# M1M1 Notes

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# 1 Functions

**Definition 1.0.1.** A function $f$ is a rule assigning every element x in a set $A$ an element $f(x)$ in another set $B$.

**Remark 1.1.**

- $A$ is called the domain of $f$ whereas $B$ is called codomain.
- The range (image) of a function is the set:

$$\begin{aligned}\text{Range(f)} &= \text{Im}(f) \subseteq \text{codomain}\\ &= \{f(x) \in B |\ \forall x \in A\}\end{aligned}$$

  It does not have to be equal to the codomain.
- In the following we will mostly consider functions of one variable (with $A = \mathbb{R}$ and $B = \mathbb{R}$, later $\mathbb{C}$).

**Example 1.1.** Polynomials, $c_i \in \mathbb{R}, \forall i \in \mathbb{N}$:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

**Definition 1.0.2.** The graph of a function $f$ (real not complex) is the set

$$\{(x, y) \,|x \in \text{dom}(f), y = f(x)\}$$

**Property 2.1.** The graph of any function intersects any vertical line at most once.

## 1.1 Rational Functions

**Definition 1.1.1.** A rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P$ and $Q$ are polynomials.

**Example 1.1.**

$$f(x) = \frac{1}{1-x^2}, \qquad \text{dom}(f) = \mathbb{R}\backslash\{1, -1\}$$

## 1.2 Exponential Function

**Definition 1.2.1.** The exponential function exp can be defined by several ways:

1. As a power of $e$:
$$\exp(x) = e^x$$
Obviously, for this definition the number $e$ must be defined.

2. As a power series:
$$\exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

3. By a ordinary differential equation (ODE):
$$\begin{aligned} \frac{d}{dx}\exp(x) &= \exp(x) \\ \exp(0) &= 1 \end{aligned}$$

4. As inverse of the natural logarithm:
$$\begin{aligned} \exp^{-1}(x) &= \log(x) \\ \log(x) &= \int_1^x \frac{du}{u} \end{aligned}$$

5. As a limit:
$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

**Property 1.1.**
$$\exp(x+y) = \exp(x)\cdot\exp(y)$$

## 1.3 Trigonometrical Functions

**Definition 1.3.1.** Similar to the exponential function, the trigonometrical functions cos and sin have several potential definitions:

1. The elementary geometric definition at a right-angled triangle with a hypotenuse of length 1.

2. Definition through Polar form – considering a point $p$ on a unit circle centred at the origin .

3. As a power series:
$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \end{aligned}$$

4. Through a system of ODEs:
$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x \\ \sin 0 = 0, &\quad \cos 0 = 1 \end{aligned}$$

5. With the help of complex numbers:
$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

**Property 1.1.**

- The addition formula:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

- Shifting:

$$\begin{aligned}
\sin\left(x+\frac{\pi}{2}\right) &= \cos x \\
\cos\left(x+\frac{\pi}{2}\right) &= -\sin x \\
\sin(x+\pi) &= \sin\left(x+\frac{\pi}{2}\right)+\frac{\pi}{2} \\
&= \cos\left(x+\frac{\pi}{2}\right) \\
\sin(x+2\pi) &= \sin x
\end{aligned}$$

**Remark 1.1.** Special values which should be memorized are

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$$

**Definition 1.3.2.** If a function $f$ has property $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$ it is called periodic. The period of $f$ is the smallest possible $a$ for which $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$ .

**Definition 1.3.3.** Other trigonometric functions can be written as a combination of sine and cosine:

$$\begin{aligned}
\sec x &= \frac{1}{\cos x} \\
\operatorname{cosec} x &= \frac{1}{\sin x} \\
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x}
\end{aligned}$$

## 1.4 Odd and Even Functions

**Definition 1.4.1.** A function $f$ is even if

$$\forall x \in \mathrm{dom}(f): \quad f(-x) = f(x)$$

A function $f$ is odd if

$$\forall x \in \mathrm{dom}(f): \quad f(-x) = -f(x)$$

**Remark 1.1.** These definitions assume that $\mathrm{dom}(f)$ is symmetric which means $x \in \mathrm{dom}(f) \ \Rightarrow\ -x \in \mathrm{dom}(f)$

**Example 1.1.** $\sin x$ is odd, $\cos x$ is even.

**Property 1.1.** A function can be neither odd nor even. However, any function can be split into a sum of even and odd functions

$$f(x) = \frac{1}{2}\left(f(x)+f(-x)\right) + \frac{1}{2}\left(f(x)-f(-x)\right)$$

The odd and even part of a function are unique.

**Example 1.2.**

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)$$

## 1.5 Hyperbolic Functions

**Definition 1.5.1.**

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$
$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

**Property 1.1.**

- Addition theorem and derivatives:

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$

- The hyperbolic functions can also be expressed through power series:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}\dots$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

- Similarly to the trigonometrical Pythagoras the following equation holds:

$$\cosh^2 x - \sinh^2 x = 1$$

**Remark 1.1.** Origin of the name:

$$x = \cosh t, \qquad t \in \mathbb{R}$$
$$y = \sinh t$$
$$x^2 - y^2 = 1$$

parametrizes a hyperbola.

## 1.6 Inverse Functions

**Definition 1.6.1.** The inverse function $f^{-1}$, if it exists, is a function $f^{-1}: B \to A$ with the properties

$$f\left(f^{-1}(y)\right) = y, \qquad \forall y \in B$$
$$f^{-1}(f(x)) = x, \qquad \forall x \in A$$

**Example 1.1.**

$$f(x) = x^2 \qquad A = [0, \infty) = B$$
$$f^{-1}(y) = \sqrt{y}$$

**Remark 1.1.**

- A necessary condition for a function to be invertible is that $f$ is injective (one-to-one).

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

  or

$$f(x_1) \neq f(x_2) \quad \Leftarrow \quad x_1 \neq x_2$$

  Graphical test: $f$ is injective if its graph intersects any horizontal line at most once.

- The graph of the inverse $f^{-1}$ is the set of the points of the graph of $f$ with the $x$ and $y$ coordinates exchanged. The graph of $f^{-1}$ can be obtained by reflecting the graph of $f$ about the line $y = x$.
- If $f$ is strictly increasing (decreasing), it is injective.

**Definition 1.6.2.** $f$ is strictly increasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

$f$ is strictly decreasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

**Example 2.1.** The exponential function is strictly increasing. (proof in problem sheet)

**Remark 2.1.**

- Any even function $f$ is not injective if $\operatorname{dom} f \not\subseteq \{0\}$.
- Any periodic function is not injective either.
- Therefore, the trigonometric functions

$$\sin, \cos, \tan$$

  are not invertible.

  In order to inverse the sin function, restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

  In order to inverse the cos funtion, restrict the domain to $[0, \pi]$.

The inverse of the exponential function is called logarithm, $\log x$.

Anaytic treatment is sometimes possible. Require the existence of $f^{-1}(x)$ and 'Solve' $y = f(x)$ to obtain $x$ in terms of $y$, $x = f^{-1}(y)$.

**Example 2.2.**

- $$\begin{aligned} f(x) &= e^{-\frac{1}{x}} \\ x &= -\frac{1}{\log y} \end{aligned}$$

- inverse hyperbolic functions

$$\begin{aligned} f(x) &= \cosh x \\ f(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) \\ e^{2x} - 2ye^x + 1 &= 0 \\ (e^x)^2 - 2ye^x + 1 &= 0 \\ e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\ e^x &= y \pm \sqrt{y^2 - 1} \\ x &= \log\left(y \pm \sqrt{y^2 - 1}\right) \end{aligned}$$

  Restrict domain of $\cosh x$ to non-negative $x$.

$$\begin{aligned} x &= \log(y + \sqrt{y^2 - 1} \\ \cosh^{-1} x &= \log\left(x + \sqrt{x^2 - 1}\right) \end{aligned}$$

$$\sinh^{-1} x = \log\left(x + \sqrt{1 + x^2}\right)$$

### 1.6.1 Derivatives of Inverse Functions

The slope of the inverse function is $\frac{1}{f'(a)}$, the reciprocal of slope of the original function.

**Example 2.3.** $f(x) = e^x = y$ so that $x = \log y$.

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{e^x} = \frac{1}{y}$$

or

$$\frac{d}{dy}\log y = \frac{1}{y}$$

### 1.6.2 Inverse Trigonometrical functions

We are going to differentiate $\sin^{-1}$, $\tan^{-1}$. We set $y = \sin x$ so that $x = \sin^{-1} y$.

$$\begin{aligned} \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{\cos x} &= \frac{1}{\sqrt{1-\sin^2 x}} \\ &= \frac{1}{\sqrt{1-y^2}} \\ \frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

Similarly,

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

**Property 2.1.**

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

**Example 2.4.** Find the mistake in the following proof.

$$\begin{aligned} \sin\left(x + \frac{\pi}{2}\right) &= \cos x \\ x &= \cos^{-1} y \\ \sin\left(\cos^{-1} y + \frac{\pi}{2}\right) &= \cos\left(\cos^{-1} y\right) \\ \sin\left(\cos^{-1} y + \frac{\pi}{2}\right) &= y \\ \cos^{-1} y + \frac{\pi}{2} &= \sin^{-1} y \\ \sin^{-1} - \cos^{-1} y &= \frac{\pi}{2} \end{aligned}$$

# 2 Limits

**Definition 2.0.3.** The symbolic notation

$$L = \lim_{x\to a} f(x) \quad \text{or} \quad f(x) \stackrel{x\to a}{\to} f(x)$$

means:

$$\forall \epsilon > 0\ \exists \delta > 0 : \ (|x-a| > \delta \wedge x \in \mathrm{dom}(f)) \Rightarrow |f(x) - L| < \epsilon$$

$f(x)$ approaches $L$ as $x$ approaches $a$.

**Remark 3.1.** It is important that $x$ approaches $a$ from both the left and the right. A one sided limit is written as follows:

$$\lim_{a\to a^+} f(x)$$

or

$$\lim_{x\to a^-} f(x)$$

**Example 3.1.**

- Let
$$f(x) = \frac{x}{|x|}, \qquad x \neq 0$$
Then
$$\lim_{x\to 0} f(x)$$
is undefined but one sided limits exist:
$$\lim_{x\to 0^+} f(x) = 1$$
$$\lim_{x\to 0^-} f(x) = -1$$

- Let
$$f(x) = x^2$$
Then
$$\lim_{x\to 2} f(x) = f(2) = 4$$

- Let
$$f(x) = \frac{x^2-1}{x-1}$$
$\lim_{x\to 1} f(x)$ is an indeterminate limit of the form $\frac{0}{0}$. However $\frac{x^2-1}{x-1} = x+1$ if $x \neq 1$. Therefore $\lim_{x\to 1} f(x) = 2$.

**Remark 3.2.** Not all indeterminate limits are meaningful.

**Example 3.2.** The limit

$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^2}$$

of the form $\frac{0}{0}$ does not exist.

## 2.1 Infinite Limits

**Example 0.3.**

$$\lim_{x\to\infty} \frac{1}{x} = 0$$
$$\lim_{x\to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$x \to \infty$ is the same as $\frac{1}{x} \to 0^+$.
$x \to -\infty$ is the same as $\frac{1}{x} \to 0^-$.

**Property 0.1.** Provided the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist we know the following rules:

- addition formula

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

- product rule

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

- quotient rule ($\lim_{x\to a} g(x) \neq 0$)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

## 2.2 Computing Limits

- manipulate function so the limit is 'obvious'
- use power-series
- L'Hopital's rule

**Example 0.4.**

-

$$\begin{aligned}
L &= \lim_{x\to 1} \frac{\sqrt{2-x}-1}{1-x} \\
&= \lim_{x\to 1} \frac{\sqrt{2-x}-1}{1-x} \cdot \frac{\sqrt{2-x}-1}{\sqrt{2-x}-1} \\
&= \lim_{x\to 1} \frac{(2-x)-1}{(1-x)(\sqrt{2-x}+1} \\
&= \lim_{x\to 1} \frac{1-x}{(1-x)\left(\sqrt{2-x}+1\right)} \\
&= \lim_{x\to 1} \frac{1}{\sqrt{2-x}+1} \\
&= \frac{1}{2}
\end{aligned}$$

Alternatively the power series can be used. Let $s = 1-x$. Then

$$L = \lim_{s\to 0} \frac{\sqrt{1+s}-1}{s}$$

**Theorem 2.2.1.** The general binomial theorem says

$$(1+s)^p = 1 + ps + \frac{p(p-1)}{2!}s^2 + \frac{p(p-1)(p-2)}{3!}s^3 + \dots$$

for $|s| < 1$.

The geometric series is a special case of the binomial theorem with $p = -1$.

If $p$ is a positive integer, the series terminates – and gives us the standard binomial theorem. If $p$ is not a positive integer, the formula continues infinitely. Hence

$$(1+s)^{\frac{1}{2}} = 1 + \frac{1}{2}s + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}s^2 + \dots$$

can be inserted in our formula and we get:

$$\lim_{s\to 0} \frac{\sqrt{1+s}-1}{s} = \lim_{s\to 0} \frac{1+\frac{1}{2}s+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}s^2+\dots-1}{s} = \frac{1}{2}$$

- Let us calculate the following well-known limit:

$$\lim_{x\to 0} \frac{\sin x}{x}$$

Using a power series we get

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) = 1$$

- Having calculated this limit we can compute

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \cos x = 1$$

- Using limits for graph sketching.

$$f(x) = \frac{x}{e^x - 1} = \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots} \overset{x\to 0}{\to} 1$$

- Let

$$f(x) = \frac{\cos\left(\frac{\pi}{2}x\right)}{1 - x^2}$$

Now we can calculate $\lim_{x\to 1} f(x)$.

$$\begin{aligned}
&\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x^2} \\
=&\frac{\cos\left(\frac{\pi}{2}(x-1) + \frac{\pi}{2}\right)}{(x-1)(x+1)} \\
=&\frac{\sin\left(\frac{\pi}{2}(x-1)\right)}{(x-1)(x+1)}
\end{aligned}$$

Substituting $s = x - 1$ that gives us

$$\begin{aligned}
=&\frac{\sin\left(\frac{\pi}{2}s\right)}{s(2+s)} \\
=&\frac{\frac{\pi}{2}s - \frac{1}{3!}\left(\frac{\pi}{2}s\right)^3 + \frac{1}{5!}\left(\frac{\pi}{2}s\right)^5 + \dots}{s(2+s)} \\
=&\frac{\frac{\pi}{2} - \frac{1}{3!}\left(\frac{\pi}{2}s\right)^3 + \frac{1}{5!}\left(\frac{\pi}{2}s\right)^5 + \dots}{2+s} \\
\overset{s\to 0}{\to}&\frac{\pi}{4}
\end{aligned}$$

- Consider the limit

$$\begin{aligned}
&\lim_{x\to\infty} x^{\frac{1}{3}}\left((x+1)^{\frac{2}{3}} - x^{\frac{2}{3}}\right) \\
=&\lim_{x\to\infty} x^{\frac{1}{3}}\left(x^{\frac{2}{3}}(1+\frac{1}{x})^{\frac{2}{3}} - x^{\frac{2}{3}}\right) \\
=&\lim_{x\to\infty} x^{\frac{1}{3}}\left(x^{\frac{2}{3}}\left(1 + \frac{2}{3}\cdot\frac{1}{x} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{1}{x}\right)^2 + \dots\right) - x^{\frac{2}{3}}\right) \\
=&\lim_{x\to\infty} x\left(1 + \frac{2}{3}\cdot\frac{1}{x} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\left(\frac{1}{x}\right)^2 + \dots\right) - x \\
=&\lim_{x\to\infty}\left(x\left(1 + \frac{2}{3}\cdot\frac{1}{x} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\left(\frac{1}{x}\right)^2 + \dots\right) - x\right) \\
=&\lim_{x\to\infty}\left(x + \frac{2}{3} + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\frac{1}{x} + \dots - x\right) \\
=&\frac{2}{3}
\end{aligned}$$

- Using limits, the equivalence of the definitions for the exponential function can be proven.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$$

Derivation:

$$\begin{aligned}
&\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x \\
=&\lim_{x\to\infty}\exp\left(\log\left(1+\frac{a}{x}\right)^x\right) \\
=&\lim_{x\to\infty}\exp\left(x\log\left(1+\frac{a}{x}\right)\right) \\
=&\exp\left(\lim_{x\to\infty}x\left(\frac{a}{x}-\frac{1}{2}\cdot\left(\frac{a}{x}\right)^2+\frac{1}{3}\left(\frac{a}{x}\right)^3-\frac{1}{4}\left(\frac{a}{x}\right)^4\dots\right)\right) \\
=&\exp(a)
\end{aligned}$$

Another possibility to derive this is

$$\begin{aligned}
\left(1+\frac{a}{x}\right)^x =& 1+x\frac{a}{x}+\frac{x(x-1)}{2!}\left(\frac{a}{x}\right)^2 \\
&+\frac{x(x-1)(x-2)}{3!}\left(\frac{a}{x}\right)^3+\dots
\end{aligned}$$

Considering the limit to infinity and therefore only the dominant powers in each fraction we get

$$=1+a+\frac{a^2}{2}+\frac{a^3}{3!}+\dots$$

## 2.3 Continuity

*Informal Definition.* A continuous function $f$ has a graph with no breaks or jumps.

**Example 0.5.** A continuous function is

$$f(x)=x^2$$

An example for a non-continuous function is the Heaviside function:

$$H(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$$

**Definition 2.3.1.** A function $f$ is continuous at $a\in\mathrm{dom}(f)$ if

$$\lim_{x\to a}f(x)=f(a)$$

As for the Heaviside function, $\lim_{x\to 0}H(x)$ doesn't exist. Nonetheless, $H(x)$ is continuous for all $x\neq 0$ .

**Example 1.1.**

$$f(x)=x\sin\left(\frac{1}{x}\right)$$

is continuous for $x\neq 0$ but not continuous at $x=0$ (because it is not defined there). However,

$$\lim_{x\to 0}f(x)=0$$

since

$$-|x|\leq f(x)\leq |x|\left|\sin\frac{1}{x}\right|\leq 1$$

Hence, we can consider the function

$$g(x)=x\begin{cases}\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

$g$ is continuous for all $x\in\mathbb{R}$.

### 2.4 List of Power Series

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\tan x &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \\
\tan^{-1} x &= x + \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots
\end{aligned}$$

To derive the power series for $\tan^{-1}$, consider

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Integrating both sides gives us

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

## 3 Differentiation

*Geometrical definition.* The derivative of a function $f$ at $x$ is the slope of the tangent to the graph $y = f(x)$ at $(x, f(x))$.

**Definition 3.0.1.**

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h)-f(x)}{h}$ denotes the slope of the secant through $(x, f(x))$ and $(x+h, f(x+h))$.

**Remark 1.1.** $f'(x)$ is also a function with $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$.

Using the limit definition to compute derivatives is called differentiation from first principles.

**Example 1.1.**

- Polynomials

$$\begin{aligned}
f(x) &= x^3 \\
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\
&= \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\
&= 3x^2 + 3hx + h^2 \\
&\stackrel{x \to 0}{\to} 3x^2
\end{aligned}$$

- The cosine function

$$\begin{aligned}
f(x) &= \cos x \\
\frac{f(x+h) - f(x)}{h} &= \frac{\cos(x+h) - \cos x}{h}
\end{aligned}$$

Let us use the trigonometrical identity

$$\cos A - \cos B = -2 \sin \frac{A-B}{2} \sin \frac{A+B}{2}$$

(Derivation:

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\cos(\alpha+\beta) - \cos(\alpha-\beta) &= -2\sin\alpha\sin\beta)
\end{aligned}$$

With $A = x + b$ and $B = x$ this gives us

$$\begin{aligned}
\frac{\cos(x+h) - \cos(x)}{h} &= \frac{-2\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)}{h} \\
&\stackrel{h\to 0}{\to} -\sin x
\end{aligned}$$

- The function $f$ with

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

  is not continuous for all $x \in \mathbb{R}$.

**Theorem 3.0.1.** If a function $f$ is differentiable at $a \in \operatorname{dom} f$ then $f$ is continuous at $a$.

This poses the question whether it is possible to find a function which is continuous but nowhere differentiable? The Fourier series are

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} = \sin(\pi x) + \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} + \dots$$

This led to the discovery of the Lacunary Fourier series

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2\pi x)}{n} = \sin(\pi x) + \frac{\sin(4\pi x)}{2} + \frac{\sin(9\pi x)}{3} + \dots$$

$R$ is not differentiable except for $x$ rational of the form $\frac{p}{q}$, $p, q$ odd.

## 3.1 Basic Derivatives

| $f(x)$ | $f('x)$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $\log x$ | $\frac{1}{x}$ |
| $\exp(x)$ | $\exp(x)$ |
| $\cosh(x)$ | $\sinh(x)$ |
| $\sinh(x)$ | $\cosh(x)$ |
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\tan^{-1}(x)$ | $\frac{1}{1+x^2}$ |
| $\sin^{-1}(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |

## 3.2 Differentiation rules

If $u, v$ and $f$ are derivable functions then the followings rules hold:

- Addition rule

$$\frac{d}{dx}(u(x) + v(x)) = u'(x) + v'(x)$$

- Multiplication rule

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x)$$

- Chain rule

$$\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$$
$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

For the derivation of the product rule consider the following quotient:

$$\begin{aligned}
&\frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
=&\frac{u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)}{h} \\
=&v(x+h)\frac{u(x+h) - u(x)}{h} + u(x)\frac{v(x+h) - v(x)}{h} \\
\overset{h\to 0}{\to}&v(x)u'(x) + u(x)v'(x)
\end{aligned}$$

The proof of chain rule will be done in spring term analysis.

### 3.3 Implicit Differentiation

**Remark 0.2.** Implicit differentiation applies the chain rule .

**Example 0.2.** Compute the slope of tangent to unit circle $x^2 + y^2 = 1$.

'Solve' to get $y = y(x)$ and use the differentiation rules.

$$y(x) = \pm\sqrt{1-x^2}$$
$$y'(x) = \pm\frac{-x}{\sqrt{1-x^2}} = \mp\frac{x}{\sqrt{1-x^2}}$$

Implicit differentiation. Treat $y^2$ as a composite function – differentiate with the chain rule.

$$\frac{d}{dx}y^2(x) = 2y(x)y'(x)$$

equation $x^2 + y^2 = 1$. Differentiate with respect to $x$

$$2x + 2yy' = 0 \quad \vee \quad y' = \frac{-x}{y}$$

**Example 0.3.**

$$y^3 - y = x^2$$
$$(3y^2 - 1)y' = 2x$$

The slope of the tangent is

$$y' = \frac{2x}{3y^2 - 1}$$

For the point $(\sqrt{6}, 2)$ we get the slope

$$y' = \frac{2\sqrt{6}}{11}$$

### 3.4 Parametric Differentiation

You can describe a curve in the $xy$ plane parametrically.

**Example 0.4.** • Let us consider a curve defined by the hyperbolic functions.

$$\begin{aligned} x(t) &= \cosh(t) \\ y(t) &= \sinh(t), \qquad t \in \mathbb{R} \end{aligned}$$

slope of the tangent

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$$

$\cdot$ denotes differentiation with respect to the parameter $t$.

$$\frac{dy}{dx} = \frac{\cosh t}{\sinh t} = \coth t$$

- The equation for a cycloid is

$$\begin{aligned} x(t) &= t - \sin t \\ y(t) &= 1 - \cos t, \qquad t \in \mathbb{R} \end{aligned}$$

The point on the edge of a rolling wheel traces a cycloid.

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin t}{1 - \cos t}$$

### 3.5 Higher Differentiation

Suppose $f$ is differentiable then consider the limit

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

If this exists, $f$ is said to be twice differentiable. The limit is called second derivative, denoted

$$f''(x) \quad \text{or} \quad \frac{d^2 f(x)}{dx^2} \quad \text{or} \quad y''(x) \quad \text{or} \quad \frac{d^2 y(x)}{dx^2}$$

This can be continued to define the $n^{th}$ derivative, denoted as

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n f(x)}{dx^n} \quad \text{or} \quad y^{(n)}(x) \quad \text{or} \quad \frac{d^n y(x)}{dx^n} \quad \text{or} \quad \left(\frac{d}{dx}\right)^n f(x)$$

$\frac{d}{dx}$ is called the differential operator.

**Example 0.5.**

$$\begin{aligned} f(x) &= \log x & f^{(1)}(x) &= \frac{1}{x} \\ f^{(2)}(x) &= -\frac{1}{x^2} & f^{(3)}(x) &= \frac{2}{x^3} \\ f^{(4)}(x) &= -\frac{2 \cdot 3}{n^4} & f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n} \end{aligned}$$

**Theorem 3.5.1.** The Leibniz' formula is

$$\left(\frac{d}{dx}\right) uv = \sum_{p=0}^{n} \binom{n}{p} u^{n-p} v^p$$

The derivation can be made through regarding the functions $u(x)v(x)$.

$$\frac{d}{dx}uv = u'v + uv'$$

Differentiating again gives us

$$\begin{aligned}\left(\frac{d}{dx}\right)^2 uv &= u''v + u'v' + u'v' + uv'' \\ &= u''v + 2u'v' + uv'' \\ \left(\frac{d}{dx}\right)^3 uv &= u''v + u''v' + 2(u''v' + u'v'') + u'v'' + uv''' \\ &= u'''v + 3u''v' + 3u'v'' + uv'''\end{aligned}$$

The coefficients are binomial coefficients. A rigorous proof can be made by induction.

**Example 0.6.**

- Leibniz is particularly useful if one term in the product is a polynomial – since the sum terminates

  $$f(x) = e^{2x}x^2$$

  Set

  $$v = x^2, \quad u = e^{2x}$$

  Then

  $$\begin{aligned}v^{(1)} = 2x, \quad v^{(2)} = 2, \quad v^{(3)} = v^{(4)} &= v^{(5)} = 0 \\ u^{(n)} &= 2^n e^{2x}\end{aligned}$$

  This gives us the $n^{th}$ derivative of $f$

  $$\begin{aligned}f^{(n)}(x) &= \binom{n}{0}u^{(n)}v^{(0)} + \binom{n}{1}u^{(n-2)}v^{(1)}\binom{n}{2}u^{(n-2)}v^{(2)} \\ &= 2^n e^{2x}x^2 + n2^{n-1}e^{2x}2x + n(n-1)2^{n-2}e^{2x}\end{aligned}$$

- Another example

  $$\begin{aligned}f(x) &= \sin^{-1} x \\ f'(x) &= \frac{1}{\sqrt{1-x^2}} \\ f''(x) &= \frac{x}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{1-x^2}f^{(1)}(x) \\ (1-x^2)f^{(2)}(x) &= xf^{(1)}(x)\end{aligned}$$

  Differentiate both sides $n$ times.

  $$\begin{aligned}\left(1-x^2\right)f^{(2+n)} + \binom{n}{1}(-2x)f^{(1+n)} + \binom{n}{2}(-2)f^{(n)} &= xf^{(n+1)} + 1f^{(n)}\binom{n}{1} \\ \left(1-x^2\right)f^{2+n} - 2nxf(1+n)(x) - n(n+1)f^{(n)} &= xf^{(n+1)} + nf^{(n)}\end{aligned}$$

  Set $x = 0$

  $$\begin{aligned}f^{(2+n)}(0) - n(n-1)f^{(n)}(0) &= nf^{(n)}(0) \\ f^{(2+n)}(0) &= n^2 f^{(n)}(0) \\ f^{(0)}(0) &= 0 \\ f^{(1)}(0) &= 1 \\ f^{(3)}(0) &= 1 \\ f^{(5)}(0) &= 9 = 3^2 \\ f^{(7)}(0) &= 3^2 5^2 = 225 \\ f^{(9)}(0) &= 3^2 5^2 7^2\end{aligned}$$