
OPERATION RESEARCH

Lecture Notes By

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SMA 2230/EMG 2513 : OPERATION RESEARCH

The **objective of Operations Research** is to provide a scientific basis to the decision maker for solving the problems involving the interaction of various components of an organization by employing a team of scientists from various disciplines, all working together for finding a solution which is in the best interest of the organization as a whole. The best solution thus obtained is known as optimal decision.

Course Description: Introductory Concepts: Definitions, scope, methodology of OR, types if OR models, concept of optimization. **Linear Programming:** Formulation of linear programming problem/models (LPP), Examples of LPP, advantages and limitations of LP. Graphical solution of LPP, Classifications of LPs. **Sensitivity analysis:** Sensitivity analysis of the problems according to the graphic solution. **The Simplex method for LP:** The Simplex method for LP: The standard form of LP, basic feasible solution, computational aspect of simplex method, cases of unique feasible solution, no feasible solution, multiple solutions and unbounded solution and degeneracy. Duality in LPP, primal-dual relationship. **Assignment problem:** Resource allocation and other applications **Transportation problem:** Transportation problem: Method for finding basic feasible solution of a transportation problem, modified distribution method for finding the optimum solution, unbalanced and maximization in a transportation problem.

Teaching Methodology: The method of instruction will be lectures, interactive tutorials, and any other presentations/demonstrations the lecturer will deem fit towards enhancing understanding of the concepts taught in class.

Instruction Materials/Equipment: Whiteboard, LCD/Overhead Projector, Handouts.

Course Evaluation: The final grade for the course will be based on a final examination at the end of the semester (70%) and the Continuous Assessment Tests(CATs) (30%) which will be based on question exams and assignments throughout the semester.

Pre-requisite: HBC 2211 Management Mathematics II

Recommended Text Books

1. J.K.Sharma: Operations Research (Theory and Application)
2. P. Rama Murthy: Operations Research, second edition.

Operations Research Lecture Schedule:

Week	Dates	Course Content to be covered:
1	21th Sept 2024	History and nature of Operations Research.
2	28th Sept 2024	Linear Programming
3	05th Oct 2024	Graphical Method of solving LPP
	12th Oct 2024	Simplex Method
4	19th Oct 2024	Simplex Method <i>Continued</i> . . .
5	26th Oct 2024	CAT I ,Sensitivity analysis of LP
6	02th Nov 2024	Sensitivity analysis of LP ,Duality of a LPP
7	02th Nov 2024	Transportation model; using NWCL, LCM, Vogel's Approximation
8	09th Nov 2024	Optimality test using MODI, Special Cases of Transportation Problem
9	16th Nov 2024	Optimality test using MODI, Special Cases of Transportation Problem
10	23th Nov 2024	Assignment model; Hungarian solution Method.
11	30th Nov 2024	Assignment model;Maximization problem.
12	19th NOV 2024	Inventory models: periodic model, quantity models, basic economic order quantity, discounts, stock-out, buffer stock activity based costing analysis, pareto analysis, just in time (JIT) systems.
13	26th NOV 2024	CAT II , Network model; deterministic, critical path analysis/critical path method, probabilistic model, programme evaluation review technique, crashing, resource levelling.

Lecture 1: Introduction to Operational Research

By Dr. Antony Ngunyi

0.1 Introduction to Operational Research

This teaching module is designed to be an entertaining and representative introduction to the subject of Operational Research. It is divided into a number of sections each covering different aspects of OR.

What is Operational Research?

This looks at the characteristics of Operational Research, how you define what OR is and why organisations might use it. It considers the scientific nature of OR and how it helps in dealing with problems involving uncertainty, complexity and conflict.

OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing.

0.1.1 Introduction to Operations Research Terminology

The British/Europeans refer to “operational research”, the Americans to “operations research” - but both are often shortened to just “OR” (which is the term we will use).

Another term which is used for this field is “management science” (“MS”). The Americans sometimes combine the terms OR and MS together and say “OR/MS” or “ORMS”. Yet other terms sometimes used are “industrial engineering” (“IE”) and “decision science” (“DS”). In recent years there has been a move towards a standardisation upon a single term for the field, namely the term ”OR”.

One thing I would like to emphasise in relation to OR is that it is (in my view) a subject/discipline that has much to offer in making a real difference in the real world. OR can help you to make better decisions and it is clear that there are many, many people and companies out there in the real world that need to make better decisions. I have tried to include throughout OR-Notes discussion of some of the real-world problems that I have personally been involved with.

The main objective of learning this course of Operations Research is that it seeks the determination of the best (optimum) course of action of a decision problem under the restriction of limited resources.

At the end of the course the student should be able to use various mathematical techniques to model and analyze decision problems. OR is an application of scientific techniques and

tools to problems involving operations of systems so as to provide those in control of operations with optimum solutions to problems. OR provides executive departments with a quantitative basis for decision making regarding the operations under their control.

0.1.2 Characteristics of Operations Research

1. OR is **inter-disciplinary**; it is developed and applied by inter-disciplinary teams. i.e. a team of scientists drawn from various disciplines e.g. mathematics, statistics, actuarial science, accounting, finance etc.
2. OR is a **system approach**; its emphasis is on the overall approach to a system in order to get the optimum decisions.
3. OR is **goal oriented**; in most cases OR tries to optimize a well defined function subject to some given constraints.
4. OR is **helpful in improving the quality of solutions**; it doesn't give perfect answers but merely gives bad answers to problems which would otherwise have worst answers.
5. OR is **scientific**; as research involves up-to-date methods.
6. OR is **model based**; it uses models built by quantitative measurements of variables concerning a given problem. Solutions are derived from the models etc.

In general, OR is the application of up-to-date scientific methods and techniques by interdisciplinary teams to problems involving control of organized systems so as to provide solutions which best serve purposes of the organization as a whole.

0.1.3 Methodology of Operations Research

Steps involved in an OR inquiry;

1. Identify the problem
2. Formulating the problem
3. Constructing the appropriate model
4. Deriving the solutions
5. Testing the validity of the solution
6. Controlling the solutions
7. Implementing the results

Note: The nature of the problem may dictate the OR method to be used.

0.1.4 Operational Research Techniques

These are methods which have been used to solve OR problems. They are determined by the nature of the problem. They include;

1. Programming - Linear, non-linear, dynamic, heuristic, integer etc.
2. Transportation and Assignment problem
3. Transshipment problem
4. Queuing theory (waiting lines)
5. Inventory control (analysis)
6. Network analysis (critical path method), shortest route problem
7. Simulation
8. Game theory
9. Ring theory
10. Decision theory

0.1.5 History of Operations Research

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was really only in the late 1930's that operational research began in a systematic fashion, and it started in the UK. As such I thought it would be interesting to give a short history of OR and to consider some of the problems faced (and overcome) by early OR workers.

Operations Research is a '**war baby**'. It is because, the first problem attempted to solve in a systematic way was concerned with *how to set the time fuse bomb to be dropped from an aircraft on to a submarine*. In fact the main origin of Operations Research was during the **Second World War**. At the time of Second World War, the military management in England invited a team of scientists to study the strategic and tactical problems related to air and land defense of the country. The problem attained importance because at that time the resources available with England was very limited and the objective was to win the war with available meager resources. The resources such as food, medicines, ammunition, manpower etc., were required to manage war and for the use of the population of the country. It was necessary to decide upon the most effective utilization of the available resources to achieve the objective. It was also necessary to utilize the military resources cautiously. Hence, the Generals of military, invited a team of experts in various walks of life such as scientists, doctors, mathematicians, business people, professors, engineers etc., and the problem of resource utilization is given to them to discuss and come out with a feasible

solution. These specialists had a brain storming session and came out with a method of solving the problem, which they coined the name “**Linear Programming**”. This method worked out well in solving the war problem.

As the name indicates, the word **Operations** is used to refer to the problems of military and the word **Research** is used for inventing new method. As this method of solving the problem was invented during the war period, the subject is given the name ‘**OPERATIONS RESEARCH**’ and abbreviated as ‘**O.R.**’ After the World War there was a scarcity of industrial material and industrial productivity reached the lowest level. Industrial recession was there and to solve the industrial problem the method **linear programming** was used to get optimal solution. From then on words, lot of work done in the field and today the subject of O.R. have numerous methods to solve different types of problems. After seeing the success of British military, the United States military management started applying the techniques to various activities to solve military, civil and industrial problems.

In industrial world, most important problem for which these techniques used is how to optimize the profit or how to reduce the costs. The introduction of Linear Programming and Simplex method of solution developed by American Mathematician George B. Dantzig in 1947 given an opening to go for new techniques and applications through the efforts and co-operation of interested individuals in academic field and industrial field. Today the scenario is totally different. A large number of Operations Research consultants are available to deal with different types of problems. In India also, we have O.R. Society of India (1959) to help in solving various problems. Today the Operations Research techniques are taught at High School levels.

In one word we can say that Operations Research play a vital role in every organization, especially in decision-making process.

Assignment I due in two weeks time from today

1. Trace the history of Operations Research.
2. Discuss the objective of Operations Research.
3. “Operations Research is a bunch of mathematical techniques to break industrial problems”. Critically comment.
4. Briefly explain the characteristics of Operations Research.
5. Discuss the various steps used in solving Operations Research. problems.
6. Discuss the scope of Operations Research.
7. Briefly explain the significance of Operations Research.
8. Briefly explain the limitations of Operations Research.

Lesson 2: Linear Programming

By Dr. Antony Ngunyi

0.2 Linear Programming

0.2.1 Introduction

Linear Programming is that branch of mathematical programming which is designed to solve optimization problems where all the constraints as well as the objectives are expressed as Linear function. It was developed by George B. Denting in 1947. Its earlier application was solely related to the activities of the **Second World War**. However soon its importance was recognized and it came to occupy a prominent place in the industry and trade.

Linear Programming is a technique for making decisions under certainty i.e., when all the courses of options available to an organization are known and the objective of the firm along with its constraints are quantified. That course of action is chosen out of all possible alternatives which yields the optimal results. Linear Programming can also be used as a verification and checking mechanism to ascertain the accuracy and the reliability of the decisions which are taken solely on the basis of manager's experience without the aid of a mathematical model.

What makes the model a Linear program?

Technically, it is a linear program because all its functions (objective function and constraints) are linear. Linearity implies that both the proportionality and additivity properties are satisfied. For example, an unusual application of LP is the marriage problem, whose optimum solution show that monogamy is the best type of marriage, though not acceptable due to cultural beliefs.

0.2.2 Some of the definitions of Linear Programming

Linear Programming is a method of planning and operations involved in the construction of a model of a real-life situation having the following elements:

- (a) Variables which denote the available choices and
- (b) the related mathematical expressions which relate the variables to the controlling conditions, reflect clearly the criteria to be employed for measuring the benefits flowing out of each course of action and providing an accurate measurement of the organizations objective. The method maybe so devised' as to ensure the selection of the best alternative out of a large number of alternative available to the organization

Linear Programming is the analysis of problems in which a Linear function of a number of variables is to be optimized (maximized or minimized) when whose variables are subject

to a number of constraints in the mathematical near inequalities.

From the above definitions, it is clear that:

- (i) Linear Programming is an optimization technique, where the underlying objective is either to maximize the profits or to minimize the cost.
- (ii) It deals with the problem of allocation of finite limited resources amongst different competing activities in the most optimal manner.
- (iii) It generates solutions based on the feature and characteristics of the actual problem or situation. Hence the scope of linear programming is very wide as it finds application in such diverse fields as marketing, production, finance and personnel etc.
- (iv) Linear Programming has been highly successful in solving the following types of problems:
 - (a) Product-mix problems
 - (b) Investment planning problems
 - (c) Blending strategy formulations and
 - (d) Marketing and Distribution management.
- (v) Even though Linear Programming has wide and diverse applications, yet all LP problems have the following properties in common:
 - (a) The objective is always the same (i.e., profit maximization or cost minimization).
 - (b) Presence of constraints which limit the extent to which the objective can be pursued/achieved.
 - (c) Availability of alternatives i.e., different courses of action to choose from, and
 - (d) The objectives and constraints can be expressed in the form of linear relation.
- (vi) Regardless of the size or complexity, all LP problems take the same form i.e. allocating scarce resources among various competing alternatives. Irrespective of the manner in which one defines Linear Programming, a problem must have certain basic characteristics before this technique can be utilized to find the optimal values.

The characteristics or the basic assumptions of linear programming are as follows:

1. **Decision or Activity Variables and Their Inter-Relationship.** The decision or activity variables refer to any activity which are in competition with other variables for limited resources. Examples of such activity variables are: services, projects, products etc. These variables are most often inter-related in terms of utilization of the scarce resources and need simultaneous solutions. It is important to ensure that the relationship between these variables be linear.

2. **Finite Objective Functions.** A Linear Programming problem requires a clearly defined, unambiguous objective function which is to be optimized. It should be capable of being expressed as a liner function of the decision variables. The single-objective optimization is one of the most important prerequisites of linear programming. Examples of such objectives can be: cost-minimization, sales, profits or revenue maximization and the idle-time minimization etc.
3. **Limited Factors/Constraints.** These are the different kinds of limitations on the available resources e.g. important resources like availability of machines, number of man hours available, production capacity and number of available markets or consumers for finished goods are often limited even for a big organisation. Hence, it is rightly said that each and every organisation function within overall constraints both internal and external. These limiting factors must be capable of being expressed as linear equations or in equations in terms of decision variables
4. **Presence of Different Alternatives.** Different courses of action or alternatives should be available to a decision maker, who is required to make the decision which is the most effective or the optimal. For example, many grades of raw material may be available, the same raw material can be purchased from different supplier, the finished goods can be sold to various markets, production can be done with the help of different machines.
5. **Non-Negative Restrictions.** Since the negative values of (any) physical quantity has no meaning, therefore all the variables must assume non-negative values. If some of the variables is unrestricted in sign, the non-negativity restriction can be enforced by the help of certain contained in the problem.
6. **Linearity Criterion.** The relationship among the various decision variables must be directly proportional. Both the objective and the constraint, must be expressed in terms of linear equations or inequalities. For example, if one of the factor inputs (resources like material, labour, plant capacity etc.) decreases, then it should result in a proportionate manner in the final output. These linear equations and in equations can graphically be presented as a straight line.
7. **Additively.** It is assumed that the total profitability and the total amount of each resource utilized would be exactly equal to the sum of the respective individual amounts. Thus the function or the activities must be additive - and the interaction among the activities of the resources does not exist.
8. **Mutually Exclusive Criterion.** All decision parameters and the variables are assumed to be mutually exclusive. In other words, the occurrence of anyone variable rules out the simultaneous occurrence of other such variables.
9. **Divisibility.** Variables may be assigned fractional values. i.e., they need not necessarily always be in whole numbers. If a fraction of a product can not be produced, an integer programming problem exists. Thus, the continuous values of the decision variables and resources must be permissible in obtaining an optimal solution.

10. **Certainty.** It's assumed that conditions of certainty exist i.e., all the relevant parameters or coefficients in the Linear Programming model are fully and completely known and that they don't change during the period. However, such an assumption may not hold good at all times.
11. **Finiteness.** Linear Programming assumes the presence of a finite number of activities and constraints without which it is not possible to obtain the best or the optimal solution.

0.2.3 Advantages of Linear Programming

The following are some of the advantages of Linear Programming approach:

1. **Scientific Approach to Problem Solving.** Linear Programming is the application of scientific approach to problem solving. Hence it results in a better and true picture of the problems-which can then be minutely analysed and solutions ascertained.
2. **Evaluation of All Possible Alternatives.** Most of the problems faced by the present organisations are highly complicated - which can not be solved by the traditional approach to decision making. The technique of Linear Programming ensures that all possible solutions are generated - out of which the optimal solution can be selected.
3. **Helps in Re-Evaluation.** Linear Programming can also be used in re-evaluation of a basic plan for changing conditions. Should the conditions change while the plan is carried out only partially, these conditions can be accurately determined with the help of Linear Programming so as to adjust the remainder of the plan for best results.
4. **Quality of Decision.** Linear Programming provides practical and better quality of decisions that reflect very precisely the limitations of the system i.e., the various restrictions under which the system must operate for the solution to be optimal. If it becomes necessary to deviate from the optimal path, Linear Programming can quite easily evaluate the associated costs or penalty.
5. **Focus on Grey-Areas.** Highlighting of grey areas or bottlenecks in the production process is the most significant merit of Linear Programming. During the periods of bottlenecks, imbalances occur in the production department. Some of the machines remain idle for long periods of time, while the other machines are unable to meet the demand even at the peak performance level.
6. **Flexibility.** Linear Programming is an adaptive and flexible mathematical technique and hence can be utilized in analyzing a variety of multi-dimensional problems quite successfully.
7. **Creation of Information Base.** By evaluating the various possible alternatives in the light of the prevailing constraints, Linear Programming models provide an important database from which the allocation of precious resources can be done rationally and judiciously.

8. **Maximum optimal Utilization of Factors of Production.** Linear Programming helps in optimal utilization of various existing factors of production such as installed capacity,. labour and raw materials etc.

0.2.4 Limitations of Linear Programming

Although Linear Programming is a highly successful having wide applications in business and trade for solving optimization' problems, yet it has certain demerits or defects. Some of the important-limitations in the application of Linear Programming are as follows:

1. **Linear Relationship.** Linear Programming models can be successfully applied only in those situations where a given problem can clearly be represented in the form of linear relationship between different decision variables. Hence it is based on the implicit assumption that the objective as well as all the constraints or the limiting factors can be stated in term of linear expressions - which may not always hold good in real life situations. In practical business problems, many objective function and constraints can not be expressed linearly. Most of the business problems can be expressed quite easily in the form of a quadratic equation (having a power 2) rather than in the terms of linear equation. Linear Programming fails to operate and provide optimal solutions in all such cases.

For example, A problem capable of being expressed in the form of: $ax^2 + bx + c = 0$ where $a \neq 0$ can not be solved with the help of Linear Programming techniques.

2. **Constant Value of objective and Constraint Equations.** Before a Linear Programming technique could be applied to a given situation, the values or the coefficients of the objective function as well as the constraint equations must be completely known. Further, Linear Programming assumes these values to be constant over a period of time. In other words, if the values were to change during the period of study, the technique of LP would lose its effectiveness and may fail to provide optimal solutions to the problem.

However, in real life practical situations often it is not possible to determine the coefficients of objective function and the constraints equations with absolute certainty. These variables in fact may, lie on a probability distribution curve and hence at best, only the likelihood of their occurrence can be predicted. Moreover, often the value's change due to extremely as well as internal factors during the period of study. Due to this, the actual applicability of Linear Programming tools may be restricted.

3. **No Scope for Fractional Value Solutions.** There is absolutely no certainty that the solution to a LP problem can always be quantified as an integer quite often, Linear Programming may give fractional-varied answers, which are rounded off to the next integer. Hence, the solution would not be the optimal one. For example, in finding out the number of men and machines required to perform a particular job, a fractional Larson-integer solution would be meaningless.

4. **Degree Complexity.** Many large-scale real life practical problems can not be solved by employing Linear Programming techniques even with the help of a computer due to highly complex and Lengthy calculations. Assumptions and approximations are required to be made so that the, given problem can be broken down into several smaller problems and, then solve separately. Hence, the validity of the final result, in all such cases, may be doubtful:
5. **Multiplicity of Goals.** The long-term objectives of an organisation are not confined to a single goal. An organisation ,at any point of time in its operations has a multiplicity of goals or the goals hierarchy - all of which must be attained on a priority wise basis for its long term growth. Some of the common goals can be Profit maximization or cost minimization, retaining market share, maintaining leadership position and providing quality service to the consumers. In cases where the management has conflicting, multiple goals, Linear Programming model fails to provide an optimal solution. The reason being that under Linear Programming techniques, there is only one goal which can be expressed in the objective function. Hence in such circumstances, the situation or the given problem has to be solved by the help of a different mathematical programming technique called the “Goal Programming”.
6. **Flexibility.** Once a problem has been properly quantified in terms of objective function and the constraint equations and the tools of Linear Programming are applied to it, it becomes very difficult to incorporate any changes in the system arising on account of any change in the decision parameter. Hence, it lacks the desired operational flexibility.

0.2.5 Mathematical model of LPP

Linear Programming is a mathematical technique for generating and selecting the optimal or the best solution for a given objective function. Technically, Linear Programming may be formally defined as a method of optimizing (i.e.; maximizing or minimizing) a linear function for a number of constraints stated in the form of linear in equations.

Mathematically the problem of Linear Programming may be stated as that of the optimization of linear objective function of the following form:

Optimize: $Z = C_1x_1 + C_2x_2 + \cdots + C_nx_n$

Subject to the Linear constraints of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\{ \leq, =, \geq \} b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\{ \leq, =, \geq \} b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\{ \leq, =, \geq \} b_m \end{aligned}$$

and the non-negative condition

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

These are called the non-negative constraints.

From the above, it is clear that a LP problem has:

- (i) Linear objective function which is to be maximized or minimized.
- (ii) Various linear constraints which are simply the algebraic statement of the limits of the resources or inputs at the disposal.
- (iii) Non-negatively constraints.

Linear Programming is one of the few mathematical tools that can be used to provide solution to a wide variety of large, complex managerial problems.

A firm which distributes products over a large territory faces an unimaginable number of different choices in deciding how best to meet demand from its network of godown and warehouses. Each warehouse has a very limited number of items and demands often can not be met from the nearest warehouse. If there are 25 warehouses and 1,000 customers, there are 25,000 possible match ups between customer and warehouse. LP can quickly recommend the shipping quantities and destinations so as to minimize the cost of total distribution.

These are just a few of the managerial problems that have been addressed successfully by

LP. A few others are described throughout this text. Project scheduling can be improved by allocating funds appropriately among the most critical task so as to most effectively reduce the overall project duration. Production planning over a year or more can reduce costs by careful timing of the use of overtime and inventory to control changes in the size of the workforce. In the short run, personnel work schedules must take into consideration not only the production, work preferences for day offs and absenteeism etc.

Besides recommending solutions to problems like these, LP can provide useful information for managerial decisions, that can be solved by Linear Programming. The application, however, rests on certain postulates and assumptions which have to hold good for the optimality of the solution to be effective during the planning period.

0.2.6 Linear Programming problem Formation

Steps In Formulating A Linear Programming Model

Linear programming is one of the most useful techniques for effective decision making. It is an optimization approach with an emphasis on providing the optimal solution for resource allocation. How best to allocate the scarce organizational or national resources among different competing and conflicting needs (or uses) forms the core of its working. The scope for application of linear programming is very wide and it occupies a central place in many diversified decisional problems. The effective use and application of linear programming requires the formulation of a realistic model which represents accurately the objectives of the decision making subject to the constraints in which it is required to be made.

The basic steps in formulating a linear programming model are as follows:

Step I. Identification of the decision variables. The decision variables (parameters) having a bearing on the decision at hand shall first be identified, and then expressed or determined in the form of linear algebraic functions or in equations.

Step II. Identification of the constraints. All the constraints in the given problem which restrict the operation of a firm at a given point of time must be identified in this stage. Further these constraints should be broken down as linear functions in terms of the pre-defined decision variables.

Step III. Identification of the objective function. In the last stage, the objective which is required to be optimized (i.e., maximized or minimized) must be dearly identified and expressed in terms of the pre-defined decision variables.

Step IV. Non negativity condition. The decision variable cannot take negative values. Hence they must carry a value greater than or equal to zero.

Step V. The LPP gets formulated when we express the objective function and the inequalities (constraints) as explained in the earlier steps.

Examples of Formulation of Linear Program problem

Example 0.2.1. High Quality furniture Ltd. Manufactures two products, tables & chairs. Both the products have to be processed through two machines M1 & M2 the total machine-hours available are: 200 hours of M1 and 400 hours of M2 respectively. Time in hours required for producing a chair and a table on both the machines is as follows:

Machine	Table	Chair
M_1	7	4
M_2	5	5

Profit from the Sale of table is Ksh.50 and that from a chair is Ksh.30 determine optimal mix of tables and chairs so as to maximize the total profit.

Solution. Let x_1 = number of tables produced and x_2 = number of chairs produced.

Step I. The objective function for maximizing the profit is given by maximize

$$Z = 50x_1 + 30x_2 \text{ (objective function)}$$

(Since profit per unit from a table and a chair is Ksh.50 and Ksh.30 respectively).

Step II. List down all the constraints.

(i) Total time on machine M1 can not exceed 200 hours.

$$7x_1 + 4x_2 \leq 200$$

(Since it takes 7 hours to produce a table and 4 hours to produce a chair on machine M1)

(ii) Total time on machine M2 cannot exceed 400 hours.

$$5x_1 + 5x_2 \leq 400$$

(Since it takes 5 hours to produce both a table & a chair on machine M2)

Step III. Presenting the problem. The given problem can now be formulated as a linear programming model as follows:

$$\text{Maximize: } Z = 50x_1 + 30x_2$$

Subject to:

$$7x_1 + 4x_2 \leq 200$$

$$5x_1 + 5x_2 \leq 400$$

$$x_1 \geq 0, x_2 \geq 0$$

Example 0.2.2. The Marangi Company Ltd owns a small paint factory that produces both interior and exterior house paints for wholesale distribution. Two basic raw materials, A and B, are used to manufacture the paints. The maximum availability of A is 6 tonnes a day; that of B is 8 tonnes a day. The daily requirements of the raw materials per tonne of interior and exterior paints are summarized in the table below:

Tonnes of raw materials per tonne of paint				
	Exterior	Interior	Max.	Availability (tonnes)
Raw material A	1	2		6
Raw material B	2	1		8

A market survey has established that the daily demand for interior paint cannot exceed that of exterior paint by more than one tonne. The survey also shows that the maximum demand for interior paint is limited to 2 tonnes daily.

The wholesale price per tonne is Ksh.3,000 for exterior paint and Ksh.2,000 for interior paint.

How much interior and exterior paints should the company produce daily to maximize gross income?

Solution. The construction of a mathematical model can be initiated by answering the following three questions.

1. What does the model seek to determine? In other words, what are the variables (unknown) of the problem?
2. What constraints must be imposed on the variables to satisfy the limitations of the modeled system?
3. What is the objective (goal) that need to be achieved?

Verbal Summary of the Problem

The company seeks to determine the amounts (in tons) of interior and exterior paints to be produced to maximize (increase as much as is feasible) the total gross income (in thousands of shillings) while satisfying the constraints of demand and raw material usage.

Mathematical Model

1. Identify variables

$$\begin{aligned}x_1 &= \text{tons produced daily of exterior paint} \\x_2 &= \text{tons produced daily of interior paint}\end{aligned}$$

2. Objective function

If we let z represent the total gross revenue (in thousands of shillings), the objective function is

$$z = 3x_1 + 2x_2$$

The goal is to get the (feasible) values of x_1 and x_2 that maximizes this criterion.

3. Constraints

(usage of raw materials by both paints) \leq (maximum raw material available)

This leads to the following restrictions (see that data for the problem).

$$\begin{aligned}x_1 + 2x_2 &\leq 6 \text{ (raw material A)} \\2x_1 + x_2 &\leq 8 \text{ (raw material B)}\end{aligned}$$

4. The demand restriction

(excess amount of interior over exterior paint) ≤ 1 ton per day

(demand for interior paint) ≤ 2 ton per day

Mathematically, these are expressed, respectively as;

$$\begin{aligned}x_2 - x_1 &\leq 1 \text{ (excess of interior over exterior paint)} \\x_2 &\leq 2 \text{ (maximum demand for interior paint)}\end{aligned}$$

An implicit (or “understood-to-be”) constraint is that the amount produced of each paint cannot be negative (less than zero).

We impose the nonnegativity restrictions:

$$\begin{aligned}x_1 &\geq 0 \text{ (exterior paint)} \\x_2 &\geq 0 \text{ (interior paint)}\end{aligned}$$

The values of the variables x_1 and x_2 are said to constitute a feasible solution if they satisfy all the constraints of the model.

The complete mathematical model for the Marangi problem may be summarized as follows:

determine the tons of interior and exterior paints, x_2 and x_1 , to be produced to

Maximize: $p = 3x_1 + 2x_2$ (Objective function)

Subject to:

$$\begin{aligned}x_1 + 2x_2 &\leq 6 \\2x_1 + x_2 &\leq 8 \\-x_1 + x_2 &\leq 1 \\x_2 &\leq 2 \\x_1 &\geq 0, x_2 \geq 0\end{aligned}$$

The two equations yield $x_1 = 3\frac{1}{3}$ and $x_2 = 1\frac{1}{3}$. the solution thus says that the daily production should be $3\frac{1}{3}$ tons of exterior paint and $1\frac{1}{3}$ tons of interior paint.

The associated revenue is

$$\begin{aligned} z &= 3\left(3\frac{1}{3}\right) + 2\left(1\frac{1}{3}\right) \\ &= 12\frac{2}{3} \text{ thousand shillings} \end{aligned}$$

Example 0.2.3. A certain company has a plant which produces two products A and B. Each unit contributes Ksh 10 and Ksh 12 respectively to their profits.

Each product passes through 3 departments of the plant and the time requirements for the products are as follows:

Department	Products		Total time Available
	A	B	
1	4	2	1500
2	3	3	1200
3	2	1	800

Set up a Linear program model to maximize the profits.

Solution. Decision variables: let x_1 and x_2 be the number of units of product A and B respectively.

Objective function is to maximize profit

$$\text{Maximize: } p = 10x_1 + 12x_2$$

Subject to:

$$4x_1 + 2x_2 \leq 1500 \text{ (constraint on time in dept 1)}$$

$$3x_1 + 3x_2 \leq 1200 \text{ (constraint on time in dept 2)}$$

$$2x_1 + x_2 \leq 800 \text{ (constraint on time in dept 3)}$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Example 0.2.4. A farmer has 50 ha of land on which to grow maize and beans. He has a capital of Ksh 54,000 and 1 ha of maize requires Ksh 600 while that of beans requires Ksh 1,200, then he has 160 employees and each ha of maize requires 2 employees to cultivate while each ha of beans requires 4 employees. The market profit is Ksh 3,000 per ha of maize and Ksh 4,000 per ha of beans. Assuming that the farmer intends to maximize his profits, formulate the underlying linear program and solve it.

Solution. Let x_1 be the number of hectares on maize and x_2 be the number of hectares on beans, then

$$p = 3000x_1 + 4000x_2$$

Therefore the linear program is

$$\text{Maximize: } p = 3000x_1 + 4000x_2$$

Subject to:

$$x_1 + x_2 \leq 50 \text{ (restriction on land)}$$

$$600x_1 + 1200x_2 \leq 54000 \text{ (restriction on capital)}$$

$$4x_1 + 2x_2 \leq 160 \text{ (restriction on employees)}$$

$$x_1 \geq 0, x_2 \geq 0$$

Example 0.2.5. A certain company produces products from mixtures of silk and wool. The number of units required and profit made per unit is represented below:

Product	Units		Profits
	Silk	Wool	
A	3	2	4
B	1	2	1
C	5	2	7

The maximum amount of wool available is 1800 units and the maximum amount of silk available is 1500 units. Formulate a linear program model for this problem (interest is to maximize profit).

Solution. Let x_1 , x_2 and x_3 be the number of units of products A, B, and C respectively to be produced.

Objective function: becomes Maximize profits

$$p = 4x_1 + x_2 + 7x_3$$

Subject to: (constraints)

$$3x_1 + x_2 + 5x_3 \leq 1500 \text{ (constraint on silk)}$$

$$2x_1 + 2x_2 + 2x_3 \leq 1800 \text{ (constraint on wool)}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \text{ (none of these values can be negative)}$$

Example 0.2.6. An Engineering company can manufacture two different products P_1 and P_2 which have to pass through two machines E_1 and E_2 . the details relating to the time required to process the products, the profits per unit and the available capacity of each machine are given in the following table.

Product	Processing Time/Unit		Profit/Unit
	Machine E_1	Machine E_2	
p_1	3	1	5
p_2	2	1	4
Available hrs/per week	50		22

Solution. In order to maximize profit a decision has to be taken as to how many units of p_1 and p_2 should be manufactured. For this purpose, we suppose as follows:

Let x_1, x_2 be the number of units of p_1 and p_2 respectively to be manufactured. Here we call x_1 and x_2 as *decision variables*.

The objective function and the constraints (inequalities) are formulated as:

$$\text{Maximize: } p = 5x_1 + 4x_2$$

Subject to:

$$3x_1 + 2x_2 \leq 50$$

$$x_1 + x_2 \leq 22$$

$$x_1 \geq 0, x_2 \geq 0$$

Example 0.2.7. The Unilever Ltd has three factories and four warehouses for a particular product. The capacity of each factory and requirement at each warehouse are given below. The cost of transporting 1 unit of the product from a factory to a warehouse is also shown.

Factory	Warehouses				Capacity
	$A_{(1)}$	$B_{(2)}$	$C_{(3)}$	$D_{(4)}$	
Nakuru	3	3	4	7	40
Mombasa	4	2	10	8	50
Eldoret	5	6	9	4	45
Required	50	20	25	40	135

Solution. Decision Variables: Let x_{ij} be the number of units to be transported from factory i ($i = 1, 2, 3$) to warehouse j ($j = 1, 2, 3, 4$).

Then, the Objective function is

$$\begin{aligned} \text{Minimize: } z &= 3x_{11} + 3x_{12} + 4x_{13} + 7x_{14} \\ &\quad + 4x_{21} + 2x_{22} + 10x_{23} + 8x_{24} \\ &\quad + 5x_{31} + 6x_{32} + 9x_{33} + 4x_{34} \end{aligned}$$

Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 40$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 45$$

$$x_{11} + x_{21} + x_{31} \geq 50$$

$$x_{12} + x_{22} + x_{32} \geq 20$$

$$x_{13} + x_{23} + x_{33} \geq 25$$

$$x_{14} + x_{24} + x_{34} \geq 40$$

$$x_{ij} \geq 0 \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4.$$

Example 0.2.8. An electronic company manufactures two radio models, each on a separate rate production line. The daily capacity of the first line is 60 radios and that of the second is 75 radios. Each unit of the first model uses 10 pieces of a certain electronic component, whereas each unit of the second model requires 8 pieces of the same component. the maximum daily availability of the special component is 800 pieces.

The profit per unit of models 1 and 2 is \$30 and \$20 respectively. determine the optimum daily production of each using the simplex method.

Solution. Let the number of units of model 1 be x and those of model 2 be y respectively.

The objective function and the constraints (inequalities) are formulated as:

$$\text{Maximize: } p = 30x + 20y$$

Subject to:

$$x \leq 60$$

$$y \leq 75$$

$$10x + 8y \leq 800$$

$$x \geq 0, y \geq 0$$

Example 0.2.9. A financial institution, The C&J bank, is in the process of formulating its loan policy for the next quarter. A total of \$12 million is allocated for that purpose. Being a full-service facility, the bank is obligated to grant loans to different clientele. the following table provides the types of loans, the interest rate charged by the bank, and the possibility of bad debts as estimated from past experience:

Type of loan	Interest rate	Probability of Bad debt
Personal	0.140	0.10
Car	0.130	0.07
Home	0.120	0.03
Farm	0.125	0.05
Commercial	0.100	0.02

Bad debts are assumed unrecoverable and hence produce no interest revenue.

Competition with other financial institutions in the area requires that the bank allocate at least 40% of the total funds to farm and commercial loans.

To assist the housing industry in the region, home loans must equal at least 50% of the personal, car, and home loans. The bank also has a stated policy specifying that the overall ratio for bad debts on all loans may not exceed 0.04.

Construct the mathematical model for the bank to maximize its net return comprised of the difference between the revenue from interest and lost funds due to bad debts.

0.2.7 The Standard Form of the Linear Programming Model

We have seen that an LP model may include constraints of the types \leq , $=$ and \geq . Moreover, the variables may be nonnegative or unrestricted in sign.

To develop a general solution method, the LP problem must be put in a common format, which we call the **standard form**. The properties of the standard Linear Program form are;

1. All the constraints are equations with nonnegative right hand side.
2. All the variables are nonnegative.
3. The objective function may be maximization or minimization.

We now show how any LP model can be put in the standard format.

Constraints

1. A constraint of the type \leq or \geq can be converted to an equation by adding a slack variable to (subtracting a surplus variable from) the left side of the constraint.

For example, ,in the constraint

$$x_1 + 2x_2 \leq 6$$

we add a slack $s_1 \geq 0$ to the left side to obtain the equation

$$x_1 + 2x_2 + s_1 = 6 \quad s_1 \geq 0$$

If the constraint represents the limit on the usage of a resource, s_1 will represent the slack or unused amount of the resource.

Next, consider the constraint;

$$3x_1 + 2x_2 - 3x_3 \geq 5$$

Since the left side is not smaller than the right side, we subtract a surplus variable, $s_2 \geq 0$ from the left side to obtain the equation.

$$3x_1 + 2x_2 - 3x_3 - s_2 = 5; \quad s_2 \geq 0.$$

2. The right side of an equation can always be made nonnegative by multiplying both sides by -1 .

For example; $2x_1 + 3x_2 - 7x_3 = -5$ is mathematically equivalent to

$$-2x_1 - 3x_2 + 7x_3 = 5.$$

3. The direction of an inequality is reversed when both sides are multiplied by -1 .

For example: Whereas $2 \leq -2 \geq -4$. Thus, the inequality $2x_1 - x_2 \leq -5$ can be replaced by

$$-2x_1 + x_2 \geq 5.$$

Variables

An unrestricted variable y_i can be expressed in terms of two nonnegative variables by using the substitution.

$$y_i = y'_i - y''_i \quad y'_i, -y''_i \geq 0$$

The substitution must be effected throughout all the constraints and in the objective function.

The LP problem is normally solved in terms of y'_i and y''_i , from which y_i is determined by reverse substitution. An interesting property of y'_i and y''_i is that in the optimal (simplex) LP solution only one of the two variables can assume a positive value, but never both.

Objective Function

Although the standard LP model can be of either the maximization or the minimization type, it is sometimes useful to convert one form to the other.

The maximization of a function is equivalent to the minimization of the negative of the same function and vice versa.

For example; Maximize: $z = 5x_1 + 2x_2 + 3x_3$ is mathematically equivalent to

$$\text{Minimize: } -z = -5x_1 - 2x_2 - 3x_3$$

Example 0.2.10. Write the following LP model in the standard form.

$$\text{Minimize: } z = 2x_1 + 3x_2$$

Subject to:

$$\begin{aligned} x_1 + x_2 &= 10 \\ -2x_1 + 3x_2 &\leq -5 \\ 7x_1 - 4x_2 &\leq 6 \\ x_1 \text{ (unrestricted)}, x_2 &\geq 0 \end{aligned}$$

Solution. To write the LP in standard form, we do the following;

1. Multiply the second constraint by -1 and subtract a surplus variable $s_2 \geq 0$ from the left side.
2. Add a slack variable $s_3 \geq 0$ to the left side of the third constraint.
3. Substitute $x_1 = x'_1 - x''_1$ where $x'_1, x''_1 \geq 0$ in the objective function and all the constraints.

Thus we get the standard form as

$$\text{Minimize: } z = 2x_1' - 2x_1'' + 3x_2$$

Subject to:

$$x_1' - x_1'' + x_2 = 10$$

$$2x_1' - 2x_1'' - 3x_2 - s_2 = 5$$

$$7x_1' - 7x_1'' - 4x_2 + s_3 = 6$$

$$x_1', x_1'', x_2, s_2, s_3 \geq 0$$

Assignment I due in two weeks time from today

Lecture 3: Methods for the Solution of the LPP

0.3 Methods for the Solution of a LPP

Linear Programming, is a method of solving the type of problem in which two or more **candidates** or **activities** are competing to utilize the available limited resources, with a view to **optimize** the **objective function** of the problem. The objective may be to maximize the **returns** or to minimize the **costs**. The various methods available to solve the problem are:

1. The **Graphical Method**. When we have two decision variables in the problem. (To deal with more decision variables by graphical method will become complicated, because we have to deal with planes instead of straight lines. Hence in graphical method let us limit ourselves to two variable problems.)
2. The **Systematic Trial and Error method**, where we go on giving various values to variables until we get optimal solution. This method takes too much of time and laborious, hence this method is not discussed here.
3. The **Vector method**. In this method each decision variable is considered as a vector and principles of vector algebra is used to get the optimal solution. This method is also time consuming, hence it is not discussed here.
4. The **Simplex method**. When the problem is having more than two decision variables, simplex method is the most powerful method to solve the problem. It has a systematic programme, which can be used to solve the problem.

0.3.1 Graphical method

In graphical method, the inequalities (structural constraints) are considered to be equations. This is because; one cannot draw a graph for inequality. Only two variable problems are considered, because we can draw straight lines in two-dimensional plane (X - axis and Y - axis). More over as we have nonnegativity constraint in the problem that is all the decision variables must have positive values always the solution to the problem lies in first quadrant of the graph. Some times the value of variables may fall in quadrants other than the first quadrant. In such cases, the line joining the values of the variables must be extended in to the first quadrant.

We seek to understand the **importance of graphical method of solution in linear programming** and seek to find out as to how the graphical method of solution be used to generate optimal solution to a Linear Programming problem. This method provides a basis for understanding the other methods of solution.

Once the Linear programming model has been formulated on the basis of the given objective and the associated constraint functions, the next step is to solve the problem and

obtain the best possible or the optimal solution various mathematical and analytical techniques can be employed for solving the Linear programming model.

The graphical solution procedure consists of the following steps:

Step I. Defining the problem. Formulate the problem mathematically. Express it in terms of several mathematical constraints and an objective function. The objective function relates to the optimization aspect is, maximisation or minimisation Criterion.

Step II. Plot the constraints Graphically. Each inequality in the constraint equation has to be treated as an equation. An arbitrary value is assigned to one variable and the value of the other variable is obtained by solving the equation. In the similar manner, a different arbitrary value is again assigned to the variable and the corresponding value of other variable is easily obtained.

These 2 sets of values are now plotted on a graph and connected by a straight line. The same procedure has to be repeated for all the constraints. Hence, the total straight lines would be equal to the total no of equations, each straight line representing one constraint equation.

Step III. Locate the solution space. Solution space or the feasible region is the graphical area which satisfies all the constraints at the same time. Such a solution point (x, y) always occurs at the **corner points of the feasible Region**. The feasible region is determined as follows:

- (a) For “greater than” and “greater than or equal to” constraints (i.e.,), the feasible region or the solution space is the area that **lies above the constraint lines**.
- (b) For “less than” and “less than or equal to” constraint (i.e.,). The feasible region or the solution space is the area that **lies below the constraint lines**.

Step IV. Selecting the graphic technique. Select the appropriate graphic technique to be used for generating the solution. Two techniques viz; Corner Point Method and Iso-profit (or Iso-cost) method may be used, however, it is easier to generate solution by using the corner point method.

Corner Point Method.

- (i) Since the solution point (x, y) always occurs at the corner point of the feasible or solution space, identify each of the extreme points or corner points of the feasible region by the method of simultaneous equations.
- (ii) By putting the value of the corner point's co-ordinates [e.g. (2,3)] into the objective function, calculate the profit (or the cost) at each of the corner points.
- (iii) In a maximisation problem, the optimal solution occurs at that corner point which gives the highest profit.

In a minimisation problem, the optimal solution occurs at that corner point which gives the lowest profit.

One problem with two variables is solved by using both graphical and simplex method, so as to enable the reader to understand the relationship between the two.

Example 0.3.1. Feasible Region

A company purchasing scrap material has two types of scarp materials available. The first type has 30% of material X, 20% of material Y and 50% of material Z by weight. The second type has 40% of material X, 10% of material Y and 30% of material Z. The costs of the two scraps are Kshs. 120 and Kshs. 160 per kg respectively. The company requires at least 240 kg of material X, 100 kg of material Y and 290 kg of material Z . Find the optimum quantities of the two scraps to be purchased so that the company requirements of the three materials are satisfied at a minimum cost.

Solution. First we have to formulate the linear programming model. Let us introduce the decision variables x_1 and x_2 denoting the amount of scrap material to be purchased. Here the objective is to minimize the purchasing cost. So, the objective function here is

Minimize

$$120x_1 + 160x_2$$

Subject to:

$$0.3x_1 + 0.4x_2 \geq 240$$

$$0.2x_1 + 0.1x_2 \geq 100$$

$$0.5x_1 + 0.3x_2 \geq 290$$

$$x_1 \geq 0; x_2 \geq 0$$

Multiply by 10 both sides of the inequalities, then the problem becomes

Multiply by 10 both sides of the inequalities, then the problem becomes

Minimize

$$120x_1 + 160x_2$$

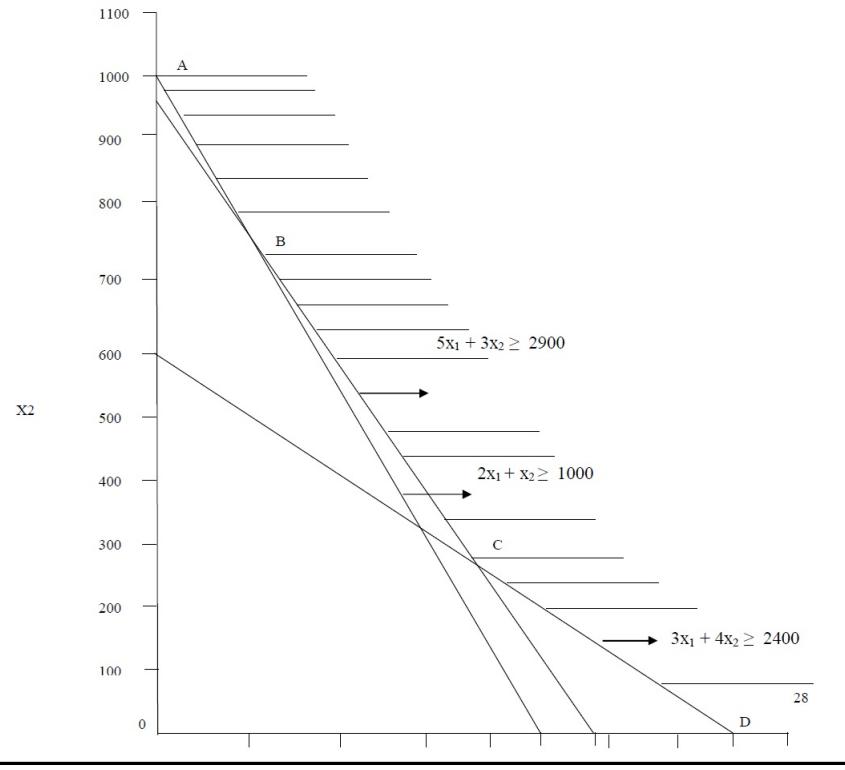
Subject to:

$$3x_1 + 4x_2 \geq 2400$$

$$2x_1 + x_2 \geq 1000$$

$$5x_1 + 3x_2 \geq 2900$$

$$x_1 \geq 0; x_2 \geq 0$$



Graph: Feasible Region, Multiple Optimal Solutions The extreme points are A, B, C, and D. One of the objective functions $120x_1 + 160x_2 = M$ family coincides with the line CD at the point C with value $M = 96000$, and the optimum value variables are $x_1 = 400$, and $x_2 = 300$. And at the point D with value $M = 96000$, and the optimum value variables are $x_1 = 800$, and $x_2 = 0$.

Thus, every point on the line CD minimizes objective function value and the problem contains multiple optimal solutions.

Example 0.3.2. Unbounded Solution

When the feasible region is unbounded, a maximization problem may don't have optimal solution, since the values of the decision variables may be increased arbitrarily. This is illustrated with the help of the following problem.

$$\begin{aligned}
 & \text{Maximize} \\
 & 3x_1 + x_2 \\
 & \text{Subject to:} \\
 & x_1 + x_2 \geq 6 \\
 & -x_1 + x_2 \leq 6 \\
 & -x_1 + 2x_2 \geq -6
 \end{aligned}$$

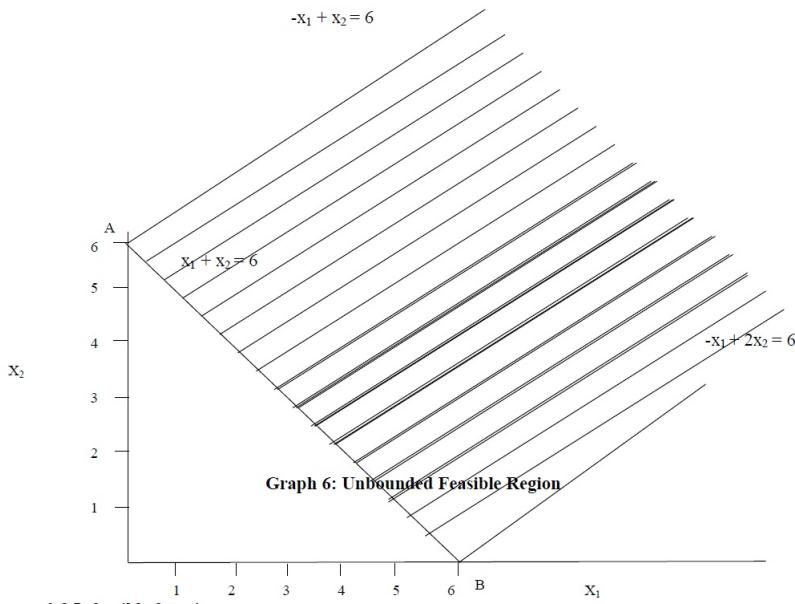
and

$$x_1, x_2 \geq 0$$

Graph: shows the unbounded feasible region and demonstrates that the objective function can be made arbitrarily large by increasing the values of x_1 and x_2 within the

unbounded feasible region. In this case, there is no point (x_1, x_2) is optimal because there are always other feasible points for which objective function is larger. Note that it is not the unbounded feasible region alone that precludes an optimal solution. The minimization of the function subject to the constraints shown in the Graph would be solved at one the extreme point (A or B).

The unbounded solutions typically arise because some real constraints, which represent a practical resource limitation, have been missed from the linear programming formulation. In such situation the problem needs to be reformulated and re-solved.



Example 0.3.3. A company manufactures two types of boxes, corrugated and ordinary cartons. The boxes undergo two major processes: cutting and pinning operations. The profits per unit are Ksh.6 and Ksh.4 respectively. Each corrugated box requires 2 minutes for cutting and 3 minutes for pinning operation, whereas each carton box requires 2 minutes for cutting and 1 minute for pinning. The available operating time is 120 minutes and 60 minutes for cutting and pinning machines. Determine the optimum quantities of the two boxes to maximize the profits.

Solution. **Key Decision:** To determine how many (number of) corrugated and carton boxes are to be manufactured.

Decision variables: Let x_1 be the number of corrugated boxes to be manufactured and x_2 be the number of carton boxes to be manufactured.

Objective Function: The objective is to maximize the profits. Given profits on corrugated box and carton box are Ksh.6 and Ksh.4 respectively.

The objective function is, $Z_{\max} = 6x_1 + 4x_2$

Constraints: The available machine-hours for each machine and the time consumed by each product are given.

Therefore, the constraints are,

$$\text{Maximize: } Z = 6x_1 + 4x_2$$

Subject to:

$$2x_1 + 3x_2 \leq 120$$

$$2x_1 + x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

Graphical Solution: As a first step, the inequality constraints are removed by replacing “equal to” sign to give the following equations:

$$2x_1 + 3x_2 = 120$$

$$2x_1 + x_2 = 60$$

Find the co-ordinates of the lines by substituting $x_1 = 0$ and $x_2 = 0$ in each equation.

The line $2x_1 + 3x_2 = 120$ passes through co-ordinates $(0, 40)$, $(60, 0)$.

The line $2x_1 + x_2 = 60$ passes through co-ordinates $(0, 60)$, $(30, 0)$.

The lines are drawn on a graph with horizontal and vertical axis representing boxes x_1 and x_2 respectively.

Now, the objective is to maximize the profit. The point that lies at the furthermost point of the feasible area will give the maximum profit. To locate the point, we need to plot the objective function (profit) line. Therefore, we conclude that to maximize profit, 15 numbers of corrugated boxes and 30 numbers of carton boxes should be produced to get a maximum profit. Substituting $x_1 = 15$ and $x_2 = 30$ in objective function, we get

$$Z_{\max} = 6x_1 + 4x_2 = 6(15) + 4(30)$$

Maximum profit: Ksh.210.00

Example 0.3.4. A company manufactures two products, X and Y by using three machines A, B, and C. Machine A has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines B and C during the coming week is 24 hours and 35 hours respectively. One unit of product X requires one hour of Machine A, 3 hours of machine B and 10 hours of machine C. Similarly one unit of product Y requires 1 hour, 8 hour and 7 hours of machine A, B and C respectively. When one unit of X is sold in the market, it yields a profit of Ksh.5/- per product and that of Y is Ksh.7/- per unit. Solve the problem by using graphical method to find the optimal product mix.

Solution. Let the company manufactures x units of X and y units of Y , and then the L.P.

model is:

$$\text{Maximize: } Z = 5x + 7y$$

Subject to:

$$x + y \leq 4$$

$$3x + 8y \leq 24$$

$$10x + 7y \leq 35$$

$$x, y \geq 0$$

As we cannot draw graph for inequalities, let us consider them as equations.

$$\text{Maximize: } Z = 5x + 7y$$

Subject to:

$$x + y = 4$$

$$3x + 8y = 24$$

$$10x + 7y = 35$$

$$x, y \geq 0$$

Let us take machine A and find the boundary conditions. If $x = 0$, machine A can manufacture $4/1 = 4$ units of y .

Similarly, if $y = 0$, machine A can manufacture $4/1 = 4$ units of x . For other machines:

Machine B When $x = 0$, $y = 24/8 = 3$ and when $y = 0$, $x = 24/3 = 8$

Machine C When $x = 0$, $y = 35/10 = 3.5$ and when $y = 0$, $x = 35/7 = 5$.

These values we can plot on a graph, taking product X on x -axis and product Y on y -axis.

Here we find the co-ordinates of corners of the closed polygon ROUVW and substitute the values in the objective function. In maximisaton problem, we select the co-ordinates giving maximum value. And in minimisaton problem, we select the co-ordinates, which gives minimum value. In the problem the co-ordinates of the corners are:

$$R = (0, 3.5), O = (0, 0), U = (3.5, 0), V = (2.5, 1.5) \quad \text{and} \quad W = (1.6, 2.4).$$

Substituting these values in objective function:

$$Z_{(1.6, 2.4)} = 5 \times 1.6 + 7 \times 2.4 = \text{Ksh.}24.80 \quad \text{at point } W$$

Hence the optimal solution for the problem is company has to manufacture 1.6 units of product X and 2.4 units of product Y , so that it can earn a maximum profit of Ksh.24.80 in the planning period.

Question 0.3.1. A company manufactures two products X and Y. The profit contribution of X and Y are Ksh.3/- and Ksh.4/- respectively. The products X and Y require the services of four facilities. The capacities of the four facilities A, B, C, and D are limited and the available capacities in hours are 200 Hrs, 150 Hrs, and 100 Hrs. and 80 hours respectively. Product X requires 5, 3, 5 and 8 hours of facilities A, B, C and D respectively. Similarly the requirement of product Y is 4, 5, 5, and 4 hours respectively on A, B, C and D. Find the optimal product mix to maximise the profit.

Question 0.3.2. An aviation fuel manufacturer sells two types of fuel A and B. Type A fuel is 25% grade 1 gasoline, 25% of grade 2 gasoline and 50% of grade 3 gasoline. Type B fuel is 50% of grade 2 gasoline and 50% of grade 3 gasoline. Available for production are 500 liters per hour grade 1 and 200 liters per hour of grade 2 and grade 3 each. Costs are 60 paise per liter for grade 1, 120 paise for grade 2 and 100 paise for grade 3. Type A can be sold at Ksh.7.50 per liter and B can be sold at Ksh.9.00 per liter. How much of each fuel should be made and sold to maximise the profit.

Question 0.3.3. A company manufactures two products X_1 and X_2 on three machines A, B, and C. X_1 require 1 hour on machine A and 1 hour on machine B and yields a revenue of Ksh.3/- . Product X_2 requires 2 hours on machine A and 1 hour on machine B and 1 hour on machine C and yields revenue of Ksh.5/- . In the coming planning period the available time of three machines A, B, and C are 2000 hours, 1500 hours and 600 hours respectively. Find the optimal product mix.

Important Theorems

While obtaining the optimum feasible solution to the linear programming problem, the statement of the following four important theorems is used:

Theorem 0.3.1. *The feasible solution space constitutes a convex set.*

Theorem 0.3.2. *Within the feasible solution space, feasible solution correspond to the extreme (or Corner) points of the feasible solution space.*

Theorem 0.3.3. *There are a finite number of basic feasible solution with the feasible solution space.*

Theorem 0.3.4. *The optimum feasible solution, if it exists. will occur at one, or more, of the extreme points that are basic feasible solutions.*

Note: Convex set is a polygon “Convex” implies that if any two points of the polygon are selected arbitrarily then straight line segment Joining these two points lies completely within the polygon. The extreme points of the convex set are the basic solution to the linear programming problem.

Important Terms

Some of the important terms commonly used in linear programming are disclosed as follows:

- (i) **Solution.** Values of the decision variable x , ($i = 1, 2, 3, \dots, n$) satisfying the constraints of a general linear programming model is known as the solution to that linear programming model.
- (ii) **Feasible solution.** Out of the total available solution, a solution that also satisfies the non-negativity restrictions of the linear programming problem is called a **feasible solution**.
- (iii) **Basic solution.** For a set of simultaneous equations in Q unknowns (pQ) a solution obtained by setting $(P - Q)$ of the variables equal to zero and solving the remaining P equation in P unknowns is known as a basic solution.

The variables which take zero values at any solution are detained as non-basic variables and remaining are known as-basic variables, often called basic.

- (iv) **Basic feasible solution.** A feasible solution to a general linear programming problem which is also basic solution is called a basic feasible solution.
- (v) **Optimal feasible solution.** Any basic feasible solution which optimizes (ie; maximise or minimises) the objective function of a linear programming modes known as the optimal feasible solution to that linear programming model.

(vi) **Degenerate Solution.** A basic solution to the system of equations is termed as degenerate if one or more of the basic variables become equal to zero.

Example 0.3.5. A farmer has 50 ha of land on which to grow maize and beans. He has a capital of Ksh 54,000 and 1 ha of maize requires Ksh 600 while that of beans requires Ksh 1,200, then he has 160 employees and each ha of maize requires 2 employees to cultivate while each ha of beans requires 4 employees. The market profit is Ksh 3,000 per ha of maize and Ksh 4,000 per ha of beans. Assuming that the farmer intends to maximize his profits, formulate the underlying linear program and solve it.

Solution. Let x_1 be the number of hectares on maize and x_2 be the number of hectares on beans, then

$$P = 3000x_1 + 4000x_2$$

Therefore the linear program is

$$\text{Maximize: } P = 3000x_1 + 4000x_2$$

Subject to:

$$x_1 + x_2 \leq 50 \text{ (restriction on land)}$$

$$600x_1 + 1200x_2 \leq 54000 \text{ (restriction on capital)}$$

$$4x_1 + 2x_2 \leq 160 \text{ (restriction on employees)}$$

$$x_1 \geq 0, x_2 \geq 0$$

Graph for inequalities

Note: The feasible region obtained is a convex set with extreme points. A, B, C, D, E . The optimal solution lies in one of the extreme points.

The feasible solution (on the extreme points) are:-

Points: (x_1, x_2)	Profits = $3000x_1 + 4000x_2$
A(0, 0)	0
B(0, 45)	180,000
C(10, 40)	190,000
D(30, 20)	170,000
E(40, 0)	120,000

Point $C(10, 40)$ provides the optimal solution.

The farmer should cultivate 10 ha of maize and 40 ha of beans in order to make the maximum profit of Ksh.190,000.

Example 0.3.6. Using the graphical method for the solution of LP problems, calculate the value of x_1 and x_2 which would maximize P , where;

$$\text{Maximize: } P = 6x_1 + 7x_2$$

Subject to:

$$2x_1 + 3x_2 \leq 12$$

$$2x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Example 0.3.7. A company manufactures two products X and Y , which require, the following resources. The resources are the capacities machine M1, M2, and M3. The available capacities are 50, 25, and 15 hours respectively in the planning period. Product X requires 1 hour of machine M2 and 1 hour of machine M3. Product Y requires 2 hours of machine M1, 2 hours of machine M2 and 1 hour of machine M3. The profit contribution of products X and Y are Ksh.5/- and Ksh.4/- respectively. Formulate a mathematical model of the problem.

Example 0.3.8. A retail store stocks two types of shirts A and B. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type A and a maximum of 300 shirts of type B. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type A shirt fetches a profit of Ksh.2/- per unit and type B a profit of Ksh.5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

Example 0.3.9. X Ltd wishes to purchase a maximum of 3600 units of a product two types of product a. & are available in the market Product a occupies a space of 3 cubic feet & cost Rs. 9 whereas occupies a space of 1 cubic feet & cost Rs. 13 per unit. The budgetary constraints of the company do not allow to spend more than Rs. 39,000. The total availability of space in the company's godown is 6000 cubic feet. Profit margin of both the product a & is Rs. 3 & Rs. 4 respectively. Formulate as a linear programming model and solve using graphical method. You are required to ascertain the best possible combination of purchase of a & so that the total profits are maximized.

0.3.2 Special Cases of LPP

In the previous lecture we have discussed some linear programming problems which may be called “well behaved” problems. In such cases, a solution was obtained, in some cases it took less effort while in some others it took a little more. But a solution was finally obtained.

Special cases of linear programming problems are:

- Alternative Optimal Solutions
- Degeneracy
- Unboundedness
- Infeasibility (or non existing) Solution

Alternative Optimal Solution

When the objective function is parallel to a binding constraint (a constraint that is satisfied in the equality sense by the optimal solution), the objective function will assume the same optimal value at more than one solution point. For this reason they are called **alternative optima**. The example below shows that normally there is infinity of such solutions. The example also demonstrates the practical significance of encountering alternative optima.

In practice, knowledge of alternative optima is useful because it gives management the opportunity to choose the solution that best suits their situation without experiencing any deterioration in the objective value. In the example, for instance, the solution at B shows that only activity 2 is at a positive level, whereas at C both activities are positive. If the example represents a product-mix situation, it may be advantageous from the standpoint of sales competition to produce two products rather than one. In this case the solution at C would be recommended.

Infeasible 2-var LP's

Consider again the original prototype example, modified by the additional requirements (imposed by the company's marketing department) that the daily production of product P_1 must be at least 30 units, and that of product P_2 should exceed 20 units. These requirements introduce two new constraints into the problem formulation, i.e.,

$$\begin{aligned}x_1 &\geq 30 \\x_2 &\geq 20\end{aligned}$$

Attempting to plot the feasible region for this new problem, we get Figure 2, which indicates that there are no points on the (x_1, x_2) -plane that satisfy all constraints, and therefore our problem is infeasible (over-constrained).

Unbounded 2-var LP's

Example 0.3.10. Fresh Products Ltd. Is engaged in the business of breeding cows quits farm. Since it is necessary to ensure a particular level of nutrients in their diet, Fresh Product Ltd. Buys two products P1 and P2 the details of nutrient constituents in each of which are as follows:

Summarizing the above discussion, I have shown that a 2-var LP can either

- have a unique optimal solution which corresponds to a “corner” point of the feasible region, or
- have many optimal solutions that correspond to an entire “edge” of the feasible region, or
- be unbounded, or be infeasible.

Lecture 4: Simplex Method

By Dr. Antony Ngunyi

0.4 Simplex Method

0.4.1 Limitations of the Graphical Method

Once a Linear programming model has been constructed on the basis of the given constraints and the objective function, it can easily be solved by using the graphical method (as discussed in earlier lectures) and the optimal solution can be generated.

However, the applicability of the graphical method is very limited in scope. This is due to the fact that it is quite simple to identify all the corner points and then test them for optimality-in the case of a two-variable problem. As a result, the graphical method can not be always employed to solve the real-life practical Linear programming models which involve more than two decision-variables.

The above limitation of the graphical method is tackled by what is known as the simplex method. Developed in 1947 by George B-Dantizg, it remains a widely applicable method for solving complex LP problems. It can be applied to any LP problem which can be expressed in terms of a Linear Objective function subject to a set of Linear Constraints. As such, no theoretical restrictions are placed on the number of decision variables or constraints contained in a linear programming problem.

0.4.2 Simplex Method

This is a step by step method which is used to solve linear programming problems with any number of decision variables. The simplex method solves linear programs in iterations where the same computational steps are repeated a number of times before the optimum is reached. The method is a perfect example of the iterative process that characterizes computations in most optimization models.

The method assumes that the linear program can be written in the form

$$\begin{aligned} \text{Maximize: } z &= \underline{c}' \underline{x} \\ \text{Subject to: } A \underline{x} &\leq \underline{b}, \\ \underline{x} &\geq 0 \end{aligned}$$

The simplex method employed in solving LP problem is discussed as under:

This method is used to solve LP problems with any number of variable or constraints as it is geared towards solving optimization problems which have constraints of less than or equal to type.

- (i) This method utilizes the property of a LP problem of having optimal solution only at the corner point of the feasible solution space. It systematically generates corner point solutions and evaluates them for optimality. The method stops when an optimal solution is found. Hence, it is an iterative (repetitive) technique.

If we get more variables and less equations, we can set extra variables equal to zero, to obtain a system of equal variables and equal equations. Such solution is called basic solution.

- (ii) The variables having positive values in a basic feasible solution are called basic variable while the variables which are set equal to zero, so as to define a corner point are called non-basic variables.
- (iii) Slack variables are the fictitious variables which indicate how much of a particular resource remains unused in any solution. These variables can not be assigned negative values. A zero value indicates that all the resources are fully used up in the production process.
- (iv) C_j column denotes the unit contribution margin.
- (v) C_j row is simply a statement of the projective function.
- (vi) Z_j row denotes the contribution margin lost if one unit is brought into the solution. Hence, it represents the opportunity cost. (Opportunity cost is the cost of sacrifice i.e., the opportunity foregone by selecting a particular course of action out of a number of different available alternatives).
- (vii) $C_j - Z_j$ row denotes the Net Potential contribution or the Net unit Margin potential, per unit.

The rules used under simplex method, for solving a linear programming problem are as follows:-

1. **Convert the LP to the following form:** Convert the given problem into Standard maximization Problem i.e. minimization problem into a maximization one (by multiplying the objective function by -1). All variables must be non-negative. All RHS values must be non-negative (multiply both sides by -1 , if needed). All constraints must be in \leq form (except the non-negativity conditions). No strictly equality or \geq constraints are allowed.
2. **Convert all \leq constraints to equalities by adding a different slack variable for each one of them.**
3. **Construct the initial simplex tableau with all slack variables in the BVS.** The last row in the table contains the coefficient of the objective function (row C_j).
4. **Determine whether the current tableau is optimal.** That is: If all RHS values are non-negative (called, the **feasibility condition**)

If all elements of the last row, that is C_j row, are non-positive (called, the **optimality condition**).

If the answers to both of these two questions are yes, then stop. The current tableau contains an **optimal solution**. Otherwise, go to the next step.

5. **If the current BVS is not optimal, determine, which non basic variable should become a basic variable and, which basic variable should become a non basic variable.** To find the new BVS with the better objective function value, perform the following tasks:

- **Identify the entering variable:** The entering variable is the one with the largest positive C_j value (In case of a tie, select the variable that corresponds to the leftmost of the columns).
- **Identify the outgoing variable:** The outgoing variable is the one with smallest non-negative column ratio (to find the column ratios, divide the RHS column by the entering variable column, wherever possible). In case of a tie select the variable that corresponds to the up most of the tied rows.

6. Generate the new tableau

- (a) Select the largest value of $C_j - Z_j$ row. The column, under which this value falls is the pivot-column.
- (b) **Pivot-row selection rule.** Find the ratio of quantity to the corresponding pivotcolumn co-efficient. The pivot-row selected is the variable having the least ratio.

Note: Rows having negative or zero co-efficient in the pivot-column are to be neglected.

- (c) The coefficient, which is in both, the pivot-row and the pivot-column is called the **pivot-element** or **pivot-number**.
- (d) Up-dating Pivot-row. Pivot-row, also called **replaced rows**, are updated as under.

All elements of old-row divided by Pivot-element.

Now, in the basic activities column, write the pivot-column variable in place of the pivot-row variable. i.e.; the pivot-row variable is to be replaced by the pivotcolumn variable.

- (e) Up-Dating all other rows. Up date all other rows by updating the formulae.

$$(\text{New element}) = (\text{Old-row element}) - (\text{Corresponding pivot column element} * \text{updated corresponding pivot row element})$$

- (f). Up-dating Z_j and $C_j - Z_j$ rows. Each Z_j , is obtained as the sum of the products of the C_j column coefficients multiplied by the corresponding coefficient in the

j th column. (i.e.) the Quantity column).

It is then subtracted from $C_j - Z_j$ row values to get $C_j - Z_j$ values.

This pivoting is to be repeated till no positive coefficients exist in the $C_j - Z_j$ row, the optimal solution is known.

What is a Standardization Maximum Problem?

A Standard Maximization Problem is the one that satisfies the following 4 conditions

1. The Objective function is to be maximized.
2. All the inequalities are of \leq type.
3. All right hand constants are non-negative.
4. All variables are non-negative.

The simplex method begins by setting each decision variable to zero and moves progressively until no improvement can be made on the objective function. Let us consider some examples to test our understanding of the solution algorithm that has been discussed so far.

Example 0.4.1. Solve the following LP using the Simplex method (Marangi Company Ltd Paint Example above)

$$\text{Maximize: } z = 3x_1 + 2x_2$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 8 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

The steps of the simplex algorithm are as follows:

Step 0: Write the LP in standard form. In this Case the LP above is already in standard form.

Step 1: Adding the slack variables S_1 , S_2 , S_3 , and S_4 to the constraints leads to

$$\text{Maximize: } z = 3x_1 + 2x_2$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 + S_1 &= 6 \\ 2x_1 + x_2 + S_2 &= 8 \\ -x_1 + x_2 + S_3 &= 1 \\ x_2 + S_4 &= 2 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Step 2: Construct the initial simplex table in which $\underline{x} = 0$ and $X = 0$ (Note: when $x_1 = x_2 = 0$, then $Z = 0$), all the variables of the objective function are set to zero.

Basic	decision variables		slack variables				Soln. Quantity
	x_1	x_2	S_1	S_2	S_3	S_4	
S_1	1	2	1	0	0	0	6
S_2	2	1	0	1	0	0	8
S_3	-1	1	0	0	1	0	1
S_4	0	1	0	0	0	1	2
z	3	2	0	0	0	0	0

The corresponding solution is:

$$x_1 = 0, x_2 = 0, S_1 = 6, S_2 = 8, S_3 = 1 \text{ and } S_4 = 2$$

The unique solution resulting from setting x_1, x_2 variables equal to zero is called **basic solution**. If a basic solution satisfies the non-negativity restrictions, it is called a **feasible basic solution**. i.e.

$S_1 = 6, S_2 = 8, S_3 = 1$ and $S_4 = 2$ is a feasible basic solution.

The variable set equal to zero, i.e. x_1 and x_2 are called **non-basic (zero) variables**, the remaining (S_1, S_2, S_3 , and S_4) are called **basic solutions**.

Step 3: Select an **entering variable** from among the current (zero) non-basic variable using the **optimality condition**.

Definition 0.4.1. The **entering variable** (EV) from among the current (zero) having the largest positive coefficient in the Z -row.

The **optimality condition** of the simplex method: The entering variable in maximization (minimization) is the non-basic variable with the most positive (negative) coefficient in the z -row. A tie is broken arbitrarily.

Note: When all the non-basic coefficients in the Z -row are non-positive (negative), the **optimum** is reached.

Step 4: To identify the **leaving (departing) variable**, we apply the feasibility condition (i.e computing the ratios and identifying the leaving variable).

The departing variable is the basic variable associated with the smallest positive ratio of solution quantity to pivot column (entering column) entries. Ties are broken arbitrarily.

In our case;

The entering variable is x_1 because 3 ia the maximum value in the Z -row.

$$\min \left[\frac{6}{1}, \frac{8}{2}, \frac{1}{-1} \text{ (ignore)}, \frac{2}{0} \text{ (ignore)} \right] = 4$$

From S_2 row the pivot value is 2.

After determining the entering and the leaving variable (by applying the optimality and feasibility conditions, the next iteration (new basic solution) is determined by applying the **Gauss-Jordan method**. The method effects a change in basis by using two types of computations.

(i). Type 1 (Pivot equation):

$$\text{New pivot equation} = \text{Old pivot equation} \div \text{Pivot element}$$

Rename it x_1 .

(ii). Type 2 (all other equations, including z):

$$\text{New equation} = \text{Old equation} - (\text{its entering column coefficient}) \times (\text{New pivot equation})$$

The Second Simplex Table:

	Basic	Decision variables		Slack variables				Soln.
		x_1	x_2	S_1	S_2	S_3	S_4	Quantity
$S_1 - 1(x_1)$	S_1	0	$3/2$	1	$-1/2$	0	0	2
$S_2/2$	x_1	1	$1/2$	0	$1/2$	0	0	4
$S_3 - (-1)x_1$	S_3	0	$3/2$	0	$1/2$	1	0	5
S_4	S_4	0	1	0	0	0	1	2
$Z - 3(x_1)$	Z	0	$1/2$	0	$-3/2$	0	0	-12

which gives the feasible solution

$$x_2 = 0, x_1 = 4, S_1 = 2, S_2 = 0, S_3 = 5, S_4 = 2, \text{ and } z = -(-12) = 12$$

which is not optimal since we still have a positive value in the z -row.

The procedure above is again repeated;

The entering variable is x_2 .

$$\max x_2 = \min \left[\frac{2}{3/2}, \quad \frac{4}{1/2}, \quad \frac{5}{3/2}, \quad \frac{2}{1} \right] = \frac{4}{3}$$

From s_1 , row the pivot value is $3/2$.

The new Pivot row values

$$x_2 = \frac{s_1}{3/2}$$

The Third Simplex Table:

	Basic	decision variables		slack variables				Soln.
		x_1	x_2	S_1	S_2	S_3	S_4	Quantity
$x_1 - \frac{1}{2}x_2$	x_2	0	1	$2/3$	$-1/3$	0	0	$4/3$
$s_3 - \frac{3}{2}x_2$	x_1	1	0	$-1/3$	$2/3$	0	0	$10/3$
$s_4 - x_1$	S_3	0	0	-1	1	1	0	3
$z - \frac{1}{2}x_1$	S_4	0	0	$-2/3$	$1/3$	0	1	$2/3$
	z	0	0	$-1/3$	$-4/3$	0	0	$-12\frac{2}{3}$

which gives the optimal solution as:

$$x_2 = \frac{4}{3}, x_1 = \frac{10}{3}, S_1 = 0, S_2 = 0, S_3 = 3, S_4 = \frac{2}{3}, \text{ and } Z = -\left(-12\frac{2}{3}\right) = 12\frac{2}{3}$$

which is the optimal since we do not have any positive value in the Z -row.

Example 0.4.2. From the Maize and Beans Farmer example above:

Step 0: Write the linear program in standard form; i.e.

$$\text{Maximize: } p = 3000x_1 + 4000x_2$$

Subject to:

$$x_1 + x_2 \leq 50 \text{ (restriction on land)}$$

$$600x_1 + 1200x_2 \leq 54000 \text{ (restriction on capital)}$$

$$4x_1 + 2x_2 \leq 160 \text{ (restriction on employees)}$$

$$x_1 \geq 0, x_2 \geq 0$$

On each inequality $\{\leq\}$ add a slack variable.

Step 1: Adding slack variables s_1 , s_2 , and s_3 to the constraints leads to;

$$\text{Maximize: } p = 3000x_1 + 4000x_2$$

Subject to:

$$x_1 + x_2 + s_1 = 50$$

$$600x_1 + 1200x_2 + s_2 = 54000$$

$$4x_1 + 2x_2 + s_3 = 160$$

$$x_1 \geq 0, x_2 \geq 0$$

Step 2: Construct the initial simplex table in which $x = 0$ and $p = 0$.

Basic	decision variables		slack variables			Soln.
	x_1	x_2	s_1	s_2	s_3	Qty.
s_1	1	1	1	0	0	50
s_2	600	1,200	0	1	0	54,000
s_3	4	2	0	0	1	160
p	3,000	4,000	0	0	0	0

The corresponding solution is $x_1 = 0$, $x_2 = 0$, $s_1 = 50$, $s_2 = 54,000$, $s_3 = 160$ and $p = 0$ (feasible solution).

We identify the entering variable using the optimality condition.

Optimality Condition

The **entering variable** (EV) is the non-basic variable, having the largest positive coefficient in the profits row (ties are broken arbitrarily).

The **optimum condition** is reached when all the entries of profit's row are negative or zero.

To identify the **departing variable** (DV) we apply the **feasibility condition**.

The departing variable is the basic variable associated with the smallest positive ratio of solution quantity to pivot column entries.

i.e. Solution quantity : Pivot column entries (ties are broken arbitrarily).

In our case, the entering variable is x_2 because 4,000 is the maximum value in the profit's row.

$$\min \left[\frac{50}{1} \quad \frac{54,000}{1200} \quad \frac{160}{2} \right] = \min [50 \quad 45 \quad 80] = 45$$

from s_2 -row.

Hence the departing variable is s_2 and pivot value = 1200.

The new pivot row = Old Row \div Pivot Value (and rename it x_2)

That is;

$$x_2 = s_2 \div 1200$$

Basic	decision	variable	slack	variables	Soln.	
	x_1	x_2	s_1	s_2	s_3	Qty.
s_1						
x_2	0.5	1	0	1/1200	0	45
s_3						
p						

Basic	decision	variable	slack	variables	Soln.	
	x_1	x_2	s_1	s_2	s_3	Qty.
s_1						
x_2						
s_3						
p						

Example 0.4.3. A farmer has 50 ha of land on which to plant maize and wheat. He has a capital of £2700. 1 ha of maize requires £60 while 1 ha of wheat requires £30. He has 160 workers and it takes 2 workers to work 1 ha of maize and 4 workers to work on 1 ha of wheat. The profit contributions are £30 from 1 ha of maize and £40 from 1 ha of wheat. Set up a LP problem and solve it.

Note: The interest is to maximize profits.

Solution. Let x_1 and x_2 be the number of ha of maize and wheat cultivated respectively. Then we need to maximize the profit.

$$\text{Maximize: } p = 30x_1 + 40x_2$$

subject to:

$$x_1 + x_2 \leq 50 \text{ (constraint on land)}$$

$$2x_1 + 4x_2 \leq 160 \text{ (constraint on workers)}$$

$$60x_1 + 30x_2 \leq 2700 \text{ (constraint on capital)}$$

$$x_1 \geq 0, x_2 \geq 0$$

$$\text{Ans: } x_1 = £20, x_2 = £30, \text{ profits} = £1800$$

The simplex method works with real numbers of decision variables. The LP is required to be in the form (standard form).

Step 1: In our case we have;

$$\text{Maximize: } p = 30x_1 + 40x_2$$

Subject to:

$$x_1 + x_2 \leq 50 \text{ (constraint on land)}$$

$$2x_1 + 4x_2 \leq 160 \text{ (constraint on workers)}$$

$$60x_1 + 30x_2 \leq 2700 \text{ (constraint on capital)}$$

$$x_1 \geq 0, x_2 \geq 0$$

which is the same as;

$$\text{Maximize: } p = 30x_1 + 40x_2$$

Subject to:

$$x_1 + x_2 \leq 50$$

$$x_1 + 2x_2 \leq 80$$

$$2x_1 + x_2 \leq 90$$

$$x_1 \geq 0, x_2 \geq 0$$

Step 2: Add the slack variables as follows;

$$\text{Maximize: } p = 30x_1 + 40x_2$$

subject to:

$$x_1 + x_2 + s_1 = 50$$

$$x_1 + 2x_2 + s_2 = 80$$

$$2x_1 + x_2 + s_3 = 90$$

$$x_1 \geq 0, x_2 \geq 0$$

s_i is the i th slack variable

Step 3: The initial simplex table in which $\underline{x} = 0$ and $\underline{z} = 0$.

Basic	decision variables		slack variables			Soln. Quantity
	x_1	x_2	s_1	s_2	s_3	
s_1	1	1	1	0	0	50
s_2	1	2	0	1	0	80
s_3	2	1	0	0	1	90
p	30	40	0	0	0	0

with the corresponding solution being $x_1 = 0$, $x_2 = 0$, $z = 0$ which is not optimal because at least one value in p row is positive.

Step 4: Identify the largest value in p -row (departing variable) in this case , 40 which corresponds to x_2 column.

The maximum value of x_2 should be equal to the minimum obtained by

$$\max x_2 = \min \left[\frac{50}{1}, \frac{80}{2}, \frac{90}{1} \right] = 40 \text{ corresponding to } s_2 \text{ row.}$$

The 2 at x_2 column and s_2 row is known as the **pivot value**.

Divide s_2 by the pivot value and re-name it x_2 .

$$\frac{s_2}{2} \rightarrow x_2 = \frac{1}{2}, \quad 1, \quad 0, \quad \frac{1}{2}, \quad 0, \quad 40$$

Reduce all the other entries of x_2 column to zero using this new row.

Basic	decision variables		slack variables			Soln. Quantity
	x_1	x_2	s_1	s_2	s_3	
s_1	1/2	0	1	-1/2	0	10
x_2	1/2	1	0	1/2	0	40
s_3	3/2	0	0	-1/2	1	50
p	10	0	0	-20	0	-1600

and the corresponding solution $x_1 = 0$, $x_2 = 40$ and $p = -(-1600) = 1600$, which is not optimal since we still have a positive value in the p -row.

Step 5: Maximum value in the p -row is 10 corresponding to x_1 column.

The maximum value of x_1 should be equal to the minimum obtained by

$$\max x_1 = \min \left[\frac{10}{1/2}, \frac{40}{1/2}, \frac{50}{3/2} \right] = 20 \text{ corresponding to } s_1 \text{ row.}$$

Step 4: The $1/2$ at x_1 column and s_1 row is known as the **pivot value**. Divide s_1 by the pivot value and re-name it x_1 .

$$\frac{s_1}{1/2} \rightarrow x_1 = 1, \quad 1, \quad 0, \quad 2, \quad -1, \quad 0, \quad 20$$

Reducing all the other entries of x_1 column to zero using this new row gives.

Basic	decision variables		slack variables			Soln. Quantity
	x_1	x_2	s_1	s_2	s_3	
x_1	1	0	2	-1	0	20
x_2	0	1	-1	1	0	30
s_3	0	0	-3	1	1	20
p	0	0	-20	0	0	-1800

Reducing other entries of x_1 column to zero using this row gives the above table. The corresponding solution is; $x_1 = \mathcal{L}20$, $x_2 = \mathcal{L}30$ and optimal $z = \mathcal{L}1800$. $s_3 = 20$ means the capital was not exhausted.

Interpretation of results

The farmer should cultivate 20 ha of maize and 30 ha of wheat in order to attain maximum profit of £1800.

Example 0.4.4. Use simplex method to solve

$$\text{Maximize: } z = 23x_1 + 39x_2 + 33x_3$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 3 \\ x_1 + x_2 + 2x_3 &\leq 2 \\ x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

Solution. Adding the slack variable to the inequalities.

Step 2:

$$\text{Maximize: } z = 23x_1 + 39x_2 + 33x_3$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 + x_3 + s_1 &= 3 \\ x_1 + x_2 + 2x_3 + s_2 &= 2 \\ x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \\ s_i &\text{ is the } i\text{th slack variable} \end{aligned}$$

Step 3: The initial simplex table

Basic	decision variables			slack variables		Soln. Quantity
	x_1	x_2	x_3	s_1	s_2	
s_1	1	2	1	1	0	3
s_2	1	1	2	0	1	2
z	23	39	33	0	0	0

The largest value in the z -row is 39 in column x_2 . The maximum value of x_1 should be equal to the minimum obtained by

$$\max x_2 = \min \left[\frac{3}{2}, \frac{2}{1} \right] = \frac{3}{2} \text{ corresponding to } s_1 \text{ row.}$$

2 is the pivot value.

Step 4: The 2 at x_2 column and s_1 row is known as the **pivot value**. Divide s_1 by the pivot value 2 and re-name it x_2 .

$$\frac{s_1}{2} \rightarrow x_2 = \frac{1}{2}, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{2}, \quad 0, \quad \frac{3}{2}$$

Reducing all the other entries of x_2 column to zero using this new row gives.

Basic	decision variables			slack variables		Soln.
	x_1	x_2	x_3	s_1	s_2	Quantity
x_2	1/2	1	1/2	1/2	0	3/2
s_2	1/2	0	3/2	-1/2	1	1/2
z	7/2	0	27/2	-39/2	0	-117/2

The maximum value in the z -row is $27/2$ corresponding to column x_3 .

$$\max x_3 = \min \left[\frac{3/2}{1/2}, \quad \frac{1/2}{3/2} \right] = \left[3, \quad \frac{1}{3} \right] = \frac{1}{3}$$

corresponding to s_2 row. $3/2$ is the pivot value.

$$\frac{s_2}{2/3} \rightarrow x_3 = \frac{1}{3}, \quad 0, \quad 1, \quad -\frac{1}{3}, \quad \frac{2}{3}, \quad \frac{1}{3}$$

Basic	decision variables			slack variables		Soln.
	x_1	x_2	x_3	s_1	s_2	Quantity
x_2	1/3	1	0	2/3	-1/3	4/3
x_3	1/3	0	1	-1/3	2/3	1/3
z	-1	0	0	-15	-9	-63

The corresponding solution is $x_1 = 0$, $x_2 = 4/3$, $x_3 = 1/3$ and the optimal $z = 63$.

Example 0.4.5. Solve the following LP problem using simplex method.

$$\text{Maximize: } p = 45x_1 + 55x_2$$

Subject to:

$$6x_1 + 4x_2 \leq 120$$

$$3x_1 + 10x_2 \leq 180$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Example 0.4.6. Solve the LP by graphical and simplex method.

$$\text{Maximize: } p = 5x_1 + 4x_2$$

Subject to:

$$3x_1 + 2x_2 \leq 50$$

$$x_1 + x_2 \leq 22$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Example 0.4.7. A company produces three products A, B, and C which contributes a profit of 80, 50 and 10 respectively. The production machine has 400 hours capacity and each product uses 2, 3, and 1 machine hour respectively.

There are 150 units available of a special component with product A using 1 unit and C using 1 unit per unit. A special alloy of 200 kg is needed in this period and product A and C uses 2 kg and 4 kg per unit.

Production of product B is limited to at most 50. Advice this company in order to maximize its profit.

Solution. Let x_1 , x_2 and x_3 be the number of units of products A, B, and C to be produced respectively. Then we have the LP as

$$\text{Maximize: } z = 80x_1 + 50x_2 + 10x_3$$

Subject to:

$$2x_1 + 3x_2 + x_3 \leq 400 \text{ (constraint on machine hours)}$$

$$x_1 + x_3 \leq 150 \text{ (constraint on special component)}$$

$$2x_1 + 4x_3 \leq 200 \text{ (constraint on alloy)}$$

$$x_2 \leq 50 \text{ (constraint on product B)}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

Introducing the slack variables. We get

$$\text{Maximize: } z = 80x_1 + 50x_2 + 10x_3$$

Subject to:

$$2x_1 + 3x_2 + x_3 + s_1 = 400$$

$$x_1 + x_3 + s_2 = 150$$

$$2x_1 + 4x_3 + s_3 = 200$$

$$x_2 + s_4 = 50$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

The initial simplex table is given by

Basic	decision variables		slack variables		Soln.			
	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Quantity
s_1	2	3	1	1	0	0	0	400
s_2	1	0	1	0	1	0	0	150
s_3	2	0	4	0	0	1	0	200
s_4	0	1	0	0	0	0	1	50
z	80	50	10	0	0	0	0	0

with solution $x_1 = x_2 = x_3 = 0$, $z = 0$ which is not optimal since there are positive values in the z -row.

The largest value in the z -row is 80 in column x_1 . The maximum value of $z = 80$ corresponding to x_1 , should be equal to the minimum obtained by

$$\max x_1 = \min \left[\frac{400}{2}, \frac{150}{1}, \frac{200}{2}, \frac{50}{0} \right] = 100$$

which corresponds to s_3 row and 2 is the pivot value.

Step 4: The 2 at x_1 column and s_3 row is known as the **pivot value**. Divide s_3 by the pivot value 2 and re-name it x_1 .

$$\frac{s_3}{2} \rightarrow x_1 = 1, 0, 2, 0, 0, \frac{1}{2}, 0, 100$$

Reducing all the other entries of x_2 column to zero using this new row gives.

Basic	decision variables			slack variables				Soln. Quantity
	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
s_1	0	3						
s_2	0	0						
x_1	1	0	2	0	0	$\frac{1}{2}$	0	100
s_4	0	1						
z	0							

whose solution is $x_1 = 100$, $x_2 = 50$, $x_3 = 0$ and $z = 10,500$ which is optimal since all the values in the z row are negative.

The company should produce 100 units of product A and 50 units of product B, in order to get maximum profit of £10,500.

Example 0.4.8. Solve the following LP using Simplex method.

$$\text{Maximize: } z = 5x_1 + 3x_2 + 4x_3$$

Subject to:

$$3x_1 + 12x_2 + 6x_3 \leq 900$$

$$6x_1 + 6x_2 + 3x_3 \leq 1350$$

$$2x_1 + 3x_2 + 3x_3 \leq 390$$

$$x_1 \geq 0, x_2 \geq 20, x_3 \geq 10$$

Since x_2 and x_3 are ≥ 0 , we adjust the constraint set to ensure that this is the case as follows:

$$\text{Maximize: } z = 5x_1 + 3x_2 + 4x_3$$

Subject to:

$$3x_1 + 12x_2 + 6x_3 \leq 900 - 12(20) - 6(10) = 600$$

$$6x_1 + 6x_2 + 3x_3 \leq 1350 - 6(20) - 3(10) = 1200$$

$$2x_1 + 3x_2 + 3x_3 \leq 390 - 3(20) - 3(10) = 300$$

which can be solved using simplex method.

The optimal table becomes

Basic	decision variables		slack variables		Soln.		
	x_1	x_2	x_3		Quantity		
s_1	0	$5/2$	$1/2$	1	0	$-1/2$	50
s_2	0	-1	-2	0	1	-1	100
x_1	1	$3/2$	$3/2$	0	0	$-1/2$	150
z	0	$-9/2$	$-7/2$	0	0	$-5/2$	-750

with the corresponding solution of $x_1 = 150$, $x_2 = 0$, $x_3 = 0$ and $\max z = 750$

But under the initial requirement the optimal solution should be

$$\begin{aligned}x_1 &= 150 \\x_2 &= 0 + 20 = 230 \\x_3 &= 0 + 10 = 10 \\\max z &= 750 + 3(20) + 4(10) = 850\end{aligned}$$

Example 0.4.9. Mozaq limited manufactures three products; tables, sideboards and chairs which require special material, timber and labour. In a given week, there are 92 units of special material, 116 pieces of timber and 140 hours of labour. The requirements of special materials are 2 units, 4 units and 2 units, for a table, sideboard and chair respectively. Manufacturing a table or a sideboard requires 2 hours of labour time while a chair requires 4 hours. Timber requirements are 4 pieces, 2 pieces and 2 pieces for a table, a sideboard and a chair respectively.

At least 2 tables and 4 chairs must be made. The profit contributions are £30, £40, £20 for a table, sideboard and chair respectively.

Formulate the underlying LP and solve it to advice Mozaq limited how to maximize profits.

Solution. Let x_1 , x_2 and x_3 be the number of tables, sideboards and chairs respectively.

$$\text{Maximize: } z = 30x_1 + 40x_2 + 20x_3$$

Subject to:

$$2x_1 + 4x_2 + 2x_3 \leq 92 \text{ constraint on special material}$$

$$2x_1 + 2x_2 + 4x_3 \leq 140 \text{ constraint on labour}$$

$$4x_1 + 2x_2 + 2x_3 \leq 116 \text{ constraint on timber}$$

$$x_1 \geq 2, \quad x_2 \geq 4, \quad x_3 \geq 0$$

Adjusting the constraint since x_1 and x_3 are greater than zero.

$$\text{Maximize: } z = 30x_1 + 40x_2 + 20x_3$$

Subject to:

$$2x_1 + 4x_2 + 2x_3 \leq 92 - 2(2) - 2(4) = 80$$

$$2x_1 + 2x_2 + 4x_3 \leq 140 - 2(2) - 4(4) = 120$$

$$4x_1 + 2x_2 + 2x_3 \leq 116 - 4(2) - 2(4) = 100$$

Introducing the slack variables, we get;

$$\text{Maximize: } z = 30x_1 + 40x_2 + 20x_3$$

Subject to:

$$2x_1 + 4x_2 + 2x_3 + s_1 \leq 80$$

$$2x_1 + 2x_2 + 4x_3 + s_2 \leq 120$$

$$4x_1 + 2x_2 + 2x_3 + s_3 \leq 100$$

Step 2: Initial simplex table

Basic	decision variables			slack variables			Soln. Quantity
	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	2	4	2	1	0	0	80
s_2	2	2	4	0	1	0	120
s_3	4	2	2	0	0	1	100
z	30	40	20	0	0	0	0

solution $x_1 = x_2 = x_3 = 0$, $z = 0$ which is not optimal as there are positive values in z .

The largest value in the z -row is 40 in column x_2 . The maximum value of $z = 80$ corresponding to x_1 , should be equal to the minimum obtained by

$$\max x_2 = \min \left[\frac{80}{4}, \frac{120}{2}, \frac{100}{2} \right] = 20$$

which corresponds to s_1 row and 4 is the pivot value.

Divide s_1 by the pivot value 4 and re-name it x_2 .

$$\frac{s_1}{4} \rightarrow x_1 = 1/2, 1, 1/2, 1/4, 0, 0, 20$$

Reducing all the other entries of x_2 column to zero using this new row gives.

Simplex Table 2

Basic	decision variables			slack variables			Soln. Quantity
	x_1	x_2	x_3	s_1	s_2	s_3	
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	0	20
$(s_2 - 2x_2)$ s_2	1	0	3	$-\frac{1}{2}$	1	0	80
$(s_3 - 2x_2)$ s_3	3	0	1	$-\frac{1}{2}$	0	1	60
$(z - 40x_2)$ z	10	0	0	-10	0	0	-800

The Solution is $x_1 = x_3 = 0$, $x_2 = 20$ and $z = 800$ which is not optimal since there are positive values in the z -row.

We repeat the above procedure;

The highest value on the z -row is $z = 10$, corresponding to x_1 .

$$\max x_1 = \min \left[\frac{20}{1/2}, \frac{80}{1}, \frac{60}{3} \right] = 20$$

Corresponds to s_3 row and 3 is the pivot value.

Divide s_3 by the pivot value 3 and re-name it x_1 .

$$\frac{s_3}{3} \rightarrow x_1 = 1, \quad 0, \quad 1/3, \quad -1/6, \quad 0, \quad \frac{1}{3}, \quad 20$$

Simplex Table 3

Basic	decision variables	slack variables	Soln.
	x_1 x_2 x_3	s_1 s_2 s_3	Quantity
$(x_2 - \frac{1}{2}x_1)$ x_2	0 1 $\frac{1}{3}$	$\frac{1}{3}$ 0 $-\frac{1}{6}$	10
$(s_2 - x_1)$ s_2	0 0 $\frac{8}{3}$	$-\frac{1}{3}$ 1 $-\frac{1}{3}$	60
x_1	1 0 $\frac{1}{3}$	$-\frac{1}{6}$ 0 $\frac{1}{3}$	20
$(z - 10x_1)$ z	0 0 $-\frac{10}{3}$	$-\frac{25}{3}$ 0 $-\frac{10}{3}$	-1000

Solution $x_1 = 20$, $x_2 = 10$ and $x_3 = 0$ and $z = 1000$ but under initial conditions the optimum solution should be

$$\begin{aligned} x_1 &= 20 + 2 = 22 \\ x_2 &= 10, \quad x_3 = 0 + 4 = 4 \\ z &= 1000 + 30(2) + 20(4) = 1140. \end{aligned}$$

Example 0.4.10. A company produces three products A, B, and C which contributes a profit of Ksh80, Ksh50, and Ksh10 respectively. The production machine has 400 hours capacity and each product uses 2, 3, and 1 machine hour respectively. There are 150 units available of a special component, with product A using 1 unit and C using 1 unit per unit. A special alloy of 200 kg is needed in this period and product A and C uses 2kg and 4 kg per unit. Production of product B is limited to at most 50. Advice the company in order to maximize its profit.

Solution. The Linear Program of the above problem is as follows:

$$\text{Maximize: } P = 80x_1 + 50x_2 + 10x_3$$

Subject to:

$$2x_1 + 3x_2 + x_3 \leq 400$$

$$x_1 + x_3 \leq 150$$

$$2x_1 + 4x_3 \leq 200$$

$$x_2 \leq 50$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Introducing the slack variables s_1, s_2, s_3 and s_4 , then we have

$$\text{Maximize: } P = 80x_1 + 50x_2 + 10x_3$$

Subject to:

$$2x_1 + 3x_2 + x_3 + s_1 = 400$$

$$x_1 + 0x_2 + x_3 + s_2 = 150$$

$$2x_1 + 0x_2 + 4x_3 + s_3 = 200$$

$$0x_1 + 1x_2 + 0x_3 + s_4 = 50$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

The initial simplex table is given as follows:

Basis	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Soln.Qty
s_1	2	3	1	1	0	0	0	400
s_2	1	0	1	0	1	0	0	150
s_3	2	0	4	0	0	1	0	200
s_4	0	1	0	0	0	0	1	50
P	80	50	10	0	0	0	0	0

The maximum value in the P row is 80 corresponding to x_1 .

$$\max x_1 = \min \left[\frac{400}{2}, \frac{150}{1}, \frac{200}{2}, \frac{50}{0} \right] = \min [200, 150, 100, (\text{ignore})] = 100$$

which corresponds to s_3 row and 2 is the pivot value.

$\frac{s_3}{2}$ rename it $x_1 = [1, 0, 2, 0, 0, \frac{1}{2}, 0, 100]$

Basis	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Soln.Qty
s_1	0	3	-3	1	0	-1	0	200
s_2	0	0	-1	0	1	$-\frac{1}{2}$	0	50
x_1	1	0	2	0	0	$\frac{1}{2}$	0	100
s_4	0	1	0	0	0	0	1	50
P	0	50	-150	0	0	-40	0	-8000

Question 0.4.1. Consider the following product mix problem:

Three machine shops A, B, C produces three types of products X, Y, Z respectively. Each product involves operation of each of the machine shops. The time required for each operation on various products is given as follows:

Machine Shops				
Product	A	B	C	Profit per unit
X	10	7	2	\$12
Y	2	3	4	\$3
Z	1	2	1	\$1
Available Hours	100	77	80	

The available hours at the machine shops A, B, C are 100, 77, and 80 only. The profit per unit of products X, Y, and Z is \$12, \$3, and \$1 respectively.

Solution.

0.4.3 Special Cases in Simplex method Application

We consider the special cases that can arise in the application of the simplex method; which include

Degeneracy

In the application of the feasibility condition, a tie for the minimum ratio may be broken arbitrarily for the purpose of determining the leaving variable. When this happens, however, one or more of the basic variables will necessarily equal zero in the next iteration. In this case, we say that the new solution is *degenerate*.

- (ii). Alternative optima
- (iii). Unbounded solution
- (iv). Non-existing (or infeasible) solutions

0.4.4 Minimizing Problems

To solve a minimizing problem using simplex method, we should first convert into a maximizing linear program and the final interpretation be done differently.

Lecture 5: Duality in Linear Programming Problems

By Dr. Antony Ngunyi

0.5 Duality in Linear Programming Problems

Most important finding in the development of Linear Programming Problems is the existence of **duality** in linear programming problems. Linear programming problems exist in pairs. That is in linear programming problem, every maximization problem is associated with a minimization problem. Conversely, associated with every minimization problem is a maximization problem. Once we have a problem with its objective function as maximization, we can write by using duality relationship of linear programming problems, its minimization version. The original linear programming problem is known as **primal problem**, and the derived problem is known as dual problem.

The concept of the **dual problem** is important for several reasons. Most important are

- (i) the variables of dual problem can convey important information to managers in terms of formulating their future plans
- (ii) in some cases the dual problem can be instrumental in arriving at the optimal solution to the original problem in many fewer iterations, which reduces the labour of computation.

The formulation of the dual problem also sometimes referred as the concept of duality is helpful for the understanding of the linear programming. The variable of the dual problem is known as the dual variables or shadow price of the various resources. The dual problem is easier to solve than the original problem. The dual problem solution leads to the solution of the original problem and thus efficient computational techniques can be developed through the concept of duality. Finally, in the competitive strategy problem solution of both the original and dual problem is necessary to understand the complete problem.

0.5.1 Dual Problem Formulation

If the original problem is in the standard form then the dual problem can be formulated using the following rules:

- The number of constraints in the original problem is equal to the number of dual variables. The number of constraints in the dual problem is equal to the number of variables in the original problem.
- The original problem profit coefficients appear on the right hand side of the dual problem constraints.
- If the original problem is a maximization problem then the dual problem is a minimization problem. Similarly, if the original problem is a minimization problem then the dual problem is a maximization problem.

- The original problem has less than or equal to (\leq) type of constraints while the dual problem has greater than or equal to (\geq) type constraints.
- The coefficients of the constraints of the original problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

The Dual Linear Programming Problem is explained with the help of the following Example.

Example 0.5.1. Minimize

$$P = x_1 + 2x_2$$

Subject to:

$$\begin{aligned} x_1 + x_2 &\geq 8 \\ 2x_1 + y &\geq 12 \\ x_1 &\geq 1 \end{aligned}$$

Solution.

Lecture 6: Transportation Model

By Dr. Antony Ngunyi

0.6 Transportation Model

The transportation model deals with a special class of linear programming problem in which the objective is to **transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost.**

To understand the problem more clearly, let us take an example and discuss the rationale of transportation problem. Three factories A, B and C manufactures sugar and are located in different regions. Factory A manufactures, b_1 tons of sugar per year and B manufactures b_2 tons of sugar per year and C manufactures b_3 tons of sugar. The sugar is required by four markets W, X, Y and Z. The requirement of the four markets is as follows: Demand for sugar in Markets W, X, Y and Z is d_1 , d_2 , d_3 and d_4 tons respectively. The transportation cost of one ton of sugar from each factory to market is given in the matrix below. The objective is to transport sugar from factories to the markets at a minimum total transportation cost. A transportation problem arises whenever there are a number of supply points and a number of demand points of a certain good. The cost of transporting one unit of the available goods from one supply point to a given destination is assumed to be known and the objective is to minimize the cost of transporting the goods.

Suppose there are three origins O_1, O_2, O_3 and three destinations D_1, D_2, D_3 . Let x_{ij} represent the quantity to be transported from O_i to D_j and also let c_{ij} represent per unit cost of transporting from O_i to D_j . If the amounts of supply from O_1, O_2, O_3 are s_1, s_2, s_3 respectively and the quantities of demand from D_1, D_2, D_3 are d_1, d_2, d_3 respectively, and the quantities of demand from D_1, D_2, D_3 are d_1, d_2, d_3 respectively, then we have the following table.

Destinations				
Origin	D_1	D_2	D_3	Supply
O_1	c_{11}	c_{12}	c_{13}	s_1
O_2	c_{21}	c_{22}	c_{23}	s_2
O_3	c_{31}	c_{32}	c_{33}	s_3
Demand	d_1	d_2	d_3	

Here the Linear programming problem is to minimize the cost.

$$C = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33}$$

Subject to the supply and demand constraints which are:

For supply:

$$x_{11} + x_{12} + x_{13} = s_1$$

$$x_{21} + x_{22} + x_{23} = s_2$$

$$x_{31} + x_{32} + x_{33} = s_3$$

and for demand:

$$x_{11} + x_{21} + x_{31} = d_1$$

$$x_{12} + x_{22} + x_{32} = d_2$$

$$x_{13} + x_{23} + x_{33} = d_3$$

Example 0.6.1. Consider the following transportation problem table for ABC company.

Destinations

Sources	Kisumu	Eldoret	Available
Industrial Area	5	3	40
Nakuru	4	7	30
Mombasa	6	6	20
Required:	25	65	

Entries within the table are unit cost of transportation.

We can formulate a LP from the table as follows:

Let x_{ij} , $i = 1, 2, 3$ $j = 1, 2$ be the units allocated to (ij) then the total cost is;

$$\text{Cost } C = 5x_{11} + 3x_{12} + 4x_{21} + 7x_{22} + 6x_{31} + 6x_{32}$$

which is to be minimized subject to;

$$x_{11} + x_{12} = 40$$

$$x_{21} + x_{22} = 30$$

$$x_{31} + x_{32} = 20$$

and for demand:

$$x_{11} + x_{21} + x_{31} = 25$$

$$x_{12} + x_{22} + x_{32} = 65$$

Note: When the total supply equals the total demand, the resulting formulation is called a **balanced transportation model**.

For the unbalanced case;

$$\begin{aligned}x_{11} + x_{12} &\leq 40 \\x_{21} + x_{22} &\leq 30 \\x_{31} + x_{32} &\leq 20\end{aligned}$$

and for demand:

$$\begin{aligned}x_{11} + x_{21} + x_{31} &\geq 25 \\x_{12} + x_{22} + x_{32} &\geq 65\end{aligned}$$

General Formulation of the Transportation Model

Let the amount of supply at source i be a_i and the demand at destination j be b_j . the unit transportation cost between source i and destination j is c_{ij} .

Let x_{ij} represent the amount transported from source i to destination j ; then the linear program model representing the transportation problem is given generally as:

$$\text{Minimize: } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

Subject to:

$$\begin{aligned}\sum_{j=1}^n x_{ij} &\leq a_i, \quad i = 1, 2, \dots, m \\\sum_{i=1}^m x_{ij} &\geq b_j, \quad j = 1, 2, \dots, n \\x_{ij} &\geq 0, \text{ for all } i \text{ and } j.\end{aligned}$$

The first set of constraints stipulate that the sum of the shipments from a source cannot exceed its supply; Similarly, the second set requires that the sum of the shipments to a destination must satisfy its demand.

For balanced transportation model, we have

$$\text{Minimize: } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

Subject to:

$$\begin{aligned}\sum_{j=1}^n x_{ij} &= a_i, \quad i = 1, 2, \dots, m \\\sum_{i=1}^m x_{ij} &= b_j, \quad j = 1, 2, \dots, n \\x_{ij} &\geq 0, \text{ for all } i \text{ and } j.\end{aligned}$$

Solution of the Transportation Problem

Note: Before you obtain the initial basic feasible solution, ensure that

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j \text{ (the problem is a balanced transportation model).}$$

If this is not the case, create a dummy source or destination and allocate the difference. the dummy costs are taken to be zero.

Example 0.6.2. A company has three supply points with a certain resource. The availability matrix is [300, 450, 520] and three destinations, with requirements matrix [250, 400, 480] respectively. the transportation costs are as given in the table below.

Destinations				
Supply Points	D_1	D_2	D_3	A_i
s_1	1	2	1	300
s_2	1	1	2	450
s_3	2	1	2	520
B_j	250	400	480	

$$\sum_{i=1}^m A_i = 1270 \quad \sum_{j=1}^n B_j = 1130$$

Since the two sums above are not equal, this is not a balanced transportation model. We therefore create a dummy destination D_4 with $(1270 - 1130) = 140$ units.

Destinations					
Supply Points	D_1	D_2	D_3	D_4	A_i
s_1	1	2	1	0	300
s_2	1	1	2	0	450
s_3	2	1	2	0	520
B_j	250	400	480	140	

Note: If the demand exceeds the supply, we create a dummy supply point.

The Transportation Technique

Step 1: Determine a starting feasible solution (initial allocation).

Step 2: Determine an entering variable from among the non basic variables. if all such variables satisfy the optimality condition (of the simplex method) stop, otherwise go to step 3.

Step 3: Determine a leaving variable (using the feasibility condition) from among the variables of the current basic solution; then find the new basic solution. return to step 2.

Step 4: Repeat steps 2 and 3 until the solution is obtained.

0.6.1 Methods of finding initial solution for a transportation problem

There are several methods of finding initial basis feasible solution. Here we shall discuss only three of them.

1. North-West Corner Rule (NWC)
2. Minimum Cost Method (LCM)
3. Vogel's Approximation Method (VAM)

North-West Corner Rule

The North West corner rule is a method for computing a basic feasible solution of a transportation problem where the basic variables are selected from **Steps:**

1. Firstly, select the upper left hand corner cell which is in the North-west corner of the table, and allocate units equal to the minimum of s_1 and d_1 against the supply and demand quantities in the respective rows and columns.
2. Make an allocation to this cell to exhaust either A_1 or B_1 or both.
3. Eliminate either the row or the column which has the smallest capacity (but not both).
4. If x_{ij} was the basic variable (occupied cell) then select the next variable to be $x_{i, j+1}$, if A_i has not been exhausted, otherwise select $x_{i+1, j}$.
5. Make an allocation to this cell and test for optimality using the **Stepping Stone method** or **Modified Distributed (MODI) method**.

The following Problems explains the procedure mentioned in the above steps.

Example 0.6.3. Solve the following transportation problem by NWC rule.

Warehouses

Plants	w_1	w_2	w_3	Capacity
P_1	7	5	9	28
P_2	12	16	10	44
P_3	8	7	6	43
Demand	34	45	36	115

Solution. The initial allocation table using the NWC rule is as follows:

Warehouses

Plants	w_1	w_2	w_3	Capacity
P_1	28 [7]	[5]	[9]	28
P_2	6 [12]	38 [16]	[10]	44
P_3	[8]	7 [7]	36 [6]	43
Demand	34	45	36	115

The initial feasible solution is,

$$\text{Cost} = (28 \times 7) + (12 \times 6) + (16 \times 38) + (7 \times 7) + (36 \times 6) = 1141$$

Example 0.6.4. Consider a case where a firm has three supply points with availability vector [55, 45, 30] and four destinations with [40, 20, 50, 20] as the requirements matrix. The cost of transporting one unit from a supply point to a destination is given as below:

Destinations

Plants	D_1	D_2	D_3	D_4	Supply A_i
s_1	12	4	9	5	55
s_2	8	1	6	6	45
s_3	1	12	4	7	30
Demand B_j	40	20	50	20	130

Solution. The initial allocation table using the NWC rule is as follows:

Destinations

Plants	D_1	D_2	D_3	D_4	Capacity A_i
s_1	40 [12]	15 [4]	[9]	[5]	55
s_2	[8]	5 [1]	40 [6]	[6]	45
s_3	[1]	[12]	10 [4]	20 [7]	30
Demand B_j	40	20	50	20	115

Which gives the transportation cost to be;

$$C = (40 \times 12) + (15 \times 4) + (5 \times 1) + (40 \times 6) + (10 \times 4) + (20 \times 7) = 965 \text{ units}$$

Least Cost Method/Minimum Matrix Method (MMM)

Matrix minimum method is a method for computing a basic feasible solution of a transportation problem where the basic variables are chosen according to the unit cost of transportation.

In this method, we take into consideration the lowest cost for the purpose of appropriate allocation. This method is useful in reducing the calculations considerably. The initial table is obtained by allocating units to cells with the smallest/minimum cost first. You should move from left to right and top to bottom as you identify/seek for these cells.

Steps:

1. We select the cell with the lowest cost amongst all the figures of cost in all the rows and columns of the given data.
2. To this selected cell (i) we allocate all the possible number of units either supply or demand as the case may be.
3. According to step (ii) when the demand gets satisfied or the supply gets exhausted, we eliminate the concerned row or column.
4. The process mentioned above in steps (i) and (ii) is repeated till the supply at various sources gets exhausted according to the demand from the concerned warehouses.

Example 0.6.5. Use the Least Cost Method for minimizing the total cost for the following data.

warehouses

Plants	w_1	w_2	w_3	Capacity (Supply)
s_1	7	5	9	28
s_2	12	16	10	44
s_3	8	7	6	43
Demand	34	45	36	115

Note: This is a balanced transportation problem.

$$\sum A_i = \sum B_j = 115$$

Thus by the least cost method, we get

Destinations

Plants	w_1	w_2	w_3	Capacity (Supply)
s_1	— [7]	28 [5]	— [9]	28
s_2	34 [12]	10 [16]	— [10]	44
s_3	— [8]	7 [7]	36 [6]	43
Demand	34	45	36	115

$$\begin{aligned}
 \text{Total Cost} &= (28 \times 5) + (34 \times 12) + (10 \times 16) + (7 \times 7) + (36 \times 6) \\
 &= 140 + 408 + 160 + 49 + 216 \\
 &= 973
 \end{aligned}$$

The following are the steps:

- (i). Starting with the cell (p_1, w_2) which has the minimum cost i.e., 5, we allocate the minimum of $(28, 45) = 28$ to the cell (p_1, w_2) . This exhausts the available capacity of Source I (s_1).

- (ii). We select the next minimum cost cell i.e., (p_3, w_3) and allocate the minimum of $(36, 43) = 36$ to this cell. The next minimum cost being 7 pertaining to the cell (p_3, w_2) , we allocate the balance $= (43 - 36) = 7$ to this cell.
- (iii). The next minimum cost is now 12 pertaining to (p_2, w_1) and we allocate the minimum of $(34, 44) = 34$ to this cell (p_2, w_1) .
- (iv). The next minimum cost is 16 which corresponds to the cell (p_2, w_2) and the balance remaining i.e., $(44 - 34) = 10$ has to be allocated to this cell (p_2, w_2) .

Example 0.6.6. Find the initial allocation table for the transportation problem. Using

(i). North-West Corner Rule

(ii). Least Cost Method

Plants	w_1	w_2	w_3	w_4	Supply
s_1	12	20	45	15	10
s_2	60	35	25	40	15
s_3	15	20	30	50	20
Demand	8	12	10	15	45

Solution. Using the following methods, we get

(i). North-West Corner Rule

Plants	w_1	w_2	w_3	w_4	Supply
s_1	8 [12]	2 [20]	— [45]	— [15]	10
s_2	— [60]	10 [35]	5 [25]	— [40]	15
s_3	— [15]	— [20]	5 [30]	15 [50]	20
Demand	8	12	10	15	45

This gives the total cost as

$$\begin{aligned}
 \text{Total Cost} &= (12 \times 8) + (2 \times 20) + (35 \times 10) + (25 \times 5) + (30 \times 5) + (50 \times 15) \\
 &= 96 + 40 + 350 + 125 + 150 + 750 \\
 &= 1511
 \end{aligned}$$

(ii). Least Cost Method

Plants	w_1	w_2	w_3	w_4	Supply
s_1	8 [12]	— [20]	— [45]	2 [15]	10
s_2	— [60]	— [35]	2 [25]	13 [40]	15
s_3	— [15]	12 [20]	8 [30]	— [50]	20
Demand	8	12	10	15	45

This gives the total cost as

$$\begin{aligned}
 \text{Total Cost} &= (8 \times 12) + (2 \times 15) + (2 \times 25) + (13 \times 40) + (12 \times 20) + (8 \times 30) \\
 &= 96 + 30 + 50 + 520 + 240 + 240 \\
 &= 1176
 \end{aligned}$$

Example 0.6.7. Find the initial table allocation and hence the minimum cost by the North West Corner method and Least Cost method.

	A	B	C	Supply
I	15	20	25	23
II	28	22	30	16
III	20	25	15	21
Demand	18	20	22	60

Example 0.6.8. From the (previous example) consider a case where a firm has three supply points with availability vector [55, 45, 30] and four destinations with [40, 20, 50, 20] as the requirements matrix. The cost of transporting one unit from a supply point to a destination is given as below:

Destinations

Plants	D_1	D_2	D_3	D_4	Supply A_i
s_1	12	4	9	5	55
s_2	8	1	6	6	45
s_3	1	12	4	7	30
Demand B_j	40	20	50	20	130

Solution. The initial allocation table using the LCM rule is as follows:

Destinations

Plants	D_1	D_2	D_3	D_4	Capacity A_i
s_1	10 [12]	— [4]	25 [9]	20 [5]	55
s_2	— [8]	20 [1]	25 [6]	— [6]	45
s_3	30 [1]	[12]	— [4]	— [7]	30
Demand B_j	40	20	50	20	130

This gives the total transportation cost as

$$\begin{aligned}
 \text{Total Cost} &= (10 \times 12) + (25 \times 9) + (20 \times 5) + (20 \times 1) + (25 \times 6) + (30 \times 1) \\
 &= 120 + 225 + 100 + 20 + 150 + 30 \\
 &= 645 \text{ units}
 \end{aligned}$$

Suppose that these costs c_{ij} 's are profits associated with transporting one unit from origin i to destination j , the problem becomes a **maximization transportation problem**.

Example 0.6.9.

Example 0.6.10. A Company has three supply points S_1, S_2 and S_3 and three warehouses (destinations), D_1, D_2 and D_3 . The capacities of supply points (A_i) are [100 150 200] and the demands of the destinations (B_j) are [70 120 210]. The cost (C_{ij}) associated with transporting one unit of the resource from a supply point i to a destination j are $C_{11} = C_{33} = 4, C_{12} = 1, C_{13} = C_{21} = C_{23} = 3, C_{22} = C_{32} = 2$, and $C_{31} = 5$.

1. Obtain an initial allocation using **North-west Corner** rule (*Do not solve*).
2. Obtain an initial allocation using **Least Cost** rule hence determine the optimal allocation that minimizes the transportation cost.

Vogel's Approximation Method (VAM)

The Vogel approximation method is an iterative procedure for computing a basic feasible solution of the transportation problem.

When compared with the earlier two methods, VAM is considered better and preferable because the initial basic feasible solution can be very close to the optimal solution or may be in itself optimal.

Steps:

1. For each row and column, we find the difference between the smallest cost and the next smallest cost in the concerned row or column. Each such difference is called a **penalty**. In the example below, we find the respective penalties.
2. We find the row or column with the largest penalty and in this row or column we select the cell having the smallest cost and allocate the maximum possible quality to this cell. The row or column for which the supply gets exhausted or the demand gets satisfied, becomes a deleted row or column. When two penalties have the same largest value, then we can arbitrarily select any one of them.
3. The process stated in steps (1) and (2) is repeated till the entire supply at the different plants gets exhausted for satisfying the demand at the various warehouses.

The example below makes the above method clear

Example 0.6.11. Use the Vogel's Approximation Method to minimize the total cost for the following data.

Plants	warehouses			Capacity (Supply)
	w_1	w_2	w_3	
s_1	7	5	9	28
s_2	12	16	10	44
s_3	8	7	6	43
Demand	34	45	36	115

Minimizing the total cost using Vogel's Approximation Method, the Procedure is as follows:

- Penalties shown in the column (I) and row (I) above are obtained by finding the difference between the smallest and the next smallest of the items in each row and column.
- With reference to the row Diff I we find that 3 is the largest and correspondingly we make the allocation of the minimum of $(36, 43) = 36$ in the cell (p_3, w_3) as the cost 6 is minimum in this cell. Thus the demand at warehouse w_3 is satisfied and the column w_3 will not be considered for further allocation. However the balance of 7 of the supply i.e., $(43 - 36) = 7$ is adjusted in the cell (p_3, w_2) as this cell has the next minimum cost, i.e., (7). After this allocation of 7 in (p_3, w_2) , the row p_3 will not be considered for further allocation.
- We now consider the row Diff II where 11 is the highest. In the column w_2 , we find that 5 is the lowest and accordingly we allocate to the cell (p_1, w_2) the highest possible and available capacity is 28. After this allocation, row p_1 will not be considered.
- Corresponding to the highest difference 4 in the Diff column II, we consider the lowest cost 12 in the cell (p_2, w_1) and to this cell we allocate the highest possible and available demand of 34 units. We adjust and allocate the balance of $(44 - 34) = 10$ units of capacity to the cell (p_2, w_2) . thereby the remaining demand units 10, in column w_2 automatically gets adjusted.

Thus, by the Vogel's Approximation Method allocation, we get:

Destinations

Plants	w_1	w_2	w_3	Capacity	Diff I	Diff II
s_1	— [7]	28 [5]	— [9]	28	2	2
s_2	34 [12]	10 [16]	— [10]	44	2	4
s_3	— [8]	7 [7]	36 [6]	43	1	-
Demand	34	45	36	115		
Diff I	1	2	3			
Diff II	5	11	—			

$$\begin{aligned}
 \text{Total Cost} &= (28 \times 5) + (34 \times 12) + (10 \times 16) + (7 \times 7) + (36 \times 6) \\
 &= 140 + 408 + 160 + 49 + 216 \\
 &= 973 \text{ units}
 \end{aligned}$$

0.6.2 Test for Optimal solution to a Transportation Problem

Test for Optimality

Once the initial feasible solution is reached, the next step is to check the optimality. An optimal solution is one where there is no other set of transportation routes (allocations) that will further reduce the total transportation cost. Thus, we'll have to evaluate each unoccupied cell (represents unused routes) in the transportation table in terms of an opportunity of reducing total transportation cost. In this process, if there is no negative opportunity cost, and the solution is an optimal solution.

It is necessary to test the initial feasible solution for optimality. For this purpose we can use any one of the following methods.

1. The Stepping Stone method
2. The Modified Distributed method (MODI)

Both these methods give the same results even though they differ in their calculations. The procedure consists in testing each unoccupied cell one at a time by calculating the cost change.

Stepping Stone Method

The following are the steps:

Modified Distributed Method (MODI)

This is a method of testing for optimality which proceed as follows:

Steps:

Step 1: Determine an initial basic feasible solution using any one of the three methods given below:

- North West Corner Rule
- Least Cost Method
- Vogel's Approximation method

Step 2: Row 1, row 2, \dots , row i of the cost matrix are assigned with variables u_1, u_2, \dots, u_i and the column 1, column 2, \dots , column j are assigned with variables v_1, v_2, \dots, v_j respectively.

Initially, assume any one of u_i values as zero and compute the values for u_1, u_2, \dots, u_i and v_1, v_2, \dots, v_j by applying the formula for occupied cells.

For the basic cells (occupied) $(m + n - 1)$ cells

We first have $c_{ij} = u_i + v_j$ where

c_{ij} is the cost of transporting one unit from i to j .

u_i is the opportunity cost at row i , ($i = 1, 2, \dots, m$)

v_j is the opportunity cost at column j , ($j = 1, 2, \dots, n$)

Solve the $(m + n - 1)$ equations by setting $u_1 = 0$ or $v_1 = 0$ and calculate u_i and v_j for all the basic cells.

Step 3: For the nonbasic cells (unoccupied) $(m+n-1)$ cells. Compute the opportunity cost using $c_{ij} - (u_i + v_j)$. The equation $I_{ij} = c_{ij} - u_i - v_j$ is the improvement index for the nonbasic cells (unoccupied cells). Then calculate the improvement indices for the nonbasic cells.

Step 4: Check the sign of each opportunity cost. If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimum. On the other hand, if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimum solution and further savings in transportation cost are possible.

Step 5: Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution. If $I_{ij} < 0$, the allocation to cell (i, j) is more economical (start with the most negative). Select the nonbasic cell corresponding to I_{ij} and adjust it appropriately as follows;

- (a). From the most economical cell trace a loop (closed loop). Draw a closed path or loop for the unoccupied cell selected in the previous step. Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

A Loop is a sequence of cells such that;

- Each pair of consecutive cells lie in either the same row or the same column.
 - No three cells lie in the same row or column.
 - The first and the last cells lie in the same row or column.
 - No cells appear more than once in the sequence.
- (b). Identify a closed loop (path) from the most economical cell ($I_{ij} < 0$), taking right angle turns at each stop until we are back to the same cell.
 - (c). Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated. The first step is positive, label the next negative, the third is positive and so on.
 - (d). Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, add this quantity to all the cells on the corner points of the closed path marked with plus signs and subtract it from those cells marked with minus signs. In this way an unoccupied cell becomes an occupied cell.

Identify all the x_{ij} values with negative signs and take the minimum of these values (say k).

Maximum allocation to this cell = Minimum (entries in the cells marked negative).

- (e). Let this allocation be k . Add k to all the cells marked positive and subtract k from every cell marked negative. The

$$\text{New cost} = \text{Previous Cost} - |I_{ij}| \times k \text{ (New allocation)}$$

where I_{ij} is the most economical cell index.

- (f). This gives a new solution which should be tested for optimality.

Step 6: Repeat the whole procedure until an optimum solution is obtained. The optimal solution is obtained when all the values of the improvement indices. $I_{ij} < 0$.

Example 0.6.12.

Solution. The solution table is;

Plants	Destinations				A_i	u_i
	D_1	D_2	D_3	D_4		
s_1	40 [12]	15 _⊖ ^[4]	— [9]	— _⊕ ^[5]	55	0
s_2	— [8]	5 _⊕ ^[1]	40 _⊖ ^[6]	— _⊕ ^[6]	45	-3
s_3	— [1]	— [12]	10 _⊕ ^[4]	20 _⊖ ^[7]	30	-5
Demand B_j	40	20	50	20	130	
v_j	12	4	9	12		

For the basic cells (occupied) ($m + n - 1$) cells

$$c_{ij} = u_i + v_j \quad \text{Let } u_1 = 0$$

$$\begin{aligned}
 c_{11} &= u_1 + v_1 & \Rightarrow 12 = 0 + v_1 & \Rightarrow v_1 = 12 \\
 c_{12} &= u_1 + v_2 & \Rightarrow 4 = 0 + v_2 & \Rightarrow v_2 = 4 \\
 c_{22} &= u_2 + v_2 & \Rightarrow 1 = u_2 + 4 & \Rightarrow u_2 = -3 \\
 c_{23} &= u_2 + v_3 & \Rightarrow 6 = -3 + v_3 & \Rightarrow v_3 = 9 \\
 c_{33} &= u_3 + v_3 & \Rightarrow 4 = u_3 + 9 & \Rightarrow u_3 = -5 \\
 c_{34} &= u_3 + v_4 & \Rightarrow 7 = -5 + v_4 & \Rightarrow v_4 = 12
 \end{aligned}$$

For the nonbasic cells (unoccupied) ($m + n - 1$) cells. The improvement indices are given by, $I_{ij} = c_{ij} - u_i - v_j$

x_{ij}	c_{ij}	u_i	v_j	I_{ij}	
x_{13}	9	0	9	0	
x_{14}	5	0	12	-7	(most economical cell)
x_{21}	8	-3	12	-1	
x_{24}	6	-3	12	-3	
x_{31}	1	-5	12	-6	
x_{32}	12	-5	4	13	(uneconomical cell)

Cells with x_{13} and x_{32} are uneconomical, since I_{14} gives the greatest improvement. We allocate to x_{14} as follows:

1. Trace a path from x_{14} through basic (occupied) cells taking right angle turns at each step until we are back to the same cell.
2. The first step is positive, second is negative, third is positive and so on.
3. Identify all the x_{ij} values with negative sign and take the minimum of these values.

$$\min(15, 40, 20) = 15$$

4. Add 15 to all the cells with positive signs and subtract 15 from all cells with negative signs to get a new solution table.
5. Test this solution for optimality.

In our example above, the next solution table is;

Destinations

Plants	D_1	D_2	D_3	D_4	A_i	u_i
s_1	40 [12]	— [4]	— [9]	15 [5]	55	
s_2	— [8]	20 [1]	25 [6]	— [6]	45	
s_3	— [1]	— [12]	25 [4]	5 [7]	30	
Demand B_j	40	20	50	20	130	
v_j						

$$\max x_{14} = \min(20, 40, 15) = 15$$

Example 0.6.13. A company is spending Ksh.1,000 on transportation of its units from these plants to four distribution centers. The supply and requirement of units, with unity cost of transportation are given as:

Distribution centres						
		D_1	D_2	D_3	D_4	Supply
Plants	P_1	19	30	50	12	7
	P_2	70	30	40	60	10
	P_3	40	10	60	20	18
Demand		5	8	7	15	

Determine the optimum solution of the above problem.

Example 0.6.14. Given below are the costs of shipping a product from various warehouses to different stores:

Warehouses	Stores (costs in Kshs.)				Supply
	S1	S2	S3	S4	
A	7	3	5	5	34
B	5	5	7	6	15
C	8	6	6	5	12
D	6	1	6	4	19
Demand	21	25	17	17	80

Generate an initial feasible solution and check it for optimality.

Question 0.6.1. A company has three factories (A, B, C) and warehouses (P, Q, R, S). It supplies the finished goods from each of its three factories to the four warehouses, the relevant shipping cost of which are as follows:

Warehouses	Factory		
	A	B	C
P	4	3	7
Q	5	8	4
R	2	4	7
S	5	8	4

Assuming the monthly production capacity of A, B, and C to be 120, 80 and 200 tons respectively, determine the optimum transportation schedule so as to minimize the total transportation costs by using Vogel's method.

The monthly requirements for the warehouses are as follows:

Warehouses	Monthly requirement (tons)		
	P	Q	R
P	60		
Q		50	
R			140
S			50
	Total 300 (tons)		

Question 0.6.2. A company has three factories and five warehouses, where the finished goods are shipped from the factories. The following relevant details are provided to you:

Factory	Warehouses					Availability
	W1	W2	W3	W4	W5	
1	3	5	8	9	11	20
2	5	4	10	7	10	40
3	2	5	8	7	5	30
Requirements	10	15	25	30	40	120/90

Solve the above transportation problem.

0.6.3 Special cases in Transportation Problems

Some variations that often arise while solving the transportation problem could be as follows:

1. Multiple Optimum Solution
2. Unbalanced Transportation Problem
3. Degeneracy in the Transportation Problem
4. Maximization in the Transportation Problem

Multiple Optimum Solution

This problem occurs when there are more than one optimal solutions. This would be indicated when more than one occupied cell have zero value for the net cost change in the optimal solution. Thus a reallocation to cell having a net cost change equal to zero will have no effect on transportation cost. This reallocation will provide another solution with same transportation cost, but the route employed will be different from those for the original optimal solution. This is important because they provide management with added flexibility in decision making.

Unbalanced Transportation Problem

If the total supply is not equal to the total demand then the problem is known as unbalanced transportation problem. If the total supply is more than the total demand, we introduce an additional column, which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply, an additional row is introduced in the table, which represents unsatisfied demand with transportation cost zero.

Degeneracy in the Transportation Problem

If the basic feasible solution of a transportation problem with m origins and n destinations has fewer than $m+n-1$ positive x_{ij} (Occupied cells), the problem is said to be a degenerate transportation problem.

Degeneracy can occur at two stages:

1. At the initial solution
2. During the testing of the optimum solution

A degenerate basic feasible solution in a transportation problem exists if and only if some partial sum of availability's (row(s)) is equal to a partial sum of requirements (column(s)).

Example 0.6.15. Consider the following transportation problem.

	Dealers				
Factory	D1	D2	D3	D4	Supply
A	2	2	2	4	1000
B	4	6	4	3	700
C	3	2	1	0	900
Requirement	900	800	500	400	

Here, $S_1 = 1000$, $S_2 = 700$, $S_3 = 900$.

$R_1 = 900$, $R_2 = 800$, $R_3 = 500$, $R_4 = 400$. Since $R_3 + R_4 = S_3$, so the given problem is a degeneracy problem.

Now we will solve the transportation problem by least cost method.

To resolve degeneracy, we make use of an artificial quantity (d). The quantity d is so small that it does not affect the supply and demand constraints.

Degeneracy can be avoided if we ensure that no partial sum of S_i (supply) and r_j (requirement) are the same. We set up a new problem where:

$$S_i = S_i + d, \quad i = 1, 2, \dots, m$$

$$r_j = r_j$$

$$r_n = r_n + md$$

	Dealers				
Factory	D1	D2	D3	D4	Supply
A	900 ^[2]	100 + d ^[2]	2	4	1000+d
B	4	700 - d ^[6]	2d ^[4]	3	700+d
C	3	2	500 - 2d ^[1]	400 + 3d ^[0]	900+d
Requirement	900	800	500	400+3d	

Substituting $d = 0$.

	Dealers				
Factory	D1	D2	D3	D4	Supply
A	900 ^[2]	100 ^[2]	2	4	1000
B	4	700 ^[6]	0 ^[4]	3	700
C	3	2	500 ^[1]	400 ^[0]	900
Requirement	900	800	500	400+3d	

Initial basic feasible solution:

$$2 \times 900 + 2 \times 100 + 6 \times 700 + 4 \times 0 + 1 \times 500 + 0 \times 400 = 6700.$$

Now degeneracy has been removed.

To find the optimum solution, you can use any one of the following: Stepping Stone method or MODI method.

Maximization in the Transportation Problem

Although transportation model is used to minimize transportation cost. However, it can also be used to get a solution with an objective of maximizing the total value or returns. In this case, **convert the maximization problem into minimization by subtracting all the unit cost from the highest unit cost given in the table and solve.**

Example 0.6.16. A firm has three factories X , Y , and Z . It supplies goods to four dealers spread all over the country. The production capacities of these factories are 200, 500 and 300 per month respectively.

Factory	A	B	C	D	Capacity
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand	180	320	100	400	

Determine suitable allocation to maximize the total net return.

Solution. Maximization transportation problem can be converted into minimization transportation problem by subtracting each transportation cost from maximum transportation cost.

Here, the maximum transportation cost is 25. So subtract each value from 25.

Factory	A	B	C	D	Capacity
X	13	7	19	0	200
Y	17	18	15	7	500
Z	11	22	14	5	300
Demand	180	320	100	400	

Now, solve the above problem by first finding the initial solution, and then testing the initial solution for optimality using MODI method.

Prohibited Routes Problems

In practice, there may be routes that are unavailable to transport units from one source to one or more destinations. The problem is said to have an unacceptable or prohibited route. To overcome such kind of transportation problems, **assign a very high cost to prohibited routes, thus preventing them from being used in the optimal solution regarding allocation of units.**

Transshipment Problem

The transshipment problem is a general case of transportation problem recognizing that it may be cheaper to ship goods through intermediate or transient points before their final destination.

Each of these nodes in turn supply to other destinations. The objective of the transshipment problem is to determine how many units should be shipped over each node so that all the demand requirements are met with the minimum transportation cost.

In order to solve a transshipment problem we need to first convert it into a regular transportation problem.

The conversion proceeds as follows:

Let the Buffer = maximum ($\sum A_i, \sum B_j$).

Identify and allocate to the points as in the table below:

Point	Allocation
Pure supply	Original supply (A_i)
Pure demand	Original demand (B_j)
Supply + Transshipment	Original supply + buffer amount
Demand + Transshipment	Original demand + buffer amount
pure Transshipment	Buffer amount

Note: If there is no route (from a certain point) linking two points then the transportation cost $c_{ij} = \infty$.

Example 0.6.17. A company has 80 boxes of jewels to transport around i.e. Nairobi to Nyeri and then to Nyahururu in amounts of 50 and 30 respectively. The cost of transporting one unit between the towns are as given below:

	Nairobi	Nyeri	Nyahururu	Nakuru
Nairobi	0	2	4	2.5
Nyeri	2	0	1	∞
Nyahururu	4	1	0	1.5
Nakuru	2.5	∞	1.5	0

Given that shipment through intermediate towns are allowed, determine the shipment schedule that minimizes the cost.

0.6.4 Transportation Problem Maximization

Lecture 7: Assignment Model

By Dr. Antony Ngunyi

0.7 Assignment Problem

The assignment problem arises whenever there are n tasks to be assigned to m resources (applicants on the basis of quantifiable attributes). It is assumed that each resource is assigned to one and only one task.

Consider the situations of assigning m jobs (or workers) to n machines. A job ($i = 1, 2, 3 \dots, m$) when assigned to machine ($j = 1, 2, 3 \dots, n$) incurs a cost c_{ij} . The objective is to assign the jobs to the machines (one job per machine) at the least total cost. This situation is known as the **assignment problem**.

0.7.1 Formulation of the Assignment problem

The table below gives a general representation of the Assignment model.

		Machines					
		1	2	3	...	n	
Jobs	I	c_{11}	c_{12}	c_{13}	...	c_{1n}	1
	II	c_{21}	c_{22}	c_{23}	...	c_{2n}	1
	III	c_{31}	c_{32}	c_{33}	...	c_{3n}	1
	\vdots						
	m	c_{m1}	c_{m2}	c_{m3}		c_{mn}	1
		1	1	1	...	1	

Before the model (problem) can be solved, by the transportation techniques it is necessary to balance the problem by adding fictitious (non-existing) jobs or machines, depending on whether $m < n$ or $m > n$. It will thus be assumed that $m = n$ without loss of generality.

The assignment model can be expressed mathematically as follows:

Let

$$x_{ij} = \begin{cases} 0, & \text{if the } j\text{th job is not assigned to the } i\text{th machine} \\ 1, & \text{if the } j\text{th job is assigned to the } i\text{th machine} \end{cases}$$

The model is thus given by

$$\text{Minimize, } z = \sum_{i=1}^n \sum_{j=1}^m c_{ij} x_{ij}$$

Subject to:

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, 3 \dots, n$$

$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, 3 \dots, n$$

$$x_{ij} = 0 \text{ or } 1$$

The assignment problem can be solved using the approach for the transportation problem but a more straight forward algorithm referred to as **Hungarian method** has been developed.

0.7.2 The Hungarian method

According to this method the following are the steps to be performed:

1. Introduce a dummy rows or column to get a square matrix if the problem is not balanced.
2. In each row, find the smallest cost element and subtract it from all the elements of the concerned row.
3. With reference to the resulting matrix, in each column subtract the smallest cost element from the rest of the elements of the concerned column.
4. The result obtained after step (2) and (3) is called the *reduced matrix* and hence we find at least one zero-element in each row and column. We cover all the zero elements by the minimum number of horizontal and vertical straight lines. If the number of lines so drawn are equal to the number of rows or columns then the solution would be optimal.

Beginning with the first row, if we find only one zero in that row, we give an assignment here by enclosing the zero in a square and cross the zero/zeros in the column of this assigned zero element.

5. We proceed in this manner with the other rows, also and if we find more than one uncovered zero in a row, we leave it without giving any assignment.
6. Now, we begin with the first column and use the same procedure of assignment as explained in step (4).

7. If the number of horizontal and vertical lines in step (4) are less than the number of rows or columns, then we find the smallest uncovered element (uncovered by the lines) and subtract it from all the elements covered twice at the intersection of any two lines and leave the elements through which only one line passes as they are. We repeat the steps mentioned in (4), (5), and (6).

The Hungarian algorithm is summarized in the flow chart below:

Example 0.7.1. A company has 5 jobs to be done. The following matrix shows the cost of assigning each job ($w_j = 1, 2, \dots, 5$) to each machine ($m_i = 1, 2, \dots, 5$). Assign the five jobs to the five machines so as to minimize the total cost.

		Jobs				
		w_1	w_2	w_3	w_4	w_5
Machines	m_1	5	11	10	12	4
	m_2	2	4	6	3	5
	m_3	3	12	14	6	-
	m_4	6	14	4	11	7
	m_5	7	9	8	12	5

Solution.

Note: The symbol ∞ is inserted in the blank corresponding to cell (m_3, w_5) .

First, we subtract the smallest number in each row from the rest of the elements of the row,

	w_1	w_2	w_3	w_4	w_5
m_1	1	7	6	8	0
m_2	0	2	4	1	3
m_3	0	9	11	3	∞
m_4	2	10	0	7	3
m_5	2	4	3	7	0

Next, in the second and fourth columns, we subtract the smallest number from the rest of the elements of the column.

	w_1	w_2	w_3	w_4	w_5
m_1	1	5	6	7	0
m_2	0	0	4	0	3
m_3	0	7	11	2	∞
m_4	2	8	0	6	3
m_5	2	2	3	6	0

Here, the number of lines are 4 which is less than the number of columns/rows which is 5.

We find that 2 is the smallest of the uncovered elements. We subtract 2 from each of the uncovered elements and add it to the elements at the 4 intersecting places.

	w_1	w_2	w_3	w_4	w_5
m_1	1	3	4	5	0
m_2	2	0	4	0	5
m_3	0	5	9	0	∞
m_4	4	8	0	6	5
m_5	2	0	1	4	0

We can now cover all the zeros by 5 lines and hence the solution has been reached as shown below:

	w_1	w_2	w_3	w_4	w_5
m_1	1	3	4	5	0
m_2	2	0	4	0	5
m_3	0	5	9	0	∞
m_4	4	8	0	6	5
m_5	2	0	1	4	0

Example 0.7.2. Five applicants are to be assigned some jobs in the finance department of a company. To determine the appropriate section for the applicant each of the applicants is given a test. The following table gives the marks obtained out of 20.

Applicants

		1	2	3	4	5
Sections	Audit	12	15	11	13	12
	Purchasing	12	13	10	14	13
	Salaries	15	13	12	15	10
	Pensions	17	17	13	19	15

Determine where each applicant should be assigned to.

Solution. Introduce a dummy row with zero marks. Let T_i —Tasks ($i = 1, 2, \dots, 5$) and A_j —Applicants ($j = 1, 2, \dots, 5$).

	A_1	A_2	A_3	A_4	A_5
T_1	12	15	11	13	12
T_2	12	13	10	14	13
T_3	15	13	12	15	10
T_4	17	17	13	19	15
T_5	0	0	0	0	0

The Objective of the problem is to assign the applicants to the sections where they score the highest mark. Hence this is a maximization problem.

First, we subtract the maximum mark in each row from the rest of the elements of the row.

	A_1	A_2	A_3	A_4	A_5
T_1	3	0	4	2	3
T_2	2	1	4	0	1
T_3	0	2	3	0	5
T_4	2	2	6	0	4
T_5	0	0	0	0	0

Next, covering all zeros with the minimum number of lines we get.

The number of lines = 4 are not equal to the number of assignments (Optimality test).

The smallest uncovered element $\theta = 1$. Adding 1 to intersection element and subtracting 1 from uncovered elements leads to;

	A_1	A_2	A_3	A_4	A_5
T_1	2	0	3	2	2
T_2	1	1	3	0	0
T_3	0	3	3	1	5
T_4	1	2	5	0	3
T_5	0	1	0	1	0

Covering all zeros with minimum number of lines leads to the table above;

Note: the lines should cover the maximum number of zeros at once.

Therefore the number of line = 5 = number of assignments. Hence the above table gives the Optimal Assignment.

	A_1	A_2	A_3	A_4	A_5
T_1	2	0	3	2	2
T_2	1	1	3	0	0
T_3	0	3	3	1	5
T_4	1	2	5	0	3
T_5	0	1	0	1	0

Tasks	Applicant	Marks	Allocation
T_1	A_2	15	Applicant 2 to Audit
T_2	A_5	13	Applicant 5 to Purchasing
T_3	A_1	15	Applicant 1 to Salaries
T_4	A_4	19	Applicant 4 to Pensions
T_5	A_3	0	Applicant 3 is NOT employed

Example 0.7.3. Assume that these estimates are not marks but the cost. Find the Optimal assignment.

Example 0.7.4. Solve the following minimal Assignment problem.

Location	1	2	3	4	5
A	21	16	18	7	4
B	14	8	6	9	3
C	9	6	5	8	10
D	3	40	21	10	7
E	15	21	6	4	9

Example 0.7.5. Solve the following assignment model.

Machines	Operators	A 1	A 2	A 3	A 4	A 5
	A	3	9	2	3	7
B	6	1	5	6	6	
C	9	4	7	10	3	
D	2	5	4	2	1	
E	9	6	2	4	6	

Lecture 8: Network Analysis

By Dr. Antony Ngunyi

0.8 Network Analysis

0.8.1 Introduction

A **Project** defines a combination of interrelated activities that must be executed in a certain order before the entire task can be completed. The activities are interrelated in a logical sequence in the sense that some activities cannot start until others are completed. An **activity** in a project is usually viewed as a job requiring time and resources for its completion. In general, a project is a one-time effort; that is, the same sequence of activities may not be repeated in the future.

In the past, the scheduling of a project (over time) was done with little planning. The best-known “planning” tool then was the **Gantt bar chart**, which specifies the start and finish times for each activity on a horizontal time scale.

A network is a representation of events and activities through a graphical procedure. In modern times, the role played by network analysis in product management is very significant.

Two important techniques called the Programme Evaluation Review Technique (PERT) and Critical Path Method (CPM) are commonly used for helping the managers in planning and controlling large construction, research, development and other projects. Delays in project completion arise on account of poor planning and inadequate preparation. In this regard, CPM and PERT which have been developed as scientific techniques, are used effectively for the purpose of analysing various aspects and thereby assist in planning, scheduling and coordinating for optimum results.

0.8.2 Critical Path Method

A project generally consists of a series of activities that involve use of labour, materials, plant machinery and money. Some of these activities may be independent, whereas others may be related. The activities need to be executed in a certain order so as to form an appropriate sequential pattern. It is necessary to identify those activities which need more time and money beyond the budgeted limits, so as to reduce the overall loss of time.

The CPM was developed in the year 1957 by *Morgan Walker* of *Du Pont* and *James Kelly* of *Remington Rand (USA)*. This technique was used for the development of a chemical plant through proper planning and scheduling of the related activities. The CPM can be advantageously used for revising the patterns of the jobs depending on the observed criticality. This technique helps the managers in determining the expected project completion times.

0.8.3 Programme Evaluation Review Technique (PERT)

This technique was developed in 1958 by the Naval Engineers of US Navy Missile Project. PERT and CPM have been independently developed, but they have many common features and hence in project analysis, both the techniques are used. Therefore the analysis that is carried out is termed as 'Critical Path Analysis' (CPA). It is also called 'Network Analysis'.

The basic difference between CPM and PERT is that PERT is used for analyzing projects scheduling problems, in which the completion times of the various activities and hence the completion time of the whole project is uncertain. While in CPM, the completion times of the various activities are known and therefore the completion time of the whole project can be determined. Further, the amount of resources required for the completion of each activity is also known. It can be said that CPM is 'deterministic' due to certainty of the completion times of activities associated with it. On the other hand, PERT is 'probabilistic' due to the uncertainty of completion times of activities associated with it.

Using these techniques many kinds of projects can be undertaken. In such projects it is assumed that each one consists of a well defined set of activities or tasks. Further, the activities are such that their start and termination times are independent of each other. Here the resources used for the various activities may not be independent of each other. There always exists a sequential pattern amongst the activities. According to the sequence of activities, there always exists a precedence relationship.

0.8.4 Construction of a Network

In PERT, the network is built with reference to events. While in the case of CPM, the network is built on the basis of activities. PERT is Event oriented and CPM is Activity oriented. PERT makes three time estimates for each activity and CPM makes only one time estimate for each activity.

Rules of Network Construction

1. Each well-defined activity can be represented by one and only one arrow in the network. Therefore no activity can be represented more than once in a network.
2. Unless the preceding activity is completed, the connecting activity cannot commence. Hence there should be a logical sequence of all the concerned activities based on the concept of predecessor activity - successor activity.
3. It should be noted that the lengths of the arrows are only representative and have no other significance.
4. The direction showed by the respective arrow has significance in regard to progression of time. Each arrow has a head and tail. The arrow head represents the completion of an activity. The starting time of an activity is represented by the tail of the arrow. The events showing the start of activities are called 'tail-events' and those showing the completion of the activities are called 'head-events'.

5. If a number of activities terminate at one single event, then it is possible for any other activity to start only from that terminating event.
 6. Integers such as 1, 2, 3, etc. are used to denote events. Activities are denoted by capital letters such as A, B, C, D , etc.
 7. For example 2-3 points out that 2 is the starting event and 3 is the terminating event. Hence in the above figure (i) activities A, B, C and D are represented by 1-4, 2-2, 3-4 and 4-5 respectively.
 8. It should be noted that a network should always have only one initial node or event and one terminal node or event.
 9. In Fig.1, event 4 is the ‘merge-event’. Fig.2 below shows a number of activities emanating from a single event. This single event from which the activities emanate is called the ‘burst-event’. Thus the event 7 in the Fig.2 below is a ‘burst-event’.
 10. The question of parallel activities between two events without any intermediary event, does not arise, as the same cannot be permitted. It should be noted that there cannot be the same ending or same beginning for two or more activities. Hence for beginning and ending of an activity there can be one and only one arrow. It would be necessary to introduce a dummy activity in between two parallel activities that have the same head or tail. The following figures 3 and 4(a) and 4(b) are self explanatory in this regard. They also show a dummy activity which is called ‘identity dummy’ and it does not consume any resources or time. The dotted line is a dummy activity.

Dummy activities are useful as they indicate the logical relationship in a network in a consistent manner. Dummy activities are independent of each other.

11. When two or more activities have the same immediate predecessor and successor activities, then the need for introduction of a dummy activity arises. However, minimum number of dummy activities should be used in order to logically and adequately represent the connective relationships. This would provide an appropriate network.

12. If an activity A precedes B and B precedes C , then C cannot precede A , which implies that a loop of this nature cannot be permitted.

The rules for constructing the arrow diagram will be summarized as:-

Rule 1. *Each activity is represented by one and only one arrow in the network.*

Rule 2. *No two activities can be identified by the same head and tail events.*

Rule 3. *To ensure the correct precedence relationship in the arrow diagram, the following questions must be answered as every activity is added to the network.*

1. *What activities must be completed immediately before this activity can start?*
2. *What activities must follow this activity?*
3. *What activities must occur concurrently with this activity?*

This rule is self-explanatory. It actually allows for checking (and rechecking) the precedence relationships as one progresses in the development of the network.

Example 0.8.1. A certain industry is about to introduce a new product (i.e product 3). One unit of product 3 is produced by assembling one unit of product 1 and one unit of product 2. Before the production begins of either product 1 or product 2, raw materials must be purchased and workers must be trained. Before product 1 and 2 can be assembled into product 3, the finished product 2 must be inspected. A list of activities and their predecessors and the duration of each activity is given below. Draw a project network for this project.

Activity	Predecessor	Duration(Days)
A-Train workers	-	6
B-Purchase raw materials	-	9
C-Produce product 1	A,B	8
D-produce product 2	A,B	7
E-Test product 2	D	10
F-Assemble product 1 and 2	C, E	12

Example 0.8.2. Draw a network on the basis of the information given below.

Activity	Immediate Predecessor
A	-
B	A
C	A
D	B
E	C
F	D, E

Example 0.8.3. For the purpose of preparing its next year's budget, a company must gather information from its sales, production, accounting, and finance departments. The table below indicates the activities and their durations. Prepare the network model of the problem and carry out the critical path computations.

Activity	Description	Predecessor (s)	Duration (days)
A	Forecast sales volume	-	10
B	Study competitive market	-	7
C	Design item and facilities	A	5
D	Prepare production schedules	C	3
E	Estimate cost of production	D	2
F	Set sales price	B, E	1
G	Prepare budget	E, F	14

Example 0.8.4. Draw a network for showing the following relationship between the activities.

Activity	Preceded by
1-2	-
2-3	1-2
2-4	1-2
3-5	2-3
4-6	2-4
5-7	3-5
6-7	4-6
7-8	5-7,6-7

Critical Path Definitions

Critical Activity - An activity is said to be critical if a delay in its start will cause a delay in the completion of the entire project.

Non-critical Activity - A non critical activity is such that the time between the earliest start and the latest completion date as allowed by the project is larger than the actual duration. In this case the non-critical activity is said to be a slack or float time.

A critical path - defines a chain of critical activities that connect the start and the finish events (nodes). It is the path with the *longest duration*. The critical path determines the time for the completion of the project. Thus if any activity on the critical path is delayed, the entire project will be delayed.

If the duration of each activity is known *with certainty*, we use the CPM to determine the *length of time* required for completion of the entire project. In other words we say that, CPM is *deterministic*. On the other hand if the duration of the activities is *not known with certainty*; we use the PERT to estimate the probability that the project will be completed within a given time. In other words PERT is *probabilistic*.

DETERMINATION OF CRITICAL PATH BY USE OF PROJECT EVALUATION AND REVIEW TECHNIQUE (PERT)

In this method the duration of an activity is not fixed but it is a random variable. Therefore a probabilistic model of the project is formulated. We assume that the time estimate for each activity is based on three different values.

1. t_0 - optimistic time required if the extension of the project is undermost favourable conditions.
2. t_p - pessimistic time required if the extension of the project is under least favourable conditions.
3. t_m - most likely time required if the execution of the project is under normal conditions.

The duration for each activity is assumed to follow a beta (β) distribution with its unimodal point occurring at t_m and its endpoints at t_0 and t_p .

If t_{ij} follows a beta distribution, then it can be shown that the mean and variance of t_{ij} , may be approximated by

$$\begin{aligned} E(t_{ij}) &= \left(\frac{t_0 + t_p + 4t_m}{2} \right) / 3 \\ &= \frac{t_0 + 4t_m + t_p}{6} \end{aligned}$$

and the variance of t_{ij} is

$$var(t_{ij}) = \left(\frac{t_p - t_0}{6} \right)^2$$

To find the critical path, the expected time estimates are determined for all the activities. After getting the time estimates, the same procedure is used as in CPM to get the critical path. For any path in the project network, the expected duration and the variance of the duration of activities are given by

$$\sum_{(i,j) \in path} E(t_{ij})$$

and

$$\sum_{(i,j) \in path} var(t_{ij})$$

respectively.

Examples:

1. The following table gives the activities and the corresponding time estimates.

Determine the critical path and its duration. therefore the corresponding network is given by

Critical Path Method and Programme Evaluation Review Technique

Introduction

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 8. It should be noted that a network should always have only one initial node or event and one terminal node or event.
 9. In Fig.1, event 4 is the ‘merge-event’. Fig.2 below shows a number of activities emanating from a single event. This single event from which the activities emanate is called the ‘burst-event’. Thus the event 7 in the Fig.2 below is a ‘burst-event’.
 10. The question of parallel activities between two events without any intermediary event, does not arise, as the same cannot be permitted. It should be noted that there cannot

be the same ending or same beginning for two or more activities. Hence for beginning and ending of an activity there can be one and only one arrow. It would be necessary to introduce a dummy activity in between two parallel activities that have the same head or tail. The following figures 3 and 4(a) and 4(b) are self explanatory in this regard. They also show a dummy activity which is called ‘identity dummy’ and it does not consume any resources or time. The dotted line is a dummy activity.

Dummy activities are useful as they indicate the logical relationship in a network in a consistent manner. Dummy activities are independent of each other.

11. When two or more activities have the same immediate predecessor and successor activities, then the need for introduction of a dummy activity arises. However, minimum number of dummy activities should be used in order to logically and adequately represent the connective relationships. This would provide an appropriate network.
12. If an activity A precedes B and B precedes C , then C cannot precede A , which implies that a loop of this nature cannot be permitted.

The rules for constructing the arrow diagram will be summarized as:-

Rule 1. *Each activity is represented by one and only one arrow in the network.*

Rule 2. *No two activities can be identified by the same head and tail events.*

Rule 3. *To ensure the correct precedence relationship in the arrow diagram, the following questions must be answered as every activity is added to the network.*

1. *What activities must be completed immediately before this activity can start?*
2. *What activities must follow this activity?*
3. *What activities must occur concurrently with this activity?*

This rule is self-explanatory. It actually allows for checking (and rechecking) the precedence relationships as one progresses in the development of the network.

Example 0.8.5. A certain industry is about to introduce a new product (i.e product 3). One unit of product 3 is produced by assembling one unit of product 1 and one unit of product 2. Before the production begins on either product 1 or product 2, raw materials

must be purchased and workers must be trained. Before product 1 and 2 can be assembled into product 3, the finished product 2 must be inspected. A list of activities and their predecessors and the duration of each activity is given below. Draw a project network for this project.

Activity	Predecessor	Duration(Days)
A-Train workers	-	6
B-Purchase raw materials	-	9
C-Produce product 1	A,B	8
D-produce product 2	A,B	7
E-Test product 2	D	10
F-Assemble product 1 and 2	C, E	12

Example 0.8.6. Draw a network on the basis of the information given below.

Activity	Immediate Predecessor
A	-
B	A
C	A
D	B
E	C
F	D, E

Example 0.8.7. For the purpose of preparing its next year's budget, a company must gather information from its sales, production, accounting, and finance departments. The table below indicates the activities and their durations. Prepare the network model of the problem and carry out the critical path computations.

Activity	Description	Predecessor (s)	Duration (days)
A	Forecast sales volume	-	10
B	Study competitive market	-	7
C	Design item and facilities	A	5
D	Prepare production schedules	C	3
E	Estimate cost of production	D	2
F	Set sales price	B, E	1
G	Prepare budget	E, F	14

Example 0.8.8. Draw a network for showing the following relationship between the activities.

Activity	Preceded by
1-2	-
2-3	1-2
2-4	1-2
3-5	2-3
4-6	2-4
5-7	3-5
6-7	4-6
7-8	5-7,6-7

Critical Path Definitions

Critical Activity - An activity is said to be critical if a delay in its start will cause a delay in the completion of the entire project.

Non-critical Activity - A non critical activity is such that the time between the earliest start and the latest completion date as allowed by the project is larger than the actual duration. In this case the non-critical activity is said to be a slack or float time.

A critical path - defines a chain of critical activities that connect the start and the finish events (nodes). It is the path with the *longest duration*. The critical path determines the time for the completion of the project. Thus if any activity on the critical path is delayed, the entire project will be delayed.

If the duration of each activity is known *with certainty*, we use the CPM to determine the *length of time* required for completion of the entire project. In other words we say that, CPM is *deterministic*. On the other hand if the duration of the activities is *not known with certainty*; we use the PERT to estimate the probability that the project will be completed within a given time. In other words PERT is *probabilistic*.

Determination of the Critical Path by use of Critical Path Method (CPM)

The critical path calculations include two phases:

(i). Forward Pass

Here the calculations begin from the start node and move towards the end node. At each node, a number is computed representing the *earliest occurrence time* of the corresponding event. These numbers are shown in squares in the figure below.

(ii). Backward Pass

Here the calculations begin from the end node and moves towards the start node. The objective of this phase is to compute the *latest completion time* for all the activities coming into a given node. The numbers computed at each node are shown in triangles.

Forward Pass

Let ES_i be the earliest start time of all the activities emanating from i . Thus, ES_i represent the *Earliest start time* of event i . If $i = 0$, is the “start” node, then by convention

$$ES_0 = 0.$$

Let d_{ij} be the duration of activity (i, j) , the forward pass calculations are thus obtained from the formula;

$$ES_j = \max_i (ES_i + d_{ij}) \quad \text{for all } (i, j) \text{ activities defined.}$$

Therefore;

$$\begin{aligned}
 ES_1 &= ES_o + d_{01} = 0 + 2 = 2 \\
 ES_2 &= ES_o + d_{02} = 0 + 3 = 3 \\
 ES_3 &= \max_{i=1,2}(ES_i + d_{i3}) = \max(2 + 2, 3 + 3) = 6 \\
 ES_4 &= \max_{i=2,3}(ES_i + d_{i4}) = \max(6 + 0, 3 + 2) = 6 \\
 ES_5 &= \max_{i=3,4}(ES_i + d_{i5}) = \max(6 + 7, 6 + 3) = 13 \\
 ES_6 &= \max_{i=3,4,5}(ES_i + d_{i6}) = \max(6 + 2, 6 + 5, 13 + 6) = 19
 \end{aligned}$$

These calculations completes the forward pass.

Backward Pass

Let LC_i be the *latest completion time* for all activities coming into node i . Thus, if $i = n$ is the end node, then

$$LC_n = ES_n$$

In general, for any node i ,

$$LC_i = \min_j(LC_j - d_{ij}) \quad \text{for all } (i,j) \text{ activities defined.}$$

Therefore;

$$\begin{aligned}
 LC_6 &= ES_6 = 19 \\
 LC_5 &= LC_j - d_{ij} = 19 - 6 = 13 \\
 LC_4 &= \min_{j=5,6}(LC_j - d_{4j}) = \min(19 - 5, 13 - 7) = 6 \\
 LC_3 &= \min_{j=4,5,6}(LC_j - d_{3j}) = \min(19 - 2, 13 - 3, 6 - 0) = 6 \\
 LC_2 &= \min_{j=3,4}(LC_j - d_{2j}) = \min(6 - 3, 6 - 2) = 3 \\
 LC_1 &= (LC_3 - d_{13}) = 6 - 2 = 4 \\
 LC_0 &= \min_{j=1,2}(LC_j - d_{0j}) = \min(3 - 3, 4 - 2) = 0
 \end{aligned}$$

The backward pass calculations are now complete.

The critical path activities can now be identified by using the results of the forward and backward pass. Note that an activity (i, j) lies on the **critical path** if it satisfies the following conditions:

- (i). $ES_i = LC_j$
- (ii). $ES_j = LC_i$
- (iii). $ES_j - ES_i = LC_j - LC_i = d_{ij}$

Therefore the critical path is given by

$$\{(0, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

Notice also that the critical path must form a chain of connected activities that span the network from “start” to “end”.

The duration of the project is

$$3 + 3 + 0 + 7 + 6 = 19.$$

Float Analysis

Non-critical activities can be delayed by a certain number of time units without the delay of the entire project. The magnitude of this flexibility is what we call **float of the activity**.

The latest start (LS) and the earliest completion time (EC) for a given activity (i, j) is given by

$$LS_{ij} = LC_j - d_{ij}$$

$$EC_{ij} = ES_i + d_{ij}$$

We have two types of floats.

(i). Total float (TF)

(ii). Free float (FF)

The **total float** (TF) for activity (i, j) i.e. TF_{ij} is the difference between the maximum time available to perform the activity and its duration.

$$\begin{aligned} TF_{ij} &= LC_j - ES_i - d_{ij} \\ &= LC_j - EC_{ij} = LS_{ij} + d_{ij} - ES_i - d_{ij} \\ &= LS_{ij} - ES_i \end{aligned}$$

The **free float** (FF) is defined by assuming that all the activities start as early as possible. In this case, the free float for the activity (i, j) is the excess of available time over its duration.

$$FF_{ij} = ES_j - ES_i - d_{ij}$$

Since $LC_j \geq ES_i \implies TF_{ij} \geq FF_{ij}$

Note: An activity is critical iff *total float is zero*. The critical path calculations together with the floats for the non-critical activities can be summarized in the following table.

Activity (i, j)	Duration (d_{ij})	Earliest Start(ES_i)	Earliest Start(EC_{ij})	Latest Start(LS_{ij})	Latest Start(LC_j)	TF	FF
(0,1)	2	0	2	2	4	2	0
(0,2)	3	0	3	0	3	0^a	0
(1,3)	2	2	4	4	6	2	0
(2,3)	3	3	6	3	6	0^a	0
(2,4)	2	3	5	4	6	1	0
(3,4)	0	6	6	6	6	0^a	0
(3,6)	2	6	8	17	19	11	0
(3,5)	3	6	9	10	13	4	0
(4,5)	7	6	13	6	13	0^a	0
(4,6)	5	6	11	14	19	8	0
(5,6)	6	13	19	13	19	0^a	0

Note: a represents critical activity.

$$TF = LS_{ij} - ES_i \text{ and } FF = ES_j - ES_i - d_{ij}$$

0.9 Inventory Control

One of the basic functions of management is to employ capital efficiently so as to yield the maximum returns. This can be done in either of two ways or by both, i.e. (a) By maximizing the margin of profit; or (b) By maximizing the production with a given amount of capital, i.e. to increase the productivity of capital. This means that the management should try to make its capital work hard as possible. However, this is all too often neglected and much time and ingenuity are devoted to make only labour work harder. In the process, the capital turnover and hence the productivity of capital is often totally neglected. Several new techniques have been developed and employed by modern management to remedy this deficiency. Among these Materials Management has become one of the most effective. In Materials Management, Inventory Control play vital role in increasing the productivity of capital.

Inventory management or Inventory Control is one of the techniques of Materials Management which helps the management to improve the productivity of capital by reducing the material costs, preventing the large amounts of capital being locked up for long periods, and improving the capital - turn over ratio. The techniques of inventory control were evolved and developed during and after the Second World War and have helped the more industrially developed countries to make spectacular progress in improving their productivity.

0.9.1 DEFINITION OF INVENTORY AND INVENTORY CONTROL

The word **inventory** means a physical stock of material or goods or commodities or other economic resources that are stored or reserved or kept in stock or in hand for smooth and efficient running of future affairs of an organization at the minimum cost of funds or capital blocked in the form of materials or goods (**Inventories**).

The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finished goods in an orderly manner to meet the objectives of maximum customer service with minimum investment and efficient (low cost) plant operation is termed as *inventory control*.

Classification of Inventories

0.9.2 COSTS ASSOCIATED WITH INVENTORY

While maintaining the inventories, we will come across certain costs associated with inventory, which are known as **economic parameters**. Most important of them are discussed below:

0.9.3 PURPOSE OF MAINTAINING INVENTORY OR OBJECTIVE OF INVENTORY COST CONTROL

The purpose of maintaining the inventory or controlling the cost of inventory is to use the available capital optimally (efficiently) so that inventory cost per item of material will be as minimum as possible. For this the materials manager has to strike a balance between the interrelated inventory costs. In the process of balancing the interrelated costs i.e. Inventory carrying cost, ordering cost or set up cost, stock out cost and the actual material cost. Hence we can say that **the objective of controlling the inventories is to enable the materials manager to place and order at right time with the right source at right price to purchase right quantity.**

0.9.4 INVENTORY MODELS: DETERMINISTIC MODELS

0.10 Programme Evaluation and Review Technique and Critical Path Method (PERT and CPM)

Programme Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are two techniques that are project is completed within the stipulated time and at minimum possible cost. Many managers, who use the PERT and CPM techniques, have claimed that these techniques drastically reduce the project completion time. But it is wrong to think that network analysis is a solution to all bad management problems.

0.10.1 PERT AND CPM

In PERT and CPM the milestones are represented as *events*. Event or node is either starting of an activity or ending of an activity. Activity is represented by means of an arrow, which is resource consuming. Activity consumes resources like time, money and materials. Event will not consume any resource, but it simply represents either starting or ending of an activity. Event can also be represented by rectangles or triangles. When all activities and events in a project are connected logically and sequentially, they form a *network*, which is the basic document in network-based management. The basic steps for writing a network are:

0.10.2 Determination of the Critical Path by use of Programme Evaluation Review Technique (PERT)

In the case of CPM, the duration of each activity in a network is specifically known. Hence the project completion time and the critical path, along with critical activities can be easily determined. In the case of PERT, the durations of the various activities are not known (not fixed) but it is a random variable. Therefore a probabilistic model of the project is formulated. Hence to specify the duration of such activities, a procedure of estimation based on three different estimates of time for each activity is adopted. These estimates of time are called *Optimistic Time* (t_o), *Most-Likely Time* (t_m) and *Pessimistic Time* (t_p) respectively.

t_o Optimistic time required if the extension of the project is undermost favourable conditions.

t_p Pessimistic time required if the execution of the project is under least favourable conditions.

t_m Most likely time required if the execution of the project is under normal conditions.

The duration for each activity is assumed to follow a beta β distribution with its unimodal point occurring at t_m and its end points at t_o and t_p . i.e It can be noted that in accordance with values of the Optimistic Time, Most-Likely Time and Pessimistic Time, the probability distribution of an activity duration may be of beta distribution type. The expected times of the different activities can be calculated on the basis

of the three estimates t_o, t_m and t_P using the weighted average method as shown below:

$$\begin{aligned} E(t_{ij}) &= \left(\frac{t_o + 4t_m + t_p}{2} \right) / 3 \\ &= \frac{t_o + 4t_m + t_P}{6} \end{aligned}$$

where

t_{ij} - Expected time of the n^{th} activity

t_o - Optimistic time

t_m - Most-Likely time

t_P - Pessimistic time

The standard deviation in respect of the completion of the ij^{th} activity is

$$\sigma_t = \frac{t_P - t_o}{6}$$

and

$$\text{Var}(t_{ij}) = \left[\frac{t_P - t_o}{6} \right]^2$$

To find the critical path, the expected time estimates are determined for all the activities. After getting the time estimates, the same procedure is used as in CPM to get the critical path. For any path in the project network, the expected duration and the project network, the expected duration and the variance of the duration of activities are given by

$$\sum_{(i,j) \in \text{path}} E(t_{ij}) \quad \text{and} \quad \sum_{(i,j) \in \text{path}} \text{Var}(t_{ij}) \quad \text{respectively.}$$

Hence according to the PERT technique, it follows that determination and control of the critical activities would provide a dependable result for project completion.

Example 0.10.1. From the information given below construct a network diagram and determine the critical path and the expected completion time of the project.

Name	Activity	Predecessor		Optimistic Time (t_o)	Most-Likely Time (t_m)	Pessimistic Time (t_P)
		Name	Activity			
A	1-2	-	-	6	9	12
B	2-3	A	1-2	6	8	10
C	2-4	A	1-2	3	7	11
D	3-5	B	2-3	12	16	20
E	4-6	C	2-4	6	11	22
F	5-7	D	3-5	12	15	24
G	6-7	E	4-6	5	7	15
H	7-8	F, G	5-7, 6-7	4	8	12

Solution. The calculations are shown in the following table.

Name	Activity	Time			$t_n = \frac{t_o+4t_m+t_P}{6}$	$\sigma_n = \frac{t_P-t_o}{6}$	$\text{Var} = \left[\frac{t_P-t_o}{6} \right]^2$
		(t_o)	(t_m)	(t_P)			
A	1-2	6	9	12	9	1	1
B	2-3	6	8	10	8	$4/6=2/3$	$4/9$
C	2-4	3	7	11	7	$8/6=4/3$	$16/9$
D	3-5	12	16	20	16	$8/6=4/3$	$16/9$
E	4-6	6	11	22	12	$16/6=8/3$	$64/9$
F	5-7	12	15	24	16	$12/6=2$	4
G	6-7	5	7	15	8	$10/6=5/3$	$25/9$
H	7-8	4	8	12	8	$8/6=4/3$	$16/9$

With reference to the above calculations of the expected time, we find the critical path as, $ABDFH$ the length of which is

$$9 + 8 + 16 + 16 + 8 = 57 \text{ hours.}$$

The variance of this project length can be calculated as

$$1 + \frac{4}{9} + \frac{16}{9} + 4 + \frac{16}{9} = 9.$$

The duration for the completion of the project is 57 hours. The standard deviation is $\sigma_n = \sqrt{9} = 3$ hours.

0.10.3 CRITICAL PATH METHOD (CPM) FOR CALCULATING PROJECT COMPLETION TIME