M1M1 Notes

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Contents

1 Functions

Definition 1.0.1. A function f is a rule assigning every element x in a set A an element $f(x)$ in another set B.

Remark 1.1.

- A is called the domain of f whereas B is called codomain.
- The range (image) of a function is the set:

$$\begin{aligned}\text{Range(f)} &= \text{Im}(f) \subseteq \text{codomain} \\ &= \{f(x) \in B |\ \forall x \in A\}\end{aligned}$$

  It does not have to be equal to the codomain.

- In the following we will mostly consider functions of one variable (with $A = \mathbb{R}$ and $B = \mathbb{R}$, later $\mathbb{C}$).

Example 1.1. Polynomials, $c_i \in \mathbb{R}, \forall i \in \mathbb{N}$:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Definition 1.0.2. The graph of a function f (real not complex) is the set

$$\{(x, y) \,| x \in \text{dom}(f), y = f(x)\}$$

Property 2.1. The graph of any function intersects any vertical line at most once.

1.1 Rational Functions

Definition 1.1.1. A rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

Example 1.1.

$$f(x) = \frac{1}{1 - x^2}, \qquad \text{dom}(f) = \mathbb{R} \backslash \{1, -1\}$$

1.2 Exponential Function

Definition 1.2.1. The exponential function exp can be defined by several ways:

1. As a power of e:

$$\exp(x) = e^x$$

   Obviously, for this definition the number e must be defined.

2. As a power series:

$$\exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

3. By a ordinary differential equation (ODE):

$$\begin{aligned}\frac{d}{dx} \exp(x) &= \exp(x) \\ \exp(0) &= 1\end{aligned}$$

4. As inverse of the natural logarithm:

$$\exp^{-1}(x) = \log(x)$$
$$\log(x) = \int_1^x \frac{du}{u}$$

5. As a limit:

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Property 1.1.

$$\exp(x+y) = \exp(x)\cdot\exp(y)$$

1.3 Trigonometrical Functions

Definition 1.3.1. Similar to the exponential function, the trigonometrical functions cos and sin have several potential definitions:

1. The elementary geometric definition at a right-angled triangle with a hypotenuse of length 1.
2. Definition through Polar form – considering a point p on a unit circle centred at the origin .
3. As a power series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

4. Through a system of ODEs:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\sin 0 = 0, \quad \cos 0 = 1$$

5. With the help of complex numbers:

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

Property 1.1.

- The addition formula:

$$\sin(x+y) = \sin x\cos y + \cos x \sin y$$

- Shifting:

$$\sin\left(x+\frac{\pi}{2}\right) = \cos x$$
$$\cos\left(x+\frac{\pi}{2}\right) = -\sin x$$
$$\sin(x+\pi) = \sin\left(x+\frac{\pi}{2}\right)+\frac{\pi}{2}$$
$$= \cos\left(x+\frac{\pi}{2}\right)$$
$$\sin(x+2\pi) = \sin x$$

Remark 1.1. Special values which should be memorized are

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$$

Definition 1.3.2. If a function f has property $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$ it is called periodic. The period of f is the smallest possible a for which $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$.

Definition 1.3.3. Other trigonometric functions can be written as a combination of sine and cosine:

$$\begin{aligned}
\sec x &= \frac{1}{\cos x} \\
\operatorname{cosec} x &= \frac{1}{\sin x} \\
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x}
\end{aligned}$$

1.4 Odd and Even Functions

Definition 1.4.1. A function f is even if

$$\forall x \in \mathrm{dom}(f) : \quad f(-x) = f(x)$$

A function f is odd if

$$\forall x \in \mathrm{dom}(f) : \quad f(-x) = -f(x)$$

Remark 1.1. These definitions assume that $\mathrm{dom}(f)$ is symmetric which means $x \in \mathrm{dom}(f) \ \Rightarrow\ -x \in \mathrm{dom}(f)$

Example 1.1. $\sin x$ is odd, $\cos x$ is even.

Property 1.1. A function can be neither odd nor even. However, any function can be split into a sum of even and odd functions

$$f(x) = \frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(x) - f(-x)\right)$$

The odd and even part of a function are unique.

Example 1.2.

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)$$

1.5 Hyperbolic Functions

Definition 1.5.1.

$$\begin{aligned}
\cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right) \\
\sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right)
\end{aligned}$$

Property 1.1.

- Addition theorem and derivatives:

$$\begin{aligned}
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\frac{d}{dx}\sinh x &= \cosh x \\
\frac{d}{dx}\cosh x &= \sinh x
\end{aligned}$$

- The hyperbolic functions can also be expressed through power series:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

- Similarly to the trigonometrical Pythagoras the following equation holds:

$$\cosh^2 x - \sinh^2 x = 1$$

Remark 1.1. Origin of the name:

$$\begin{aligned} x &= \cosh t, & t \in \mathbb{R} \\ y &= \sinh t \\ x^2 - y^2 &= 1 \end{aligned}$$

parametrizes a hyperbola.

1.6 Inverse Functions

Definition 1.6.1. The inverse function f^{-1}, if it exists, is a function $f^{-1} : B \to A$ with the properties

$$\begin{aligned} f\left(f^{-1}(y)\right) &= y, & \forall y \in B \\ f^{-1}(f(x)) &= x, & \forall x \in A \end{aligned}$$

Example 1.1.

$$\begin{aligned} f(x) &= x^2 & A = [0, \infty) = B \\ f^{-1}(y) &= \sqrt{y} \end{aligned}$$

Remark 1.1.

- A necessary condition for a function to be invertible is that f is injective (one-to-one).

  $$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

  or

  $$f(x_1) \neq f(x_2) \quad \Leftarrow \quad x_1 \neq x_2$$

  Graphical test: f is injective if its graph intersects any horizontal line at most once.

- The graph of the inverse f^{-1} is the set of the points of the graph of f with the x and y coordinates exchanged. The graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$.

- If f is strictly increasing (decreasing), it is injective.

Definition 1.6.2. f is strictly increasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

f is strictly decreasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

Example 2.1. The exponential function is strictly increasing. (proof in problem sheet)

Remark 2.1.

- Any even function f is not injective if $\operatorname{dom} f \not\subseteq \{0\}$.

- Any periodic function is not injective either.
- Therefore, the trigonometric functions

  $$\sin, \cos, \tan$$

  are not invertible.

  In order to inverse the sin function, restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

  In order to inverse the cos funtion, restrict the domain to $[0, \pi]$.

The inverse of the exponential function is called logarithm, $\log x$.

Anaytic treatment is sometimes possible. Require the existence of $f^{-1}(x)$ and 'Solve' $y = f(x)$ to obtain x in terms of y, $x = f^{-1}(y)$.

Example 2.2.

- $$f(x) = e^{-\frac{1}{x}}$$
  $$x = -\frac{1}{\log y}$$

- inverse hyperbolic functions

  $$f(x) = \cosh x$$
  $$f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$
  $$e^{2x} - 2ye^x + 1 = 0$$
  $$(e^x)^2 - 2ye^x + 1 = 0$$
  $$e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$
  $$e^x = y \pm \sqrt{y^2 - 1}$$
  $$x = \log\left(y \pm \sqrt{y^2 - 1}\right)$$

  Restrict domain of $\cosh x$ to non-negative x.

  $$x = \log(y + \sqrt{y^2 - 1}$$
  $$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$$

  $$\sinh^{-1} x = \log\left(x + \sqrt{1 + x^2}\right)$$

1.6.1 Derivatives of Inverse Functions

The slope of the inverse function is $\frac{1}{f'(a)}$, the reciprocal of slope of the original function.

Example 2.3. $f(x) = e^x = y$ so that $x = \log y$.

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{e^x} = \frac{1}{y}$$

or

$$\frac{d}{dy} \log y = \frac{1}{y}$$

1.6.2 Inverse Trigonometrical functions

We are going to differentiate $\sin^{-1}$, $\tan^{-1}$. We set $y = \sin x$ so that $x = \sin^{-1} y$.

$$\begin{aligned}\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{\cos x} &= \frac{1}{\sqrt{1-\sin^2 x}} \\ &= \frac{1}{\sqrt{1-y^2}} \\ \frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Similarly,

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

Property 2.1.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Example 2.4. Find the mistake in the following proof.

$$\begin{aligned}\sin\left(x + \frac{\pi}{2}\right) &= \cos x \\ x &= \cos^{-1} y \\ \sin\left(\cos^{-1} y + \frac{\pi}{2}\right) &= \cos\left(\cos^{-1} y\right) \\ \sin\left(\cos^{-1} y + \frac{\pi}{2}\right) &= y \\ \cos^{-1} y + \frac{\pi}{2} &= \sin^{-1} y \\ \sin^{-1} - \cos^{-1} y &= \frac{\pi}{2}\end{aligned}$$

2 Limits

Definition 2.0.3. The symbolic notation

$$L = \lim_{x\to a} f(x) \quad \text{or} \quad f(x) \overset{x\to a}{\to} f(x)$$

means:

$$\forall \epsilon > 0\ \exists \delta > 0 : \ (|x-a| > \delta \wedge x \in \mathrm{dom}(f)) \Rightarrow |f(x) - L| < \epsilon$$

$f(x)$ approaches L as x approaches a.

Remark 3.1. It is important that x approaches a from both the left and the right. A one sided limit is written as follows:

$$\lim_{a\to a^+} f(x)$$

or

$$\lim_{x\to a^-} f(x)$$

Example 3.1.

- Let

$$f(x) = \frac{x}{|x|}, \qquad\qquad x \neq 0$$

Then

$$\lim_{x\to 0} f(x)$$

is undefined but one sided limits exist:

$$\lim_{x\to 0^+} f(x) = 1$$
$$\lim_{x\to 0^-} f(x) = -1$$

- Let

$$f(x) = x^2$$

Then

$$\lim_{x\to 2} f(x) = f(2) = 4$$

- Let

$$f(x) = \frac{x^2-1}{x-1}$$

$\lim_{x\to 1} f(x)$ is an indeterminate limit of the form $\frac{0}{0}$. However $\frac{x^2-1}{x-1} = x+1$ if $x \neq 1$. Therefore $\lim_{x\to 1} f(x) = 2$.

Remark 3.2. Not all indeterminate limits are meaningful.

Example 3.2. The limit

$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^2}$$

of the form $\frac{0}{0}$ does not exist.

2.1 Infinite Limits

Example 0.3.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$
$$\lim_{x\to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$x \to \infty$ is the same as $\frac{1}{x} \to 0^+$.
$x \to -\infty$ is the same as $\frac{1}{x} \to 0^-$.

Property 0.1. Provided the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist we know the following rules:

- addition formula

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

- product rule

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

- quotient rule ($\lim_{x\to a} g(x) \neq 0$)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

2.2 Computing Limits

- manipulate function so the limit is 'obvious'
- use power-series
- L'Hopital's rule

Example 0.4.

- $$\begin{aligned} L &= \lim_{x\to 1} \frac{\sqrt{2-x}-1}{1-x} \\ &= \lim_{x\to 1} \frac{\sqrt{2-x}-1}{1-x} \cdot \frac{\sqrt{2-x}-1}{\sqrt{2-x}-1} \\ &= \lim_{x\to 1} \frac{(2-x)-1}{(1-x)(\sqrt{2-x}+1} \\ &= \lim_{x\to 1} \frac{1-x}{(1-x)\left(\sqrt{2-x}+1\right)} \\ &= \lim_{x\to 1} \frac{1}{\sqrt{2-x}+1} \\ &= \frac{1}{2} \end{aligned}$$

  Alternatively the power series can be used. Let $s = 1-x$. Then

  $$L = \lim_{s\to 0} \frac{\sqrt{1+s}-1}{s}$$

  Theorem 2.2.1. The general binomial theorem says

  $$(1+s)^p = 1 + ps + \frac{p(p-1)}{2!}s^2 + \frac{p(p-1)(p-2)}{3!}s^3 + \dots$$

  for $|s| < 1$.

  The geometric series is a special case of the binomial theorem with $p = -1$.

  If p is a positive integer, the series terminates – and gives us the standard binomial theorem. If p is not a positive integer, the formula continues infinitely. Hence

  $$(1+s)^{\frac{1}{2}} = 1 + \frac{1}{2}s + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}s^2 + \dots$$

  can be inserted in our formula and we get:

  $$\lim_{s\to 0} \frac{\sqrt{1+s}-1}{s} = \lim_{s\to 0} \frac{1 + \frac{1}{2}s + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}s^2 + \dots - 1}{s} = \frac{1}{2}$$

- Let us calculate the following well-known limit:

  $$\lim_{x\to 0} \frac{\sin x}{x}$$

  Using a power series we get

  $$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) = 1$$

- Having calculated this limit we can compute

  $$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \cos x = 1$$

- Using limits for graph sketching.

$$f(x) = \frac{x}{e^x - 1} = \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots} \overset{x \to 0}{\to} 1$$

- Let

$$f(x) = \frac{\cos\left(\frac{\pi}{2}x\right)}{1 - x^2}$$

Now we can calculate $\lim_{x\to 1} f(x)$.

$$\begin{aligned}
&\frac{\cos\left(\frac{\pi}{2}x\right)}{1 - x^2} \\
=&\frac{\cos\left(\frac{\pi}{2}(x-1) + \frac{\pi}{2}\right)}{(x-1)(x+1)} \\
=&\frac{\sin\left(\frac{\pi}{2}(x-1)\right)}{(x-1)(x+1)}
\end{aligned}$$

Substituting $s = x - 1$ that gives us

$$\begin{aligned}
=&\frac{\sin\left(\frac{\pi}{2}s\right)}{s(2+s)} \\
=&\frac{\frac{\pi}{2}s - \frac{1}{3!}\left(\frac{\pi}{2}s\right)^3 + \frac{1}{5!}\left(\frac{\pi}{2}s\right)^5 + \dots}{s(2+s)} \\
=&\frac{\frac{\pi}{2} - \frac{1}{3!}\left(\frac{\pi}{2}s\right)^3 + \frac{1}{5!}\left(\frac{\pi}{2}s\right)^5 + \dots}{2+s} \\
\overset{s\to 0}{\to}&\frac{\pi}{4}
\end{aligned}$$

- Consider the limit

$$\begin{aligned}
&\lim_{x\to\infty} x^{\frac{1}{3}} \left((x+1)^{\frac{2}{3}} - x^{\frac{2}{3}}\right) \\
=&\lim_{x\to\infty} x^{\frac{1}{3}} \left(x^{\frac{2}{3}}(1+\frac{1}{x})^{\frac{2}{3}} - x^{\frac{2}{3}}\right) \\
=&\lim_{x\to\infty} x^{\frac{1}{3}} \left(x^{\frac{2}{3}} \left(1 + \frac{2}{3}\cdot\frac{1}{x} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{1}{x}\right)^2 + \dots\right) - x^{\frac{2}{3}}\right) \\
=&\lim_{x\to\infty} x \left(1 + \frac{2}{3}\cdot\frac{1}{x} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\left(\frac{1}{x}\right)^2 + \dots\right) - x \\
=&\lim_{x\to\infty} \left(x \left(1 + \frac{2}{3}\cdot\frac{1}{x} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\left(\frac{1}{x}\right)^2 + \dots\right) - x\right) \\
=&\lim_{x\to\infty} \left(x + \frac{2}{3} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\frac{1}{x} + \dots - x\right) \\
=&\frac{2}{3}
\end{aligned}$$

- Using limits, the equivalence of the definitions for the exponential function can be proven.

$$\lim_{x\to\infty}\left(1 + \frac{a}{x}\right)^x = e^a$$

Derivation:

$$\begin{aligned}
&\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x\\
&=\lim_{x\to\infty}\exp\left(\log\left(1+\frac{a}{x}\right)^x\right)\\
&=\lim_{x\to\infty}\exp\left(x\log\left(1+\frac{a}{x}\right)\right)\\
&=\exp\left(\lim_{x\to\infty}x\left(\frac{a}{x}-\frac{1}{2}\cdot\left(\frac{a}{x}\right)^2+\frac{1}{3}\left(\frac{a}{x}\right)^3-\frac{1}{4}\left(\frac{a}{x}\right)^4\dots\right)\right)\\
&=\exp(a)
\end{aligned}$$

Another possibility to derive this is

$$\begin{aligned}
\left(1+\frac{a}{x}\right)^x =&1+x\frac{a}{x}+\frac{x(x-1)}{2!}\left(\frac{a}{x}\right)^2\\
&+\frac{x(x-1)(x-2)}{3!}\left(\frac{a}{x}\right)^3+\dots
\end{aligned}$$

Considering the limit to infinity and therefore only the dominant powers in each fraction we get

$$=1+a+\frac{a^2}{2}+\frac{a^3}{3!}+\dots$$

2.3 Continuity

Informal Definition. A continuous function f has a graph with no breaks or jumps.

Example 0.5. A continuous function is

$$f(x)=x^2$$

An example for a non-continuous function is the Heaviside function:

$$H(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$$

Definition 2.3.1. A function f is continuous at $a\in\operatorname{dom}(f)$ if

$$\lim_{x\to a}f(x)=f(a)$$

As for the Heaviside function, $\lim_{x\to 0}H(x)$ doesn't exist. Nonetheless, $H(x)$ is continuous for all $x\neq 0$.

Example 1.1.

$$f(x)=x\sin\left(\frac{1}{x}\right)$$

is continuous for $x\neq 0$ but not continuous at $x=0$ (because it is not defined there). However,

$$\lim_{x\to 0}f(x)=0$$

since

$$-|x|\leq f(x)\leq |x|\left|\sin\frac{1}{x}\right|\leq 1$$

Hence, we can consider the function

$$g(x)=x\begin{cases}\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

g is continuous for all $x\in\mathbb{R}$.

2.4 List of Power Series

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\tan x &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \\
\tan^{-1} x &= x + \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\
\log(1+x) &= x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots
\end{aligned}$$

To derive the power series for $\tan^{-1}$, consider

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Integrating both sides gives us

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

3 Differentiation

Geometrical definition. The derivative of a function f at x is the slope of the tangent to the graph $y = f(x)$ at $(x, f(x))$.

Definition 3.0.1.

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h)-f(x)}{h}$ denotes the slope of the secant through $(x, f(x))$ and $(x+h, f(x+h))$.

Remark 1.1. $f'(x)$ is also a function with $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$.

Using the limit definition to compute derivatives is called differentiation from first principles.

Example 1.1.

- Polynomials

$$\begin{aligned}
f(x) &= x^3 \\
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\
&= \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\
&= 3x^2 + 3hx + h^2 \\
&\overset{x \to 0}{\to} 3x^2
\end{aligned}$$

- The cosine function

$$\begin{aligned}
f(x) &= \cos x \\
\frac{f(x+h) - f(x)}{h} &= \frac{\cos(x+h) - \cos x}{h}
\end{aligned}$$

Let us use the trigonometrical identity

$$\cos A - \cos B = -2 \sin \frac{A-B}{2} \sin \frac{A+B}{2}$$

(Derivation:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \cos(\alpha+\beta) - \cos(\alpha-\beta) &= -2\sin\alpha\sin\beta)\end{aligned}$$

With $A = x + b$ and $B = x$ this gives us

$$\begin{aligned}\frac{\cos(x+h) - \cos(x)}{h} &= \frac{-2\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)}{h} \\ &\overset{h\to 0}{\to} -\sin x\end{aligned}$$

- The function f with

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

is not continuous for all $x \in \mathbb{R}$.

Theorem 3.0.1. If a function f is differentiable at $a \in \operatorname{dom} f$ then f is continuous at a.

This poses the question whether it is possible to find a function which is continuous but nowhere differentiable? The Fourier series are

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} = \sin(\pi x) + \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} + \dots$$

This led to the discovery of the Lacunary Fourier series

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2\pi x)}{n} = \sin(\pi x) + \frac{\sin(4\pi x)}{2} + \frac{\sin(9\pi x)}{3} + \dots$$

R is not differentiable except for x rational of the form $\frac{p}{q}$, p, q odd.

3.1 Basic Derivatives

$f(x)$	$f('x)$
x^n	nx^{n-1}
$\log x$	$\frac{1}{x}$
$\exp(x)$	$\exp(x)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$

3.2 Differentiation rules

If u, v and f are derivable functions then the followings rules hold:

- Addition rule

$$\frac{d}{dx}(u(x) + v(x)) = u'(x) + v'(x)$$

- Multiplication rule

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x)$$

- Chain rule

$$\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$$
$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$$

For the derivation of the product rule consider the following quotient:

$$\begin{aligned}
&\frac{u(x+h)v(x+h)-u(x)v(x)}{h}\\
=&\frac{u(x+h)v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x)}{h}\\
=&v(x+h)\frac{u(x+h)-u(x)}{h}+u(x)\frac{v(x+h)-v(x)}{h}\\
\overset{h\to 0}{\to}&v(x)u'(x)+u(x)v'(x)
\end{aligned}$$

The proof of chain rule will be done in spring term analysis.

3.3 Implicit Differentiation

Remark 0.2. Implicit differentiation applies the chain rule .

Example 0.2. Compute the slope of tangent to unit circle $x^2+y^2=1$.

'Solve' to get $y=y(x)$ and use the differentiation rules.

$$y(x) = \pm\sqrt{1-x^2}$$
$$y'(x) = \pm\frac{-x}{\sqrt{1-x^2}} = \mp\frac{x}{\sqrt{1-x^2}}$$

Implicit differentiation. Treat y^2 as a composite function – differentiate with the chain rule.

$$\frac{d}{dx}y^2(x) = 2y(x)y'(x)$$

equation $x^2+y^2=1$. Differentiate with respect to x

$$2x+2yy'=0 \quad\vee\quad y'=\frac{-x}{y}$$

Example 0.3.

$$y^3-y=x^2$$
$$(3y^2-1)y'=2x$$

The slope of the tangent is

$$y'=\frac{2x}{3y^2-1}$$

For the point $(\sqrt{6},2)$ we get the slope

$$y'=\frac{2\sqrt{6}}{11}$$

3.4 Parametric Differentiation

You can describe a curve in the xy plane parametrically.

Example 0.4. • Let us consider a curve defined by the hyperbolic functions.

$$x(t) = \cosh(t)$$
$$y(t) = \sinh(t), \qquad t \in \mathbb{R}$$

slope of the tangent

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$$

$\cdot$ denotes differentiation with respect to the parameter t.

$$\frac{dy}{dx} = \frac{\cosh t}{\sinh t} = \coth t$$

- The equation for a cycloid is

$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t, \qquad t \in \mathbb{R}$$

The point on the edge of a rolling wheel traces a cycloid.

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin t}{1 - \cos t}$$

3.5 Higher Differentiation

Suppose f is differentiable then consider the limit

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

If this exists, f is said to be twice differentiable. The limit is called second derivative, denoted

$$f''(x) \quad \text{or} \quad \frac{d^2 f(x)}{dx^2} \quad \text{or} \quad y''(x) \quad \text{or} \quad \frac{d^2 y(x)}{dx^2}$$

This can be continued to define the n^{th} derivative, denoted as

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n f(x)}{dx^n} \quad \text{or} \quad y^{(n)}(x) \quad \text{or} \quad \frac{d^n y(x)}{dx^n} \quad \text{or} \quad \left(\frac{d}{dx}\right)^n f(x)$$

$\frac{d}{dx}$ is called the differential operator.

Example 0.5.

$$f(x) = \log x \qquad f^{(1)}(x) = \frac{1}{x}$$
$$f^{(2)}(x) = -\frac{1}{x^2} \qquad f^{(3)}(x) = \frac{2}{x^3}$$
$$f^{(4)}(x) = -\frac{2 \cdot 3}{n^4} \qquad f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

Theorem 3.5.1. The Leibniz' formula is

$$\left(\frac{d}{dx}\right) uv = \sum_{p=0}^{n} \binom{n}{p} u^{n-p} v^p$$

The derivation can be made through regarding the functions $u(x)v(x)$.

$$\frac{d}{dx} uv = u'v + uv'$$

Differentiating again gives us

$$\begin{aligned}\left(\frac{d}{dx}\right)^2 uv &= u''v + u'v' + u'v' + uv'' \\ &= u''v + 2u'v' + uv'' \\ \left(\frac{d}{dx}\right)^3 uv &= u''v + u''v' + 2(u''v' + u'v'') + u'v'' + uv''' \\ &= u'''v + 3u''v' + 3u'v'' + uv'''\end{aligned}$$

The coefficients are binomial coefficients. A rigorous proof can be made by induction.

Example 0.6. • Leibniz is particularly useful if one term in the product is a polynomial – since the sum terminates

$$f(x) = e^{2x}x^2$$

Set

$$v = x^2, \quad u = e^{2x}$$

Then

$$\begin{aligned}v^{(1)} = 2x, \quad v^{(2)} = 2, \quad v^{(3)} = v^{(4)} &= v^{(5)} = 0 \\ u^{(n)} &= 2^n e^{2x}\end{aligned}$$

This gives us the n^{th} derivative of f

$$\begin{aligned}f^{(n)}(x) &= \binom{n}{0}u^{(n)}v^{(0)} + \binom{n}{1}u^{(n-2)}v^{(1)}\binom{n}{2}u^{(n-2)}v^{(2)} \\ &= 2^n e^{2x}x^2 + n2^{n-1}e^{2x}2x + n(n-1)2^{n-2}e^{2x}\end{aligned}$$

• Another example

$$\begin{aligned}f(x) &= \sin^{-1} x \\ f'(x) &= \frac{1}{\sqrt{1-x^2}} \\ f''(x) &= \frac{x}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{1-x^2}f'(x) \\ (1-x^2)f^{(2)}(x) &= xf^{(1)}(x)\end{aligned}$$

Differentiate both sides n times.

$$\begin{aligned}\left(1-x^2\right)f^{(2+n)} + \binom{n}{1}(-2x)f^{(1+n)} + \binom{n}{2}(-2)f^{(n)} &= xf^{(n+1)} + 1f^{(n)}\binom{n}{1} \\ \left(1-x^2\right)f^{2+n} - 2nxf(1+n)(x) - n(n+1)f^{(n)} &= xf^{(n+1)} + nf^{(n)}\end{aligned}$$

Set $x = 0$

$$\begin{aligned}f^{(2+n)}(0) - n(n-1)f^{(n)}(0) &= nf^{(n)}(0) \\ f^{(2+n)}(0) &= n^2 f^{(n)}(0) \\ f^{(0)}(0) &= 0 \\ f^{(1)}(0) &= 1 \\ f^{(3)}(0) &= 1 \\ f^{(5)}(0) &= 9 = 3^2 \\ f^{(7)}(0) &= 3^2 5^2 = 225 \\ f^{(9)}(0) &= 3^2 5^2 7^2\end{aligned}$$

4 Graphs

Definition 4.0.1. The graph of a function f is defined by $y = f(x)$

Definition 4.0.2. $a \in \mathrm{dom}(f)$ is a *stationary point* if $f'(a) = 0$

Remark 2.1. A stationary point can be a local minimum, a local maximum or a point of inflection with horizontal tangent.

Suppose a is a stationary point.

1. If $f''(a) > 0$, then a is a local minimum
2. If $f''(a) < 0$, then a is a local maximum
3. If $f''(a) = 0$ gives no information.

This test is called the 2nd Derivative Test.

Example 2.1.

$$\begin{aligned} f(x) &= x^4 \\ f'(x) &= 4x^3 \end{aligned}$$

$x = 0$ is a stationary point because $f''(x) = 12x^2 = 0$.

Geometrical definition. A point of inflection is a point where the graph crosses its own tangent
A sufficient condition for a point of inflection (p_0, I) is: If $f''(a) = 0$ and $f'''(0) \neq 0$, then a is a point of inflection. This is not a necessary condition.

Example 2.2.

$$\begin{aligned} f(x) &= x^5 \\ f'(x) &= 5x^4 \\ f''(x) &= 20x^3 \\ f'''(x) &= 60x^2 \end{aligned}$$

The sufficient condition does not work for this example at $x = 0$ but $(0, 0)$ is a point of inflection.

A point of inflection is not necessarily a stationary point.

Example 2.3.

$$\begin{aligned} f(x) &= x^4 - 2x^2 \\ f'(x) &= 4x^3 - 4x = 4x\left(x^2 - 1\right) \\ &= 4x(x-1)(x+1) \end{aligned}$$

There are 3 stationary points at $x = 0$ and $x = \pm 1$.

$$\begin{aligned} f''(x) &= 12x^2 - 4 \\ f''(0) &= -4 < 0 \end{aligned}$$

So $x = 0$ is a local maximum. Furthermore,

$$f''(\pm 1) = 12 - 4 = 8 > 0$$

Hence, $x = \pm 1$ is a local minimum. Consider the following equation to find points of inflection

$$f''(x) = 12x^2 - 4 = 0$$

This holds if $x^2 = \frac{1}{3}$. i.e. $x = \pm 1/\sqrt{7}$ are points of inflection since $f'''(x) = 24x \neq 0$ at these points.

4.1 Curve sketching

There is no correct way to sketch the graph of a function – in some cases the graph is too complicated to sketch it by hand. In this case try using a computer. (e.g. Riemann's Lacunary Fourier series.) However, the following often helps:

1. Does the graph have any special features (e.g. odd, even or periodic)?
2. Does the graph intersect the x or y axes?
3. Does the graph have stationary points or points of inflection?
4. Does the graph have linear asymptotes?

Rational functions often have linear asymptotes.

Example 0.4.

$$f(x) = \frac{x^3}{1-x^2}$$

- At $x = \pm 1$ the graph has vertical asymptotes.
- For $x \to \pm\infty$ the graph has the linear asymptote $y = x$.
- For x small $f(x) \approx x^3$.

4.1.1 Polar Coordinates

We can represent curves via functions, equations or parametrically – yet another way is through polar coordinates. The Idea is to replace the cartesian coordinates x and y with polar coordinates r and θ. r is the distance to the origin, θ is the angle measured anti-clockwise from the x axis. Replacing θ by $\theta + 2\pi$ has no effect.

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Now we are able to represent curves using equations involving r and θ instead of x and y.

Example 0.5.

- If r is a constant greater than 0, the equations represent a circle.
- Let l be a positive constant and e be a non-negative constant.

$$r = \frac{l}{1 + e\cos\theta}$$

gives a conic section where e is the 'eccentricity'.

4.1.2 Conic sections

In general consider $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, F are constants.

Degenerate cases are:

- point $x^2 + y^2 = 0$
- line $y = 0$
- two lines $x^2 - y^2 = 0 \Leftrightarrow x = \pm y$
- two parallel lines

All other possibilities are of three types ellipse, parabola, hyperbola.

Definition 4.1.1. An *ellipse* is a curve defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or any translation or rotation of this curve. $a = b$ reduces to a circle.

Definition 4.1.2. A Parabola is a curve of the form

$$y = ax^2$$

or any translation or rotation of this curve.

Definition 4.1.3. A Hyperbola is a curve of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or any translation or rotation of this curve.

The equation

$$r = \frac{l}{1 + e\cos\theta}$$

- gives us an ellipse for $0 \le e < 1$.
- gives us a parabola for $e = 1$.
- gives us a hyperbola for $e > 1$.

Set $l = 1$ for all cases: To obtain the equation for a parabola set $e = 1$:

$$r = \frac{1}{1 + \cos\theta}$$

To obtain the equation for a parabola set $e = \frac{1}{2}$:

$$r = \frac{1}{1 + \frac{1}{2}\cos\theta}$$

To obtain the equation for a parabola set $e = 2$:

$$r = \frac{1}{1 + 2\cos\theta}$$
$$-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$$

There are two different conventions for dealing with negative r:

1. Discard any θ values leading to negative r.
2. Retain θ values leading to negative r.

$$x = r\cos\theta \qquad y = r\sin\theta$$

Allow r to be negative. In case of r negative, flip the sign of r, i.e. flip the sign of x and y. This is equivalent to shifting θ by π (or $-\pi$)

$$\sin(\theta \pm \pi) = \sin\theta$$
$$\cos(\theta \pm \pi) = \cos\theta$$

Using this prescription

$$r = \frac{l}{1 + e\cos\theta}$$

gives a full hyperbola (both branches) for $e > 1$.

5 Power Series

Definition 5.0.4. A polynomial of degree n is a function of the form

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^2$$

where $c_0, c_1, \ldots, c_n$ are constants, $c_n \neq 0$.

Remark 4.1. It is easy to see that $c_0 = f(0)$. Differentiating gives us

$$f'(x) = c_1 + 2c_2 x + \cdots + nc_n x^{n-1}$$

And by repeated differentiation we gain

$$c_m = \frac{f^{(m)}(0)}{m!}$$

This gives us the following formula for any polynomial:

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(0)}{m!} x^m$$

If the function is not a polynomial, the formula is evidently not correct but represents an approximation for the function. This polynomial approximation is called a Maclaurin series. It works near $x = 0$ as $f(x)$ and its first n derivatives agree with the polynomial at this value.

We can shift the expansion point from $x = 0$ to another point a:

$$S(x) \approx \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} (x-a)^x$$

This is now called a Taylor series, approximating $f(x)$ near $x = a$.

The error of the approximation can be specified. In fact, if we write

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} (x-a)^x + R_n(x)$$

there are the following exact formulas for $R_n(x)$:

1. Lagrange form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

where c is between a and x.

2. Cauchy form

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!} (x-a)(x-c)^n$$

where c is between a and x.

3. Integrated form

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt$$

Summarized this is called the Taylor Theorem:

Theorem 5.0.1. *Taylor Theorem.* (with Lagrange form of remainder)

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} (x-a)^m + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

For the derivation let us first consider Rolle's theorem:

Theorem 5.0.2. *Rolle's Theorem.*
Suppose f is differentiable on (a,b) and continuous on $[a,b]$ with $f(a)=f(b)$. Then there is a $c\in(a,b)$ such that $f'(c)=0$.

Since the proof for this requires the *intermediate value theorem* whose proof would lead to a long chain of required theory which has not been dealt with in this course, we assume Rolle's Theorem to be obvious.

A generalization of Rolle's Theorem is the mean value theorem:

Theorem 5.0.3. *Mean Value Theorem.* (MVT)
Suppose f is differentiable on (a,b) and continuous on $[a,b]$. Then there is a $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Proof. Define the function g with

$$g(x)=f(x)-\frac{f(b)-f(a)}{b-a}(x-a)$$

Hence,

$$g(a)=g(b)=f(a) \qquad \wedge \qquad g'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$$

Therefore, we can apply Rolle's theorem on g. I.e. there exists a $c\in(a,b)$ with

$$\begin{aligned} & g(c)=0=f'(c)-\frac{f(b)-f(a)}{b-a} \\ \Rightarrow \qquad & f'(c)=\frac{f(b)-f(a)}{b-a} \end{aligned}$$

$\square$

Having dealt with these basic properties, we can proof the different forms of the remainders at least for $n=0$.

Then our approximation is

$$f(x)=f(a)+R_0(x)$$

The integral form of $R_0(x)$ is

$$R_0(x)=\int_a^x f'(t)dt=f(x)-f(a)$$

which obviously fits into our formula.

The Lagrange and Cauchy form are the same:

$$f(x)=f(a)+f'(c)(x-a)$$

where c lies between a and c. The respective c exists according to the MVT.

Example 4.1.

- Consider

$$\begin{aligned} f(x)&=e^x \\ f^{(n)}(x)&=e^x \\ f^{(n)}(0)&=1 \end{aligned}$$

We approximate about $x=0$ with $n=3$ (Maclaurin series):

$$\begin{aligned} f(x)&=1+x+\frac{x^2}{3!}+\frac{x^3}{3!}+R_3(x) \\ R_3(x)&=\frac{e^cx^4}{4!}, \qquad\qquad 0\le c\le x \end{aligned}$$

Suppose x is negative so c is negative. Then $|e^c|<1$. Hence,

$$\left|e^x-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\right|<\frac{x^4}{4!}$$

- For the trigonometrical function this works a bit better:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -sinx \\ f'''(x) &= -\cos x \\ f^{(4)} &= \sin x \end{aligned}$$

Choose $a = 0$ as the expansion point.

$$\begin{aligned} f^{(2)}(0) = f^{(2)}(0) = f^{(2)}(0) &= 0 \\ f^{(1)}(0) = f^{(5)}(0) &= 1 \\ f^{(3)}(0) = f^{(8)}(0) &= -1 \end{aligned}$$

Apply the result with $n = 4$

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + R_4(x) \\ R(4)(x) &= \frac{f^{(5)}(c)x^5}{5!} \\ &= \frac{x^5 \cos x}{5!} \end{aligned}$$

but $|\cos c| \leq 1$.

$$|R_4(x)| = \left| \sin x - \left(x - \frac{x^3}{3!} \right) \right| \leq \frac{|x|^5}{5!}$$

This is true for all x. Thus, for x approaching 1, we get the approximation

$$\sin x \approx x - \frac{x^3}{6}$$

5.1 Infinite Taylor series

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m + R_n(x)$$

In some cases $\lim_{n\to\infty} R_n(x) = 0$ for some or all x (x fixed when taking $n \to \infty$ limit). In this case

$$f(x) = \lim_{n\to\infty} \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!}(x-a)^m$$

Example 0.2.

- Again, consider

$$\begin{aligned} f(x) &= e^x \\ f^{(m)}(x) &= e^x \\ f^{(m)}(0) &= 1 \end{aligned}$$

Expand about $a = 0$:

$$\begin{aligned} f(x) &= \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} + R_n(x) \\ R_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c x^{n+1}}{(n+1)!}, \qquad 0 \leq c \leq x \end{aligned}$$

We claim $R_n(x) \to 0$ as $n \to \infty$ for any fixed x. For example let us consider $x = 1000$:

$$\begin{aligned} R_n &= \frac{e^c 1000^{n+1}}{(n+1)!} \\ &\leq \frac{e^{1000} 10^{3(n+1)}}{(n+1)!} \\ &\stackrel{n\to\infty}{\to} 0 \end{aligned}$$

(Factorials always grow faster than exponentials.) In this example $R_n(x) \to 0$ as $n \to \infty$ for all fixed x. Hence, we get

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

which is the infinite Maclaurin series for the exponential function.

- In our last example the approximation worked for all $x \in \mathbb{R}$. However, in some cases $R_n(x) \to 0$ as $n \to \infty$ works for a range of x values but not all of them. For this purpose regard the geometric series:

$$f(x) = \frac{1}{1-x}$$

The respective Maclaurin series expanded at $a = 0$ is

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots + x^n + R_n(x) \\ R_n(x) &= \frac{1}{1-x} - \left(1 + x + x^2 + \cdots + x^n\right) \\ &= \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} = \frac{x^{n-1}}{1-x} \end{aligned}$$

If $-1 < x < 1$, $R_n(x) \to 0$ as $n \to \infty$ but if $x \geq 1$ or $x \leq -1$, then $\lim_{x\to\infty} R_n(x)$ is undefined.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

is only valid for $-1 < x < 1$.

- Consider the binomial expansion

$$f(x) = (1+x)^p$$

where p is constant. Expand this about $a = 0$.

$$\begin{aligned} f^{(1)}(x) &= p(1+x)^{p-1} \\ f^{(2)}(x) &= p(p-1)(1+x)^{p-2} \\ &\ldots \\ f^{(m)}(x) &= p(p-1)\ldots(p-m+1)(1+x)^{p-m} \end{aligned}$$

The Maclaurin series is

$$\begin{aligned} f(x) &= 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}x^n + R_n(x) \\ R_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \\ &= \frac{p(p-1)\ldots(p-n)(1+c)^{p-n}}{(n+1)!}x^{n+1}, \qquad 0 \leq c \leq x \end{aligned}$$

We claim that $R_n(x) \to 0$ as $x \to \infty$ if $-1 < x < 1$. If $x > 0$, c between 0 and x. If $x > 0$ then $1 + c < 1$. The solution in this case is to use the Cauchy form of the remainder for negative x:

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(c)}{n!}x(x-c)^n \\ &= \frac{p(p-1)\ldots(p-n)}{n!}(1+c)^{p-n}(x-c)^n x \end{aligned}$$

The fraction is a constant and hence unimportant for the limit as $x \to \infty$.

$$\begin{aligned}(1+x)^{p-n}(x-c)^n &= (1+c)^p\left(\frac{x-c}{1+c}\right)^n \\ &= (1+c)^p\left(\frac{x-c}{(c-x)+(1+x)}\right)^n \\ &\leq (1+c)^p\left(\frac{-1}{1+(1+x)}\right)^n\end{aligned}$$

Because $1 > c-x > 0$. This has the limit 0 as $x \to \infty$

5.2 Manipulating Infinite Maclaurin Series

We can multiply and compose infinite power series.

Example 0.3. • Assume we wanted to calculate the first three nonzero terms of the power series for $\tan x$:

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x} \\ &= \left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots\right)\cdot\left(1+\underbrace{\frac{x^2}{2!}+\frac{x^4}{4!}+\dots}_{d}\right)^{-1} \qquad (1)\\ &= \left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots\right)\cdot x\left(1-\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)^2-\dots\right) \\ &= \left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots\right)\left(1+x^2\left(-\frac{1}{2}\right)+x^4\left(-\frac{1}{24}+\frac{1}{4}\right)+\dots\right) \qquad (2)\end{aligned}$$

(1) holds because of the general binomial theorem

$$(1+d)^{-1} = 1-d+d^2-d^3+\dots$$

The 2nd bracket of (2) can be simplified to

$$1-\frac{1}{2}x^2+\frac{5}{24}+\dots$$

Hence the product of the 2 brackets of (2) is:

$$\begin{aligned}&x^3\left(\frac{1}{6}-\frac{1}{2}\right)+x^5l+\left(\frac{3}{100}-\frac{1}{12}+\frac{5}{24}\right)+\dots \\ =&x-\frac{1}{3}x^3+\frac{2}{15}x^5+\dots\end{aligned}$$

In order to get to this result, we could have also used

$$f(x) = f(0)+f''(0)x+\frac{f''(0)x^2}{2!}+\dots+\frac{f^{(5)}(0)x^5}{5!}$$

• Find the first two nonzero terms im the Maclaurin series of $\log(\cos x)$.

$$\begin{aligned}\log(\cos x) &= \log\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right) \\ &= \left(-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)-\frac{1}{2}\left(-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)^2+\dots\end{aligned}$$

because

$$\log(1+X) = X-\frac{X^2}{2}+\frac{X^3}{3}-\dots$$

where in our case

$$X = -\frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

So we get

$$\begin{aligned}\log(\cos x) &= 0 - \frac{1}{2}x^2 + x^4\left(\frac{1}{24} - \frac{1}{8}\right) + \dots \\ &= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots\end{aligned}$$

We can integrate and differentiate power series

Example 0.4.

- Through differentiating the power series it is possible to calculate the first derivative of the hyperbolic tan function:

$$\begin{aligned}\tanh^{-1} &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \dots \\ \frac{d}{dx}\tanh^{-1} x &= 1 + x^2 + x^4 + x^6 + \dots \\ &= \frac{1}{1-x^2}\end{aligned}$$

- Similarly we can obtain the well-known first derivative of $\sin x$:

$$\begin{aligned}\frac{d}{dx}\sin x &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\ &= 1 - x - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ &= \cos x\end{aligned}$$

We argued that for certain infinite Taylor series the remainder term is absent (vanishes in the $n \to \infty$ limit). The reverse procedure also works. We can define functions as power series without a remainder term.

Take as a starting point

$$f(x) = \sum_{m=0}^{\infty} c_m x^m, \qquad c_0, c_1, c_2, \dots \in \mathbb{R}$$

For $c_m = \frac{1}{m!}$ we get the exponential function. Taking $c_m = 1$ for all m gives us

$$f(x) = \frac{1}{1-x}$$

The problem is that we can always define a function this way but we don't know whether it is convergent if the sum is convergent.

Example 0.5.

- $c_m = \frac{1}{m}$ gives us $f(x) = e^x$. This formula is valid for any x
- The geometric series:

$$c_m = 1 \qquad f(x) = \frac{1}{1-x}$$

  This expansion is only valid if $-1 < x < 1$.

- An extreme example is $c_m = m!$.

$$f(x) = \sum_{m=0}^{\infty} m! x^m$$

  This only converges if $x = 0$.

- For the infinite sum

$$c_m = \frac{1}{(m!)^2} \qquad f(x) = \sum_{m=0}^{\infty} \frac{x^m}{(m!)^2}$$

  we do not know for which x it converges.

We would like to know under what conditions

$$f(x) = \sum_{m=0}^{\infty} c_m x^m = \lim_{n\to\infty} \sum_{m=0}^{n} c_m x^m$$

converges and if so for what range of x.

For this purpose we have to take a step back. Consider convergence of numerical series. That means a sum of the form

$$\sum_{m=0}^{\infty} a_m$$

where $a_0, a_1, a_2, \dots$ is an infinite list of numbers.

Example 0.6. • Consider

$$\begin{aligned} a_m &= \frac{1}{m^4}, \qquad m \geq 1 \\ a_0 &= 0 \\ \sum_{m=1}^{\infty} \frac{1}{m^4} &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} = \xi(4) = \frac{\pi^4}{90} \end{aligned}$$

  is a convergent series.

- The harmonic series:

$$\begin{aligned} a_m &= \frac{1}{m}, \qquad m \geq 1, a_0 = 0 \\ \sum_{m=1}^{\infty} \frac{1}{m} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots \end{aligned}$$

  does not converge.

- The alternating harmonic series:

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \\ &= \log 2 \end{aligned}$$

- The following series oscillates between 1 and -1 but does not have a limit for $n \to \infty$

$$\begin{aligned} &\sum_{m=0}^{\infty} (-1)^m, \qquad a_m = (-1)^m \\ =&1 - 1 + 1 - 1 + 1 \dots \\ =&\sum_{m=0}^{n} (-1)^m = \frac{(-1)^n + 1}{2} \end{aligned}$$

How to decide whether a numerical series converges?

- Evaluate it!
- Use some standard tests.

5.3 Tests

1. *Preliminary Test (Easy Test)*
   If $a_m \not\to 0$ as $m \to \infty$ then

   $$\sum_{m=0}^{\infty} a_m$$

   does not converge. If the series converges, then $a_m \to 0$ as $m \to \infty$.

   Example 0.7.
   - $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m^4}$ and $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m}$
     both pass the preliminary test since $a_m \to 0$ as $m \to \infty$.
   - $\displaystyle\sum_{m=0}^{\infty} = 0 + 1 + 2 + 3 + 4 + \dots$
     does not pass the preliminary test and hence diverges.

2. *Alternating Series Test*
   Suppose a_m is alternating and $|a_m|$ is strictly decreasing, $|a_m + 1| < |a_m|$ and a_m passes the preliminary test, then

   $$\sum_m a_m$$

   is convergent.

   Example 0.8. Alternating Harmonic series.

   $$\begin{aligned} s_m &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{5} - \dots \\ a_m &= \frac{(-1)^{m+1}}{m} \qquad m \geq 1 \\ \lim_{m\to\infty} s_m &= \ln 2 \end{aligned}$$

Definition 5.3.1. *Absolute Convergence*
If $\displaystyle\sum_m |a_m|$ converges, then the series $\displaystyle\sum_m a_m$ is said to be absolutely convergent.

Remark 1.1. Absolute convergence is stronger than convergence. I.e. every absolutely convergent series is convergent but not vice versa.

Example 1.1. The alternating harmonic series is convergent but not absolutely convergent.

5.4 Tests for Absolute Convergence

1. *Comparison test*
   Suppose $|a_m| \leq |b_m|$ and $\displaystyle\sum_m b_m$ is absolutely convergent. Then so is $\displaystyle\sum_m a_m$.

   Example 0.2. The series

   $$\sum_{m=0}^{\infty} \frac{1}{m^2 + e^{-m}}$$

   converges because

   $$\frac{1}{m^2 + e^{-m}} < \frac{1}{m^2}$$

   and this series converges to $\frac{\pi^2}{6}$.

2. *Integral Test*
   Suppose $|a_m| = f(m)$ where f is a decreasing positive function. Consider the integral

$$I = \int_N^\infty f(x)dx.$$

If I diverges for any N, then the series $\sum_m a_m$ is not absolutely convergent and if I exists or is finite for some N, then the series is absolutely convergent.

Example 0.3. • The series

$$\sum_{m=1}^\infty \frac{1}{m}$$
$$\int_n^\infty \frac{x}{dx} = [\log x]_{x=N}^{x=\infty} = \log\infty - \log N$$

is meaningless or infinite. This shows once again that

$$\sum_{m=1}^\infty \frac{1}{m}$$

diverges.

•

$$\sum_{m=1}^\infty \frac{1}{m^2}$$
$$\int_N^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{x=N}^{x=\infty}$$
$$= 0 + \frac{1}{N} = \frac{1}{N}$$

is finite unless $N = 0$. By the integral test, the sum

$$\sum_{m=1}^\infty \frac{1}{m^2}$$

is absolutely convergent.

3. *Ratio and root tests*
   The idea is to approximate a series $\sum_m a_m$ with a geometric series.
   Ratio test
   Consider

$$L = \lim_{n\to\infty}\left|\frac{a_{m+1}}{a_m}\right|$$

Root test
Consider

$$L = \lim_{m\to\infty}|a_m|^{\frac{1}{m}}$$

Using either test, if $L > 1$ the series diverges. if $L < 1$ the series is absolutely convergent. If $L = 1$ the test is indecisive.

Example 0.4. Consider

$$\sum_{m=0}^\infty me^{-m}$$
$$a_m = me^{-m}$$

converges due to exponential decay. The ratio is

$$\begin{aligned}\left|\frac{a_{m+1}}{a_m}\right| &= \frac{(m+1)e^{m+1}}{me^{-1}} \\ &= \frac{m+1}{m}e^{-1} \\ &\overset{m\to\infty}{\to} \frac{1}{e} < 1.\end{aligned}$$

Hence, the series converges due to ratio test. The root test gives us

$$\begin{aligned}|a_m|^{\frac{1}{m}} &= (me^{-m})^{\frac{1}{m}} \\ &= m^{\frac{1}{m}}e^{-1} \\ &= e^{\frac{1}{m}\log m}e^{-1} \\ &\overset{m\to\infty}{\to} e^0e^{-1} = \frac{1}{e}.\end{aligned}$$

By the root test, the series is convergent.

In both cases $L = \frac{1}{e} < 1$. The ratio and root test are comparing the series with the geometric series

$$\sum_m L^m$$

where L is a constant. If $L > 1$, the series is not convergent.

Example 0.5.

- exponential series

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

converges for all x.

$$a_m = \frac{x^m}{m!}$$

The ratio test gives us

$$\begin{aligned}\left|\frac{a_{m+1}}{a_m}\right| &= \frac{\frac{|x|^{m+1}}{(m+1)!}}{\frac{|x|^m}{m!}} \\ &= \frac{m!|x|}{(m+1)!} \\ &= \frac{|x|}{m+1} \\ &\overset{m\to\infty}{\to} 0.\end{aligned}$$

The same can be done for the root test:

$$|a_m|^{\frac{1}{m}} = \frac{|x|}{(m!)^{\frac{1}{m}}}$$

We need to consider $(m!)^{\frac{1}{m}}$ for large m. This can be done using Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

describes the behaviour of factorial for large numbers.

$$\begin{aligned}(m!)^{\frac{1}{m}} &\approx (2\pi m)^{\frac{1}{2m}}\frac{m}{e} \\ &= e^12m\log(2\pi m) \\ &\overset{m\to\infty}{\to} 1\end{aligned}$$

Hence, $L = 0$ and by the root test the series converges for all x.

Claim
Any power series $\sum_m c_m x$ converges for all x or converges for $|x| < R$ and diverges for $|x| > R$ where R is a non-negative constant (which is different for each power series and is called radius of convergence). If series converges for all x we can write $R = \infty$.

Example 0.6.

- $\sum_{m=0}^{\infty} 2^m m x^m$

  To find R for this series use the ratio test:

$$\begin{aligned} a_m &= 2^m m x^m \\ \left|\frac{a_{m+1}}{a_m}\right| &= \frac{2^{m+1}(m+1)|x|^{m+1}}{2^m m|x|^m} \\ &= \frac{2(m+1)}{m}|x| \\ &\overset{m\to\infty}{\to} 2|x| \\ L &= 2|x| \end{aligned}$$

  This is smaller than 1 for $x < \frac{1}{2}$ so the series converges for $-\frac{1}{2} < x < \frac{1}{2}$ and $R = \frac{1}{2}$.

-

$$\begin{aligned} \sin^{-1} x &= \sum_{p=0}^{\infty} \frac{4^{-p}(2p)!x^{2p+1}}{(2p+1)(p!)^2} \\ a_p &= \frac{4^{-p}(2p)!x^{2p+1}}{(2p+1)(p!)^2} \end{aligned}$$

  The ratio test results in

$$\begin{aligned} \left|\frac{a_{p+1}}{a_p}\right| &= \frac{4^{-(p+1)}}{4^{-p}} \cdot \frac{(2p+2)!}{(2p)!} \cdot \frac{(p!)^2}{((p+1)!)^2} \cdot \frac{2p+1}{2p+3}|x|^2 \\ &= \frac{1}{4}(2p+1)(2p+2)\frac{1}{(p+1)^2} \cdot \frac{2p+1}{2p+3}|x|^2 \\ &\overset{p\to\infty}{\to} \left|x^2\right| \end{aligned}$$

  For convergence $|x|^2 < 1$ hence $R = 1$.

The root test is not as convenient as the ratio test in computing R but we can use the root test to obtain a formula for R.

$$\begin{aligned} &\sum_m c_m x^m \\ a_m &= c_m x^m \\ &\overset{m\to\infty}{\to} |x| \cdot \lim_{m\to\infty} |c_m|^{\frac{1}{m}} \\ R &= \frac{1}{\lim_{m\to\infty} |c_m|^{\frac{1}{m}}} \end{aligned}$$

works if the limit exists. Otherwise the formula is obviously wrong. In this case take the *Hadamard formula*:

$$R = \frac{1}{\limsup |c_m|^{\frac{1}{m}}}$$

$\limsup$ is called the limit superior.

$$\begin{aligned} \limsup a_n &= \lim_{m\to\infty} b_m \\ b_m &= \sup_{n\geq m} a_n \end{aligned}$$

Example 0.7.

$$
\begin{aligned}
c_m &= \begin{cases} 1, & m \text{ prime} \\ 0, & \text{otherwise} \end{cases} \\
f(x) &= x^2 + x^3 + x^5 + x^7 + x^{11} + \dots \\
R &= 1
\end{aligned}
$$

Definition 5.4.1. Consider

$$
\sum_{m=0}^{\infty} c_m z^m
$$

where $c_0, c_1, c_2, \dots$ is an infinite list of complex numbers $z = x + iy$.

Claim.
A complex power series converges for $|z| < R$ where R is a non-negative constant (radius of convergence).
If a complex function is analytic in a disc, the complex Taylor series always converges within the disc.

Idea: Consider the complex function

$$
\begin{aligned}
f(z) &= \frac{1}{e^2 + 1}, \qquad z = x + yi \\
&= \sum_{m=0}^{\infty} c_m z^m
\end{aligned}
$$

We don't know what the coefficients c_m are. We find out where $f(z)$ is singular in the complex plane. Recipe: R is the distance between the origin and the nearest singularity in the complex plane. This results in $R = \pi$.

5.5 L'Hôpital's Rule

L'Hôpital's Rule is a formula for indeterminate limits:

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
$$

if

$$
\lim_{x \to a} f(x) = \lim g(x) = 0
$$

Example 0.1.

- $$
\begin{aligned}
\lim_{x \to 1} \frac{\cos(\frac{\pi}{2}x)}{1 - x^2} &= \lim_{x \to 1} \frac{-\frac{\pi}{2} \sin \frac{\pi x}{2}}{-2x} \\
&= \frac{-\frac{\pi}{x}}{-2} = \frac{\pi}{4}
\end{aligned}
$$

- $$
\begin{aligned}
\lim_{x \to 0} \frac{\sin^2 x}{x^2} &= \lim_{x \to 0} \frac{2 \sin x \cos x}{2x} \\
&= \lim_{x \to 0} \frac{2 \sin 2x}{2x} \\
&= \lim_{x \to 0} \frac{2 \cos 2x}{2} = 1
\end{aligned}
$$

Justification:

- Applied maths approach: Consider the Taylor series for $f(x)$ and $g(x)$ expanded about $x = a$.

$$
\frac{f(x)}{g(x)} = \frac{f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \dots}{g(a) + g'(x)(x-a) + \frac{1}{2!} g''(a)(x-a)^2 + \dots}
$$

What is the limit for $x \to a$? If $g(a) \neq 0$ the limit is

$$\frac{f(a)}{g(a)}.$$

Suppose $g(a) = f(a) = 0$:

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{f'(a) + f''(a)(x-a) + \frac{1}{2!}f'''(a)(x-a)^2 + \ldots}{g'(a) + g''(x)(x-a) + \frac{1}{2!}g'''(a)(x-a)^2 + \ldots} \\ &\overset{x\to a}{\to} \frac{f'(x)}{g'(x)}\end{aligned}$$

If $g'(a) \neq 0$. If $g'(a) = f'(a) = 0$, limit is

$$\frac{f''(a)}{g''(a)}$$

if $g''(a) \neq 0$.

- Pure maths prove

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

Apply Cauchy's MVT:

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g(c)}$$
$$\ldots$$

6 Complex numbers

Definition 6.0.1. A complex number z is of the form

$$z = x + iy$$

where x and y are a pair of real numbers. i is the imaginary unit with basic property $i^2 = -1$.

Geometrically a complex number is a point in the complex plane.
Abbreviations:

Expression	Abbreviation
$x + i0$	x
$0 + iy$	iy
$0 + i0$	0

We can use polar coordinates:

$$x = r\cos\theta$$
$$y = ri\sin\theta$$

r is the distance from the origin and θ is the angle from the positive x-axis (anti-clockwise).

Definition 6.0.2. The *modulus* $|z|$ of a complex number

$$z = x + iy = r\cos\theta + i\sin\theta$$

is

$$|z| = \sqrt{x^2 + y^2} = r$$

Definition 6.0.3. An *argument* of z is the polar angle θ. This is ambiguous since replacing θ with $\theta \pm 2\pi$ has no effect on z.

Example 3.1.

$$\begin{aligned} z &= 0 + 1i = i \\ |i| &= 1 \\ \arg(i) &= \frac{\pi}{2} \end{aligned}$$

or $\frac{5\pi}{2}$, $\frac{9\pi}{2}$.

Arg$(z) = \theta$ denotes the principal value of the argument with $-\pi < \theta \leq \pi$.

Remark 3.1.

$$|z_1 z_2| = |z_1||z_2|$$

The rules of algebra of complex numbers are the same as for reals (remember that $i^2 = -1$).

Definition 6.0.4. If $z = x + iy$, the *reciprocal* is defined as

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Remark 4.1.

$$z \cdot \frac{1}{z} = 1$$

In polar coordinates the reciprocal can by obtained by replacing r with $\frac{1}{r}$ and θ with $-\theta$. If $z = r\cos\theta + ir\sin\theta$, then

$$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).$$

Example 4.1.

$$\begin{aligned} z &= 1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ \frac{1}{z} &= \frac{1 - i\sqrt{3}}{1 + 3} \\ &= \frac{1 - i\sqrt{3}}{4} \\ &= \frac{1}{2}\left(\cos\frac{\pi}{3} - \sin\frac{\pi}{3}\right) \end{aligned}$$

Integer powers can be defined by multiplication

$$\begin{aligned} z^2 &= z \cdot z \quad z^3 = z \cdot z \cdot z \quad \ldots \\ z^2 &= (x + iy)(x + iy) \\ &= x^2 - y^2 + 2ixy \end{aligned}$$

Negative integer powers are defined through the reciprocal.

$$\begin{aligned} z^{-1} &= \frac{1}{z} \\ z^{-2} &= \frac{1}{z} \cdot \frac{1}{z} \\ &\ldots \end{aligned}$$

6.1 Power series

Definition 6.1.1. A *complex power series* has the form

$$\sum_{m=0}^{\infty} c_m z^m, \qquad c_i \in \mathbb{C},\ i \in \mathbb{N}.$$

The ratio test, the root test and the radius of convergence still work, except that instead of absolute values the complex modulus is used.

Example 1.1. An important example is the complex exponential function. Recall the real exponential has a power series expansion

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

with $R = 0$. We can define the complex exponential

$$\exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

with $R = 0$. The right-hand side is convergent for any complex number z.

The addition formula for the complex exponential is still true:

$$\exp(z + w) = \exp(z)\exp(w)$$

Theorem 6.1.1. *Euler's Formula*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof. Insert $z = i\theta$ into the exponential power series:

$$\begin{aligned}
\exp(i\theta) &= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \dots + \frac{i\theta}{1!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \dots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots + i\left(\frac{i\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

□

Hence, we can rewrite the polar form of $z = r\cos\theta + ir\sin\theta$ as $z = re^{i\theta}$

Example 1.2.

- $$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\pi 6\right) = e^{\frac{i\pi}{6}}$$

- $$\begin{aligned}
& & z &= -1 \\
& & |z| &= 1, \quad \arg(z) = \pi \\
&\Rightarrow & -1 &= e^{i\pi} \\
&\Leftrightarrow & 1 + e^{i\pi} &= 0
\end{aligned}$$

We can use Euler to derive trigonometrical formulas. For example the addition formulas:

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\
\cos(\alpha + \beta) &= \cos\alpha\cos\alpha + \cos\beta\cos\beta
\end{aligned}$$

Proof. Apply Euler to the three exponentials.

$$\begin{aligned}
&\cos(\alpha + \beta) + i\sin(\alpha + \beta) \\
=&(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
=&\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)
\end{aligned}$$

The real part is the addition formula for cosine and the imaginary part gives us the addition formula for the sine function. □

Theorem 6.1.2. *De Moivre's Theorem*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof. Apply Euler to two exponentials $e^{i\theta}$ and $e^{in\theta}$. □

Euler's formula expresses a complex exponential in terms of trigonometrical functions. We can also do the reverse, i.e. express trigonometrical functions in terms of complex exponentials.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

The derivation can be made using Euler and

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

We can generalise cosine and sine to complex arguments.

Definition 6.1.2. For any complex number z

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} + e^{-iz}}{2i}.$$

Example 2.1.

$$\sin i = \frac{e^{i^2} - e^{-i^2}}{2i} = \frac{e^{-1} - e}{2i}$$

Definition 6.1.3. The complex conjugate of a complex number

$$z = x + iy = re^{i\theta}$$

is defined as

$$\overline{z} = x - iy = re^{-i\theta}$$

Geometrically, complex conjugation is a reflection about the real axis.
Useful formulas are

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}$$
$$\mathrm{Im}(z) = \frac{z - \overline{z}}{2}$$
$$|z| = (z\overline{z})^{\frac{1}{2}}$$
$$\overline{ab} = \overline{a} \cdot \overline{b}$$

6.2 Complex Polynomials

Definition 6.2.1. A complex Polynomial of degree n has the form

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n, \qquad c_0, c_1, \ldots, c_n \in \mathbb{C}, c_n \neq 0.$$

Theorem 6.2.1. *Fundamental Theorem of Algebra*
A complex polynomial has at least one root.

A polynomial of degree n has therefore n roots, (where roots may be repeated). A complex polynomial can be factorized into linear factors. I.e. we can write

$$P(z) = c_n(z - a_1)(z - a_2)\ldots(z - a_n)$$

where $a_1, a_2, \ldots, a_n$ are the n roots of P (which can be repeated, e.g.. $a_1 = a_2$).
If the coefficients $c_0, c_1, \ldots, c_n$ of the polynomial are real, the roots can still be complex.

Proposition 1. If the coefficients are real, the roots are either real or appear in complex conjugate pairs.

Proof. Suppose a is a root of P, i.e.

$$P(a) = c_0 + c_1 a + c_2 a^2 + \cdots + c_n a^n = 0.$$

We take the complex conjugate

$$\overline{c}_0 + \overline{c}_1 \overline{a} + \overline{c}_2 \overline{a}^2 + \cdots + \overline{c}_n \overline{a}^n = 0.$$

However, we assumed that the coefficients are real. Hence

$$c_0 + c_1 \overline{a} + c_2 \overline{a}^2 + \cdots + c_n \overline{a}^n = P(\overline{z}) = 0.$$

□

Example 1.1.

- $P(z) = z^6 + 7z^3 - 8$ has six complex roots.

$$\begin{aligned} P(z) &= (z^3)^2 + 7z^3 - 8 \\ &= (z^3 + 8)(z^3 - 1) \end{aligned}$$

Hence, the roots are solutions of $z^3 - 1 = 0$ or $z^3 + 8 = 0$. Let us first consider the former:

$$\begin{aligned} & z^3 = 1 = e^{2\pi i} \\ \Leftrightarrow \quad & z = 1 \quad \vee \quad z = e^{\frac{2\pi i}{3}} \quad \vee \quad z = e^{\frac{4\pi i}{3}} = e^{\frac{-2\pi i}{3}} \end{aligned}$$

We can do the same with the other equation:

$$\begin{aligned} & z^3 = -8 = 8e^{\pi i} \\ \Leftrightarrow \quad & z = -2 \quad \vee \quad z = 2e^{\frac{\pi i}{3}} \quad \vee \quad z = e^{\frac{5\pi i}{3}} = e^{\frac{-\pi i}{3}} \end{aligned}$$

As a result, we can factorize $P(z)$ as

$$(z-1)(z-2)(z-e^{\frac{2\pi i}{3}})(z-e^{\frac{-2\pi i}{3}})(z-2e^{\frac{\pi i}{3}})(z-2e^{\frac{-\pi i}{3}})$$

- Similarly, we solve the following equation:

$$\begin{aligned} z^3 - i &= 0 \\ z^3 &= i = e^{i\frac{\pi}{2}} = e^{5i\frac{\pi}{2}} = e^{9i\frac{\pi}{2}} \\ z &= e^{i\frac{\pi}{6}}, z = e^{5i\frac{\pi}{6}}, z = e^{9i\frac{\pi}{6}} \end{aligned}$$

Definition 6.2.2. A *real Polynomial* of degree n has the form

$$P(x) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n, \qquad c_0, c_1, \ldots, c_n \in \mathbb{R}, c_n \neq 0.$$

A real polynomial can be factorized into real linear and real quadratic factors. I.e. we can write

$$P(x) = c_n (x - a_1)(x - a_2) \ldots (x - a_n)$$

where $a_1, a_2, \ldots, a_n$ are the roots of the complex polynomial

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n.$$

We can combine c_1, c_2 conjugate pairs into a real quadratic term.

Example 2.1. We already found out that the complex polynomial P can be factorised in the following way:

$$\begin{aligned} P(z) &= z^6 + 7z^3 - 8 \\ &= (z^3 + 8)(z^3 - 1) \\ &= (z-1)(z+2)\left(z - e^{2i\frac{\pi}{3}}\right)\left(z - e^{-2i\frac{\pi}{3}}\right)\left(z - e^{i\frac{\pi}{3}}\right)\left(z - e^{-i\frac{\pi}{3}}\right) \end{aligned}$$

Now, consider the real polynomial

$$\begin{aligned} P(x) &= z^6 + 7z^3 - 8 \\ &= (x-1)(x+2)\left(x^2 + x\left(e^{2i\frac{\pi}{3}} + e^{-2i\frac{\pi}{3}}\right) + 1\right)\left(x^2 + x\left(e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}\right) + 4\right) \\ &= (x-1)(x-2)(x^2 + x + 1)(x^2 - 2x + 4). \end{aligned}$$

6.3 Complex Functions

Definition 6.3.1. A *function f of a complex variable* assigns a complex number $f(z)$ to every z in $\text{dom}(f) \subseteq \mathbb{C}$.

Example 1.1.

- complex polynomials
- complex power series
- complex exponential
- complex trigonometrical functions
- complex hyperbolic functions

6.3.1 Complex Logarithm

Definition 6.3.2. Define log with

$$\exp(\log z) = z$$

for $z \in \mathbb{C}$. We can use the polar form $z = re^{i\theta}$ to simplify that:

$$\begin{aligned} z &= e^{\log r + i\theta} \\ \log z &= \log r + i\theta \end{aligned}$$

The ambiguity can be omitted through

$$\log z = \log |z| + i \operatorname{Arg}(z).$$

The complex logarithm has a singularity at zero and the "branch cut" of discontinuity at the negative real axis.

6.3.2 Inverse Tangent Function

Definition 6.3.3. We define the *inverse tangent function* through the power series

$$tan^{-1}u = z - \frac{z^2}{3} + \frac{z^5}{5} + \ldots.$$

In the complex plane the inverse tangent function has two branch cuts. A nice way to see this is

$$\begin{aligned} \tanh^{-1} x &= \frac{1}{2} \log \frac{1+x}{1-x} \\ \tan^{-1} x &= \frac{1}{2i} \log \frac{1+iz}{1-iz}. \end{aligned}$$

Since there are 2 logarithms involved there are two Branch cuts.

6.3.3 Powers

Definition 6.3.4. We define

$$z^p = e^{p \log z}.$$

$$\begin{aligned} \log z &= \log r + i\theta \\ z^p &= e^{p(\log + i\theta)} \\ &= r^p e^{ip\theta} \end{aligned}$$

is ambiguous unless p is an integer. Try

$$\log z^p = e^{p \log z}$$

This again has a Branch cut on the negative real axis.

7 Integration

Definition 7.0.5. There are 3 approaches to integrals:

1. *Geometrical approach*
   Define symbol

$$\int_a^b f(x)dx, \qquad b > a$$

   as an area under the graph $y = f(x)$ between $x = a$ and $x = b$. If the graph falls below the x-axis, the area above the graph and below the x-axis counts negatively.

2. *Analytical approach*
   Riemann integral Start with a partition P of the interval $a \leq x \leq b$. Consider any numbers

$$x_1, x_2, \dots x_{N-1}$$

   with the property

$$a < x_1 < x_2 < \dots < x_{N-1} < b.$$

   We can write $x_0 = a$ and $x_N = b$. The upper Riemann sum is

$$U(f, P) = \sum_{i=1}^{N} p_i(x_i - x_{i-1})$$

   where $p_i = \sup\{f(x)|x_i > x > x_{i-1}\}$. We define the lower Riemann sum

$$L(f, P) = \sum_{i=1}^{N} q_i(x_i - x_{i-1})$$

   where $q_i = \inf\{f(x)|x_i > x > x_{i-1}\}$. Clearly

$$U(f, P) \geq L(f, P).$$

   Define the upper Riemann integral as

$$\overline{\int_a^b} f(x)\ dx = \inf_P U(f, P).$$

   Define the lower Riemann integral as

$$\underline{\int_a^b} f(x)\ dx = \sup_P L(f, P).$$

   If both integrals exist and are equal to each other, we say that the function is Riemann integrable and the value of the Riemann integral is equal to them.

3. *Fundamental Theorem of Calculus approach*
   Suppose

$$F'(x) = f(x)$$

   then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. Suppose that $F'(x) = f(x)$. Furthermore, let P be any partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$. Then

$$\begin{aligned}
F(b) - F(a) &= F(x_N) - F(x_{N-1}) + F(x_{N-1}) - F(x_{N-2}) + \cdots + F(x_1) - F(x_0) \\
&= \sum_{i=1} n(F(x_i) - F(x_{i-1})) \\
&= \sum_{i=1}^{n} \frac{F(x_i) - F(x_i - 1)}{x_i - x_{i-1}} \cdot (x_i - x_i - 1)
\end{aligned}$$

Use the mean value theorem.

$$\begin{aligned}
&\sum_{i=1}^{N} F'(c_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})
\end{aligned}$$

where c_i is between x_i and x_{i-1}. By definition

$$\begin{aligned}
q_i \leq f(c_i) \leq p_i \\
L(f, p) \leq F(b) - F(a) \leq U(f, p)
\end{aligned}$$

For all partitions P. If f is Riemann integrable then $F(b) - F(a) = \int_a^b f(x)dx$. □

Example 5.1.

- Geometrically, it can be seen that

$$\int_{-1}^{1} \sqrt{1 - x^2} = \frac{\pi}{2}$$

as it is half the area below the unit circle.

- Similarly, it is obvious that

$$\int_{-\pi}^{\pi} \sin x \; dx = 0$$

Since the area below the x-axis is the same as above.

- Let

$$f(x) = \begin{cases} 1, & x \notin \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}.$$

Then the Riemann integral

$$\int_0^1 f(x) \; dx$$

does not exist since

$$1 = \overline{\int_0^1} f(x) \; dx \neq \underline{\int_0^1} f(x) \; dx = 0.$$

- The integral

$$\int_0^1 x^2 \; dx = \frac{1}{3}$$

can be calculated by the Riemann definition by approximating the area with rectangles but needs a long time.

Definition 7.0.6. An *indefinite Integral* written

$$\int f(x)\ dx$$

is the set of functions such that

$$\frac{d}{dx}\int f(x)\ dx = dx.$$

To emphasize that a set of function is meant, it is customary to include a constant of integration in tables of indefinite integrals.

7.1 Basic Integrals

$f(x)$	$\int f(x)\ dx$
x^n	$\frac{x^{n+1}}{n+1}+C$
$\frac{1}{x}$	$\log\lvert x\rvert + C$
$\cos x$	$-\sin x + C$
$\sin x$	$\cos x + C$
e^x	$e^x + C$
$\cosh x$	$\sinh x + C$
$\sinh x$	$\cosh x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$

7.2 Integration Techniques

General techniques are

1. *Inspection*

   Example 0.1.
   - If

     $$f(x) = \frac{g'(x)}{g(x)}$$

     then

     $$\int f(x)\ dx = \log g(x) + C.$$

   - $$\int xe^{-x^2}\ dx = e^{-x^2} + C$$

   - $$\int (4+x)^{\frac{1}{5}}\ dx = \frac{5}{6}(4+x)^{\frac{6}{5}} + C$$

   - $$\int \frac{x}{1+x^2}\ dx = \frac{1}{2}\log(1+x^2) + C$$

- $$\int \frac{dx}{\cos x} = \int \frac{\cos x}{cos^2 x}\ dx$$
$$= \frac{\cos x}{1-\sin^2 x}\ dx$$

  Let us first consider
$$\int \frac{\cos x}{1+\sin^2 x}\ dx = \tan^{-1}(\sin x).$$
  Now it is easy to spot that
$$\frac{\cos x}{1-\sin^2 x}\ dx = \tanh^{-1}(\sin x).$$

2. *Integration by parts*
The product rule for differentiation is
$$\frac{d}{dx}uv = u'v + uv'.$$
Rewrite that as
$$\int (u'v + uv')\ dx = uv + C$$
or
$$\int uv'\ dx = uv - \int u'v\ dx$$
which is the integration by parts formula. The method is only useful if the second integral is easier to compute than the first one. It is natural to apply the method to products but there are exceptions.

Example 0.2.

- $$\int \underbrace{x}_{u}\underbrace{e^x}_{v}\ dx = xe^x - \int 1e^x\ dx$$
$$= xe^x - e^x + c$$
$$v' = e^x \quad u = x$$
$$v = e^x \quad u' = 1$$

- $$\int \underbrace{x^2}_{u}\underbrace{e^x}_{v}\ dx = x^2e^x - \int (2x)e^x\ dx$$
$$= xe^x - e^x + c$$
$$v = e^x \quad u' = x$$
  and this has already been computed in the example above.

- $$\int \underbrace{x}_{v'}\underbrace{\tan^{-1} x}_{u}\ dx = uv - \int u'v\ dx$$
$$= \frac{x^2}{2}\tan^{-1} x - \int \frac{\frac{1}{2}x^2}{1+x^2}\ dx$$
$$= \frac{x^2}{2}\tan^{-1} - \frac{1}{2}\int \frac{(1+x^2)-1}{1+x^2}\ dx$$
$$u' = \frac{1}{1+x^2} \quad v = \frac{x^2}{2}$$
$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{\tan^{-1}}{x} + C$$
$$= \left(\frac{x^2}{2} + \frac{1}{x}\right)\tan^{-1} x - \frac{1}{2}x + C$$

3. *Substitution* or *Change of Variables*

$$\int_a^b f(x)dx$$

The Idea is to replace x with a function of a new variable u. Write $x = h(u)$ where u is a new variable.

$$\int_{h(c)}^{h(d)} f(x)dx = \int_c^d f(h(u))\frac{dh(u)}{du}du$$

To derive this apply the Fundamental theorem of calculus to $F(h(u))$.

$$\begin{aligned}
\frac{d}{du}F(h(u)) &= F'(h(u))h'(u) \\
&= f(h(u))h'(u) \\
\int_c^d f(h(u))h'(u)du &= F(h(d)) - F(h(c)) \\
&= \int_{h(c)}^{h(d)} f(x)dx
\end{aligned}$$

If h is invertible set $d = h^{-1}(b)$ and $c = h^{-1}(a)$

$$\int_a^b f(x) \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u))\frac{dh(u)}{du}du$$

This can be rewritten as

$$\int_a^b f(x)dx = \int_{u(a)}^{u(b)} f(x(u))\frac{dx(u)}{du}du$$

$x = x(u)$, $u = u(x)$ inverse function. For an indefinite integral

$$\int f(x)dx = f(x(u))\frac{dx(u)}{du}du$$

$f(x)$ has to be replaced $f(x(u))$, x with $x(u)$ and dx with $\frac{dx}{du}$. Furthermore we integrate with respect to u. In the end the expression has to be rewritten back in x.

Example 0.3.

- $$\begin{aligned}
\int \frac{dx}{1+x^2} &= \int du = u + c \qquad x = \tan u, \quad u = \tan^{-1} x \\
\frac{1}{1+x^2} &= \frac{1}{1+\tan^2 u} = \frac{1}{\sec^2 u} \\
&= \cos^2 u \\
dx &= \frac{dx}{du}\ du = \sec^2 u\ du
\end{aligned}$$

- $$\begin{aligned}
\int \sqrt{1+x^2}\ dx &= \int \sqrt{1+\sinh^2 u}\cosh u\ du \qquad x = \sinh u, \quad dx = \cosh u\ du \\
&= \int \cosh^2 u\ du \\
&= \int \frac{1+\cosh 2u}{2}\ du \\
&= \frac{u}{2} + \frac{\sinh 2u}{4} + C \\
&= \frac{\sinh^{-1} x}{2} + \frac{2\sinh u\cosh u}{4} + C \\
&= \frac{1}{2}\sinh^{-1} x + \frac{1}{2}x\sqrt{1+x^2} + C
\end{aligned}$$

- If $f(x)$ is rational in $\sin x$ or $\cos x$ or both, use a substitution $x = 2\tan^{-1}$ or $u = \tan\frac{x}{2}$.

$$\begin{aligned}\sin x &= \frac{2u}{1+u^2}\\ \cos x &= \frac{1-u^2}{1+u^2}\\ dx &= \frac{2du}{1+u^2}\end{aligned}$$

Let us consider a particular integral:

$$\begin{aligned}\int \frac{1}{\sin x}dx &= \int \frac{\frac{2}{1+u^2}}{\frac{2u}{1+u^2}}du\\ &= \int \frac{du}{u}\\ &= \log u\\ &= \log\left(\tan\frac{x}{2}\right) + C\end{aligned}$$

Techniques which can be very helpful in particular situation but are not as general as the ones above are:

4. *Partial Fractions* (for rational functions)
Partial fractions are useful if the function is rational, i.e. of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials.
The simplest case is the one where the order of P is less than the order of Q and Q has no repeated roots. So

$$Q(x) = c(x-a_1)(x-a_2)\dots(x-a_n)$$

where $a_1, a_2, \dots a_n$ are distinct. Then $f(x)$ can be written in the form

$$\frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \dots + \frac{c_n}{x-a_n}.$$

Example 0.4.

- $$\frac{x+1}{x(x-1)(x-2)} = \frac{c_1}{x} + \frac{c_2}{x-1} + \frac{c_3}{x-2}$$

In order to work out the coefficients we can use the *cover-up rule*. This means that we set x to the root in the denominator of the respective fraction and evaluate the original fraction for this x, *covering up the term involving the respective root.* For our example we get

$$\begin{aligned}\frac{x+1}{x(x-1)(x-2)} &= \frac{\frac{0+1}{(0-1)(0-1)}}{x} + \frac{\frac{2}{1\cdot(-1)}}{x-1} + \frac{\frac{2}{2\cdot 1}}{x-2}\\ &= \frac{1}{2x} + \frac{-2}{x-1} + \frac{3}{2(x-2)}\end{aligned}$$

So

$$\int \frac{x+1}{x(x-1)(x-2)}\,dx = \frac{1}{2}\log x - 2\log(x-1) + \frac{3}{2}\log(x-2) + C$$

- $$\begin{aligned}\int \frac{dx}{1+x^2} &= \int \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)\\ &= \frac{1}{2i}(\log(x-i) - \log(x+i))\\ &= \frac{1}{2i}\left(\log\frac{x-i}{x+i}\right)\\ &= \frac{1}{2i}\left(\log\frac{1+ix}{1-ix}\right) + C\end{aligned}$$

- If there are complex roots it may seem easier to avoid partial fractions:

$$\frac{1}{1+x^2} = \frac{Ax+B}{1+x^2} + \frac{c}{x}$$

Let us now consider some examples with non-distinct roots:

Example 0.5.

$$\begin{aligned}
\frac{1}{(x-1)^2(x-2)} &= \frac{1}{x-1} \cdot \frac{1}{(x-1)(x-2)} \\
&= \frac{1}{x-1}\left(\frac{-1}{x-1} + \frac{1}{x-2}\right) \\
&= -\frac{1}{(x-1)^2} + \frac{1}{(x-1)(x-2)} \\
&= -\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-2}
\end{aligned}$$

If the order of P is greater or equal to the order of Q Write

$$P(x) = A(x)Q(x) + R(x)$$

where A is a polynomial and R is the remainder polynomial whose order is less than the order of Q.

Example 0.6.

$$\begin{aligned}
f(x) &= \frac{x^3}{x^2-1} \\
x^3 &= x(x^2-1) + x
\end{aligned}$$

5. *Using Complex Numbers*

Example 0.7.

- $$\int_{-\pi}^{\pi} \cos^4 x \; dx$$

Using $\cos = \frac{1}{2}(e^{ix} + e^{-ix})$ we get

$$\cos^4 x = \frac{1}{2^4}(e^{4ix} + 4e^{2ix} + 6 + 4^{-2ix} + e^{-4ix}$$

Because $\int_{-\pi}^{\pi} e^{inx} \; dx = 0$ if n is a non zero integer we can simplify our integral to

$$\begin{aligned}
\frac{1}{2^4}\int_{-\pi}^{\pi} 6 \; dx &= \frac{12\pi}{16} \\
&= \frac{3}{4}\pi
\end{aligned}$$

- $$\begin{aligned}
&\int_{-\pi}^{\pi} \cos^{2n} x \; dx, \qquad n \in \mathbb{N} \\
&= \int_{-\pi}^{\pi} \frac{1}{4^n}\left(e^{2inx} + \binom{2n}{1} e^{2i(n-1)x} + \cdots + e^{-2inx}\right)
\end{aligned}$$

All exponential integrate to zero. The only remaining term is the constant $\frac{1}{4n}\binom{2n}{n}$:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^{2n} x &= \frac{2\pi}{4^n}\binom{2n}{n} \\
&= \frac{2\pi}{4^n}\frac{(2n)!}{(n!)^2}
\end{aligned}$$

where n is a positive integer.

- $$\int_{-\pi}^{\pi} \cos^{2n} x \; dx = \frac{2\pi}{4^n} \frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}$$

6. *Differentiation under the Integral*

Example 0.8. •
$$\int x^n e^x \; dx$$

Consider instead this integral:

$$\int x^n e^{ax} \; dx$$

We use $x^n e^{ax} = \frac{d^n}{da^n} e^{ax}$:

$$\begin{aligned} \int x^n e^{ax} \; dx &= \frac{d^n}{da^n} \int e^{ax} \; dx \\ &= \frac{d^n}{da^n} \left(\frac{e^{ax}}{a} + C \right) \end{aligned}$$

We use Leibniz rule and get

$$\frac{d^n}{da^n} \underbrace{a{-}1}_{u(a)} \underbrace{e^{ax}}_{v(a)}$$

$$\begin{aligned} u^{(n)}(a) = (-1)^n a^{-1-n} n! \qquad & v^{(n)}(a) = x^n e^{ax} \\ \int x^n e^{ax} \; dx &= \sum_{m=0}^{n} \binom{n}{m} u^{(m)}(a) v(n-m)(a) \\ &= \sum_{m=0}^{n} \binom{n}{m} (-1)^m a^{-1-m} m! x^{n-m} e^{ax} \\ &= \sum_{m=0}^{n} \frac{1}{(n-m)!} (-1)^n a^{-1-m} x^{n-m} + C \end{aligned}$$

- Another application is

$$\int_0^{\pi} \frac{dx}{a - \cos x} = \frac{1}{2} \frac{d^2}{da^2} \int_0^{\pi} \frac{dx}{a - \cos x} \; dx$$

We use the tangens substitution

$$\begin{aligned} u &= \tan \frac{x}{2} \\ dx &= \frac{2du}{1+u^2} \\ \cos x &= \frac{1-u^2}{1+u^2}. \end{aligned}$$

7.3 Definite Integrals and Improper Integrals

Example 0.9. • A definite integral is

$$\int_0^1 e^x dx$$

We use anti-derivative

$$\int_0^1 e^x \; dx = [e^x]_0^1$$

where

$$[F(x)]_a^b$$

means

$$F(a) = F(b)$$

- Improper integrals are integrals such as

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx$$
$$\int_0^{\infty} \frac{\sin x}{x}$$
$$\int_0^1 x^{-\frac{1}{2}}\ dx$$

with infinite limits or unbounded integrands.

$$\begin{aligned}\int \frac{dx}{(a+x^2)^2} &= -\frac{d}{da}\int \frac{dx}{a+x^2} \qquad x = \tan u \\ &= -\frac{d}{da}\cdot\frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}\end{aligned}$$

The method relies on order independence of partial differentiation:

$$\frac{\delta}{\delta x}\cdot\frac{\delta}{\delta a}f(x,a) = \frac{\delta}{\delta a}\cdot\frac{\delta}{\delta x}f(x,a)$$

Definition 7.3.1. *Improper integrals* are integrals with infinite limits or integrals with unbounded integrands.

Example 1.1.

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx$$
$$\int_0^1 x^{-\frac{1}{2}}\ dx$$

Such integrals can be understood as limits of Riemann integrals.

$$\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty}\int_a^b f(x)\ dx$$

Example 1.2.

- $$\begin{aligned}\int_0^{\infty} e^{-x}\ dx &= \lim_{b\to\infty}\int_0^b e^{-x}\ dx \\ &= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b \\ &= \lim_{b\to\infty}\left(-e^{-b}+1\right) \\ &= 1\end{aligned}$$

- $$\begin{aligned}\int_0^1 x^{-\frac{1}{2}}\ dx &= \lim_{a\to 0^+}\int_a^1 x^{-\frac{1}{2}}\ dx \\ &= \lim_{a\to 0^+}[2x^{\frac{1}{2}}]_a^1 \\ &= \lim_{a\to 0^+}(2-2a^{\frac{1}{2}}) = 2\end{aligned}$$

This procedure fails for some integrals.

$$\begin{aligned}\int_0^1 x^{-\frac{3}{2}}\ dx &= \lim_{a\to 0^+}\int_a^1 x^{-\frac{3}{2}}\ dx\\ &= \lim_{a\to 0^+}\left[-2x^{-\frac{1}{2}}\right]_a^1\\ &= \lim_{a\to 0^+} -2+2a^{-\frac{1}{2}}\end{aligned}$$

This limit does not exist so the integral is meaningless or does not exist.

We would like to know whether an integral is defined before attempting to compute it. It is not necessary to compute the integral – we need to consider the integrand at stress points. (limits $x\to\infty$, $x\to-\infty$ or points where $f(x)$ is unbounded.)

Example 1.3.

- $$\int_0^\infty x^{-\frac{3}{2}}\tan^{-1}x\ dx$$

  Does this exist?

  $$f(x) = x^{-\frac{3}{2}}\tan^{-1}x$$

  Stress points are $x=0$ and $x\to\infty$.
  $x=0$

  $$\begin{aligned}f(x) &= x^{-\frac{3}{2}}\left(x-\frac{x^3}{3}+\dots\right)\\ &\approx x^{-\frac{1}{2}}\quad \text{near } x=0\end{aligned}$$

  Since

  $$\int_0^b x^{-\frac{1}{2}}\ dx$$

  is well defined, $f(x)$ has an integrable singularity at $x=0$.
  $x\to\infty$
  For

  $$f(x) = x^{-\frac{3}{2}}\frac{\pi}{2}$$

  $x\to\infty$ is not a problem. Therefore

  $$\int_0^\infty x^{-\frac{3}{2}}x^{-\frac{3}{2}}\tan^{-1}x\ dx$$

  is finite.

- $$\int_0^{\frac{\pi}{2}} x\tan x\ dx$$

  Does it exist? The stress point is $x=\frac{\pi}{2}$. Look at $f(x)=x\tan x$ near $x=\frac{\pi}{2}$.

  $$\begin{aligned}\tan x &= \frac{\sin x}{\cos x}\\ \frac{1}{\cos x} &= \frac{1}{\cos((x-\frac{\pi}{2})+\frac{\pi}{2})}\\ &= \frac{1}{-\sin(x-\frac{\pi}{2})}\\ &\approx -\frac{1}{x-\frac{\pi}{2}}\\ f(x) &\approx -\frac{\pi}{2}\cdot\frac{1}{x-\frac{\pi}{2}}\quad \text{near } x\\ &= \frac{\pi}{2}\end{aligned}$$

$f(x)$ has a non-integrable singularity.

Or approximate $f(x) \approx \frac{\pi}{2} \tan x$ (works near $x = \frac{\pi}{2}$).

$$\frac{\pi}{2} \int_a^{\frac{\pi}{2}} \tan x \; dx = \frac{\pi}{2} \int_a^{\frac{\pi}{2}} \frac{\sin x}{\cos x} \; dx$$

- $$\int_{-1}^{1} \frac{dx}{x}$$

this is not an improper integral. It is undefined and can be any number. We can fix the integral to be zero – define it as

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{x} &= \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{2\epsilon}^{1} \frac{dx}{x} \right) \\
&= \lim_{\epsilon \to 0^+} (\log \epsilon - \log 2\epsilon) \\
&= \lim_{\epsilon \to 0^+} (-\log 2) = -\log 2.
\end{aligned}$$

However, according to different definitions, the integral can take any value. We therefore say that the integral is not defined as an improper integral because

$$\int_{-1}^{0} \frac{dx}{x} \quad \text{and}$$
$$\int_{0}^{1} \frac{dx}{x}$$

are no improper integrals.

- $$\int_0^{\frac{\pi}{2}} \frac{x}{\cos x} \; dx$$
$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{x}{\cos x}} \; dx$$
$$\frac{x}{\cos x} \stackrel{x \to \frac{\pi}{2}}{\to} \infty$$
$$\begin{aligned}
\frac{1}{\cos x} &= \frac{1}{\cos\left(\left(x - \frac{\pi}{2}\right) + \right)} \\
&= \frac{-1}{\sin\left(x - \frac{\pi}{2}\right)} \\
&\approx \frac{-1}{x - \frac{\pi}{2}}
\end{aligned}$$

- $$\int_a^{\frac{\pi}{2}} \frac{1}{x - \frac{\pi}{2}} \; dx = \left[\log\left(x - \frac{\pi}{2}\right) \right]_a^{\frac{\pi}{2}}$$
$$\frac{1}{\sqrt{\cos x}} \approx \frac{1}{\cos x} = \frac{1}{\cos\left(x - \frac{\pi}{2}\right)}$$
$$\int_0^1 x^{-\frac{1}{2}} \; dx$$

is well defined.

- $$\int_0^b \frac{1}{x^n} \; dx$$

If $n < 1$, $\frac{1}{x^n}$ integrable at $x = 0$.

8 Lengths, Areas and Volumes

We already know that

$$\int_a^b f(x)\ dx$$

represents the area under the graph $y = f(x)$ between $x = a$ and $x = b$. But sometimes we may want to compute the length of a graph between $x = a$ and $x = b$.

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

or

$$L = \int_a^b \sqrt{1 + (y'(x))^2}\ dx$$

Derivation: Consider a small segment of a graph.

$$\delta l = \sqrt{(\delta x)^2 + (\delta y)^2}$$

and

$$\begin{aligned} \delta y &= f'(x)\delta x \\ \delta l &\approx \sqrt{(\delta x)^2 + (f'(x)\delta x)^2} \\ &= \delta x\sqrt{1 + (f'(x))^2} \end{aligned}$$

Adding up a large number of small segments

$$\to L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

Because of the square root, length integrals can be difficult!

Example 1.4. • Length of a sine arch.

$$\begin{aligned} y' &= \cos x \\ L &= \int_0^\pi \sqrt{1 + \cos^2 x}\ dx \end{aligned}$$

- Length of a logarithmic curve

$$\begin{aligned} y &= \log x \\ y' &= \frac{1}{x} \\ L &= \int_1^a \sqrt{1 + \frac{1}{x^2}}\ dx \\ = \int_1^a \frac{\sqrt{x^2 + 1}}{x}\ dx \qquad x &= \sinh u \end{aligned}$$

What about $x = \tan u$? $dx = \sec^2 du$

$$\begin{aligned} \int_1^a \frac{\sqrt{\tan^2 u + 1}}{\tan u} \sec^2 u\ du & \\ = \int_1^a \frac{\sec^3 u}{\tan u}\ du & \\ = \int_1^a \frac{du}{\sin u \cos^2 u} & \end{aligned}$$

postpone try $x = \sinh u, dx = \cosh u du$

$$
\begin{aligned}
\int_1^a \frac{\sqrt{x^2+1}}{x}\, dx &= \int_1^a \frac{\cosh^2 u}{\sinh u}\, du \\
&= \int_1^a \frac{1+\sinh^2 u}{\sinh u}\, du \\
&= \int \left(\frac{1}{\sinh u} + \sinh u\right)\, du \\
&= \log\left(\tanh\frac{u}{2}\right) + \cosh u + C \\
&= \sqrt{1+x^2} + \log\left(\frac{\sinh frac u2}{\cosh\frac{u}{2}}\right) + C \\
&= \sqrt{1+x^2} + \log\frac{\sinh\frac{u}{2}\cos\frac{u}{2}}{2\cosh^2\frac{u}{2}} \\
&= \sqrt{1+x^2} + \log\frac{sinhu}{1+\cosh u} + C \\
&= \sqrt{1+x^2} + \log\frac{x}{1+\sqrt{1+x^2}} + C \\
L &= \int_1^a \frac{\sqrt{1+x^2}}{x}\, dx \\
&= \sqrt{1+a^2} + \log\frac{a}{1+\sqrt{1+a^2}} - \sqrt{2}\log\frac{1}{\sqrt{2}+1}
\end{aligned}
$$

to compute this recall

$$\int \frac{dx}{\sin x} = \log\left(\tan\frac{x}{2}\right)$$

Other integral formulas

$$L = \int_a^b \sqrt{1+y'^2(x)} dx$$

We rotated the graph $y = f(x)$ about the x-axis to give a surface a revolution. Area of surface of revolution.

$$A = 2\pi \int_a^b y(x)\sqrt{1+(y'(x))^2}\, dx$$

Example 1.5. •

$$y(x) = \cosh x$$

between $x = -1$ and $x = 1$.

Volume enclosed by the surface of revolution and $x = a$ and $x = b$ planes

$$V = \pi \int_a^b (y(x))^2 dx$$

There are also alternative length formulas: for a parametrized curve:

$$
\begin{aligned}
x &= x(t) \\
y = y(t) \qquad t_< &\le t \le t_b
\end{aligned}
$$

For a cycloid for example

$$
\begin{aligned}
x(t) &= t - \sin t \\
y(t) &= 1 - \cos t
\end{aligned}
$$

Then the length is

$$L = \int_{t_a}^{t_b} \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

$\dot{x}$ denotes differentiation with respect to t. Polar form

We can describe a curve via a polar equation

$$r = r(\theta), \qquad \theta_a \leq \theta \leq \theta_b$$

(eg

$$r = 1 + \cos\theta$$

)

$$L = \int_{\theta_a}^{\theta_b} \sqrt{r^2(\theta) + r'^2(\theta)}\ \theta$$

Example 1.6. Consider the parabola

$$r = \frac{1}{1+\cos\theta}$$

We want to calculate the length from $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} r' &= \frac{\sin\theta}{(1+\cos\theta)^2} \\ r^2 + r'^2 &= \frac{1}{(1+\cos\theta)^2} + \frac{\sin\theta}{(1+\cos\theta)^2} \\ &= \frac{(1+\cos^2\theta)^2 + \sin^2\theta}{(1+\cos\theta)^4} \\ &= \frac{2}{(1+\cos\theta)^3} \\ L &= -\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{(1+\cos\theta)^{\frac{3}{2}}} \end{aligned}$$

Use $u = \tan\frac{\theta}{2}$ substitution:

$$\begin{aligned} &\sqrt{2}\int_{-1}^{1} \frac{\frac{2du}{1+u^2}}{\left(1+\frac{1-u^2}{1+u^2}\right)^{\frac{3}{2}}} \\ &= \sqrt{2}\int_{-1}^{1} \frac{2du}{1+u^2}\frac{1}{\left(\frac{2}{1+u^2}\right)^{\frac{3}{2}}} \\ &= \sqrt{2}2\int_{-1}^{1} \frac{\sqrt{1+u^2}du}{2^{\frac{3}{2}}} \end{aligned}$$

8.1 Multiple Integration

A function of 2 variables f is a rule assigning a real number $f(x, y)$ to a pair of real numbers (x, y). The domain of f is a subset of $\mathbb{R}^2$.

We can consider the surface defined by $z = f(x, y)$. For example

$$f(x, y) = x^2 + y^2$$

The surface $z = x^2 + x^2$ (paraboloid) For one variable:

$$\int_a^b f(x)dx = \text{area under the graph } x = f(x)$$

$$\int\int_S f(x, y)\ dx\ dy$$

where $S \subseteq \mathbb{R}^2$ can be defined a the volume under the surface $z = f(x, y)$ above S in the xy-plane. If the surface falls below the $z = 0$ plane, the volume counts negatively.

Example 0.7.

$$\int\int_S \sqrt{1-x^2-y^2}\ dx\ dy$$

S is the unit disc $x^2+y^2 \leq 1$.

$$z = \sqrt{1-x^2-y^2}$$

is a hemisphere of unit radius ($x^2+y^2+z^2=1$). The volume of a sphere of radius r is $\frac{4}{3}\pi R^3$.

$$\int\int_S \sqrt{1-x^2-y^2}\ dx\ dy = \frac{2\pi}{3}$$

half the volume of the unit sphere.

$$\int\int_S 1\ dx\ dy$$

gives us the area of S. Just as

$$\int_a^b 1\ dx = b-a = L$$

is the length of the interval. Integrating over a rectangle. the integral over the rectangle can be written as

$$\int_c^d \left(\int_a^b f(x,y)\ dx\right)\ dx$$

or

$$int_a^b \left(\int_c^d f(x,y)\ dy\right)\ dx$$

Both integrals agree (by Fubini's theorem). Without brackets write as

$$\int_c^d dy \int_a^b f(x,y)$$

or

$$\int_a^b dx \int_c^d dy f(x,y)$$

1 variable can write

$$\int_{-\infty}^{\infty} dx e^{-x^2}$$

instead of

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx$$

Can interate over

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx f(x,y)$$

or

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy f(x,y)$$

For one variable we can make a substitution or a change of variables. We can do this for multiple integrals (theory of Jacobians) – for polar coordinates it is quite simple. We want to compute

$$\int\int_f f(x,)\ dx\ dy$$

Suppose the boundary of S can be represented as a polar curve.

$$r = h(\theta)$$

and

$$f(x,y) = g(r,\theta), \qquad x = r\cos\theta, y = r\sin\theta$$

eg

$$\begin{aligned} f(x,y) = e^{-x^2-y^2} &= e^{-r^2} \\ g(r,\theta) &= e^{-r^2} \\ \int\int_S f(x,y)\ dx\ dy \\ = \int_0^{2\pi} d\theta \int_0^{h(\theta)} dr\ rg(r,\theta) \end{aligned}$$

The element of area in polar coordinates $\delta A = r\delta r\delta\theta$ (cartesians $\delta A = \delta x\delta y$).

$$A = \int\int_S 1\ dx\ dy$$

transform to polar coordinates to compute A. Centroid $(\overline{x}, \overline{y})$.

$$\overline{x} = \frac{\int\int_S\ dx\ dy}{\int\int_S dxdY}$$

$$\overline{y} = \frac{\int\int_S y\ dx\ dy}{\int\int_S\ dx\ dy}$$

9 ODEs

Definition 9.0.1. An *ordinary differential equation* (ODE) is an equation involving a function y and one or more of its derivatives.

The general form is

$$h(y, y', y'', \dots y^{(n)}, x) = 0$$

This has order n where n is the highest derivative present. There are linear and non-linear ODEs.

Definition 9.0.2. A *linear ODE* can be written as

$$p_0(x)y + p_1(x)y' + \cdots + p_n(x)y^{(n)} = q(x)$$

where $p_0, p_1, \dots, p_n$ are functions of x.

Example 2.1.

- A first order linear equation is
$$y' = \alpha y \qquad \alpha \in \mathbb{R}.$$
- A first order non-linear equation is
$$y' = \alpha y^2.$$
- The following equation is second order and non-linear
$$y'' = -1 - (y')^2$$
- A second order linear equation is
$$x^2y'' + xy' + y = \tan(e^x)$$

9.1 First order ODEs

The general form is

$$h(y, y', x) = 0.$$

The simplest possible 1st order ODE is

$$y'(x) = f(x)$$

The solution is

$$y(x9 = \int f(x)\ dx.$$

Example 0.2.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

has the general solution

$$y(x) = \sin^{-1} x + C$$

The general linear ODE of order one looks like

$$y'(x) + p(x)y(x) = q(x)$$

where p and q are arbitrary functions. (Integrating the equation does not always work:

$$y(x) + \int p(x)y(x)\ dx = \int q(x)\ dx$$

)

The integrating factor method helps. Multiply the ODE by

$$e^{\int p(x)\ dx}$$

We get

$$e^{\int p(x)\ dx}(y'(x) + p(x)y(x)) = q(x)e^{\int p(x)\ dx}$$

The LHS can be written as

$$\begin{aligned} \frac{d}{dx}\left(y(x)e^{\int p(x)\ dx}\right) &= q(x)e^{\int p(x)\ dx} \\ \frac{d}{dx}e^{\int p(x)\ dx} &= p(x)e^{\int p(x)\ dx} \end{aligned}$$

Now directly integrate

$$\begin{aligned} y(x)e^{\int p(x)\ dx} &= \int q(x)e^{\int p(x)\ dx}\ dx \\ y(x) = e^{-\int p(x)\ dx} &\cdot \int q(x)e^{\int p(x)\ dx}\ dx \end{aligned}$$

Example 0.3.

$$y' + xy = x \qquad p(x) = x, q(x) = x$$

The integrating factor is

$$e^{\int p(x)\ dx} = e^{\int x\ dx} = e^{\frac{1}{2}x^2}$$

We multiply the PDE by $e^{\frac{1}{2}x^2}$.

$$\begin{aligned} e^{\frac{1}{2}x^2}(y' + xy) &= xe^{\frac{1}{2}x^2} \\ \frac{d}{dx}\left(ye^{\frac{1}{2}x^2}\right) &= xe^{\frac{1}{2}x^2} \end{aligned}$$

We integrate both sides

$$\begin{aligned} ye^{\frac{1}{2}x^2} &= \int xe^{\frac{1}{2}x^2}\ dx \\ &= e^{\frac{1}{2}x^2} + C \\ y &= e^{-\frac{1}{2}x^2}\left(e^{\frac{1}{2}x^2} + C\right) \\ &= 1 + Ce^{-\frac{1}{2}x^2} \end{aligned}$$

where C is an arbitrary constant.

9.2 Separation of Variables

We can solve certain linear and non-linear ODEs

Example 0.4.

- $$\frac{dy}{dx} = y \sin x$$

  Separate the x and y dependence.

  $$\begin{aligned} \frac{dy}{y} &= \int \sin x\ dx \\ \log y &= -\cos x + C \\ y &= Ae^{-\cos x} \end{aligned}$$

  where A is an arbitrary constant.

- $$\begin{aligned} \frac{dy}{dx} &= y^2 \sin x \\ \int \frac{dy}{y^2} &= \int \sin x\ dx \\ -\frac{1}{y} &= -\cos x + C \\ \frac{1}{y} &= \cos x + d \\ y &= \frac{1}{d + \cos x} \end{aligned}$$

  where d is an arbitrary constant.

9.3 Homogeneous ODEs

Definition 9.3.1. A *homogeneous ODE* of first order is something like

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

where f is an arbitrary function.

Example 1.1.

$$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$$

Use a new function $v = \frac{y}{x}$ and work with that instead of y.

$$\begin{aligned} y &= xv \\ y' &= xv' + v \end{aligned}$$

So we can write the ODE as

$$xv' + v = f(v)$$

which is always separable.

$$v' = \frac{f(v) - v}{x}$$

$$\begin{aligned}
xv' + v &= v - \tan v \\
x\frac{dv}{dx} &= -\tan v \\
\frac{dv}{\tan v} &= -\frac{dx}{x} \\
\int \frac{\cos v \ dv}{\sin v} &= -\int \frac{dx}{x} \\
\log(\sin v) &= -\log x + C \\
\sin v &= \frac{A}{x}
\end{aligned}$$

where A is an arbitrary constant.

Bernoulli (non-linear)

$$y' + py = qy^n$$

Use the trick substitution $z = y^{1-n}$

$$z' = (1-n)y^{-n}y'$$

multiply equation by $(1-n)y^{-n}$:

$$\begin{aligned}
(1-n)y^{-n}y' + p(1-n)y^{1-n} &= q(1-n) \\
z' + (1-n)pz &= q(1-n)
\end{aligned}$$

which is linear.

9.4 Second Order equations

Definition 9.4.1. A general second order linear equation is

$$y'' + p(x)y' + q(x)y = r(x)$$

where p, q, and r are arbitrary functions of x.

There is no general solution known. The form of a general solution is

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_{PI}(x)$$

were C_1 and C_2 are arbitrary constants. $y_1(x)$ and $y_2(x)$ are independent solutions of

$$y'' + p(x)y' + q(x)y = 0$$

(r is set to zero – called homogeneous form of ODE) independent means that $y_1(x)$ $y_2(x)$ are not proportional to each other. $y_{PI}(x)$ is one solution of the full ODE.

In general it is difficult to find $y_1(x)$ and $y_2(x)$. However, special cases can be solved. A useful special case is the one with constant coefficients. (p and q but not necessarily $r(x)$ are constant.) The constant coefficient case is usually written

$$ay'' + by' + cy = r(x)$$

where a, b, c are constants. this is easy to solve (the solution is usually exponential). $y = e^{\lambda x}$ will work for some constant λ. The homogeneous equation

$$\begin{aligned} ay'' + by' + cy &= 0 \\ y &= e^{\lambda x} \\ y' = \lambda e^{\lambda x} \qquad y'' &= \lambda^2 e^{\lambda x} \\ \Rightarrow \qquad a\lambda^2 + b\lambda + c &= 0 \end{aligned}$$

quadratic equation for λ.

$$y_1(x) = e^{\lambda_1 x} \qquad \lambda_2(x) = e^{\lambda_2}$$

auxialiary equation

$$a\lambda^2 + b\lambda + c = 0$$

(if $\lambda_1 = \lambda_2$ see next term) hard bit to find $y_{PI}(x)$ one solution of full equation.

$$y(x)C_2 e^{\lambda_1 x} + C_2 e^{lambda_2 x} + y_{PI}(x)$$

An important application of ODEs is:

- Mechanics
  Motion in one dimension.
  Newton's second law (find position as a function of time t)

  $$ax\ddot{x} = F(x, \dot{x}, t)$$

  Example 1.1. A boulder falls from a cliff. z measures the height above the cliff top – z is negative during the fall

  $$m\ddot{z} = -mq + \alpha(\dot{z})^2$$

  ODE

  $$\ddot{z} = -q + \beta\dot{z}^2$$

  $\beta = \frac{\alpha}{m} =$ constant. This is second order but first order if we substitute $w = \dot{z}$ in w.