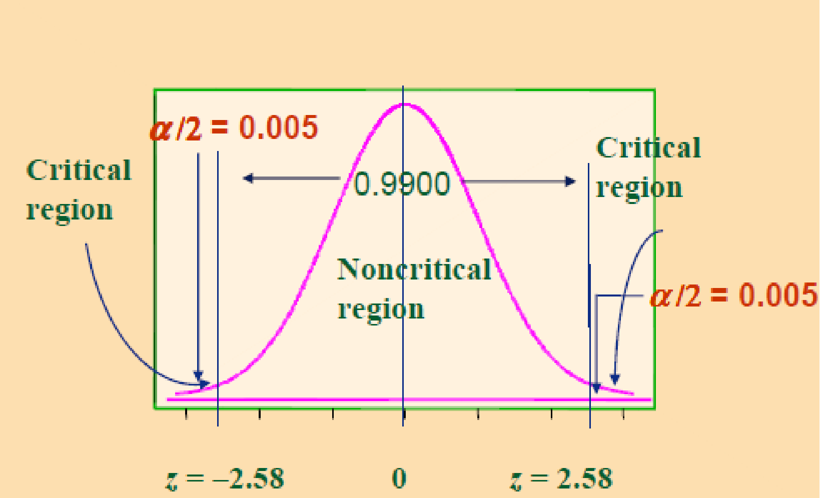
A one-tailed test (right or left) indicates that the null hypothesis should be rejected when the calculated value is in the critical region.

Critical value for $\alpha = 0.01$ (Right-Tailed Test):



In a two-tailed test, the null hypothesis should be rejected when the calculated value is in either of the two critical regions.

Critical value for $\alpha = 0.01$ (Two-Tailed Test):



Large Sample:

- If the sample size n is greater than or equal to 30 ($n \geq 30$), the sample is called a **Large Sample**.
- The z-test is a statistical test for the mean of a population. It can be used for large sample or when the population is normally distributed and σ is known.

Procedure for Hypothesis Testing

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Step 4: Choose appropriate formula and calculate test statistic, that is, z-value.
Step 5: Comparision and Conclusion.

- If $|z| < |z_\alpha|, H_0$ is accepted or H_1 is rejected,
  that is, there is no significant difference at $\alpha\%$ LOS.
- If $|z| > |z_\alpha|, H_0$ is rejected or H_1 is accepted,,
  that is, there is significant difference at $\alpha\%$ LOS.

Outline

Case 1: Test of significance of the difference between sample proportion and population proportion.

P - Population Proportion

p - Sample Proportion

n - Sample Size

(i) Test Statistic $z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}}$, where $Q = 1 - P$.

(ii) When P is not known, the 95 percent confidents limits for P are given by

$$p - 1.96\sqrt{\frac{pq}{n}} \leq P \leq p + 1.96\sqrt{\frac{pq}{n}}$$

Problem 1:

The CEO of a large electric utility claims that atleast 80 percent of his 10,00,000 customers are very satisfied with the service they receive. To test this claim, the local newspaper surveyed 100 customers, using simple random sampling. Among the sampled customers, 73 percent say they are very satisified. Based on these findings, can we reject the CEO's hypothesis that 80 percent of the customers are very satisfied? Use a 0.05 level of significance.

Solution:

In this problem, $P = 0.8, p = 0.73$ and $n = 100$.

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Alternative hypothesis $(H_1) : P < 0.80$

Solution:

In this problem, $P = 0.8, p = 0.73$ and $n = 100$.
Step 1: Null hypothesis $(H_0) : P \geq 0.80$, that is, atleat 80 percent customers are satisfied.
Alternative hypothesis $(H_1) : P < 0.80$
Step 2: Note that these hypotheses constitute a one-tailed (left-tailed) test.

Solution:

In this problem, $P = 0.8, p = 0.73$ and $n = 100$.

Step 1: Null hypothesis $(H_0) : P \geq 0.80$, that is, atleat 80 percent customers are satisfied.

Alternative hypothesis $(H_1) : P < 0.80$

Step 2: Note that these hypotheses constitute a one-tailed (left-tailed) test.

Step 3: As it is given $\alpha = 0.05$ and one-tailed test, we have $z_\alpha = -1.645$.

Solution:

In this problem, $P = 0.8, p = 0.73$ and $n = 100$.

Step 1: Null hypothesis $(H_0) : P \geq 0.80$, that is, atleat 80 percent customers are satisfied.

Alternative hypothesis $(H_1) : P < 0.80$

Step 2: Note that these hypotheses constitute a one-tailed (left-tailed) test.

Step 3: As it is given $\alpha = 0.05$ and one-tailed test, we have $z_\alpha = -1.645$.

Step 4: Test Statistic $z = \dfrac{p-P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{0.73-0.8}{\sqrt{\dfrac{0.8 \times 0.2}{100}}} = -1.75.$

Solution:

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Step 5: It can be viewed that $|z| = 1.75 > 1.645 = |z_\alpha|$. Therefore, we reject the null hypothesis H_0, that is, H_1 is accepted.

Solution:

In this problem, $P = 0.8, p = 0.73$ and $n = 100$.

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$\therefore$ The CEO's claim is wrong.

Problem 2:

Experience has showen that 20 percent of a manufactured product is of top quality. In one day's production 400 articles, only 50 are of top quality. Show that either the production of the day chosen was not a representative sample or the hypothesis 20 percent was strong.

Solution:

Step 1: Null Hypothesis $(H_0) : P = \dfrac{1}{5}$, that is, 20 percent of the products manufactured is of top quality.

Solution:

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Solution:

Step 1: Null Hypothesis $(H_0) : P = \frac{1}{5}$, that is, 20 percent of the products manufactured is of top quality.

Alternative Hypothesis $(H_1) : P \neq \frac{1}{5}$, that is, 20 percent of the products manufactured is not of top quality.

Step 2: Assume that $\alpha = 5\%$ and one can note that the type of test is two-tailed based on H_1.

Solution:

Step 1: Null Hypothesis $(H_0) : P = \frac{1}{5}$, that is, 20 percent of the products manufactured is of top quality.

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Step 2: Assume that $\alpha = 5\%$ and one can note that the type of test is two-tailed based on H_1.

Step 3: Since it is two-tailed test and $\alpha = 5\%, z_\alpha = 1.96$.

Solution:

Step 1: Null Hypothesis $(H_0) : P = \frac{1}{5}$, that is, 20 percent of the products manufactured is of top quality.

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Step 4: Test Statistic $= z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{\dfrac{1}{8} - \dfrac{1}{5}}{\sqrt{\dfrac{1}{5} \times \dfrac{4}{5} \times \dfrac{1}{400}}} = -3.75.$

Solution:

Step 1: Null Hypothesis $(H_0) : P = \frac{1}{5}$, that is, 20 percent of the products manufactured is of top quality.

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Step 5: Now, $|z| = 3.75 > 1.96 = |z_\alpha|$ which implies that H_0 is rejected (or H_1 is accepted).

Solution:

Step 1: Null Hypothesis $(H_0) : P = \frac{1}{5}$, that is, 20 percent of the products manufactured is of top quality.

Alternative Hypothesis $(H_1) : P \neq \frac{1}{5}$, that is, 20 percent of the products manufactured is not of top quality.

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Step 5: Now, $|z| = 3.75 > 1.96 = |z_\alpha|$ which implies that H_0 is rejected (or H_1 is accepted).

$\therefore$ The production of the particular day chosen was not a representative sample.

Problem 3:

A recent article in a weekly magazine reported that a job awaits 33% of new college graduates. A survey of 200 recent graduates from your college revealed that 80 students had jobs. At a 99% level of confidence, can we conclude that a larger proportion of students at your college have jobs?

Solution:

Step 1: $H_0 : P = 0.33$ and $H_1 : P > 0.33$
Step 2: It is one-tailed (right-tailed) test.
Step 3: Here, $\alpha = 1\%$. So $z_\alpha = 2.33$.
Step 4:

$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{0.4 - 0.33}{\sqrt{0.33 \times 0.67 \times \frac{1}{200}}} = 2.1021.$$

Step 5: $|z| = 2.1021 < 2.33 = |z_\alpha|$ implies that H_0 is not rejected. Therefore, we do not have enough evidence to state that a larger proportion of students at our college have jobs.

Exercise:

1. A salesman in a departmental store claims that at most 60 percent of the shoppers entering the store leaves without making a purchase. A random sample of 50 shoppers showed that 35 percent of them left without making a purchase. Are these sample results consistent with the claim of the salesman at a level of significance of 0.05?
2. A cubical die is thrown 900 times and a throw of three or four is observed 3240 times. Show that the die cannot be regarded as unbiased one and find the extreme limits between which the probability of a throw of three or four lies.

Outline

Case 2: Test of significance of the difference between two sample proportions.

$$\text{Test Statistic } z = \frac{(p_1 - p_2)}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where $P = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$ and $Q = 1 - P$.

Problem 4:

Suppose the RK Drug Company develops a new drug, designed to prevent Covid19. The company states that the drug is more effective for women than for men. To test this claim, they choose a a simple random sample of 100 women and 200 men from a population of 100,000 volunteers.

At the end of the study, 38% of the women caught a Covid19 and 51% of the men caught a Covid19. Based on these findings, can we conclude that the drug is more effective for women than for men? Use a 0.01 level of significance.

Solution:

Assume that p_1 represents the effectiveness of drug on women and p_2 represents the effectiveness of drug on men. In this problem, $n_1 = 100, p_1 = 0.38, n_2 = 200$ and $p_2 = 0.51$.

Solution:

Assume that p_1 represents the effectiveness of drug on women and p_2 represents the effectiveness of drug on men. In this problem, $n_1 = 100, p_1 = 0.38, n_2 = 200$ and $p_2 = 0.51$.

Step 1: Null hypothesis (H_0): $p_1 \geq p_2$

Alternative hypothesis (H_1): $p_1 < p_2$.

Solution:

Assume that p_1 represents the effectiveness of drug on women and p_2 represents the effectiveness of drug on men. In this problem, $n_1 = 100, p_1 = 0.38, n_2 = 200$ and $p_2 = 0.51$.

Step 1: Null hypothesis (H_0): $p_1 \geq p_2$

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Step 2: Note that these hypotheses constitute a left-tailed test.

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Assume that p_1 represents the effectiveness of drug on women and p_2 represents the effectiveness of drug on men. In this problem, $n_1 = 100, p_1 = 0.38, n_2 = 200$ and $p_2 = 0.51$.

Step 1: Null hypothesis (H_0): $p_1 \geq p_2$

Alternative hypothesis (H_1): $p_1 < p_2$.

Step 2: Note that these hypotheses constitute a left-tailed test.

Step 3: As we have left-tailed test and the significance level is 0.01, $z_\alpha = -2.33$.

Solution:

Assume that p_1 represents the effectiveness of drug on women and p_2 represents the effectiveness of drug on men. In this problem, $n_1 = 100, p_1 = 0.38, n_2 = 200$ and $p_2 = 0.51$.

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Step 4: Test Statistic $(z) = \dfrac{(p_1 - p_2)}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$,

where $P = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$.

Solution:

Assume that p_1 represents the effectiveness of drug on women and p_2 represents the effectiveness of drug on men. In this problem, $n_1 = 100, p_1 = 0.38, n_2 = 200$ and $p_2 = 0.51$.

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Step 4: Test Statistic $(z) = \dfrac{(p_1 - p_2)}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$,

where $P = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$.

By simple calculation, one can note that $P = 0.467$ and $Q = 0.533$ which yields $z = -2.13$.

Step 5: Comparing z and z_α values, we obtain

$$|z| = 2.13 < 2.33 = |z_\alpha|$$

which shows that H_0 is accepted. That is, the claim is true.

Exercise:

(1) Consider the previous problem (ie, problem 4) with same data and test the claim that the drug is equally effective for men and women.

Problem 5:

In a large city A, 20 percent of a random sample of 900 school days had a slight physical defect. In another large city B, 18.5 percent of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant? Use the level of significance 5%.

Solution:

Given: $n_1 = 900, p_1 = 0.2, n_2 = 1600, p_2 = 0.185$,

Step 1: $H_0 : p_1 = p_2$

$H_1 : p_1 \neq p_2$.

Step 2: It is a two-tailed test.

Step 3: Since it is two-tailed and $\alpha = 0.05, z_\alpha = 1.96$.

Step 4: Test Statistic:

$$z = \frac{(p_1 - p_2)}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \text{ where } P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}.$$

This implies that $z = \dfrac{(0.2 - 0.185)}{\sqrt{(0.1904)(0.8096)\left(\frac{1}{900} + \frac{1}{1600}\right)}} = 0.92$.

Step 5: Clearly, $|z| < |z_\alpha|$. Thus, H_0 is accepted, that is, H_1 is rejected.

Problem 6:

956 children were born in a city A in one year out of which 52.5% were male, while 1406 children were born in cities A and B both out of which proportion of male was 0.496. Is the difference in the proportion of male children in two cities significant?

Solution:

Here, $n_1 = 956, p_1 = 52.5\%, n_2 = 1406 - 956 = 450$ and $P = 0.496 \Rightarrow Q = 1 - P = 0.504$.
We have,

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} \Rightarrow 0.496 = \frac{(956)(0.525) + (450)(p_2)}{956 + 450} \Rightarrow p_2 = 0.432$$

Step 1: $H_0 : p_1 = p_2$ and $H_1 : p_1 \neq p_2$.
Step 2: Two-tailed Test will be used in this case.
Step 3: Let $\alpha = 5\%$, then $z_\alpha = 1.96$. as the type of the test is two-tailed.
Step 4: Test Statistic:

$$z = \frac{(p_1 - p_2)}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = 3.268.$$

Step 5: Note that $|z| > |z_\alpha|$ which shows that H_1 is accepted.

Exercise:

1. Before an increase in excise duty on tea, 800 people out of a sample of 1000 were consumers of tea. After the increase in duty, 800 people were consumers of tea in a sample of 1200 persons. Find whether there is significant decrease in the consumption of tea after the increase in duty.
2. 15.5 percent of a random sample of 1600 undergraduates were smokers, whereas 20 percent of a random sample 900 postgraduates were smokers in a state. Can we conclude that less number of undergraduates were smokers than the postgraduates?
3. One thousand articles from a factory are examined and 30 were found defective. 1500 similar articles from the second factory were examined where 300 were found defective. Can it be reasonably concluded that the products of the first factory are inferior to the second?

Outline

Case 3: Test of significance of the difference between sample mean and population mean.

$\overline{x}$ - Sample Mean, n - Sample Size, μ - Population Mean and σ - Population Standard Deviation (S.D.)

$$\text{Test Statistic } z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Note: If σ is not known, the sample S.D. 's' can be used as 's' tends to σ when n is sufficiently large.

Problem 7:

An insurance agent has claimed that the average age of policy holders who insure through him is less than the average for all agents which is 30.5 years. A random sample of 100 policy holders who had insured through him reveal that the mean and S.D. are 28.8 years and 6.35 years respectively. Test his claim at 5% level of significance.

Solution:

Here, $n = 100, \mu = 30.5, s = 6.35, \overline{x} = 28.8$ and $\alpha = 5\%$.

Step 1: Null Hypothesis $(H_0) : \mu = 30.5$

Alternative Hypothesis $(H_0) : \mu < 30.5$

Step 2: Type of test is one-tailed (left-tailed).

Step 3: Note that $z_\alpha = -1.645$ as it is one-tailed and $\alpha = 0.05$.

Step 4: The test statistic:

$$z = \frac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{28.8 - 30.5}{6.23/10} = -2.68.$$

Step 5: The computed value $z = -2.68$ falls in the critical/rejection area, that is, $|z| > |z_\alpha|$. So we reject H_0.

Problem 8:

A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160 cm. Can it be reasonably regarded that, in the population, the mean height is 165 cm and S.D. is 10 cm? LOS is 1%.

Solution:

Here, $n = 160, \overline{x} = 160, \mu = 165$ and $\sigma = 10$.

Step 1: $H_0 : \overline{x} = \mu$ and $H_1 : \overline{x} \neq \mu$.

Step 2: In this case, we will use two-tailed test based on H_1.

Step 3: Type of test and LOS value imply that $|z_\alpha| = 2.58$.

Step 4: The test statistic:

$$z = \frac{\overline{x} - \mu}{\sigma\sqrt{n}} = \frac{160 - 165}{10/\sqrt{100}} = 5.$$

Step 5: Comapring the tabulated and calculated values of z, we have $|z| > |z_\alpha|$. That is, we reject H_0 at $\alpha = 0.01$.

Problem 9:

The guaranteed average life of a certain type of electric light bulbs is 1000 hours with a S.D. of 125 hours. It is decided to sample the output so as to ensure that 90 percent of the bulbs do not fall short of the guaranteed average by more than 2.5 percent. What must be the minimum size of the sample?

Solution:

Given: $\mu = 1000$ hours and $\sigma = 125$ hours.
Since we do not want the sample mean to be less than the guaranteed average, ie, $\mu = 1000$, by more than 2.5 percent, we have,

$$\overline{x} > 1000 - 2.5\% \text{ of } 1000 = 1000 - 25 = 975.$$

Let n be the size of the sample then

$$z = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}} \text{ follows } N(0,1), \text{ as sample is sufficiently large.}$$

We want

$$z = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}} > \frac{975 - 1000}{125/\sqrt{n}} > -\frac{\sqrt{n}}{5}.$$

According to the given condition, we have

$$P(z > -\sqrt{n}/5) = 0.90 \Rightarrow P(0 < z < -\sqrt{n}/5) = 0.40$$

$\therefore \sqrt{n}/5 = 1.28$ (From Normal Probability Table) which implies $n = 41$ approximately.

Excercise :

1. A sample of 900 members has a mean 3.4 cm and S.D. 2.61 cm. Is the sample from a large population of mean 3.25 cm and S.D. of 2.61 cm? (Test at 5% level of significance.)
2. A normal population has a mean of 6.48 and S.D. of 1.5. In a sample of 400 members, mean is 6.75. Is the difference significant?
3. A principal at a college claims that the students in his college are above average intelligence. A random sample of 30 students' IQ level scores have a mean score of 112.5. Is there sufficient evidence to support the principal's claim? The mean population IQ is 100 with a standard deviation of 15. LOS is 5%.
4. The average number of defective articles per day in a certain factory is claimed to be less than the average of all the factories. The average of all factories is 30.5.

A random sample of 100 days showed the following distribution:

Class limits:	16 – 20	21 – 25	26 – 30	31 – 35	36 – 40
No. of days:	12	22	20	30	16

Is the average less than the figure for all the factories? LOS is 1%.

Outline

Case 2: Test of significance of the difference between two sample means.

$$\text{Test Statistic } z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Note:

1. If the samples are drawn from the same population, that is, $\sigma = \sigma_1 = \sigma_2$, then

$$z = \frac{\overline{x_1} - \overline{x_2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

2. If σ_1 and σ_2 are not known, then σ_1 and σ_2 can be approximated by the sample S.D.'s s_1 and s_2. Now,

$$z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Problem 10:

The means of two large samples of 1000 and 2000 members are 67.5 inches and 68 inches respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 inches?

Solution:

Given: $n_1 = 1000, \overline{x_1} = 67.5, n_2 = 2000, \overline{x_2} = 68$ and $\sigma_1 = \sigma_2 = 2.5$.

Step 1: $H_0 : \overline{x_1} = \overline{x_2}$ and $H_1 : \overline{x_1} \neq \overline{x_2}$.

Step 2: Here, type of test is two-tailed.

Step 3: At $\alpha = 5\%$, tabulated value $z_\alpha = 1.96$ for two-tailed.

Step 4: The test statistic is given by

$$z = \frac{\overline{x_1} - \overline{x_2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{67.5 - 68}{2.5\sqrt{\frac{1}{1000} + \frac{1}{2000}}} = -5.16.$$

Step 5: Note that $|z| > |z_\alpha|$ which shows that H_0 is rejected.

Problem 11:

The average marks scored by 32 boys is 72 with a S.D. of 8, while that for 36 girls is 70 with a S.D. of 6. Test at 1% level of significance whether the boys perform better than girls.

Solution:

Here, $n_1 = 32, \overline{x_1} = 72, s_1 = 8, n_2 = 36, \overline{x_2} = 70$ and $s_2 = 6$.

Step 1: $H_0 : \mu_1 = \mu_2$
$H_1 : \mu_1 > \mu_2$.

Step 2: Use one-tailed (right-tailed) test.

Step 3: The tabulated $z = 2.33$ at $\alpha = 0.01$ when the nature of test is right-tailed.

Step 4: The test statistic is given by

$$z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{72 - 70}{\sqrt{\frac{8^2}{32} + \frac{6^2}{36}}} = 1.15.$$

Step 5: Since $|z| = 1.15 < 2.33 = |z_\alpha|, H_0$ is accepted.

$\therefore$ The boys do not perform better than girls.

Problem 12:

The mean height of 50 male students who showed above average participation in college atheletics was 68.2 inches with a S.D. of 2.5 inches while 50 male students showed no interest in such participation had a mean height of 67.5 inches with a S.D. of 2.8 inches.

(a) Test the hypothesis that male students who participate in college athletics are taller than other male students

(b) By how much should the sample size of each of the two groups be increased in order that the observed difference of 0.7 inches in the mean heights be significant at the 5% level of significance.

Solution:

In this problem, $n_1 = 50, \overline{x_1} = 68.2, s_1 = 2.5, n_2 = 50$, $\overline{x_2} = 67.5, s_2 = 2.8$.

(a) **Step 1:** $H_0 : \overline{x_1} = \overline{x_2}$
$H_1 : \overline{x_1} > \overline{x_2}$.

Step 2: The nature of the test is one-tailed (right-tailed) according to H_1.

Step 3: Observe that $z_\alpha = 1.645$ at $\alpha = 5\%$.

Step 4: The test statistic is

$$z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{68.2 - 67.5}{\sqrt{\frac{(2.5)^2}{50} + \frac{(2.8)^2}{50}}} = 1.32.$$

Step 5: As $|z| = 1.32 < 1.645 = |z_\alpha|$, we accept H_0, that is, we conclude that the college athletes are not taller than other male students.

(b) The difference between the mean heights of two groups, each of size n will be significant at 5% level of significance if $z \geq 1.645$. That is,

$$\begin{aligned}
\frac{68.2 - 67.5}{\sqrt{\frac{(2.5)^2}{n} + \frac{(2.8)^2}{n}}} &\geq 1.645 \\
\frac{0.7}{\sqrt{\frac{14.09}{n}}} &\geq 1.645 \\
\frac{0.7\sqrt{n}}{3.754} &\geq 1.645 \\
n &\geq 77.83 \approx 78.
\end{aligned}$$

Hence, the sample size of the two groups should be increased by atleast $78 - 50 = 28$, in order that the difference between the mean heights of the two groups is significant.

Exercise:

1. A sample of 100 bulbs of brand A gave a mean lifetime of 1200 hours with a S.D. of 70 hours, while another sample of 120 bulbs of brand B gave a mean lifetime of 1150 hours with a S.D. of 85 hours. Can we conclude that brand A bulbs are superior to brand B bulbs?
2. Two samples drawn from two different populations gave the following results:

	Size	*Mean*	*S.D.*
Sample I	100	582	24
Sample II	100	540	28

   Test the hypothesis, at 5% level of significance, that the difference of the means of the population is 35.
3. A sample of 1000 students from VIT was taken and their average weight was found to be 112 pounds with a S.D. of 20 pounds. Could the mean weight of the students in the population be 120 pounds?

Outline

P-value

- Besides listing an a value, many computer statistical packages give a P-value for hypothesis tests.
- The P-value is the actual probability of getting the sample mean value or a more extreme sample mean value in the direction of the alternative hypothesis ($>$ or $<$) if the null hypothesis is true.
- The P-value is the actual area under the standard normal distribution curve (or other curve, depending on what statistical test is being used) representing the probability of a particular sample mean or a more extreme sample mean occurring if the null hypothesis is true.

P-Values

Decision Rules:

- If P-value $\leq \alpha$, reject the null hypothesis.
- If P-value $> \alpha$, do not reject the null hypothesis.

Problem 13:

A genetic experiment involving peas yielded one sample of offspring consisting of 434 green peas and 173 yellow peas. Use a 0.05 significance level to test the claim that under the same circumstances, 24% of offspring peas will be yellow. Identify the null hypothesis, alternative hypothesis, test statistic, P-value, conclusion about the null hypothesis, and final conclusion that addresses the original claim. Use the P-value method.

Solution:

- The researcher wants to test the claim that 24% of offspring peas will be yellow at the significance level 0.05.
- Total number of green peas (y) is 434.
- Total number of yellow peas (x) is 173.
- Total number of peas in the experiment (n) is $434 + 173 = 607$.

The estimated proportion of yellow peas (p) is given as,

$$p = x/n = 173/607 = 0.285.$$

Let $\tilde{P}$ be the population proportion of yellow peas. Then the statistical hypothesis is, $H_0 : \tilde{P} = 0.24$ and $H_1 : \tilde{P} \neq 0.24$ which implies that the test is two-tailed. Now the test statistic is

$$z = \frac{p - \tilde{P}}{\sqrt{\frac{\tilde{P}Q}{n}}} = \frac{0.285 - 0.24}{\sqrt{\frac{0.24 \times 0.76}{607}}} = 2.596.$$

The P-value can be obtained by using the test statistic 2.596 by from the standard normal table as,

$$\begin{aligned} P\text{-value} &= 2P(Z > z) = 2P(Z > 2.596) \\ &= 2(1 - P(Z < 2.596)) = 2(1 - 0.9953) \\ &= 0.0094. \end{aligned}$$

Since the P-value (0.0094) is less than the significance level 0.05, so the decision is to reject the null hypothesis (H_0).

Therefore, in conclusion, there is not enough evidence to support the claim that 24% of offspring peas will be yellow.

Problem 14:

The distribution of blood cholesterol levels in the population of young men aged 20 to 34 years is close to Normal with mean 188 milligrams per deciliter (mg/dl) and standard deviation 41 mg/dl. You measure the blood cholesterol of 14 cross-country runners. The mean level is $\overline{x} = 172$ mg/dl. Assume that σ is the same as in the general population. You increase the sample of cross-country runners from 14 to 56. Suppose that this larger sample gives the same mean level, $\overline{x} = 172$ mg/dl. Test whether the test is significant at $\alpha = 0.10$.

Solution:

- The distribution of blood cholesterol levels in the population of young men aged 20 to 34 years is close to Normal with mean 188 milligrams per deciliter (mg/dl) and standard deviation 41 mg/dl.
- The sample mean is 172 mg/dl ($\bar{x}$)
- New sample size is 56(n).

The hypothesis is formulated as,

$$H_0 : \mu = 188$$
$$H_1 : \mu \neq 188$$

The test statistic is:

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{172 - 188}{\frac{41}{\sqrt{56}}} = -2.92$$

Hence the test statistic is -2.92.

The P-value is given as,

$$P\text{-value} = 2P\left(Z < -2.92\right) = 2\left(0.0018\right) = 0.0036$$

Since, P-value $< \alpha$, H_0 is rejected.

Exercise:

(1) researcher suspects that the actual prevalence of depression among children and adolescents is higher than was previously reported. The previously reported prevalence for depression among children and adolescents was 12.5%, and the researcher conducts a study to test the accuracy of the previously reported prevalence of depression among children and adolescents by recruiting 100 total children and adolescents and tests them for depression using DSM-5. The researcher determines that the prevalence of depression from the study is 12.9% and the standard deviation was 0.4. What should the researcher's conclusion be for a 5% significance level?

(2) A sample of 150 homes for sale in Katpadi showed a mean asking price of Rs. 2,33,000, but the city claimed that the mean asking price for the population was Rs. 2,55,000. The population standard deviation of all homes for sale was Rs. 11,000.

Use the P-value approach to conduct a full hypothesis test (all steps) that can be used to determine whether the mean asking price is significantly less than Rs. 2,55,000. Let alpha = 0.10.

(3) In a simple random sample of size 100, there are 65 individuals in the category of interest. It is desired to test the hypotheses $H_0 : \mu = 0.70$ versus $H_1 : \mu < 0.70$.

(a) Compute the sample proportion p.

(b) Compute the z-test statistic.

(c) Do you reject H_0 at the $\alpha = 0.05$ level? State why or why not.

(4) In 2007, the National Fire Protection Association reported that 22% of all fireworks injuries affected the eyes. The administrator of a local hospital claims that in his community the percentage is actually higher. His ER reported 40 eye injuries in the 154 patients who presented with fireworks injuries. Test the administrator's claim at the $\alpha = 0.05$ level of significance.

(5) A national magazine claims that the average college student watches less television than the general public. The national average is 29.4 hours per week, with a standard deviation of 2 hours. A sample of 30 college students has a mean of 27 hours. Is there enough evidence to support the claim at a = 0.01?