M1M1 Notes

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Contents

1 Functions

Definition 1.0.1. A function f is a rule assigning every element x in a set A an element $f(x)$ in another set B.

Remark 1.1.

- A is called the domain of f whereas B is called codomain.
- The range (image) of a function is the set:

$$\begin{aligned}\text{Range(f)} &= \text{Im}(f) \subseteq \text{codomain} \\ &= \{f(x) \in B |\ \forall x \in A\}\end{aligned}$$

  It does not have to be equal to the codomain.
- In the following we will mostly consider functions of one variable (with $A = \mathbb{R}$ and $B = \mathbb{R}$, later $\mathbb{C}$).

Example 1.1. Polynomials, $c_i \in \mathbb{R}, \forall i \in \mathbb{N}$:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Definition 1.0.2. The graph of a function f (real not complex) is the set

$$\{(x, y) \,|x \in \text{dom}(f), y = f(x)\}$$

Property 2.1. The graph of any function intersects any vertical line at most once.

1.1 Rational Functions

Definition 1.1.1. A rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

Example 1.1.

$$f(x) = \frac{1}{1 - x^2}, \qquad \text{dom}(f) = \mathbb{R}\backslash\{1, -1\}$$

1.2 Exponential Function

Definition 1.2.1. The exponential function exp can be defined by several ways:

1. As a power of e:
$$\exp(x) = e^x$$
Obviously, for this definition the number e must be defined.

2. As a power series:
$$\exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

3. By a ordinary differential equation (ODE):
$$\begin{aligned} \frac{d}{dx}\exp(x) &= \exp(x) \\ \exp(0) &= 1 \end{aligned}$$

4. As inverse of the natural logarithm:
$$\begin{aligned} \exp^{-1}(x) &= \log(x) \\ \log(x) &= \int_1^x \frac{du}{u} \end{aligned}$$

5. As a limit:
$$\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Property 1.1.
$$\exp(x+y) = \exp(x)\cdot\exp(y)$$

1.3 Trigonometrical Functions

Definition 1.3.1. Similar to the exponential function, the trigonometrical functions cos and sin have several potential definitions:

1. The elementary geometric definition at a right-angled triangle with a hypotenuse of length 1.

2. Definition through Polar form – considering a point p on a unit circle centred at the origin .

3. As a power series:
$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \end{aligned}$$

4. Through a system of ODEs:
$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x \\ \sin 0 = 0, &\quad \cos 0 = 1 \end{aligned}$$

5. With the help of complex numbers:
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Property 1.1.

- The addition formula:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

- Shifting:

$$\begin{aligned}
\sin\left(x+\frac{\pi}{2}\right) &= \cos x \\
\cos\left(x+\frac{\pi}{2}\right) &= -\sin x \\
\sin(x+\pi) &= \sin\left(x+\frac{\pi}{2}\right)+\frac{\pi}{2} \\
&= \cos\left(x+\frac{\pi}{2}\right) \\
\sin(x+2\pi) &= \sin x
\end{aligned}$$

Remark 1.1. Special values which should be memorized are

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$$

Definition 1.3.2. If a function f has property $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$ it is called periodic. The period of f is the smallest possible a for which $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$.

Definition 1.3.3. Other trigonometric functions can be written as a combination of sine and cosine:

$$\begin{aligned}
\sec x &= \frac{1}{\cos x} \\
\operatorname{cosec} x &= \frac{1}{\sin x} \\
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x}
\end{aligned}$$

1.4 Odd and Even Functions

Definition 1.4.1. A function f is even if

$$\forall x \in \mathrm{dom}(f): \quad f(-x) = f(x)$$

A function f is odd if

$$\forall x \in \mathrm{dom}(f): \quad f(-x) = -f(x)$$

Remark 1.1. These definitions assume that $\mathrm{dom}(f)$ is symmetric which means $x \in \mathrm{dom}(f) \;\Rightarrow\; -x \in \mathrm{dom}(f)$

Example 1.1. $\sin x$ is odd, $\cos x$ is even.

Property 1.1. A function can be neither odd nor even. However, any function can be split into a sum of even and odd functions

$$f(x) = \frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(x) - f(-x)\right)$$

The odd and even part of a function are unique.

Example 1.2.

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)$$

1.5 Hyperbolic Functions

Definition 1.5.1.

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$
$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

Property 1.1.

- Addition theorem and derivatives:

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$

- The hyperbolic functions can also be expressed through power series:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

- Similarly to the trigonometrical Pythagoras the following equation holds:

$$\cosh^2 x - \sinh^2 x = 1$$

Remark 1.1. Origin of the name:

$$x = \cosh t, \qquad t \in \mathbb{R}$$
$$y = \sinh t$$
$$x^2 - y^2 = 1$$

parametrizes a hyperbola.

1.6 Inverse Functions

Definition 1.6.1. The inverse function f^{-1}, if it exists, is a function $f^{-1} : B \to A$ with the properties

$$f\left(f^{-1}(y)\right) = y, \qquad \forall y \in B$$
$$f^{-1}(f(x)) = x, \qquad \forall x \in A$$

Example 1.1.

$$f(x) = x^2 \qquad A = [0, \infty) = B$$
$$f^{-1}(y) = \sqrt{y}$$

Remark 1.1.

- A necessary condition for a function to be invertible is that f is injective (one-to-one).

  $$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

  or

  $$f(x_1) \neq f(x_2) \quad \Leftarrow \quad x_1 \neq x_2$$

  Graphical test: f is injective if its graph intersects any horizontal line at most once.

- The graph of the inverse f^{-1} is the set of the points of the graph of f with the x and y coordinates exchanged. The graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$.
- If f is strictly increasing (decreasing), it is injective.

Definition 1.6.2. f is strictly increasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

f is strictly decreasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

Example 2.1. The exponential function is strictly increasing. (proof in problem sheet)

Remark 2.1.

- Any even function f is not injective if $\operatorname{dom} f \not\subseteq \{0\}$.
- Any periodic function is not injective either.
- Therefore, the trigonometric functions

$$\sin, \cos, \tan$$

  are not invertible.

  In order to inverse the sin function, restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

  In order to inverse the cos funtion, restrict the domain to $[0, \pi]$.

The inverse of the exponential function is called logarithm, $\log x$.
Anaytic treatment is sometimes possible. Require the existence of $f^{-1}(x)$ and 'Solve' $y = f(x)$ to obtain x in terms of y, $x = f^{-1}(y)$.

Example 2.2.

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$$\begin{aligned} f(x) &= e^{-\frac{1}{x}} \\ x &= -\frac{1}{\log y} \end{aligned}$$

- inverse hyperbolic functions

$$\begin{aligned} f(x) &= \cosh x \\ f(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) \\ e^{2x} - 2ye^x + 1 &= 0 \\ (e^x)^2 - 2ye^x + 1 &= 0 \\ e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\ e^x &= y \pm \sqrt{y^2 - 1} \\ x &= \log\left(y \pm \sqrt{y^2 - 1}\right) \end{aligned}$$

  Restrict domain of $\cosh x$ to non-negative x.

$$\begin{aligned} x &= \log(y + \sqrt{y^2 - 1} \\ \cosh^{-1} x &= \log\left(x + \sqrt{x^2 - 1}\right) \end{aligned}$$

$$\sinh^{-1} x = \log\left(x + \sqrt{1 + x^2}\right)$$

1.6.1 Derivatives of Inverse Functions

The slope of the inverse function is $\frac{1}{f'(a)}$, the reciprocal of slope of the original function.

Example 2.3. $f(x) = e^x = y$ so that $x = \log y$.

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{e^x} = \frac{1}{y}$$

or

$$\frac{d}{dy} \log y = \frac{1}{y}$$

1.6.2 Inverse Trigonometrical functions

We are going to differentiate $\sin^{-1}$, $\tan^{-1}$. We set $y = \sin x$ so that $x = \sin^{-1} y$.

$$\begin{aligned}\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{\cos x} &= \frac{1}{\sqrt{1-\sin^2 x}} \\ &= \frac{1}{\sqrt{1-y^2}} \\ \frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Similarly,

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

Property 2.1.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Example 2.4. Find the mistake in the following proof.

$$\begin{aligned}\sin\left(x + \frac{\pi}{2}\right) &= \cos x \\ x &= \cos^{-1} y \\ \sin\left(\cos^{-1} y + \frac{\pi}{2}\right) &= \cos\left(\cos^{-1} y\right) \\ \sin\left(\cos^{-1} y + \frac{\pi}{2}\right) &= y \\ \cos^{-1} y + \frac{\pi}{2} &= \sin^{-1} y \\ \sin^{-1} - \cos^{-1} y &= \frac{\pi}{2}\end{aligned}$$

2 Limits

Definition 2.0.3. The symbolic notation

$$L = \lim_{x\to a} f(x) \quad \text{or} \quad f(x) \overset{x\to a}{\to} f(x)$$

means:

$$\forall \epsilon > 0\ \exists \delta > 0 : \ (|x-a| > \delta \wedge x \in \mathrm{dom}(f)) \Rightarrow |f(x) - L| < \epsilon$$

$f(x)$ approaches L as x approaches a.

Remark 3.1. It is important that x approaches a from both the left and the right. A one sided limit is written as follows:

$$\lim_{a \to a^+} f(x)$$

or

$$\lim_{x \to a^-} f(x)$$

Example 3.1.

- Let

  $$f(x) = \frac{x}{|x|}, \qquad x \neq 0$$

  Then

  $$\lim_{x \to 0} f(x)$$

  is undefined but one sided limits exist:

  $$\lim_{x \to 0^+} f(x) = 1$$
  $$\lim_{x \to 0^-} f(x) = -1$$

- Let

  $$f(x) = x^2$$

  Then

  $$\lim_{x \to 2} f(x) = f(2) = 4$$

- Let

  $$f(x) = \frac{x^2 - 1}{x - 1}$$

  $\lim_{x \to 1} f(x)$ is an indeterminate limit of the form $\frac{0}{0}$. However $\frac{x^2-1}{x-1} = x + 1$ if $x \neq 1$. Therefore $\lim_{x \to 1} f(x) = 2$.

Remark 3.2. Not all indeterminate limits are meaningful.

Example 3.2. The limit

$$\lim_{x \to 1} \frac{x^2 - 1}{(x-1)^2}$$

of the form $\frac{0}{0}$ does not exist.

2.1 Infinite Limits

Example 0.3.

$$\lim_{x \to \infty} \frac{1}{x} = 0$$
$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$x \to \infty$ is the same as $\frac{1}{x} \to 0^+$.
$x \to -\infty$ is the same as $\frac{1}{x} \to 0^-$.

Property 0.1. Provided the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist we know the following rules:

- addition formula

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

- product rule

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

- quotient rule ($\lim_{x\to a} g(x) \neq 0$)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

2.2 Computing Limits

- manipulate function so the limit is 'obvious'
- use power-series
- L'Hopital's rule

Example 0.4.

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$$\begin{aligned}
L &= \lim_{x\to 1}\frac{\sqrt{2-x}-1}{1-x}\\
&= \lim_{x\to 1}\frac{\sqrt{2-x}-1}{1-x}\cdot\frac{\sqrt{2-x}-1}{\sqrt{2-x}-1}\\
&= \lim_{x\to 1}\frac{(2-x)-1}{(1-x)(\sqrt{2-x}+1}\\
&= \lim_{x\to 1}\frac{1-x}{(1-x)\left(\sqrt{2-x}+1\right)}\\
&= \lim_{x\to 1}\frac{1}{\sqrt{2-x}+1}\\
&= \frac{1}{2}
\end{aligned}$$

Alternatively the power series can be used. Let $s = 1-x$. Then

$$L = \lim_{s\to 0}\frac{\sqrt{1+s}-1}{s}$$

General Binomial expansion is

$$(1+s)^p = 1 + ps + \frac{p(p-1)}{2!}s^2 + \frac{p(p-1)(p-2)}{3!}s^3 + \ldots$$

If p is a positive integer, the series terminates – and gives us the standard binomial theorem. If p is not a positive integer, the formula is valid for $|s| < 1$. Hence

$$(1+s)^{\frac{1}{2}} = 1 + \frac{1}{2}s + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}s^2 + \ldots$$

can be inserted in our formula and we get:

$$\lim_{s\to 0}\frac{\sqrt{1+s}-1}{s} = \lim_{s\to 0}\frac{1+\frac{1}{2}s+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}s^2+\cdots-1}{s} = \frac{1}{2}$$

- Let us calculate the following well-known limit:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

Using a power series we get

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\dots\right)=1$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}$$

Using limits for graph sketching.

$$f(x)=\frac{x}{e^x-1}=\frac{1}{1+\frac{x}{2!}+\frac{x^2}{3!}+\dots}\overset{x\to 0}{\to}1$$

- Let

$$f(x)=\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x^2}$$

Now we can calculate $\lim_{x\to 1}f(x)$.

$$\begin{aligned}&\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x^2}\\ =&\frac{\cos\left(\frac{\pi}{2}(x-1)+\frac{\pi}{2}\right)}{(x-1)(x+1)}\\ =&\frac{\sin\left(\frac{\pi}{2}(x-1)\right)}{(x-1)(x+1)}\end{aligned}$$

Substituting $s=x-1$ that gives us

$$\begin{aligned}=&\frac{\sin\left(\frac{\pi}{2}s\right)}{s(2+s)}\\ =&\frac{\frac{\pi}{2}s-\frac{1}{3!}\left(\frac{\pi}{2}s\right)^3+\frac{1}{5!}\left(\frac{\pi}{2}s\right)^5+\dots}{s(2+s)}\\ =&\frac{\frac{\pi}{2}-\frac{1}{3!}\left(\frac{\pi}{2}s\right)^3+\frac{1}{5!}\left(\frac{\pi}{2}s\right)^5+\dots}{2+s}\\ \overset{s\to 0}{\to}&\frac{\pi}{4}\end{aligned}$$

- Using limits, the equivalence of the definitions for the exponential function can be proven.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=e^a$$

Derivation:

$$\begin{aligned}&\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x\\ =&\lim_{x\to\infty}\exp\left(\log\left(1+\frac{a}{x}\right)^x\right)\\ =&\lim_{x\to\infty}\exp\left(x\log\left(1+\frac{a}{x}\right)\right)\\ =&\exp\left(\lim_{x\to\infty}x\left(\frac{a}{x}-\frac{\frac{a}{x}^2}{2}+\frac{\frac{a}{x}^3}{3}-\frac{\frac{a}{x}^4}{4}\dots\right)\right)\\ =&\exp(a)\end{aligned}$$

$$\left(1+\frac{a}{x}\right)^x = 1 + x\frac{a}{x} + \frac{x(x-1)}{2!}\left(\frac{a}{x}\right)^2 + \frac{x(x-1)(x-2)}{3!}\left(\frac{a}{x}\right)^3 + \dots$$

Considering the limit to infinity and therefore only the dominant powers in each fraction.

$$= 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

For $x = -x$ this is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + C$$

$C = 0$ matching at $x = 0$.

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Integrating both sides gives us

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Geometric series is a special case of the binomial theorem with $p = -1$

2.3 Continuity

Informal Definition. A continuous function f has a graph with no breaks or jumps.

Example 0.5. A continuous function is

$$f(x) = x^2$$

An example for a non-continuous function is the Heaviside function:

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Definition 2.3.1. A function f is continuous at $a \in \mathrm{dom}(f)$ if

$$\lim_{x\to a} f(x) = f(a)$$

As for the Heaviside function, $\lim_{x\to 0} H(x)$ doesn't exist. Nonetheless, $H(x)$ is continuous for all $x \neq 0$.

Example 1.1.

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

is continuous for $x \neq 0$ but not continuous at $x = 0$ (because it is not defined there). However,

$$\lim_{x\to 0} f(x) = 0$$

since

$$-|x| \le f(x) \le |x| \left|\sin\frac{1}{x}\right| \le 1$$

Hence, we can consider the function

$$g(x) = x \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

g is continuous for all $x \in \mathbb{R}$.

3 Differentiation

Geometrical definition The derivative of a function f at x is the slope of the tangent to the graph $y = f(x)$ at $(x, f(x))$.

Definition 3.0.2.

$$f'(x) = \lim_{x\to 0} \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h)-f(x)}{h}$ denotes the slope of the secant through $(x, f(x))$ and $(x+h, f(x+h))$.

Remark 2.1. $f'(x)$ is also a function with $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$.

Differentiation from first principles. Using the limit definition to compute derivatives is called differentiation from first principles.

Example 2.1.

- Polynomials

$$\begin{aligned} f(x) &= x^3 \\ \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\ &= \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\ &= 3x^2 + 3hx + h^2 \\ &\overset{x\to 0}{\to} 3x^2 \end{aligned}$$

- The cosine function

$$f(x) = \cos x$$

Differentiation by first principles:

$$\frac{f(x+h)-f(x)}{h} = \frac{\cos(x+h) - \cos x}{h}$$

Let us use the trigonometrical identity

$$\cos A - \cos B = -2\sin\frac{A-B}{2}\sin\frac{A+B}{2}$$

(Derivation:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \cos(\alpha+\beta) - \cos(\alpha-\beta) &= -2\sin\alpha\sin\beta) \end{aligned}$$

With $A = x + b$ and $B = x$ this gives us

$$\begin{aligned} \frac{\cos(x+h) - \cos(x)}{h} &= \frac{-2\sin\frac{h}{2}\sin\left(x + \frac{h}{2}\right)}{h} \\ &\overset{h\to 0}{\to} -\sin x \end{aligned}$$

- The function f with

$$f(x) = 1, x \in \mathbb{Q} \quad 0, 0x \in \mathbb{Q}$$

is not continuous for all $x \in \mathbb{R}$.

Theorem 3.0.1. If a function f is differentiable at $a \in \operatorname{dom} f$ then f is continuous at a.

Is it possible to find a function which is continuous but nowhere differentiable?
Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} = \sin(\pi x) + \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} + \dots$$

Lacunary Fourier series

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2\pi x)}{n} = \sin(\pi x) + \frac{\sin(4\pi x)}{2} + \frac{\sin(9\pi x)}{3} + \dots$$

R is not differentiable except for x rational of the form $\frac{p}{q}$, p, q odd.

3.1 Basic Derivatives

$f(x)$	$f('x)$
x^n	nx^{n-1}
$\log x$	$\frac{1}{x}$
$\exp(x)$	$\exp(x)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$

3.2 Differentiation rules

If u, v and f are derivable functions then the followings rules hold:

- Addition rule

$$\frac{d}{dx}(u(x) + v(x)) = u'(x) + v'(x)$$

- Multiplication rule

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x)$$

- Chain rule

$$\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

For the derivation of the product rule consider the following quotient:

$$\begin{aligned}
&\frac{u(x+h)v(x+h)-u(x)v(x)}{h} \\
=&\frac{u(x+h)v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x)}{h} \\
=&\frac{v(x+h)u(x+h)-u(x)}{h}+u(x)\frac{v(x+h)-v(x)}{h} \\
\overset{h\to 0}{\to}& v(x)u'(x)+u(x)v'(x)
\end{aligned}$$

Proof of chain rule see spring term analysis.

3.3 Implicit Differentiation

Remark 0.2. Implicit differentiation applies the chain rule .

Example 0.2. Compute te slope of tangent to unit circle $x^2+y^2=1$.

'Solve' to get $y=y(x)$ and use the differentiation rules.

$$\begin{aligned}
y(x) &= \pm\sqrt{1-x^2} \\
y'(x) = \pm\frac{-x}{\sqrt{1-x^2}} &= \mp\frac{x}{\sqrt{1-x^2}}
\end{aligned}$$

Implicit differentiation. Treat y^2 as a composite function – differentiate with the chain rule.

$$\frac{d}{dx}y^2(x)=2y(x)y'(x)$$

equation $x^2+y^2=1$. Differentiate with respect to x

$$2x+2yy'=0 \quad \vee \quad y'=\frac{-x}{y}$$

Example 0.3.

$$\begin{aligned}
y^3-y &= x^2 \\
(3y^2-1)y' &= 2x
\end{aligned}$$

The slope of the tangent is

$$y'=\frac{2x}{3y^2-1}$$

For the point $(\sqrt{6},2)$ we get the slope

$$y'=\frac{2\sqrt{6}}{11}$$

3.4 Parametric Differentiation

You can describe a curve in the xy plane parametrically.

Example 0.4.

$$\begin{aligned}
x(t) &= \cosh(t) \\
y(t) &= \sinh(t), \qquad t\in\mathbb{R}
\end{aligned}$$

slope of the tangent

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\dot{y}}{\dot{x}}$$

$\cdot$ denotes differentiation with respect to parameter t

$$\frac{dy}{dx} = \frac{\cosh t}{\sinh t} = \coth t$$

Cycloid

$$\begin{aligned} x(t) &= t - \sin t \\ y(t) &= 1 - \cos t \qquad t \in \mathbb{R} \end{aligned}$$

The point on the edge of a rolling wheel traces a cycloid.

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin t}{1 - \cos t}$$

3.5 Higher Differentiation

Suppose f is differentiable then consider the limit

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

If this exists, f is said to be twice differentiable. The limit is called second derivative, denoted $f''(x)$ or $\frac{d^2 f(x)}{dx^2}$ $y''(x)$ or $\frac{d^2 y(x)}{dx^2}$

This can be continued to define the n^{th} derivative, denoted as $\frac{d^n f(x)}{dx^n}$ or $f^{(n)}(x)$ or $\frac{d^n y(x)}{dx^n}$ or $y^{(n)}(x)$.

Example 0.5.

$$\begin{aligned} &f(x) = \log x \\ f^{(1)}(x) = \frac{1}{x} \qquad &f^{(2)}(x) = -\frac{1}{x^2} \\ f^{(3)}(x) = \frac{2}{x^3} \qquad &f^{(4)}(x) = -\frac{2 \cdot 3}{n^4} \\ f^{(n)}(x) = &\frac{(-1)^{n+1}(n-1)!}{x^n} \end{aligned}$$

Notation: The n^{th} derivative can be written as $f^{(n)}(x) = \left(\frac{d}{dx}\right)^n f(x)$. $\frac{d}{dx}$ is called the differential operator.

Leibniz' Rule – formula for the n^{th} derivative of a product of 2 functions $u(x)v(x)$.

$$\frac{d}{dx} uv = u'v + uv'$$

Differentiating again gives us

$$\begin{aligned} \left(\frac{d}{dx}\right)^2 uv &= u''v + u'v' + u'v' + uv'' \\ &= u''v + 2u'v' + uv'' \\ \left(\frac{d}{dx}\right)^3 uv &= u''v + u''v' + 2(u''v' + u'v'') + u'v'' + uv''' \\ &= u'''v + 3u''v' + 3u'v'' + uv''' \end{aligned}$$

The coefficients are binomial coefficients. The Leibniz' formula is

$$\left(\frac{d}{dx}\right) uv = \sum_{p=0}^{n} \binom{n}{p} u^{n-p} v^p$$

proof by induction.

Example 0.6. Leibniz is particularly useful if one term i n the product is a polynomial – since the sum terminates

$$\begin{aligned}
f(x) &= e^{2x}x^2 \\
v = x^2 \quad u &= e^{2x} \\
v^{(1)} = 2xv^{(2)} = 2, v^{(3)} = v^{(4)} = v^{(5)} &= 0 \\
u^{(n)} &= 2^n e^{2x}
\end{aligned}$$

$$\begin{aligned}
f^{(n)}(x) &= \binom{n}{0}u^{(n)}v^{(0)} + \binom{n}{1}u^{(n-2)}v^{(1)}\binom{n}{2}u^{(n-2)}v^{(2)} \\
&= 2^n e^{2x}x^2 + n2^{n-1}e^{2x}2x + n(n-1)2^{n-2}e^{2x}
\end{aligned}$$

Another example

$$\begin{aligned}
f(x) &= \sin^{-1} x \\
f'(x) &= \frac{1}{\sqrt{1-x^2}} \\
f''(x) &= \frac{x}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{1-x^2}f^{(1)}(x) \\
(1-x^2)f^{(2)}(x) = xf^{(1)}(x)
\end{aligned}$$

Differentiate both sides n times.

$$\begin{aligned}
(1-x^2)f^{(2+1)} + \binom{n}{1}(-2x)f^{(1+n)} + \binom{n}{2}(-2)f^{(n)} &= xf^{(n+1)} + 1f^{(n)}\binom{n}{1} \\
(1-x^2)f^{2+x} - 2nxf(1-n)(x) - n(n+1)f^{(n)} &= xf^(n+1)nf^{(n)}
\end{aligned}$$

Set $x = 0$

$$\begin{aligned}
f(2+n) - n(n-1)f(n) = nf(n) &= nf(n) \\
f(2+n) &= n^2 f(n) \\
f(0) &= 0 \\
f(1) &= 1 \\
f(3) &= 1 \\
f(5) &= 9 = 3^2 \\
f(7) &= 3^2 5^2 = 225 \\
f(9) &= 3^2 5^2 7^2
\end{aligned}$$