M1M1 Notes

David Burgschweiger

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Contents

1 Functions

Definition 1.0.1. A function f is a rule assigning every element x in a set A an element $f(x)$ in another set B.

Remark 1.1.

- A is called the domain of f whereas B is called codomain.
- The range (image) of a function is the set:

$$\begin{aligned}\text{Range(f)} &= \text{Im}(f) \subseteq \text{codomain}\\ &= \{f(x) \in B | \ \forall x \in A\}\end{aligned}$$

  It does not have to be equal to the codomain.
- In the following we will mostly consider functions of one variable (with $A = \mathbb{R}$ and $B = \mathbb{R}$, later $\mathbb{C}$).

Example 1.1. Polynomials, $c_i \in \mathbb{R}, \forall i \in \mathbb{N}$:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

Definition 1.0.2. The graph of a function f (real not complex) is the set

$$\{(x, y) \,| x \in \text{dom}(f), y = f(x)\}$$

Property 2.1. The graph of any function intersects any vertical line at most once.

1.1 Rational Functions

Definition 1.1.1. A rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

Example 1.1.

$$f(x) = \frac{1}{1-x^2}, \qquad \text{dom}(f) = \mathbb{R}\backslash\{1,-1\}$$

1.2 Exponential Function

Definition 1.2.1. The exponential function exp can be defined by several ways:

1. As a power of e:

$$\exp(x) = e^x$$

   Obviously, for this definition the number e must be defined.

2. As a power series:

$$\exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

3. By a ordinary differential equation (ODE):

$$\begin{aligned} \frac{d}{dx}\exp(x) &= \exp(x) \\ \exp(0) &= 1 \end{aligned}$$

4. As inverse of the natural logarithm:

$$\begin{aligned} \exp^{-1}(x) &= \log(x) \\ \log(x) &= \int_1^x \frac{du}{u} \end{aligned}$$

5. As a limit:

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Property 1.1.

$$\exp(x+y) = \exp(x)\cdot\exp(y)$$

1.3 Trigonometrical Functions

Definition 1.3.1. Similar to the exponential function, the trigonometrical functions cos and sin have several potential definitions:

1. The elementary geometric definition at a right-angled triangle with a hypotenuse of length 1.

2. Definition through Polar form – considering a point p on a unit circle centred at the origin .

3. As a power series:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \end{aligned}$$

4. Through a system of ODEs:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\sin 0 = 0, \quad \cos 0 = 1$$

5. With the help of complex numbers:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Property 1.1.

- The addition formula:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

- Shifting:

$$\begin{aligned}
\sin\left(x + \frac{\pi}{2}\right) &= \cos x \\
\cos\left(x + \frac{\pi}{2}\right) &= -\sin x \\
\sin(x+\pi) &= \sin\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2} \\
&= \cos\left(x + \frac{\pi}{2}\right) \\
\sin(x+2\pi) &= \sin x
\end{aligned}$$

Remark 1.1. Special values which should be memorized are

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$$

Definition 1.3.2. If a function f has property $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$ it is called periodic. The period of f is the smallest possible a for which $f(x+a) = f(x),\ \forall x \in \mathrm{dom}(f)$.

Definition 1.3.3. Other trigonometric functions can be written as a combination of sine and cosine:

$$\begin{aligned}
\sec x &= \frac{1}{\cos x} \\
\operatorname{cosec} x &= \frac{1}{\sin x} \\
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x}
\end{aligned}$$

1.4 Odd and Even Functions

Definition 1.4.1. A function f is even if

$$\forall x \in \mathrm{dom}(f): \quad f(-x) = f(x)$$

A function f is odd if

$$\forall x \in \mathrm{dom}(f): \quad f(-x) = -f(x)$$

Remark 1.1. These definitions assume that $\mathrm{dom}(f)$ is symmetric which means $x \in \mathrm{dom}(f) \ \Rightarrow\ -x \in \mathrm{dom}(f)$

Example 1.1. $\sin x$ is odd, $\cos x$ is even.

Property 1.1. A function can be neither odd nor even. However, any function can be split into a sum of even and odd functions

$$f(x) = \frac{1}{2}\left(f(x) + f\left(-x\right)\right) + \frac{1}{2}\left(f(x) - f\left(-x\right)\right)$$

The odd and even part of a function are unique.

Example 1.2.

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)$$

1.5 Hyperbolic Functions

Definition 1.5.1.

$$\begin{aligned}\cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right)\\ \sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right)\end{aligned}$$

Property 1.1.

- Addition theorem and derivatives:

$$\begin{aligned}\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y\\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y\\ \frac{d}{dx}\sinh x &= \cosh x\\ \frac{d}{dx}\cosh x &= \sinh x\end{aligned}$$

- The hyperbolic functions can also be expressed through power series:

$$\begin{aligned}\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!}\dots\\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\end{aligned}$$

- Similarly to the trigonometrical Pythagoras the following equation holds:

$$\cosh^2 x - \sinh^2 x = 1$$

Remark 1.1. Origin of the name:

$$\begin{aligned}x &= \cosh t, & t \in \mathbb{R}\\ y &= \sinh t\\ x^2 - y^2 &= 1\end{aligned}$$

parametrizes a hyperbola.

1.6 Inverse Functions

Definition 1.6.1. The inverse function f^{-1}, if it exists, is a function $f^{-1} : B \to A$ with the properties

$$\begin{aligned}f\left(f^{-1}(y)\right) &= y, & \forall y \in B\\ f^{-1}(f(x)) &= x, & \forall x \in A\end{aligned}$$

Example 1.1.

$$f(x) = x^2 \qquad A = [0, \infty) = B$$
$$f^{-1}(y) = \sqrt{y}$$

Remark 1.1.

- A necessary condition for a function to be invertible is that f is injective (one-to-one).

  $$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

  or

  $$f(x_1) \neq f(x_2) \quad \Leftarrow \quad x_1 \neq x_2$$

  Graphical test: f is injective if its graph intersects any horizontal line at most once.

- The graph of the inverse f^{-1} is the set of the points of the graph of f with the x and y coordinates exchanged. The graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$.

- If f is strictly increasing (decreasing), it is injective.

Definition 1.6.2. f is strictly increasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

f is strictly decreasing if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

Example 2.1. The exponential function is strictly increasing. (proof in problem sheet)

Remark 2.1.

- Any even function f is not injective if $\operatorname{dom} f \not\subseteq \{0\}$.
- Any periodic function is not injective either.
- Therefore, the trigonometric functions

  $$\sin, \cos, \tan$$

  are not invertible.

  In order to inverse the sin function, restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

  In order to inverse the cos funtion, restrict the domain to $[0, \pi]$.

The inverse of the exponential function is called logarithm, $\log x$.
Anaytic treatment is sometimes possible. Require the existence of $f^{-1}(x)$ and 'Solve' $y = f(x)$ to obtain x in terms of y, $x = f^{-1}(y)$.

Example 2.2.

- $$f(x) = e^{-\frac{1}{x}}$$
  $$x = -\frac{1}{\log y}$$

- inverse hyperbolic functions

$$
\begin{aligned}
f(x) &= \cosh x \\
f(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) \\
e^{2x} - 2ye^x + 1 &= 0 \\
(e^x)^2 - 2ye^x + 1 &= 0 \\
e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\
e^x &= y \pm \sqrt{y^2 - 1} \\
x &= \log\left(y \pm \sqrt{y^2 - 1}\right)
\end{aligned}
$$

Restrict domain of $\cosh x$ to non-negative x.

$$
\begin{aligned}
x &= \log(y + \sqrt{y^2 - 1} \\
\cosh^{-1} x &= \log\left(x + \sqrt{x^2 - 1}\right)
\end{aligned}
$$

$$
\sinh^{-1} x = \log\left(x + \sqrt{1 + x^2}\right)
$$

1.6.1 Derivatives of Inverse Functions

The slope of the inverse function is $\frac{1}{f'(a)}$, the reciprocal of slope of the original function.

Example 2.3. $f(x) = e^x = y$ so that $x = \log y$.

$$
\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{e^x} = \frac{1}{y}
$$

or

$$
\frac{d}{dy}\log y = \frac{1}{y}
$$

1.6.2 Inverse Trigonometrical functions

We are going to differentiate $\sin^{-1}$, $\tan^{-1}$. We set $y = \sin x$ so that $x = \sin^{-1} y$.

$$
\begin{aligned}
\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{\cos x} &= \frac{1}{\sqrt{1 - \sin^2 x}} \\
&= \frac{1}{\sqrt{1 - y^2}} \\
\frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}
$$

Similarly,

$$
\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}
$$

Property 2.1.

$$
\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}
$$

Example 2.4. Find the mistake in the following proof.

$$\begin{aligned}
\sin\left(x+\frac{\pi}{2}\right) &= \cos x \\
x &= \cos^{-1} y \\
\sin\left(\cos^{-1} y+\frac{\pi}{2}\right) &= \cos\left(\cos^{-1} y\right) \\
\sin\left(\cos^{-1} y+\frac{\pi}{2}\right) &= y \\
\cos^{-1} y+\frac{\pi}{2} &= \sin^{-1} y \\
\sin^{-1} - \cos^{-1} y &= \frac{\pi}{2}
\end{aligned}$$

2 Limits

Definition 2.0.3. The symbolic notation

$$L = \lim_{x\to a} f(x) \quad \text{or} \quad f(x) \overset{x\to a}{\to} f(x)$$

means:

$$\forall\epsilon > 0\ \exists\delta > 0 :\ (|x-a| > \delta \wedge x \in \text{dom}(f)) \Rightarrow |f(x) - L| < \epsilon$$

$f(x)$ approaches L as x approaches a.

Remark 3.1. It is important that x approaches a from both the left and the right. A one sided limit is written as follows:

$$\lim_{a\to a^+} f(x)$$

or

$$\lim_{x\to a^-} f(x)$$

Example 3.1.

- Let

  $$f(x) = \frac{x}{|x|}, \qquad x \neq 0$$

  Then

  $$\lim_{x\to 0} f(x)$$

  is undefined but one sided limits exist:

  $$\lim_{x\to 0^+} f(x) = 1$$
  $$\lim_{x\to 0^-} f(x) = -1$$

- Let

  $$f(x) = x^2$$

  Then

  $$\lim_{x\to 2} f(x) = f(2) = 4$$

- Let

  $$f(x) = \frac{x^2-1}{x-1}$$

  $\lim_{x\to 1} f(x)$ is an indeterminate limit of the form $\frac{0}{0}$. However $\frac{x^2-1}{x-1} = x+1$ if $x \neq 1$. Therefore $\lim_{x\to 1} f(x) = 2$.

Remark 3.2. Not all indeterminate limits are meaningful.

Example 3.2. The limit

$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^2}$$

of the form $\frac{0}{0}$ does not exist.

2.1 Infinite Limits

Example 0.3.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$
$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$x \to \infty$ is the same as $\frac{1}{x} \to 0^+$.
$x \to -\infty$ is the same as $\frac{1}{x} \to 0^-$.

Property 0.1. Provided the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist we know the following rules:

- addition formula

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

- product rule

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

- quotient rule ($\lim_{x\to a} g(x) \neq 0$)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

2.2 Computing Limits

- manipulate function so the limit is 'obvious'
- use power-series
- L'Hopital's rule

Example 0.4.

-

$$\begin{aligned} L &= \lim_{x\to 1} \frac{\sqrt{2-x}-1}{1-x} \\ &= \lim_{x\to 1} \frac{\sqrt{2-x}-1}{1-x} \cdot \frac{\sqrt{2-x}-1}{\sqrt{2-x}-1} \\ &= \lim_{x\to 1} \frac{(2-x)-1}{(1-x)(\sqrt{2-x}+1} \\ &= \lim_{x\to 1} \frac{1-x}{(1-x)\left(\sqrt{2-x}+1\right)} \\ &= \lim_{x\to 1} \frac{1}{\sqrt{2-x}+1} \\ &= \frac{1}{2} \end{aligned}$$

Alternatively the power series can be used. Let $s = 1-x$. Then

$$L = \lim_{s\to 0} \frac{\sqrt{1+s}-1}{s}$$

Theorem 2.2.1. The general binomial theorem says

$$(1+s)^p = 1 + ps + \frac{p(p-1)}{2!}s^2 + \frac{p(p-1)(p-2)}{3!}s^3 + \dots$$

for $|s| < 1$.

The geometric series is a special case of the binomial theorem with $p = -1$.

If p is a positive integer, the series terminates – and gives us the standard binomial theorem. If p is not a positive integer, the formula continues infinitely. Hence

$$(1+s)^{\frac{1}{2}} = 1 + \frac{1}{2}s + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}s^2 + \dots$$

can be inserted in our formula and we get:

$$\lim_{s\to 0}\frac{\sqrt{1+s}-1}{s} = \lim_{s\to 0}\frac{1+\frac{1}{2}s+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}s^2+\dots-1}{s} = \frac{1}{2}$$

- Let us calculate the following well-known limit:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

Using a power series we get

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) = 1$$

- Having calculated this limit we can compute

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\cos x = 1$$

- Using limits for graph sketching.

$$f(x) = \frac{x}{e^x-1} = \frac{1}{1+\frac{x}{2!}+\frac{x^2}{3!}+\dots} \stackrel{x\to 0}{\to} 1$$

- Let

$$f(x) = \frac{\cos\left(\frac{\pi}{2}x\right)}{1-x^2}$$

Now we can calculate $\lim_{x\to 1} f(x)$.

$$\begin{aligned}&\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x^2}\\ =&\frac{\cos\left(\frac{\pi}{2}(x-1)+\frac{\pi}{2}\right)}{(x-1)(x+1)}\\ =&\frac{\sin\left(\frac{\pi}{2}(x-1)\right)}{(x-1)(x+1)}\end{aligned}$$

Substituting $s = x - 1$ that gives us

$$\begin{aligned}=&\frac{\sin\left(\frac{\pi}{2}s\right)}{s(2+s)}\\ =&\frac{\frac{\pi}{2}s-\frac{1}{3!}\left(\frac{\pi}{2}s\right)^3+\frac{1}{5!}\left(\frac{\pi}{2}s\right)^5+\dots}{s(2+s)}\\ =&\frac{\frac{\pi}{2}-\frac{1}{3!}\left(\frac{\pi}{2}s\right)^3+\frac{1}{5!}\left(\frac{\pi}{2}s\right)^5+\dots}{2+s}\\ \stackrel{s\to 0}{\to}&\frac{\pi}{4}\end{aligned}$$

- Consider the limit

$$\begin{aligned}
&\lim_{x\to\infty} x^{\frac{1}{3}}\left((x+1)^{\frac{2}{3}}-x^{\frac{2}{3}}\right)\\
=&\lim_{x\to\infty} x^{\frac{1}{3}}\left(x^{\frac{2}{3}}(1+\frac{1}{x})^{\frac{2}{3}}-x^{\frac{2}{3}}\right)\\
=&\lim_{x\to\infty} x^{\frac{1}{3}}\left(x^{\frac{2}{3}}\left(1+\frac{2}{3}\cdot\frac{1}{x}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{1}{x}\right)^2+\dots\right)-x^{\frac{2}{3}}\right)\\
=&\lim_{x\to\infty} x\left(1+\frac{2}{3}\cdot\frac{1}{x}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\left(\frac{1}{x}\right)^2+\dots\right)-x\\
=&\lim_{x\to\infty}\left(x\left(1+\frac{2}{3}\cdot\frac{1}{x}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\left(\frac{1}{x}\right)^2+\dots\right)-x\right)\\
=&\lim_{x\to\infty}\left(x+\frac{2}{3}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\cdot\frac{1}{x}+\dots-x\right)\\
=&\frac{2}{3}
\end{aligned}$$

- Using limits, the equivalence of the definitions for the exponential function can be proven.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=e^a$$

Derivation:

$$\begin{aligned}
&\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x\\
=&\lim_{x\to\infty}\exp\left(\log\left(1+\frac{a}{x}\right)^x\right)\\
=&\lim_{x\to\infty}\exp\left(x\log\left(1+\frac{a}{x}\right)\right)\\
=&\exp\left(\lim_{x\to\infty} x\left(\frac{a}{x}-\frac{1}{2}\cdot\left(\frac{a}{x}\right)^2+\frac{1}{3}\left(\frac{a}{x}\right)^3-\frac{1}{4}\left(\frac{a}{x}\right)^4\dots\right)\right)\\
=&\exp(a)
\end{aligned}$$

Another possibility to derive this is

$$\begin{aligned}
\left(1+\frac{a}{x}\right)^x=&1+x\frac{a}{x}+\frac{x(x-1)}{2!}\left(\frac{a}{x}\right)^2\\
&+\frac{x(x-1)(x-2)}{3!}\left(\frac{a}{x}\right)^3+\dots
\end{aligned}$$

Considering the limit to infinity and therefore only the dominant powers in each fraction we get

$$=1+a+\frac{a^2}{2}+\frac{a^3}{3!}+\dots$$

2.3 Continuity

Informal Definition. A continuous function f has a graph with no breaks or jumps.

Example 0.5. A continuous function is

$$f(x)=x^2$$

An example for a non-continuous function is the Heaviside function:

$$H(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$$

Definition 2.3.1. A function f is continuous at $a \in \text{dom}(f)$ if

$$\lim_{x\to a} f(x) = f(a)$$

As for the Heaviside function, $\lim_{x\to 0} H(x)$ doesn't exist. Nonetheless, $H(x)$ is continuous for all $x \neq 0$.

Example 1.1.

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

is continuous for $x \neq 0$ but not continuous at $x = 0$ (because it is not defined there). However,

$$\lim_{x\to 0} f(x) = 0$$

since

$$-|x| \leq f(x) \leq |x| \left|\sin\frac{1}{x}\right| \leq 1$$

Hence, we can consider the function

$$g(x) = x \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

g is continuous for all $x \in \mathbb{R}$.

2.4 List of Power Series

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\tan x &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \\
\tan^{-1} x &= x + \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\
\log(1+x) &= x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots
\end{aligned}$$

To derive the power series for $\tan^{-1}$, consider

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Integrating both sides gives us

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

3 Differentiation

Geometrical definition. The derivative of a function f at x is the slope of the tangent to the graph $y = f(x)$ at $(x, f(x))$.

Definition 3.0.1.

$$f'(x) = \lim_{x\to 0} \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h)-f(x)}{h}$ denotes the slope of the secant through $(x, f(x))$ and $(x+h, f(x+h))$.

Remark 1.1. $f'(x)$ is also a function with $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$.

Using the limit definition to compute derivatives is called differentiation from first principles.

Example 1.1.

- Polynomials

$$\begin{aligned} f(x) &= x^3 \\ \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\ &= \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\ &= 3x^2 + 3hx + h^2 \\ &\overset{x\to 0}{\to} 3x^2 \end{aligned}$$

- The cosine function

$$\begin{aligned} f(x) &= \cos x \\ \frac{f(x+h)-f(x)}{h} &= \frac{\cos(x+h) - \cos x}{h} \end{aligned}$$

Let us use the trigonometrical identity

$$\cos A - \cos B = -2\sin\frac{A-B}{2}\sin\frac{A+B}{2}$$

(Derivation:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \cos(\alpha+\beta) - \cos(\alpha-\beta) &= -2\sin\alpha\sin\beta) \end{aligned}$$

With $A = x + b$ and $B = x$ this gives us

$$\begin{aligned} \frac{\cos(x+h) - \cos(x)}{h} &= \frac{-2\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)}{h} \\ &\overset{h\to 0}{\to} -\sin x \end{aligned}$$

- The function f with

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

is not continuous for all $x \in \mathbb{R}$.

Theorem 3.0.1. If a function f is differentiable at $a \in \mathrm{dom}\, f$ then f is continuous at a.

This poses the question whether it is possible to find a function which is continuous but nowhere differentiable? The Fourier series are

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} = \sin(\pi x) + \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} + \dots$$

This led to the discovery of the Lacunary Fourier series

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2\pi x)}{n} = \sin(\pi x) + \frac{\sin(4\pi x)}{2} + \frac{\sin(9\pi x)}{3} + \dots$$

R is not differentiable except for x rational of the form $\frac{p}{q}$, p, q odd.

3.1 Basic Derivatives

$f(x)$	$f('x)$
x^n	nx^{n-1}
$\log x$	$\frac{1}{x}$
$\exp(x)$	$\exp(x)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$

3.2 Differentiation rules

If u, v and f are derivable functions then the followings rules hold:

- Addition rule

$$\frac{d}{dx}(u(x)+v(x)) = u'(x)+v'(x)$$

- Multiplication rule

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x)+u(x)v'(x)$$

- Chain rule

$$\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$$
$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$$

For the derivation of the product rule consider the following quotient:

$$\begin{aligned}
&\frac{u(x+h)v(x+h)-u(x)v(x)}{h} \\
=&\frac{u(x+h)v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x)}{h} \\
=&v(x+h)\frac{u(x+h)-u(x)}{h}+u(x)\frac{v(x+h)-v(x)}{h} \\
\overset{h\to 0}{\to}&v(x)u'(x)+u(x)v'(x)
\end{aligned}$$

The proof of chain rule will be done in spring term analysis.

3.3 Implicit Differentiation

Remark 0.2. Implicit differentiation applies the chain rule .

Example 0.2. Compute the slope of tangent to unit circle $x^2+y^2=1$.

'Solve' to get $y = y(x)$ and use the differentiation rules.

$$y(x) = \pm\sqrt{1-x^2}$$
$$y'(x) = \pm\frac{-x}{\sqrt{1-x^2}} = \mp\frac{x}{\sqrt{1-x^2}}$$

Implicit differentiation. Treat y^2 as a composite function – differentiate with the chain rule.

$$\frac{d}{dx}y^2(x) = 2y(x)y'(x)$$

equation $x^2 + y^2 = 1$. Differentiate with respect to x

$$2x + 2yy' = 0 \quad \vee \quad y' = \frac{-x}{y}$$

Example 0.3.

$$\begin{aligned} y^3 - y &= x^2 \\ (3y^2 - 1)y' &= 2x \end{aligned}$$

The slope of the tangent is

$$y' = \frac{2x}{3y^2 - 1}$$

For the point $(\sqrt{6}, 2)$ we get the slope

$$y' = \frac{2\sqrt{6}}{11}$$

3.4 Parametric Differentiation

You can describe a curve in the xy plane parametrically.

Example 0.4.

- Let us consider a curve defined by the hyperbolic functions.

$$\begin{aligned} x(t) &= \cosh(t) \\ y(t) &= \sinh(t), \qquad t \in \mathbb{R} \end{aligned}$$

slope of the tangent

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$$

$\cdot$ denotes differentiation with respect to the parameter t.

$$\frac{dy}{dx} = \frac{\cosh t}{\sinh t} = \coth t$$

- The equation for a cycloid is

$$\begin{aligned} x(t) &= t - \sin t \\ y(t) &= 1 - \cos t, \qquad t \in \mathbb{R} \end{aligned}$$

The point on the edge of a rolling wheel traces a cycloid.

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin t}{1 - \cos t}$$

3.5 Higher Differentiation

Suppose f is differentiable then consider the limit

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

If this exists, f is said to be twice differentiable. The limit is called second derivative, denoted

$$f''(x) \quad \text{or} \quad \frac{d^2 f(x)}{dx^2} \quad \text{or} \quad y''(x) \quad \text{or} \quad \frac{d^2 y(x)}{dx^2}$$

This can be continued to define the n^{th} derivative, denoted as

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n f(x)}{dx^n} \quad \text{or} \quad y^{(n)}(x) \quad \text{or} \quad \frac{d^n y(x)}{dx^n} \quad \text{or} \quad \left(\frac{d}{dx}\right)^n f(x)$$

$\frac{d}{dx}$ is called the differential operator.

Example 0.5.

$$f(x) = \log x \qquad f^{(1)}(x) = \frac{1}{x}$$
$$f^{(2)}(x) = -\frac{1}{x^2} \qquad f^{(3)}(x) = \frac{2}{x^3}$$
$$f^{(4)}(x) = -\frac{2 \cdot 3}{n^4} \qquad f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

Theorem 3.5.1. The Leibniz' formula is

$$\left(\frac{d}{dx}\right) uv = \sum_{p=0}^{n} \binom{n}{p} u^{n-p} v^p$$

The derivation can be made through regarding the functions $u(x)v(x)$.

$$\frac{d}{dx}uv = u'v + uv'$$

Differentiating again gives us

$$\begin{aligned}\left(\frac{d}{dx}\right)^2 uv &= u''v + u'v' + u'v' + uv'' \\ &= u''v + 2u'v' + uv'' \\ \left(\frac{d}{dx}\right)^3 uv &= u''v + u''v' + 2(u''v' + u'v'') + u'v'' + uv''' \\ &= u'''v + 3u''v' + 3u'v'' + uv'''\end{aligned}$$

The coefficients are binomial coefficients. A rigorous proof can be made by induction.

Example 0.6.

- Leibniz is particularly useful if one term in the product is a polynomial – since the sum terminates

  $$f(x) = e^{2x}x^2$$

  Set

  $$v = x^2, \quad u = e^{2x}$$

  Then

  $$v^{(1)} = 2x, \quad v^{(2)} = 2, \quad v^{(3)} = v^{(4)} = v^{(5)} = 0$$
  $$u^{(n)} = 2^n e^{2x}$$

  This gives us the n^{th} derivative of f

  $$\begin{aligned}f^{(n)}(x) &= \binom{n}{0}u^{(n)}v^{(0)} + \binom{n}{1}u^{(n-2)}v^{(1)}\binom{n}{2}u^{(n-2)}v^{(2)} \\ &= 2^n e^{2x}x^2 + n2^{n-1}e^{2x}2x + n(n-1)2^{n-2}e^{2x}\end{aligned}$$

- Another example

  $$\begin{aligned}f(x) &= \sin^{-1} x \\ f'(x) &= \frac{1}{\sqrt{1-x^2}} \\ f''(x) &= \frac{x}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{1-x^2}f'(x) \\ (1-x^2)f^{(2)}(x) &= xf^{(1)}(x)\end{aligned}$$

Differentiate both sides n times.

$$\left(1-x^2\right)f^{(2+n)}+\binom{n}{1}(-2x)f^{(1+n)}+\binom{n}{2}(-2)f^{(n)}=xf^{(n+1)}+1f^{(n)}\binom{n}{1}$$
$$\left(1-x^2\right)f^{2+n}-2nxf(1+n)(x)-n(n+1)f^{(n)}=xf^{(n+1)}+nf^{(n)}$$

Set $x=0$

$$\begin{aligned}
f^{(2+n)}(0)-n(n-1)f^{(n)}(0)&=nf^{(n)}(0)\\
f^{(2+n)}(0)&=n^2f^{(n)}(0)\\
f^{(0)}(0)&=0\\
f^{(1)}(0)&=1\\
f^{(3)}(0)&=1\\
f^{(5)}(0)&=9=3^2\\
f^{(7)}(0)&=3^25^2=225\\
f^{(9)}(0)&=3^25^27^2
\end{aligned}$$

4 Graphs

Definition 4.0.1. The graph of a function f is defined by $y=f(x)$

Definition 4.0.2. $a\in\text{dom}(f)$ is a *stationary point* if $f'(a)=0$

Remark 2.1. A stationary point can be a local minimum, a local maximum or a point of inflection with horizontal tangent.

Suppose a is a stationary point.

1. If $f''(a)>0$, then a is a local minimum
2. If $f''(a)<0$, then a is a local maximum
3. If $f''(a)=0$ gives no information.

This test is called the 2nd Derivative Test.

Example 2.1.

$$\begin{aligned}
f(x)&=x^4\\
f'(x)&=4x^3
\end{aligned}$$

$x=0$ is a stationary point because $f''(x)=12x^2=0$.

Geometrical definition. A point of inflection is a point where the graph crosses its own tangent
A sufficient condition for a point of inflection (p_0,I) is: If $f''(a)=0$ and $f'''(0)\neq 0$, then a is a point of inflection. This is not a necessary condition.

Example 2.2.

$$\begin{aligned}
f(x)&=x^5\\
f'(x)&=5x^4\\
f''(x)&=20x^3\\
f'''(x)&=60x^2
\end{aligned}$$

The sufficient condition does not work for this example at $x=0$ but $(0,0)$ is a point of inflection.

A point of inflection is not necessarily a stationary point.

Example 2.3.

$$
\begin{aligned}
f(x) &= x^4 - 2x^2 \\
f'(x) = 4x^3 - 4x &= 4x\left(x^2 - 1\right) \\
&= 4x(x-1)(x+1)
\end{aligned}
$$

There are 3 stationary points at $x = 0$ and $x = \pm 1$.

$$
\begin{aligned}
f''(x) &= 12x^2 - 4 \\
f''(0) &= -4 < 0
\end{aligned}
$$

So $x = 0$ is a local maximum. Furthermore,

$$f''(\pm 1) = 12 - 4 = 8 > 0$$

Hence, $x = \pm 1$ is a local minimum. Consider the following equation to find points of inflection

$$f''(x) = 12x^2 - 4 = 0$$

This holds if $x^2 = \frac{1}{3}$. i.e. $x = \pm 1/\sqrt{7}$ are points of inflection since $f'''(x) = 24x \neq 0$ at these points.

4.1 Curve sketching

There is no correct way to sketch the graph of a function – in some cases the graph is too complicated to sketch it by hand. In this case try using a computer. (e.g. Riemann's Lacunary Fourier series.) However, the following often helps:

1. Does the graph have any special features (e.g. odd, even or periodic)?
2. Does the graph intersect the x or y axes?
3. Does the graph have stationary points or points of inflection?
4. Does the graph have linear asymptotes?

Rational functions often have linear asymptotes.

Example 0.4.

$$f(x) = \frac{x^3}{1 - x^2}$$

- At $x = \pm 1$ the graph has vertical asymptotes.
- For $x \to \pm\infty$ the graph has the linear asymptote $y = x$.
- For x small $f(x) \approx x^3$.

4.1.1 Polar Coordinates

We can represent curves via functions, equations or parametrically – yet another way is through polar coordinates. The Idea is to replace the cartesian coordinates x and y with polar coordinates r and θ. r is the distance to the origin, θ is the angle measured anti-clockwise from the x axis. Replacing θ by $\theta + 2\pi$ has no effect.

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Now we are able to represent curves using equations involving r and θ instead of x and y.

Example 0.5.

- If r is a constant greater than 0, the equations represent a circle.
- Let l be a positive constant and e be a non-negative constant.

  $$r = \frac{l}{1 + e\cos\theta}$$

  gives a conic section where e is the 'eccentricity'.

4.1.2 Conic sections

In general consider $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, F are constants.

Degenerate cases are:

- point $x^2 + y^2 = 0$
- line $y = 0$
- two lines $x^2 - y^2 = 0 \Leftrightarrow x = \pm y$
- two parallel lines

All other possibilities are of three types ellipse, parabola, hyperbola.

Definition 4.1.1. An *ellipse* is a curve defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or any translation or rotation of this curve. $a = b$ reduces to a circle.

Definition 4.1.2. A Parabola is a curve of the form

$$y = ax^2$$

or any translation or rotation of this curve.

Definition 4.1.3. A Hyperbola is a curve of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or any translation or rotation of this curve.

The equation

$$r = \frac{l}{1 + e\cos\theta}$$

- gives us an ellipse for $0 \leq e < 1$.
- gives us a parabola for $e = 1$.
- gives us a hyperbola for $e > 1$.

Set $l = 1$ for all cases: To obtain the equation for a parabola set $e = 1$:

$$r = \frac{1}{1 + \cos\theta}$$

To obtain the equation for a parabola set $e = \frac{1}{2}$:

$$r = \frac{1}{1 + \frac{1}{2}\cos\theta}$$

To obtain the equation for a parabola set $e = 2$:

$$r = \frac{1}{1 + 2\cos\theta}$$

$$-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$$

There are two different conventions for dealing with negative r:

1. Discard any θ values leading to negative r.

2. Retain θ values leading to negative r.

$$x = r\cos\theta \qquad y = r\sin\theta$$

Allow r to be negative. In case of r negative, flip the sign of r, i.e. flip the sign of x and y. This is equivalent to shifting θ by π (or $-\pi$)

$$\sin(\theta \pm \pi) = \sin\theta$$
$$\cos(\theta \pm \pi) = \cos\theta$$

Using this prescription

$$r = \frac{l}{1 + e\cos\theta}$$

gives a full hyperbola (both branches) for $e > 1$.

5 Power Series

Definition 5.0.4. A polynomial of degree n is a function of the form

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^2$$

where $c_0, c_1, \ldots, c_n$ are constants, $c_n \neq 0$.

Remark 4.1. It is easy to see that $c_0 = f(0)$. Differentiating gives us

$$f'(x) = c_1 + 2c_2x + \cdots + nc_nx^{n-1}$$

And by repeated differentiation we gain

$$c_m = \frac{f^{(m)}(0)}{m!}$$

This gives us the following formula for any polynomial:

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(0)}{m!}x^m$$

If the function is not a polynomial, the formula is evidently not correct but represents an approximation for the function. This polynomial approximation is called a Maclaurin series. It works near $x = 0$ as $f(x)$ and its first n derivatives agree with the polynomial at this value.

We can shift the expansion point from $x = 0$ to another point a:

$$S(x) \approx \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^x$$

This is now called a Taylor series, approximating $f(x)$ near $x = a$.

The error of the approximation can be specified. In fact, if we write

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^x + R_n(x)$$

there are the following exact formulas for $R_n(x)$:

1. Lagrange form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is between a and x.

2. Cauchy form

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-a)(x-c)^n$$

where c is between a and x.

3. Integrated form

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$$

Summarized this is called the Taylor Theorem:

Theorem 5.0.1. *Taylor Theorem.* (with Lagrange form of remainder)

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

For the derivation let us first consider Rolle's theorem:

Theorem 5.0.2. *Rolle's Theorem.*
Suppose f is differentiable on (a,b) and continuous on $[a,b]$ with $f(a) = f(b)$. Then there is a $c \in (a,b)$ such that $f'(c) = 0$.

Since the proof for this requires the *intermediate value theorem* whose proof would lead to a long chain of required theory which has not been dealt with in this course, we assume Rolle's Theorem to be obvious.

A generalization of Rolle's Theorem is the mean value theorem:

Theorem 5.0.3. *Mean Value Theorem.* (MVT)
Suppose f is differentiable on (a,b) and continuous on $[a,b]$. Then there is a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof. Define the function g with

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Hence,

$$g(a) = g(b) = f(a) \qquad \wedge \qquad g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

Therefore, we can apply Rolle's theorem on g. I.e. there exists a $c \in (a,b)$ with

$$\begin{aligned} & g(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a} \\ \Rightarrow \qquad & f'(c) = \frac{f(b)-f(a)}{b-a} \end{aligned}$$

$\square$

Having dealt with these basic properties, we can proof the different forms of the remainders at least for $n = 0$.

Then our approximation is

$$f(x) = f(a) + R_0(x)$$

The integral form of $R_0(x)$ is

$$R_0(x) = \int_a^x f'(t)dt = f(x) - f(a)$$

which obviously fits into our formula.

The Lagrange and Cauchy form are the same:

$$f(x) = f(a) + f'(c)(x - a)$$

where c lies between a and c. The respective c exists according to the MVT.

Example 4.1. • Consider

$$\begin{aligned} f(x) &= e^x \\ f^{(n)}(x) &= e^x \\ f^{(n)}(0) &= 1 \end{aligned}$$

We approximate about $x = 0$ with $n = 3$ (Maclaurin series):

$$\begin{aligned} f(x) &= 1 + x + \frac{x^2}{3!} + \frac{x^3}{3!} + R_3(x) \\ R_3(x) &= \frac{e^c x^4}{4!}, && 0 \leq c \leq x \end{aligned}$$

Suppose x is negative so c is negative. Then $|e^c| < 1$. Hence,

$$\left| e^x - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \right| < \frac{x^4}{4!}$$

• For the trigonometrical function this works a bit better:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -sinx \\ f'''(x) &= -\cos x \\ f^{(4)} &= \sin x \end{aligned}$$

Choose $a = 0$ as the expansion point.

$$\begin{aligned} f^{(2)}(0) = f^{(2)}(0) = f^{(2)}(0) &= 0 \\ f^{(1)}(0) = f^{(5)}(0) &= 1 \\ f^{(3)}(0) = f^{(8)}(0) &= -1 \end{aligned}$$

Apply the result with $n = 4$

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + R_4(x) \\ R(4)(x) &= \frac{f^{(5)}(c)x^5}{5!} \\ &= \frac{x^5 \cos x}{5!} \end{aligned}$$

but $|\cos c| \leq 1$.

$$|R_4(x)| = \left| \sin x - \left(x - \frac{x^3}{3!}\right) \right| \leq \frac{|x|^5}{5!}$$

This is true for all x. Thus, for x approaching 1, we get the approximation

$$\sin x \approx x - \frac{x^3}{6}$$

5.1 Infinite Taylor series

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m + R_n(x)$$

In some cases $\lim_{n\to\infty} R_n(x) = 0$ for some or all x (x fixed when taking $n \to \infty$ limit). In this case

$$f(x) = \lim_{n\to\infty} \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!}(x-a)^m$$

Example 0.2.

- Again, consider

$$\begin{aligned} f(x) &= e^x \\ f^{(m)}(x) &= e^x \\ f^{(m)}(0) &= 1 \end{aligned}$$

Expand about $a = 0$:

$$\begin{aligned} f(x) &= \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} + R_n(x) \\ R_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c x^{n+1}}{(n+1)!}, \qquad 0 \le c \le x \end{aligned}$$

We claim $R_n(x) \to 0$ as $n \to \infty$ for any fixed x. For example let us consider $x = 1000$:

$$\begin{aligned} R_n &= \frac{e^c 1000^{n+1}}{(n+1)!} \\ &\le \frac{e^{1000} 10^{3(n+1)}}{(n+1)!} \\ &\overset{n\to\infty}{\to} 0 \end{aligned}$$

(Factorials always grow faster than exponentials.) In this example $R_n(x) \to 0$ as $n \to \infty$ for all fixed x. Hence, we get

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

which is the infinite Maclaurin series for the exponential function.

- In our last example the approximation worked for all $x \in \mathbb{R}$. However, in some cases $R_n(x) \to 0$ as $n \to \infty$ works for a range of x values but not all of them. For this purpose regard the geometric series:

$$f(x) = \frac{1}{1-x}$$

The respective Maclaurin series expanded at $a = 0$ is

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots + x^n + R_n(x) \\ R_n(x) &= \frac{1}{1-x} - \left(1 + x + x^2 + \cdots + x^n\right) \\ &= \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} = \frac{x^{n-1}}{1-x} \end{aligned}$$

If $-1 < x < 1$, $R_n(x) \to 0$ as $n \to \infty$ but if $x \ge 1$ or $x \le -1$, then $\lim_{x\to\infty} R_n(x)$ is undefined.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

is only valid for $-1 < x < 1$.

- Consider the binomial expansion

$$f(x) = (1+x)^p$$

where p is constant. Expand this about $a = 0$.

$$\begin{aligned} f^{(1)}(x) &= p(1+x)^{p-1} \\ f^{(2)}(x) &= p(p-1)(1+x)^{p-2} \\ &\dots \\ f^{(m)}(x) &= p(p-1)\dots(p-m+1)(1+x)^{p-m} \end{aligned}$$

The Maclaurin series is

$$\begin{aligned} f(x) &= 1 + px + \frac{p(p-1)}{2!}x^2 + \dots + \frac{p(p-1)\dots(p-n+1)}{n!}x^n + R_n(x) \\ R_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \\ &= \frac{p(p-1)\dots(p-n)(1+c)^{p-n}}{(n+1)!}x^{n+1}, \qquad 0 \le c \le x \end{aligned}$$

We claim that $R_n(x) \to 0$ as $x \to \infty$ if $-1 < x < 1$. If $x > 0$, c between 0 and x. If $x > 0$ then $1 + c < 1$. The solution in this case is to use the Cauchy form of the remainder for negative x:

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(c)}{n!}x(x-c)^n \\ &= \frac{p(p-1)\dots(p-n)}{n!}(1+c)^{p-n}(x-c)^n x \end{aligned}$$

The fraction is a constant and hence unimportant for the limit as $x \to \infty$.

$$\begin{aligned} (1+x)^{p-n}(x-c)^n &= (1+c)^p\left(\frac{x-c}{1+c}\right)^n \\ &= (1+c)^p\left(\frac{x-c}{(c-x)+(1+x)}\right)^n \\ &\le (1+c)^p\left(\frac{x-c}{1+(1+x)}\right)^n \end{aligned}$$

Because $1 > c - x > 0$. This has the limit 0 as $x \to \infty$

5.2 Manipulating Infinite Maclaurin Series

We can multiply and compose infinite power series.

Example 0.3. • Assume we wanted to calculate the first three nonzero terms of the power series for $\tan x$:

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ &= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \cdot \left(1 + \underbrace{\frac{x^2}{2!} + \frac{x^4}{4!} + \dots}_{d}\right)^{-1} \qquad (1) \\ &= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \cdot x\left(1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)^2 - \dots\right) \\ &= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)\left(1 + x^2\left(-\frac{1}{2}\right) + x^4\left(-\frac{1}{24} + \frac{1}{4}\right) + \dots\right) \qquad (2) \end{aligned}$$

(1) holds because of the general binomial theorem

$$(1+d)^{-1} = 1 - d + d^2 - d^3 + \dots$$

The 2nd bracket of (2) can be simplified to

$$1 - \frac{1}{2}x^2 + \frac{5}{24} + \dots$$

Hence the product of the 2 brackets of (2) is:

$$\begin{aligned} & x^3\left(\frac{1}{6} - \frac{1}{2}\right) + x^5 l + \left(\frac{3}{100} - \frac{1}{12} + \frac{5}{24}\right) + \dots \\ = & x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \end{aligned}$$

In order to get to this result, we could have also used

$$f(x) = f(0) + f''(0)x + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(5)}(0)x^5}{5!}$$

- Find the first two nonzero terms im the Maclaurin series of $\log(\cos x)$.

$$\begin{aligned} \log(\cos x) &= \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) \\ &= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)^2 + \dots \end{aligned}$$

because

$$\log(1 + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \dots$$

where in our case

$$X = -\frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

So we get

$$\begin{aligned} \log(\cos x) &= 0 - \frac{1}{2}x^2 + x^4\left(\frac{1}{24} - \frac{1}{8}\right) + \dots \\ &= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots \end{aligned}$$

We can integrate and differentiate power series

Example 0.4.
- Through differentiating the power series it is possible to calculate the first derivative of the hyperbolic tan function:

$$\begin{aligned} \tanh^{-1} &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \dots \\ \frac{d}{dx}\tanh^{-1} x &= 1 + x^2 + x^4 + x^6 + \dots \\ &= \frac{1}{1 - x^2} \end{aligned}$$

- Similarly we can obtain the well-known first derivative of $\sin x$:

$$\begin{aligned} \frac{d}{dx}\sin x &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\ &= 1 - x - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ &= \cos x \end{aligned}$$

We argued that for certain infinite Taylor series the remainder term is absent (vanishes in the $n \to \infty$ limit). The reverse procedure also works. We can define functions as power series without a remainder term.

Take as a starting point

$$f(x) = \sum_{m=0}^{\infty} c_m x^m, \qquad c_0, c_1, c_2, \dots \in \mathbb{R}$$

For $c_m = \frac{1}{m!}$ we get the exponential function. Taking $c_m = 1$ for all m gives us

$$f(x) = \frac{1}{1-x}$$

The problem is that we can always define a function this way but we don't know whether it is convergent if the sum is convergent.

Example 0.5.

- $c_m = \frac{1}{m}$ gives us $f(x) = e^x$. This formula is valid for any x
- The geometric series:

  $$c_m = 1 \qquad f(x) = \frac{1}{1-x}$$

  This expansion is only valid if $-1 < x < 1$.
- An extreme example is $c_m = m!$.

  $$f(x) = \sum_{m=0}^{\infty} m! x^m$$

  This only converges if $x = 0$.
- For the infinite sum

  $$c_m = \frac{1}{(m!)^2} \qquad f(x) = \sum_{m=0}^{\infty} \frac{x^m}{(m!)^2}$$

  we do not know for which x it converges.

We would like to know under what conditions

$$f(x) = \sum_{m=0}^{\infty} c_m x^m = \lim_{n\to\infty} \sum_{m=0}^{n} c_m x^m$$

converges and if so for what range of x.

For this purpose we have to take a step back. Consider convergence of numerical series. That means a sum of the form

$$\sum_{m=0}^{\infty} a_m$$

where $a_0, a_1, a_2, \dots$ is an infinite list of numbers.

Example 0.6.

- Consider

  $$a_m = \frac{1}{m^4}, \qquad m \geq 1$$
  $$a_0 = 0$$
  $$\sum_{m=1}^{\infty} \frac{1}{m^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} = \xi(4) = \frac{\pi^4}{90}$$

  is a convergent series.

- The harmonic series:

$$a_m = \frac{1}{m}, \qquad m \geq 1, a_0 = 0$$
$$\sum_{m=1}^{\infty} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots$$

does not converge.

- The alternating harmonic series:

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
$$= \log 2$$

- The following series oscillates between 1 and -1 but does not have a limit for $n \to \infty$

$$\sum_{m=0}^{\infty} (-1)^m, \qquad a_m = (-1)^m$$
$$=1 - 1 + 1 - 1 + 1 \dots$$
$$= \sum_{m=0}^{n} (-1)^m = \frac{(-1)^n + 1}{2}$$

How to decide whether a numerical series converges?

- Evaluate it!
- Use some standard tests.

5.3 Tests

1. Preliminary Test (Easy Test)
   If $a_m \not\to 0$ as $m \to \infty$ then

$$\sum_{m=0}^{\infty} a_m$$

   does not converge. If the series converges, then $a_m \to 0$ as $m \to \infty$.

   Example 0.7. • $\sum_{m=1}^{\infty} \frac{1}{m^4}$ and $\sum_{m=1}^{\infty} \frac{1}{m}$
   both pass the preliminary test since $a_m \to 0$ as $m \to \infty$.

   - $\sum_{m=0}^{\infty} = 0 + 1 + 2 + 3 + 4 + \dots$
     does not pass the preliminary test and hence diverges.

2. Alternating Series Text
   Suppose a_m is alternating and $|a_m|$ is strictly decreasing, $|a_m + 1| < |a_m|$ and a_m passes the preliminary test, then

$$\sum_m a_m$$

   is convergent.