M1F Notes

David Burgschweiger

October 15, 2015

Contents

1 Sets

Definition 1.0.1. A set S is a collection of objects (called *elements* of the set). If x is an *element* of S let us write $x \in S$ otherwise $x \notin S$.

Remark 1.1. The order of the elements or any repetition is unimportant.

Example 1.1.

$$\{1,3\} = \{3,1,1\}$$

Definition 1.0.2. For two sets S and T let us write $S \subseteq T$ (S is contained in T) if

$$x \in S \Rightarrow s \in T$$

Result 1.0.1. $S = T$ iff $S \subseteq T$ and $T \subseteq S$.

Remark 2.1. $S \notin S$ (Foundation Axiom)
Nonetheless, elements can be sets.

Definition 1.0.3. $\emptyset$ is the set with no elements.

Property 3.1. $\emptyset \subseteq S$ and $S \subseteq S$ for all sets S

1.1 Set Operators

Definition 1.1.1. The intersection $S \cap T$ of two sets S and T is

$$\{x |\ x \in S \,\text{and}\, x \in T\}$$

Definition 1.1.2. The union $S \cup T$ of two sets S and T is

$$\{x |\ x \in S \,\text{or}\, x \in T\}$$

Definition 1.1.3. The difference $S \backslash T$ of two sets S and T is

$$(S \cup T) \backslash (S \cap T)$$

Definition 1.1.4. The symmetric difference $S \triangle T$ of two sets S and T is

$$\{x |\ x \in S \,\text{and}\, x \in T\}$$

Definition 1.1.5. In $A \subseteq \Omega$ then

$$A^C = \{x \in \Omega | x \notin A\} \qquad = \Omega \backslash A$$

Remark 5.1. The complement is only used when the reference set Ω is clear.

Some sets we will work with in this course are

$$\begin{aligned}
&\mathbb{N} = \{0, 1, 2, \dots\} \\
&\mathbb{Z} = \{0, 1, -1, 2, -2\} \\
&\mathbb{Q} = \{\frac{p}{q} | P \in \mathbb{N}, q \in \mathbb{Z}\backslash\{0\}\} \\
&\mathbb{R} \,\text{reals} \\
&\mathbb{C} \,\text{complex numbers}
\end{aligned}$$

Definition 1.1.6. $\mathbb{N}$ is defined by two axioms:

1. $0 = \emptyset \in \mathbb{N}$
2. If $n \in \mathbb{N}$ then $n + 1 \stackrel{def}{=} n \cup \{n\} \in \mathbb{N}$

Example 6.1.

$$1 = 0 + 1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}$$
$$2 = 1 + 1\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$$

1.2 Intervals in $\mathbb{R}$

Definition 1.2.1. If $a \leq b$:

$$\begin{aligned}
[a, b] &= \{t \in \mathbb{R} | a \leq t \leq b\} \\
(a, b) &= \{t \in \mathbb{R} | a < t < b\} \\
[a, b) &= \{t \in \mathbb{R} | a \leq t < b\} \\
(a, b] &= \{t \in \mathbb{R} | a < t \leq b\} \\
[a, \infty) &= \{t \in \mathbb{R} | a < t\} \\
(-\infty, b] &= \{t \in \mathbb{R} | t \leq b\}
\end{aligned}$$

1.3 Infinite Unions and Intersections

Definition 1.3.1. Suppose that, for all $n \in \mathbb{N}$, we are given a set A_n.

$$\begin{aligned}
\bigcup_{n=a}^{\infty} A_n &= \{x | \text{ there exists a } n \in \mathbb{N}, n \geq a : x \in A_n\} \\
\bigcap_{n=a}^{\infty} A_n &= \{x | \text{ for all } n \in \mathbb{N}, n \geq a : x \in A_n\}
\end{aligned}$$

Example 1.1.

$$\begin{aligned}
\bigcup_{n=1}^{\infty} [0, 1 - \frac{1}{n}] &= [0, 1) \\
\bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n}) &= \{1\}
\end{aligned}$$

2 Proofs

2.1 Elements of the propositional calculus

Definition 2.1.1. A statement (proposition) is an assertion that can be either true (T) or false (F).

Remark 1.1. In maths such an assertion usually takes the form: "If such and such assumptions are made, then we can infer such and such conclusions"

Example 1.1.

- $n = 3$
- $(A + B)^2 = A^2 + 2AB + B^2$
- If it n^2 is odd, then n is odd too.
- If it rains, then it is cloudy.
- For all real numbers ≥ 0 there exists a square root.

Definition 2.1.2. A proof is a chain of statements linked by logical implications (inferences) that establish the truth of the last statement. In the course of the proof one is allowed to "call up"

- assumptions that are made.
- statements proven previously.
- axioms (statements that are generally accepted and never proven).

"Grammar elements" of mathematical statements are Quantifiers:

Type	Sign	Meaning
Existential	$\exists$ $\exists_1$	there exists there exists a unique
Universal	$\forall$	for all
	$:, \mid$	such that

Ways to form new statements from old ones:

- If P is a statement then $notP$ "non-P" is the statement which is true if P is false and false if P is true.
- If P and Q are statements then we can form:

Sign	Meaning
$P \wedge Q$, $P\&Q$	P and Q.
$P \vee Q$	either P or Q or both.
$P \veebar Q$	either P or Q but not both.
$P \Rightarrow Q$	If P then Q.
$P \Leftrightarrow Q$	P if and only if Q.

Remark 2.1. $P \Rightarrow Q$ means any of the following:

- If P then Q.
- Q if P.
- P is true only if Q is true.
- P only if Q.
- P is sufficient for Q.
- Q is necessary for P.
- If Q is false then P is false.
- $notQ \Rightarrow notP$

Similarly, $P \Leftrightarrow Q$ means any of the following:

- $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- P if and only if Q. P is necessary and sufficient for Q.

The rigorous definition of $P \wedge Q$, $P \Rightarrow Q$ can be made through a truth table

Definition 2.1.3. $P \wedge Q$ is defined by:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 2.1.4. Also, $P \Rightarrow Q$ is defined by:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 4.1. The statement "If $x \in \{n \in \mathbb{N} |\ n^2 < 0\}$ then x is a sheep." is true as well as the statement "If $x \in \{n \in \mathbb{N} | n^2 < 0\}$ then x is not a sheep."

2.2 Inference rules

Example 0.2. *Premise 1.* If it is raining then it's cloudy.
Premise 2. It's raining.
Conclusion. It is cloudy.

We can write this more abstractly as follows:
P: it is raining
Q: it is cloudy
In this form:
Premise 1. $P \Rightarrow Q$
Premise 2. P
Conclusion. Q

This is an example of an inference rule which we write like this:

$$((P \Rightarrow Q) \wedge P) \Rightarrow Q$$

There are other inference rules

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$
$$\big((P \vee Q) \wedge \overline{P}\big) \Rightarrow Q$$
$$(P \wedge Q) \Rightarrow P$$
$$((P \Rightarrow Q) \vee (P \Rightarrow R)) \Rightarrow P \Rightarrow (Q \vee R)$$
$$((P \vee Q) \wedge (P \Rightarrow (P \wedge Q))) \Rightarrow (R \Rightarrow R)$$
$$\big((P \Rightarrow Q) \wedge \big(P \Rightarrow \overline{Q}\big)\big) \Rightarrow \overline{P}$$

Exercise proof that

$$\forall n \in \mathbb{N}, n^2 \,\mathrm{odd} \Rightarrow n \,\mathrm{odd}$$