M1F Notes

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Contents

1 Sets

Definition 1.0.1. A set S is a collection of objects (called *elements* of the set). If x is an *element* of S let us write $x \in S$ otherwise $x \notin S$.

Remark 1.1. The order of the elements or any repetition is unimportant.

Example 1.1.

$$\{1, 3\} = \{3, 1, 1\}$$

Definition 1.0.2. For two sets S and T let us write $S \subseteq T$ (S is contained in T) if

$$x \in S \Rightarrow s \in T$$

Result 1.0.1. $S = T$ iff $S \subseteq T$ and $T \subseteq S$.

Remark 2.1. $S \notin S$ (Foundation Axiom)
Nonetheless, elements can be sets.

Definition 1.0.3. $\emptyset$ is the set with no elements.

Property 3.1. $\emptyset \subseteq S$ and $S \subseteq S$ for all sets S

1.1 Set Operators

Definition 1.1.1. The intersection $S \cap T$ of two sets S and T is

$$\{x|\ x \in S \,\text{and}\, x \in T\}$$

Definition 1.1.2. The union $S \cup T$ of two sets S and T is

$$\{x|\ x \in S \,\text{or}\, x \in T\}$$

Definition 1.1.3. The difference $S \backslash T$ of two sets S and T is

$$(S \cup T) \backslash (S \cap T)$$

Definition 1.1.4. The symmetric difference $S \triangle T$ of two sets S and T is

$$\{x|\ x \in S \,\text{and}\, x \in T\}$$

Definition 1.1.5. In $A \subseteq \Omega$ then

$$A^C = \{x \in \Omega | x \notin A\} \qquad = \Omega \backslash A$$

Remark 5.1. The complement is only used when the reference set Ω is clear.

Some sets we will work with in this course are

$$\mathbb{N} = \{0, 1, 2, \dots\}$$
$$\mathbb{Z} = \{0, 1, -1, 2, -2\}$$
$$\mathbb{Q} = \{\frac{p}{q} | P \in \mathbb{N}, q \in \mathbb{Z} \backslash \{0\}\}$$
$$\mathbb{R} \, \text{reals}$$
$$\mathbb{C} \, \text{complex numbers}$$

Definition 1.1.6. $\mathbb{N}$ is defined by two axioms:

1. $0 = \emptyset \in \mathbb{N}$
2. If $n \in \mathbb{N}$ then $n + 1 \overset{def}{=} n \cup \{n\} \in \mathbb{N}$

Example 6.1.

$$1 = 0 + 1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}$$
$$2 = 1 + 1\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$$

1.2 Intervals in $\mathbb{R}$

Definition 1.2.1. If $a \leq b$:

$$[a, b] = \{t \in \mathbb{R} | a \leq t \leq b\}$$
$$(a, b) = \{t \in \mathbb{R} | a < t < b\}$$
$$[a, b) = \{t \in \mathbb{R} | a \leq t < b\}$$
$$(a, b] = \{t \in \mathbb{R} | a < t \leq b\}$$
$$[a, \infty) = \{t \in \mathbb{R} | a < t\}$$
$$(-\infty, b] = \{t \in \mathbb{R} | t \leq b\}$$

1.3 Infinite Unions and Intersections

Definition 1.3.1. Suppose that, for all $n \in \mathbb{N}$, we are given a set A_n.

$$\bigcup_{n=a}^{\infty} A_n = \{x | \, \text{there exists a} \, n \in \mathbb{N}, n \geq a : x \in A_n\}$$
$$\bigcap_{n=a}^{\infty} A_n = \{x | \, \text{for all} \, n \in \mathbb{N}, n \geq a : x \in A_n\}$$

Example 1.1.

$$\bigcup_{n=1}^{\infty} [0, 1 - \frac{1}{n}] = [0, 1)$$
$$\bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n}) = \{1\}$$

2 Proofs

2.1 Elements of the propositional calculus

Definition 2.1.1. A statement (proposition) is an assertion that can be either true (T) or false (F).

Remark 1.1. In maths such an assertion usually takes the form: "If such and such assumptions are made, then we can infer such and such conclusions"

Example 1.1.
- $n = 3$
- $(A + B)^2 = A^2 + 2AB + B^2$
- If it n^2 is odd, then n is odd too.
- If it rains, then it is cloudy.
- For all real numbers ≥ 0 there exists a square root.

Definition 2.1.2. A proof is a chain of statements linked by logical implications (inferences) that establish the truth of the last statement. In the course of the proof one is allowed to "call up"

- assumptions that are made.
- statements proven previously.
- axioms (statements that are generally accepted and never proven).

"Grammar elements" of mathematical statements are Quantifiers:

Type	Sign	Meaning
Existential	$\exists$	there exists
	$\exists_1$	there exists a unique
Universal	$\forall$	for all
	:, $\vert$	such that

Ways to form new statements from old ones:

- If P is a statement then $notP$ "non-P" is the statement which is true if P is false and false if P is true.
- If P and Q are statements then we can form:

Sign	Meaning
$P \wedge Q$, $P\&Q$	P and Q.
$P \vee Q$	either P or Q or both.
$P \veebar Q$	either P or Q but not both.
$P \Rightarrow Q$	If P then Q.
$P \Leftrightarrow Q$	P if and only if Q.

Remark 2.1. $P \Rightarrow Q$ means any of the following:

- If P then Q.
- Q if P.
- P is true only if Q is true.
- P only if Q.
- P is sufficient for Q.
- Q is necessary for P.
- If Q is false then P is false.
- $notQ \Rightarrow notP$

Similarly, $P \Leftrightarrow Q$ means any of the following:

- $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- P if and only if Q. P is necessary and sufficient for Q.

The rigorous definition of $P \wedge Q$, $P \Rightarrow Q$ can be made through a truth table

Definition 2.1.3. $P \wedge Q$ is defined by:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 2.1.4. Also, $P \Rightarrow Q$ is defined by:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 4.1. The statement "If $x \in \{n \in \mathbb{N}|\ n^2 < 0\}$ then x is a sheep." is true as well as the statement "If $x \in \{n \in \mathbb{N}|n^2 < 0\}$ then x is not a sheep."

2.2 Inference rules

Example 0.2. *Premise 1.* If it is raining then it's cloudy.
Premise 2. It's raining.
Conclusion. It is cloudy.

We can write this more abstractly as follows:
P: it is raining
Q: it is cloudy
In this form:
Premise 1. $P \Rightarrow Q$
Premise 2. P
Conclusion. Q

This is an example of an inference rule which we write like this:

$$((P \Rightarrow Q) \wedge P) \Rightarrow Q$$

There are other inference rules:

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$
$$\left((P \vee Q) \wedge \overline{P}\right) \Rightarrow Q$$
$$(P \wedge Q) \Rightarrow P$$
$$((P \Rightarrow Q) \vee (P \Rightarrow R)) \Rightarrow P \Rightarrow (Q \vee R)$$
$$((P \vee Q) \wedge (P \Rightarrow (P \wedge Q))) \Rightarrow (R \Rightarrow R)$$
$$\left((P \Rightarrow Q) \wedge \left(P \Rightarrow \overline{Q}\right)\right) \Rightarrow \overline{P}$$
$$P \wedge (Q \vee R) \Rightarrow (P \wedge Q) \vee (P \wedge R)i$$

Exercise 1. Proof that

$$\forall n \in \mathbb{N}, n^2 \,\text{odd} \Rightarrow n \,\text{odd} \quad .$$

Example 0.3. Is the following a valid argument:

1. If a movie is not worth seeing, then it is not made in the UK.
2. A movie is worth seeing only if Prof Corti reviews it.
3. "The Maths Graves" was not not reviewed by Prof Corti.

4. Therefore, "The Maths Graves" is not made in the UK.

In order to determine this, let us rewrite the argument in a more formal way:

Variable	Meaning
M	the set of all movies
$W(x)$	"x is worth seeing"
$UK(x)$	"x is made in the UK"
$C(x)$	"Prof Corti reviews x"
m	"The Maths Games" $\in M$

Now the argument can be expressed as:

$$\forall x \in M: \quad \overline{W(x)} \Rightarrow \overline{UK(x)} \tag{1}$$
$$\forall x \in M: \quad W(x) \Rightarrow C(x) \tag{2}$$
$$\overline{C(m)} \tag{3}$$
$$((1) \wedge (2) \wedge (3)) \Rightarrow \quad \overline{UK(x)} \tag{4}$$

Yes it is a valid argument. Indeed, it is the same as:

$$\forall x \in M: \quad UK(x) \Rightarrow W(x)$$
$$\forall x \in M: \quad W(x) \Rightarrow C(x)$$

Then you say:

$$\forall x \in M \quad \overline{C(x)} \Rightarrow \overline{UK(x)}$$
$$\overline{C(m)}$$
$$\Rightarrow \overline{UK(m)}$$

Result 2.2.1. What can we learn from this? If we want to be understood, we have to learn to present our arguments better. For instance, try to put everything in the positive. Use "if then" throughout. A better way of writing would be:

1. If x is made in the UK, then x is worth seeing.
2. If x is worth seeing then Prof Corti reviews it.
3. Prof Corti did not review m.
4. Therefore m is not made in the UK.

2.3 Proof-Practice

Proposition Let A, B, C, Ω be sets with $a, B \in \Omega$. Then:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{5}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{6}$$
$$(A \cup B)^C = A^C \cap B^C \tag{7}$$
$$(A \cap B)^C = A^C \cup B^C \tag{8}$$

Exercise 2. Draw pictures of these statements.

Proof. Consider (1). We show first:

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

Suppose that $x \in A \cap (B \cup C)$ then $x \in A$ and $B \cup C$.

That is:

$$\begin{aligned} & x \in A \wedge (x \in B \vee x \in C) \\ \Leftrightarrow \quad & x \in A \cap B \vee x \in A \cap C \\ \Leftrightarrow \quad & x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

This shows $\subseteq$. Now we show:

$$A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$$

Suppose $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap B$ or $x \in A \cap C$. We now distinguish between two cases:

1. $x \in A \cap B$. Then $x \in A$ and $x \in B$. Therefore, $x \in A$ and $x \in B \cup C$. Hence, $x \in A \cap (B \cup C)$
2. $x \in A \cap C$. Then $x \in A$ and $x \in C$. Therefore, $x \in A$ and $x \in B \cup C$. Hence, $x \in A \cap (B \cup C)$.

Remark 0.1. We split the proof of C in two cases. In doing so we used the inference rule:

$$(P \vee Q, P \Rightarrow R, Q \Rightarrow R) \Rightarrow R$$

Please finish the proof in your own $\square$