M1M1 Notes

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Contents

1 Functions

Definition 1.0.1. A function f is a rule assigning every element x in a set A an element $f(x)$ in another set B.

Remark 1.1.

- A is called the domain of f whereas B is called codomain.
- In the following we will mostly consider functions of one variable (with $A = \mathbb{R}$ and $B = \mathbb{R}$, later $\mathbb{C}$).

Remark 1.2. In the following we will mostly consider functions of one variable (with $A = \mathbb{R}$ and $B = \mathbb{R}$, later $\mathbb{C}$).

Example 1.1. Polynomials:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Definition 1.0.2. The graph of a function f (real not complex) is the set

$$\{(x, y)\,|x \in \mathrm{dom}(f), y = f(x)\}$$

Property 2.1. The graph of any function intersects any vertical line at most once.

1.1 Rational Functions

Definition 1.1.1. A rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)} \qquad P, Q \text{ polynomials}$$

Example 1.1.

$$f(x) = \frac{1}{1 - x^2}, \qquad \mathrm{dom}(f) = \mathbb{R}\{1; -1\}$$

1.2 Exponential Function

Definition 1.2.1. The exponential function *exp* can be defined by several ways:

1. As a power of e:

$$exp(x) = e^x$$

Obviously, for this definition the number e must be defined.

2. As a power series:

$$exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

3. By a ordinary differential equation (ODE):

$$\frac{d}{dx} exp(x) = exp(x)$$
$$exp(0) = 1$$

4. As inverse of the natural logarithm:

$$exp^{-1}(x) = \log(x)$$
$$\log(x) = \int_1^x \frac{du}{u}$$

5. As a limit:

$$exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Property 1.1.

$$exp(x+y) = exp(x) \times exp(y)$$

1.3 Trigonometrical Functions

Definition 1.3.1. Similarly to the exponential function, the trigonometrical functions cosine and sine have several potential definitions:

1. The elementary geometric definition at right-angled triangle with a hypotenuse of length 1.

2. Definition through Polar form – considering a point p on a unit circle centred at the origin .

3. As a power series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$$

4. Through a system of ODEs:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\sin 0 = 0, \cos 0 = 1$$

5. With the help of complex numbers:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Property 1.1.

- The addition formula:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

- Shifting:

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$
$$\cos\left(x + \frac{\pi}{2}\right) = -\sin x$$
$$\sin(x+\pi) = \sin\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}$$
$$= \cos\left(x + \frac{\pi}{2}\right)$$

Also

$$\cos(x + 2\pi) = \cos x$$

Remark 1.1. Special values which should be learned by heart are

$$x = 0; \frac{\pi}{6}; \frac{\pi}{4}; \frac{\pi}{3}; \frac{\pi}{2}$$

Definition 1.3.2. If a function f has property $f(x+a) = f(x)$ for all x it is called periodic. The period of f is the smallest possible a for which $f(x+a) = f(x), \quad \forall x \in \text{dom}(f)$

Definition 1.3.3. Other trigonometric functions can be written as a combination of sine and cosine:

$$\begin{aligned}
\sec x &= \frac{1}{\cos x} \\
\operatorname{cosec} x &= \frac{1}{\sin x} \\
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{sinx}
\end{aligned}$$

1.4 Odd and Even Functions

Definition 1.4.1. A function f is even if:

$$f(-x) = f(x)$$

Definition 1.4.2. A function f is odd if

$$f(-x) = -f(x)$$

Remark 2.1. These definitions assume that $\text{dom}(f)$ is symmetric which means $x \in \text{dom}(f) \;\Rightarrow\; -x \in \text{dom}(f)$

Example 2.1. $\sin x$ is odd, $\cos x$ is even.

Property 2.1. A function can be neither odd nor even. Any function can be split into a sum of even and off functions

$$f(x) = \frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(x) - f(-x)\right)$$

The odd and even part of a function are unique.

Example 2.2.

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)$$

1.5 Hyperbolic Functions

Definition 1.5.1.

$$\begin{aligned}
\cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right) \\
\sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right)
\end{aligned}$$

Property 1.1.

- $$\begin{aligned}
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\frac{d}{dx} \sinh x &= \cosh x \\
\frac{d}{dx} \cosh x &= \sinh x
\end{aligned}$$

- The hyperbolic functions can also be expressed through power series:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

- Similarly to the trigonometrical Pythagoras the following equation holds:

$$\cosh^2 x - \sinh^2 x = 1$$

Remark 1.1. Origin of the name:

$$x = \cosh t, \qquad t \in \mathbb{R}$$
$$y = \sinh t$$
$$x^2 - y^2 = 1$$

parametrizes a hyperbola.

1.6 Inverse Functions

Definition 1.6.1. The inverse function f^{-1}, if it exists, is a function $f^-1 : B \to A$ with the properties

$$f(f^{-1}(y)) = y, \qquad \forall y \in B$$
$$f^{-1}(f(x)) = x, \qquad \forall x \in A$$

Example 1.1.

$$f(x) = x^2 \qquad A = [0; \infty) = B$$
$$f^{-1}(y) = \sqrt{y}$$

Remark 1.1.

- A necessary condition for a function to be invertible is that f is injective (one-to-one).

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

or

$$f(x_1) \neq f(x_2) \quad \Leftarrow \quad x_1 \neq x_2$$

Graphical test: f is injective if its graph intersects any horizontal line at most once.

- The graph of the inverse $f - 1$ is the set of the points of the graph of f with the x and y coordinates exchanged.
graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$.

- If f is strictly increasing (decreasing) it is injective.

Definition 1.6.2. f is strictly increasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

f is strictly decreasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

Example 2.1. The exponential function is strictly increasing. (proof in problem sheet)

Remark 2.1.

- Any even function is not injective.
- Any periodic function is not injective either.
- Therefore, the trigonometric functions

$$\sin, \cos, \tan$$

are not invertible. In order to inverse the sin function, restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The inverse of the exponential function is called logarithm, $\log x$.