Distributions

Binomial distribution:-

The probability of 'x' successes in n trails is given by

$$P(x \text{ successes}) = {}^{n}c_{x}\, p^{x} q^{n-x}$$

(or) $P(X=x) = {}^{n}c_{x}\, p^{x} q^{n-x}$, $x = 0,1,2,\dots n$, $p+q=1$

where n, p are the parameters.

Note:- Here ${}^{n}c_{x}\, p^{x} q^{n-x}$ is the $(x+1)^{th}$ term in the expansion of $(p+q)^{n}$ and hence the distribution is called Binomial distribution.

Note:- Let an experiment constitutes n trails. If experiment is repeated N times, the frequency function of the binomial distribution is given by

$$f(x) = N\, p(x)$$

$$= N \cdot {}^{n}c_{x}\, p^{x} q^{n-x}, \quad x = 0,1,2,3,\dots n.$$

Note:- ① The probability of a success is the same for each trail.

② The n trails are independent.

Note:- Mean and variance of the binomial distribution are np, and npq

$S.D = \sqrt{npq}$.

Note:- Moment generating function of a binomial distribution about origin is

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= (q+pe^{t})^{n}$$

$n(q+pe^{t}) \cdot p$

$n(q+p) \cdot p = np$

Ex:- The mean and variance of a binomial distribution are 4 and $4/3$. find $P(X \geq 1)$.

Sol:- we know that the mean ~~and variance~~ of a binomial distribution is np and variance is npq.

Given $np = 4$ — ①

$npq = 4/3$ — ②

$\Rightarrow q = 1/3$

$p+q=1$

$q = 1-p = 1 - 1/3 = 2/3$

② $\Rightarrow n \cdot \frac{2}{3} = 4$

$n = 6$.

Now $P(X \geq 1) = 1 - P(X<1)$

$= 1 - P(0)$

$= 1 - {}^{n}c_{0}\, p^{0} q^{6-0}$

$= 1 - 1 \cdot \left(\frac{1}{3}\right)^{6} = 0.998$.

Ex:- Find the binomial distribution for which the mean is 4 and variance is 3.

Sol: $P(x=x) = 16c_x\left(\frac{1}{4}\right)^x \cdot \left(\frac{3}{4}\right)^{n-x}$.

Ex:- 10 coins are thrown simultaneously. find the probability of getting atleast 7 heads.

Sol:- $0.171875 = P(x \geq 7)$

Ex:- A machine manufacturing screws is known to produce 5% defective. In a random sample of 15 screws, what is the probability that are (i) exactly three defectives (ii) not more than three defectives.

Sol:- Given $p = 5\% = 5/100 = 1/20 = 0.05$

$q = 1-p = 1-0.05 = 0.95$

$n = 15$

(i) P(exactly three defectives) $= P(x=3)$

$= n_{c_3} p^3 q^{n-3}$

$= 15_{c_3} (0.05)^3 (0.95)^{12}$

$= 0.0307$

(ii) p(not more than 3 defectives) $= P(x \leq 3)$

$= P(0) + P(1) + P(2) + P(3)$

$= 15_{c_0} (0.05)^0 (0.95)^{15} + 15_{c_1} (0.05)^1 (0.95)^{14}$

$+ \dots - 15_{c_3} (0.05)^1 (0.95)^{12}$

$= 0.994$

Ex:- In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. find the probability that (i) All are good bulbs (ii) At most ~~three are~~ there are 3 defective bulbs, (iii) Exactly there are three defective bulbs.

Sol:- Have $p = \frac{10}{100} = \frac{1}{10} = 0.1$, $q = 1-p = 0.9$, $n = 20$

(i) P(all are good bulbs) = P(none are defective) $= P(0) = 0.1216$

(ii) P(at most there are 3 defective bulbs) $= P(x \leq 3) = 0.8666$

(iii) P(exactly 3 defective bulbs) $= P(3) = 0.19$

Ex:- 6 dices are thrown 729 times. How many times do you expect atleast three dice to show a five or a six?

Solu:- p = probability of getting 5 or 6 with a die

$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$.

$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$.

P (atleast three dice showing five or six)

$= P(X \geq 3)$

$= P(3) + P(4) + P(5) + P(6)$

$= 6c_3 \left(\frac{1}{3}\right)^3 \cdot \left(\frac{2}{3}\right)^3 + \cdots + 6c_6 \left(\frac{1}{3}\right)^0 \cdot \left(\frac{2}{3}\right)^6 = \frac{233}{3^6}$

for 729 times, the expected number of times at least 3 dice showing five or six.

$= N \times \frac{233}{3^6}$

$= 729 \cdot \frac{233}{3^6} = 233$ times.

Ex:- An irregular six faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws each, would you expect it to give no even number.

Solu:- Here $n = 10$

p (getting 'x' even number) $= P(X = x)$

$= 10c_x \, p^x q^{n-x}$

Given that P(getting 5 even numbers) = 2 P(getting 4 even numbers)

$\Rightarrow P(5) = 2P(4)$

$10c_5 \, p^5 q^5 = 2 \times 10c_4 \, p^4 q^6$

$\Rightarrow \frac{p}{5} = \frac{q}{3}$

$3p = 5q$

$3p = 5(1-p)$

$8p = 5 \Rightarrow p = 5/8, \quad q = 3/8$

$\therefore$ P(getting 'x' even numbers) = P(x)

$$= {}^{10}C_x (5/8)^x (3/8)^{10-x}$$

$\therefore$ The required no. of times that in 10,000 sets of 10 throws each we get no even number = 10000 × P(0)

$$= 10000 (3/8)^{10}$$

$$=$$

Ex:- With the usual notation find p for a binomial random variate 'x' if $n = 6$ and if $9P(x=4) = P(x=2)$

Sol: $p = 0.25$

Ex:- If 'x' is a random variate following binomial distribution with mean 2.4 and variance 1.44, find $P(x \geq 5)$.

Poisson Distribution:

poisson distribution is a limiting case of binomial distribution under the following assumptions:

(i) The no. of trails 'n' should be indefinitely large ie $n \to \infty$

(ii) The probability of success 'p' for each trail is indefinitely small

(iii) $np = \lambda$, should be finite where λ is a constant.

we know that the binomial distribution is

$$P(X=x) = {}^nC_x \, p^x q^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} \cdot p^x (1-p)^{n-x}$$

$$= \frac{1\cdot 2\cdot 3 \cdots (n-x) \cdots n}{1\cdot 2\cdot 3 \cdots (n-x)\cdot x!} \left(\frac{p}{1-p}\right)^x \cdot (1-p)^n$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \left(\frac{\frac{\lambda}{n}}{1-\lambda/n}\right)^x \cdot (1-\lambda/n)^n$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \frac{\lambda^x}{n^x} \cdot \frac{1}{(1-\lambda/n)^x} \cdot (1-\lambda/n)^n$$

$$= \frac{n^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\left(\frac{x-1}{n}\right)\right)}{x!} \cdot \frac{\lambda^x}{n^x} \left(1-\lambda/n\right)^{n-x}$$

$$\therefore P(X=x) = \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!} \cdot \lambda^x \left(1-\lambda/n\right)^{n-x} \quad \text{--- (1)}$$

we know that $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x} = e^{-\lambda}$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{2}{n}\right) = 1$$

$\therefore$ when $n\to\infty$, ~~R.H.S of~~ (1) gives

$$P(X=x) = \frac{\lambda^x}{x!} \cdot e^{-\lambda}$$

$$\therefore P(X=x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \cdots \infty$$

The Probability function of a r.v 'x' which follows poisson distribution is given by $P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$, $x = 0, 1, 2, \cdots \infty$.

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Note: Moment generating function of the poisson distribution is

$$m_x(t) = \sum_{x=0}^{\infty} e^{tx} \cdot p(x) = e^{\lambda(e^t - 1)}$$

Note: Mean and variance of the poisson distribution are λ

Ex: If x is a poisson variate $p(x=2) = 9\,p(x=4) + 90\,p(x=6)$ find mean and variance.

Sol: $p(X=x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$

Given $p(x=2) = 9\,p(x=4) + 90\,p(x=6)$

$$\frac{e^{-\lambda} \cdot \lambda^2}{2!} = 9\,\frac{e^{-\lambda} \cdot \lambda^4}{4!} + 90 \cdot \frac{e^{-\lambda} \cdot \lambda^6}{6!}$$

$$\frac{\lambda^2}{2!} = \frac{9\lambda^4}{4!} + \frac{90 \cdot \lambda^6}{6!}$$

$$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda^2 = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2}$$

$\lambda^2 = 1$ | $\lambda^2 = -4$

$\lambda = \pm 1$ | $\lambda = \pm 2i$

$\therefore$ mean $= \lambda = 1$, variance $= \lambda = 1$.

Ex: If x is a poisson variate such that $p(x=1) = \frac{3}{10}$, $p(x=2) = \frac{1}{5}$ find $p(x=0)$ and $p(x=3)$.

Ans: $\lambda = \frac{4}{3}$, $p(x=0) = e^{-4/3}$, $p(x=3) = \dfrac{e^{-4/3} \cdot \left(\frac{4}{3}\right)^3}{3!}$

Ex: In a certain factory turning razor blades there is a small chance of $\frac{1}{500}$ for any blade to be defective. The blades are in packets of 10. Use poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) one defective (iii) 2 defective blades respectively in a consignment of 10,000 packets.

Sol: Given $p = \frac{1}{500}$, $n = 10$

$N = 10,000$

mean $= \lambda = np$

$= 10 \times \frac{1}{500} = \frac{1}{50} = 0.02$

The poisson distribution is $P(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} = \frac{e^{-0.02} \cdot (0.02)^x}{x!}$

(i) $P(\text{no defective}) = P(0)$

$= \frac{e^{-0.02} \cdot (0.02)^0}{0!} = 0.980199$

∴ The total no. of packets containing no defective blades in 10,000 packets $= N \cdot P(0)$

$= 10000 \times 0.98019 = 9802$ packets.

(ii) $P(\text{one defective}) = P(1) = \frac{e^{-0.02} \cdot (0.02)^1}{1!} = 0.01960$

∴ No. of packets containing one defective in 10,000 packets.

$= N \times P(1) = 10000 \times 0.01960 = 196$ packets

(iii) $P(\text{two defective}) = P(2)$

$= \frac{e^{-0.02} \cdot (0.02)^2}{2!} = 0.000196$

∴ number of packets containing 2 defective $= N \times P(2)$

$= 10000 \times 0.000196$

om

Ex:- Find the probability that atmost 5 defective fuses will be found of 200 fuses if experience shows that 2% of such fuses are defective.

Sol :- Given $n = 200$, $p = 2\% = \frac{2}{100} = \frac{1}{50} = 0.02$

mean $\lambda = np = 200 \times 0.02 = 4$

Poisson distribution is

$$p(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} = \frac{e^{-4} \cdot 4^x}{x!}$$

Now p(atmost 5 defective fuses) $= p(x \leq 5)$

$= p(0) + p(1) + p(2) + p(3) + p(4) + p(5)$

$= 0.785$

Ex:- A manufacturer know that the condensers he makes contain on the average 1% of defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers.

Sol $p = 1\% = 0.01$, $n = 100$

$\lambda = np = 1$.

p(3 or more faulty condensers) $= p(x \geq 3)$

$= 1 - p(x < 3)$

$= 1 - [p(0) + p(1) + p(2)]$

$= 0.08025$

Ex:- Out of 1000 balls 50 are red and the rest white. If 60 balls are picked at random, what is the probability of picking up (i) 3 red balls (ii) not more than 3 red balls in the sample ($e^{-3} = 0.0498$)

Sol:- Since out of 1000 balls 50 are red, probability of a drawing a red ball $p = \frac{50}{1000} = 1/20$, Given $n = 60$, $\lambda = np = 60 \times \frac{1}{20} = 3$.

$p(x \text{ red balls}) = \frac{e^{-3} \cdot 3^x}{x!}$

(i) p(3 red balls) $= p(3) = 0.2241$

p(not more than 3 red balls)

$= p(x \leq 3)$

$= p(0) + p(1) + p(2) + p(3)$

$= 0.6474$.

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(5)

Ex:- fit a poisson distribution to the following data and calculate the theoretical frequencies.

Dealty	0	1	2	3	4
frequency	122	60	15	2	1

Sol:-

x	f	$f \cdot x$	theoretical frequency
0	122	0	121
1	60	60	61
2	15	30	15
3	2	6	3
4	1	4	0
	$N = 200$	$\sum fx = 100$	200

$$\text{mean} = \bar{x} = \frac{\sum fx}{N} = \frac{100}{200} = 0.5$$

Theoriotical distribution is given by

$$f(x) = N \times P(X=x) = 200 \times \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$(1) \quad = 200 \times \frac{e^{-0.5} \cdot (0.5)^x}{x!}$$

put $x = 0, 1, 2, 3, 4$ in (1), we get

$$f(0) = \frac{200 \times e^{-0.5} \cdot (0.5)^0}{0!} \simeq 121$$

$$f(1) = \frac{200 \times e^{-0.5} \cdot (0.5)^1}{1!} \simeq 61$$

$$f(2) = \frac{200 \times e^{-0.5} \times (0.5)^2}{2!} \simeq 15$$

$$f(3) = \frac{200 \times e^{-0.5} \times (0.5)^3}{3!} \simeq 3$$

$$f(4) = \frac{200 \times e^{-0.5} (0.5)^4}{4!} \simeq 0$$

Ex:- Letters were received in an office on each of 100 days. Assuming the following data to form a random sample from a poisson distribution, find the expected frequencies to correct nearest unit.

No. letters	0	1	2	3	4	5	6	7	8	9	10
frequency	1	4	15	22	21	20	8	6	2	0	1

Ans:- $N = \sum f = 100$

$\sum f(x) = 400$

$\text{mean} = \bar{x} = \frac{\sum fx}{n} = \frac{400}{100} = 4$

The expected frequencies are given by

$$f(x) = N\,p(x) = 100 \times \frac{e^{-\lambda} \cdot \lambda^x}{x!}.$$

Normal distribution:-

A random variable 'X' is said to follow normal distribution with mean μ and variance σ^2, if its density function is given by the probability law

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{--- (1)}$$

$-\infty < x < \infty,\ -\infty < \mu < \infty$

The total area bounded by the above curve is 1.

Sol:- $\text{Area} = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

put $\frac{x-\mu}{\sigma} = z$, $\sigma\,dz = dx$

$x = \infty \Rightarrow z = \infty$

$x = -\infty \Rightarrow z = -\infty$

$$\therefore \text{Area} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\,dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2}\,dz$$

put $\frac{z}{\sqrt{2}} = u$

$dz = \sqrt{2}\,du$

$$\therefore \text{Area} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-u^2} \cdot \sqrt{2}\,du$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} e^{-u^2}\,du \qquad \left(\because \int_0^{\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1.$$

Note:- $P(x_1 < x < x_2) = \int_{x_1}^{x_2} f(x)\,dx$

$$= \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

put $z = \frac{x-\mu}{\sigma}$ | $x = x_1 \Rightarrow z = \frac{x_1-\mu}{\sigma} = z_1$

$\sigma dz = dx$ | $x = x_2 \Rightarrow z = \frac{x_2-\mu}{\sigma} = \frac{x_2-\mu}{\sigma} = z_2$

$$\therefore P(z_1 < z < z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz = \phi(z)$$

The integral $\frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{z^2}{2}} dz$ is called the probability integral. The values of these integrals for different values of z are given table.

The curve given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{z^2}{2}} dz \text{ is called standard normal curve}$$

and it is bell shaped and symmetrical about the line $z=0$.

Note:- ① mean, median and mode of the normal distribution coincide.

② Q.D : MD : SD = 10 : 12 : 15

moment generating function of normal distribution:-

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

put $z = \frac{x-\mu}{\sigma}$

$\sigma dz = dx$

$$\therefore M_x(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \cdot e^{-\frac{z^2}{2}} \sigma\,dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2t\sigma z)}{2}} dz$$ om

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2 + \frac{\sigma^2 t^2}{2}} dz$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du, \qquad u = z - \sigma t, \quad du = dz$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} e^{-u^2/2} du$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi}$$

$$= e^{\mu t + \frac{t^2 \sigma^2}{2}}$$

$$\therefore m_x(t) = e^{\mu t + \frac{t^2 \sigma^2}{2}}$$

Ex:- The weekly wages of 1000 workmen are normally distributed around a mean of Rs 70 with a S.D of Rs. 5. Estimate the no. of workers whose weekly wages will be (i) between Rs. 69 and Rs. 72 (ii) less than Rs. 69 (iii) more than Rs. 72.

Sol:- Given $\mu = 70$
$\sigma = 5$

$$z = \frac{x-\mu}{\sigma}$$

(i) $P(69 < x < 72)$

when $x = 69$, $z = \frac{x-\mu}{\sigma} = \frac{69-70}{5} = -0.2$

when $x = 72$, $z = \frac{x-\mu}{\sigma} = \frac{72-70}{5} = 0.4$

$\therefore P(69 < x < 72) = P(-0.2 < z < 0.4)$

$= P(-0.2 < z < 0) + P(0 < z < 0.4)$

$= P(0 < z < 0.2) + P(0 < z < 0.4)$ ($\because$ Normal distribution is symmetric)

$= 0.0793 + 0.1554$ (from table)

$= 0.2347$

out of 1000, the no of workers whose wages lie between 69/- and 72/- $= 1000 \times P(69 < x < 72)$

$= 1000 \times 0.2347 = 234.7 \approx 235$

(ii) $P(\text{less than } 69/-) = P(x < 69)$

$P(x < 69) = P(z < -0.2)$

$= 0.5 - P(0 < z < 0.2)$

$= 0.5 - 0.0793$

$= 0.4207$

out of 1000 workmen, the no. of workers whose wages are less than Rs 69

$= 1000 \times P(z < -0.2) = 1000 \times 4207 = 420.7 = 421$.

(iii) $P(\text{more than } 72) = P(x > 72)$

when $x = 72$, $z = \frac{x-\mu}{\sigma} = \frac{72-70}{5} = \frac{2}{5} = 0.4$

$P(x > 72) = P(z > 0.4)$

$= 0.5 - 0.1554 = 0.3446$

out of 1000 workmen, the no. of workers whose wages are greater than 72.

$= 1000 \times P(z > 0.4)$

$= 1000 \times 0.3446$

$= 344.6 \approx 345$.

Ex:- A manufacturer produces air mail envelopes where weight is normal with mean $\mu = 1.950$ gm and S.D, $\sigma = 0.025$ gm. The envelopes are sold in lots of 1000. How many envelopes in a lot may be heavier than 2 grams?

Sol: $\mu = 1.950$, $\sigma = 0.025$

$P(\text{the envelopes heavier than } 2) = P(X > 2)$

we know that $z = \frac{X-\mu}{\sigma}$

when $x = 2$, $z = \frac{2-1.95}{0.025} = 2$

$\therefore P(X > 2) = P(z > 2)$

$= 0.5 - P(0 < z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$

The no. of envelopes heavier than 2 gm $= 1000 \times P(X > 2)$

$= 1000 \times 0.0228$

$= 22.8$

$\approx \boxed{23}$ (approx)

Ex:- Assuming that the diameter of 1000 plugs taken consecutively from a machine form a normal distribution with mean 0.7515cm and S.D 0.0020cm, how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm.

Sol: The approved diameter ranges from $0.752 + 0.004$ to $0.752 - 0.004$

(i) 0.748 to 0.756

Given $\mu = 0.7515$, $\sigma = 0.0020$

$P(0.748 < X < 0.756) = ?$

we know that $z = \frac{X-\mu}{\sigma}$

when $X = 0.748$, $z_1 = \frac{0.748 - 0.7515}{0.002} = -1.75$

when $x = 0.756$, $z_2 = \frac{0.756 - 0.7515}{0.002} = 2.25$

$\therefore P(0.748 < X < 0.756) = P(-1.75 < z < 2.25)$

$= P(-1.75 < z < 0) + P(0 < z < 2.25)$

$= P(0 < z < 1.75) + P(0 < z < 2.25)$

$= 0.4599 + 0.4878$

$= 0.9477$

The no. of plugs a approved diameters $= 1000 \times 0.9477$

$= 947.7 = 948$ (aprox)

⑧ OM

Ex: In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D of 60 hours. Estimate the no of bulbs likely to burn for (i) more than 2150 hours (ii) less than 1950 hours and (iii) more than 1920 but less than 2160 hours.

Sol:- Given $\mu = 2040$ hours

$\sigma = 60$ hours.

(i) P(more than 2150 hours) = $P(X > 2150)$

when $x = 2150$, $z = \frac{x-\mu}{\sigma} = 1.833$

$\therefore P(X > 2150) = P(Z > 1.833) = 0.5 - P(0 < Z < 1.833)$

$= 0.5 - 0.4664 = 0.0336$

$\therefore$ The no of bulbs expected to burn for more than 2150 hours $= 2000 \times 0.0336$

$= \boxed{67}$ (appro)

(ii) P(less than 1950 hours) = $P(X < 1950)$, $z = \frac{x-\mu}{\sigma} = -1.5$

$= P(Z < -1.5)$

$= 0.5 - (0 < Z < 1.5)$

$= 0.5 - 0.4332 = 0.0668$

$\therefore$ The no of bulbs expected to burn for less than 1950 hours $= 2000 \times 0.0668$

$= \boxed{134}$ (approx.)

(iii) P(more than 1920 hours but less than 2160 hours)

$= P(1920 < X < 2160)$

$= P(-2 < Z < 2)$

$= 2(0 < Z < 2)$

$= 2 \times 0.4773 = 0.9546$.

$\therefore$ The no of bulbs expected to burn for more than 1920 hours but less than 2160 hours $= 2000 \times 0.9546 = \boxed{1909}$ (appr).

Ex:- 'x' is normally distributed and the mean of x is 12 and the S.D is 4. Find out the probability of the following. (i) $x > 20$ (ii) $x \le 20$ (iii) $0 \le x \le 12$.

Sol: (i) 0.0228 (ii) 0.9772 (iii) 0.4987.

Ex:- Assume that mean height of soldiers to be 68.22 inches with a variance of 10.8 inches. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall.

Sol: $P(x > 6 \text{ feet}) = P(X > 72 \text{ inches})$, $x = 72 \Rightarrow z = 1.1503$

$P(x > 72) = P(Z > 1.1503) = 0.5 - P(0 < Z < 1.1503) = 0.5 - 0.3749$

$= \boxed{0.1251}$

om

Gamma distribution :-

The continuous r.v 'x' is said to follow a Gamma distribution with parameter 'λ' if its probability function is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} \cdot x^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0,\ 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Note:- A continuous r.v 'x' whose probability function is

$$f(x) = \frac{a^{\lambda} \cdot e^{-ax} \cdot x^{\lambda-1}}{\Gamma(\lambda)}, \quad a > 0,\ \lambda > 0,\ 0 < x < \infty$$ is called

a Gamma distribution with two parameters 'a' and 'λ'.

Moment generating function of Gamma distribution :-

$$M_x(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{e^{-x} \cdot x^{\lambda-1}}{\Gamma(\lambda)}\,dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-(1-t)x} \cdot x^{\lambda-1}\,dx$$

Put $(1-t)x = u$ $\qquad x = 0 \Rightarrow u = 0$

$(1-t)\,dt = du,$ $\qquad x = \infty \Rightarrow u = \infty$

$$\therefore M_x(t) = \frac{1}{\Gamma(\lambda)} \cdot \int_0^{\infty} e^{-u} \left(\frac{u}{1-t}\right)^{\lambda-1} \cdot \frac{du}{\lambda-1}$$

$$= \frac{1}{\Gamma(\lambda)} \cdot \frac{1}{(1-t)^{\lambda}} \int_0^{\infty} e^{-u} \cdot u^{\lambda-1}\,du$$

$$= \frac{1}{(1-t)^{\lambda}} \cdot \frac{1}{\Gamma(\lambda)} \cdot \Gamma(\lambda)$$

$$= \frac{1}{(1-t)^{\lambda}}$$

$$\therefore M_x(t) = (1-t)^{-\lambda}, \quad |t| < 1$$

To find mean and variance of Gamma distribution :-

$$M_x(t) = (1-t)^{-\lambda}$$

$$M_x'(t) = -\lambda(1-t)^{\lambda-1}(-1) = \lambda(1-t)^{\lambda-1}$$

$$\text{mean} = \mu_1' = M_x'(0) = \lambda$$

$$M_x''(t) = \lambda(-\lambda-1)(1-t)^{-\lambda-2}(-1)$$

$$\mu_2' \text{ (about origin)} = M_x''(0) = \lambda(\lambda+1)$$

$$\therefore \text{Variance} = \mu_2' - \mu_1'^2$$

$$= \lambda(\lambda+1) - \lambda^2$$

$$= \lambda.$$

Hence mean and variance of Gamma distribution $= \lambda$..

The exponential distribution:- ^{om} (9)

A continuous r.v 'x' is said to follow exponential distribution with parameter $\lambda>0$ if its probability function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Moment generating function of Exponential distribution:-

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-t)x} dx = \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^\infty$$

$$= \frac{\lambda}{-(\lambda-t)} \left(e^{-\infty} - e^{0} \right) = \frac{\lambda}{\lambda - t}, \quad \lambda > t.$$

$$= \frac{1}{1-\frac{t}{\lambda}} = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$M_x(t) = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$E(x) = M_x'(t) = -1\left(1-\frac{t}{\lambda}\right)^{-2} \cdot \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2}$$

$$\text{mean} = \mu_1' = M_x'(0) = \frac{1}{\lambda}$$

$$M_x''(t) = \frac{-2}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2-1} \cdot \left(-\frac{1}{\lambda}\right) = \frac{2}{\lambda^2}\left(1-t/\lambda\right)^{-3}$$

$$\mu_2' = M_x''(0) = \frac{2}{\lambda^2}$$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\therefore \text{variance} = 1/\lambda^2.$$

Om

Memoryless property of the Exponential distribution:-

If X is exponentially distributed, then

$P(X > s+t / X > s) = P(X > t)$, for any $s, t > 0$

Proof:- $P(X > k) = \int_k^\infty \lambda e^{-\lambda x} dx$

$$= \lambda \left(\frac{e^{-\lambda x}}{-\lambda}\right)_k^\infty = -e^{-\infty} + e^{-\lambda k}$$

$$= e^{-\lambda k}$$

Now $P(X > s+t / X > s) = \dfrac{P(X > s+t \text{ and } X > s)}{P(X > s)}$

$$= \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P(X > t).$$

Note:- The converse of the above property is also true.

(i.e) If $P(X > s+t / X > s) = P(X > t)$, then X follows as exponential distribution.

~~Ex:- Let X represents the time to repair the machine.~~

Ex:- The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$

(a) what is the probability that the repair time exceeds 2h?

(b) what is the conditional probability that a repair takes 11h given that its direction exceeds 8h?

Sol:- Let X represents the time to repair the machine.

The density function of X is $f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{1}{2}x}$, $x > 0$

(a) $P(X > 2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx = \frac{1}{2}\left(\frac{e^{-1/2 x}}{-1/2}\right)_2^\infty = -e^{-\infty} + e^{-1/2(2)} = e^{-1}.$

(b) $P(X > 11 / X > 8) = P(X > 3)$ ($\because P(X > s+t / X > s) = P(X > t)$ by memoryless property)

$$= \int_3^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx = \frac{1}{2}\left(\frac{e^{-1/2 x}}{-1/2}\right)_3^\infty = -e^{-\infty} + e^{-3/2} = e^{-3/2}.$$

Ex:- with mean 120 days, find the probability that such a watch will (i) have to be set in less than 24 days
(ii) not have to be reset in atleast 180 days.

Sol: Let x be the r.v

$$\text{mean} = 120$$

$$\frac{1}{\lambda} = 120, \quad \lambda = \frac{1}{120}$$

pdf of x is given by

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{120} e^{-\frac{1}{120}x}$$

(i) $P(x<24) = \int_0^{24} \frac{1}{120} e^{-\frac{1}{120}x} = 0.1813$

(ii) $P(x \geq 180) = \int_{180}^{\infty} \frac{1}{120} e^{-\frac{x}{120}} dx$.

Ex:- The mileage which car owners get with certain kind of radial tyre is r.v having an exponential distribution with mean 4,000 km. Find the probability that one of these tires will last
(i) at least 2,000 km (ii) at most 3,000 km.

Sol:- Let x denote the mileage obtained with the tire.

$$f(x) = \frac{1}{4000} \cdot e^{-\frac{x}{4000}}, \quad x>0,$$

mean = 4000

$$\frac{1}{\lambda} = 4000 \Rightarrow \lambda = \frac{1}{4000}$$

(i) $P(x>2000) = \int_{2000}^{\infty} \frac{1}{4000} e^{-\frac{x}{4000}} dx = e^{-0.5} = 0.6065$

(ii) $P(x \leq 3000) = \int_0^{3000} \frac{1}{4000} e^{-x/4000} = 1 - e^{-0.75} = 0.5270$.

om

Weibull Distribution:-

A r.v 'x' is said to follow weibull distribution with two parameters $\beta > 0$, $\alpha > 0$ if the probability density function is given by

$$f(x) = \alpha\beta\, x^{\beta-1} \cdot e^{-\alpha x^{\beta}}, \quad x > 0$$

Mean and variance of the weibull distribution:-

$$E(x^r) = \int_0^\infty x^r \alpha\beta\, x^{\beta-1} \cdot e^{-\alpha x^{\beta}}\, dx$$

$u = \alpha x^{\beta}$

$du = \alpha\beta\, x^{\beta-1} dx$

$$= \alpha\beta \int_0^\infty x^{r+\beta-1} \cdot e^{-\alpha x^{\beta}}\, dx$$

$$= \alpha\beta \int_0^\infty x^{r+\beta-1} \cdot e^{-u} \cdot \frac{du}{\alpha\beta x^{\beta-1}}$$

$$= \int_0^\infty \left(\frac{u}{\alpha}\right)^{\frac{r}{\beta}} \cdot e^{-u}\, du$$

$$= \frac{1}{\alpha^{r/\beta}} \int_0^\infty u^{r/\beta}\, e^{-u}\, du$$

$$= \frac{\Gamma\left(\frac{r}{\beta}+1\right)}{\alpha^{r/\beta}}$$

$$E(x) = \text{mean} = \alpha^{-1/\beta} \cdot \Gamma\left(\frac{1}{\beta}+1\right)$$

$$E(x^2) = \alpha^{-2/\beta} \cdot \Gamma\left(\frac{2}{\beta}+1\right)$$

$$\text{Variance} = E(x^2) - (E(x))^2 = \alpha^{-2/\beta}\, \Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right) \cdot \alpha^{-1/\beta}\right)^2$$

$$= \alpha^{-2/\beta}\left\{\Gamma\left(\frac{2}{\beta}+1\right) - \left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2\right\}$$

Ex:- The life time of a component measured in hours follows weibull distribution with parameter $\alpha = 0.2$, $\beta = 0.5$. Find the mean life time of the component.

soln:- for a weibull distribution the mean is given by

$$E(x) = \alpha^{-1/\beta}\, \Gamma\left(\frac{1}{\beta}+1\right)$$

Given $\alpha = 0.2$, $\beta = 0.5$

$$\therefore E(x) = (0.2)^{-1/0.5} \cdot \Gamma\left(1 + \frac{1}{0.5}\right) = 50 \text{ hours}.$$

om

Ex:- If the life X (in years) of a certain type of car has a weibull distribution with the parameter $\beta=2$. find the value of the parameter α given that probability that the life of the car exceeds 5 years is $e^{-0.25}$. for these values of α,β, find mean and variance of X.

Sol:- The pdf of a weibull distribution is given by

$$f(x) = \alpha\beta\, x^{\beta-1}\, e^{-\alpha x^{\beta}}$$

Given $\beta=2$,

$$f(x) = 2\alpha x\, e^{-\alpha x^2}$$

Put $x^2=t$

$2x\,dx = dt$

$x=5,\ t=25$

$x=\infty,\ t=\infty$

Now $$P(X>5) = \int_5^\infty 2\alpha x\, e^{-\alpha x^2}\,dx$$

$$= \int_{25}^\infty \alpha\, e^{-\alpha t}\,dt$$

$$= \alpha\left(\frac{e^{-\alpha t}}{-\alpha}\right)_{25}^\infty = e^{-25\alpha}$$

But, given that

$$P(X>5) = e^{-0.25}$$

$$e^{-25\alpha} = e^{-0.25}$$

$$25\alpha = 0.25$$

$$\alpha = \frac{1}{100}$$

mean $= E(X) = \alpha^{-\frac{1}{\beta}}\,\Gamma\left(\frac{1}{\beta}+1\right) = \left(\frac{1}{100}\right)^{-\frac{1}{2}}\cdot\Gamma\left(\frac{1}{2}+1\right)$ $\quad(\because \Gamma(n+1)=n\Gamma(n))$

$$= 10\cdot\frac{1}{2}\Gamma(1/2) = 5\sqrt{\pi}$$

$$var(X) = \alpha^{-\frac{2}{\beta}}\left[\Gamma\left(\frac{2}{\beta}+1\right) - \left\{\Gamma\left(\frac{1}{\beta}+1\right)\right\}^2\right]$$

$$= \left(\frac{1}{100}\right)^{-1}\left[\Gamma(2) - (\Gamma(3/2))^2\right]$$

$$= 100\left[1-\left(\frac{1}{2}\sqrt{\pi}\right)^2\right] = 100\left(1-\frac{\pi}{4}\right).$$

Om

Normal Distribution:- A random variable x is said to follow normal distribution with mean μ and variance σ^2, if its density function is given by the probability law

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{——— (1)}$$

$$-\infty < x < \infty, \quad -\infty < \mu < \infty$$

Note:- The total area bounded by the above curve is 1.

$$\text{Area} = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

put $\frac{x-\mu}{\sigma} = z$, $\sigma dz = dx$

When $x = \infty$, $z = \infty$

$x = -\infty$, $z = -\infty$

$$\therefore \text{Area} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$$

put $\frac{z}{\sqrt{2}} = u$, $dz = \sqrt{2}\,du$

$$\therefore \text{Area} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-u^2} \cdot \sqrt{2}\,du = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1 \quad \left(\because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2} \right)$$