

# M1M1 Notes

David Burgschweiger

October 24, 2015

# Contents

<b>1</b>	<b>Functions</b>	<b>2</b>
1.1	Rational Functions . . . . .	2
1.2	Exponential Function . . . . .	3
1.3	Trigonometrical Functions . . . . .	3
1.4	Odd and Even Functions . . . . .	4
1.5	Hyperbolic Functions . . . . .	5
1.6	Inverse Functions . . . . .	5
1.6.1	Derivatives of Inverse Functions . . . . .	7
1.6.2	Inverse Trigonometrical functions . . . . .	7
<b>2</b>	<b>Limits</b>	<b>7</b>
2.1	Infinite Limits . . . . .	8
2.2	Computing Limits . . . . .	9

## 1 Functions

**Definition 1.0.1.** A function  $f$  is a rule assigning every element  $x$  in a set  $A$  an element  $f(x)$  in another set  $B$ .

**Remark 1.1.**

- $A$  is called the domain of  $f$  whereas  $B$  is called codomain.
- The range (image) of a function is the set:

$$\begin{aligned}\text{Range}(f) &= \text{Im}(f) \subseteq \text{codomain} \\ &= \{f(x) \in B \mid \forall x \in A\}\end{aligned}$$

It does not have to be equal to the codomain.

- In the following we will mostly consider functions of one variable (with  $A = \mathbb{R}$  and  $B = \mathbb{R}$ , later  $\mathbb{C}$ ).

**Example 1.1.** Polynomials,  $c_i \in \mathbb{R}, \forall i \in \mathbb{N}$ :

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

**Definition 1.0.2.** The graph of a function  $f$  (real not complex) is the set

$$\{(x, y) \mid x \in \text{dom}(f), y = f(x)\}$$

**Property 2.1.** The graph of any function intersects any vertical line at most once.

### 1.1 Rational Functions

**Definition 1.1.1.** A rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where  $P$  and  $Q$  are polynomials.

**Example 1.1.**

$$f(x) = \frac{1}{1-x^2}, \quad \text{dom}(f) = \mathbb{R} \setminus \{1, -1\}$$

## 1.2 Exponential Function

**Definition 1.2.1.** The exponential function  $\exp$  can be defined by several ways:

1. As a power of  $e$ :

$$\exp(x) = e^x$$

Obviously, for this definition the number  $e$  must be defined.

2. As a power series:

$$\exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

3. By a ordinary differential equation (ODE):

$$\begin{aligned} \frac{d}{dx} \exp(x) &= \exp(x) \\ \exp(0) &= 1 \end{aligned}$$

4. As inverse of the natural logarithm:

$$\begin{aligned} \exp^{-1}(x) &= \log(x) \\ \log(x) &= \int_1^x \frac{du}{u} \end{aligned}$$

5. As a limit:

$$\exp(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

**Property 1.1.**

$$\exp(x + y) = \exp(x) \cdot \exp(y)$$

## 1.3 Trigonometrical Functions

**Definition 1.3.1.** Similar to the exponential function, the trigonometrical functions  $\cos$  and  $\sin$  have several potential definitions:

1. The elementary geometric definition at a right-angled triangle with a hypotenuse of length 1.
2. Definition through Polar form – considering a point  $p$  on a unit circle centred at the origin .
3. As a power series:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{aligned}$$

4. Through a system of ODEs:

$$\begin{aligned} \frac{d}{dx} \sin x &= \cos x \\ \frac{d}{dx} \cos x &= -\sin x \\ \sin 0 &= 0, \quad \cos 0 = 1 \end{aligned}$$

5. With the help of complex numbers:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

**Property 1.1.**

- The addition formula:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

- Shifting:

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

$$\cos\left(x + \frac{\pi}{2}\right) = -\sin x$$

$$\begin{aligned}\sin(x + \pi) &= \sin\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2} \\ &= \cos\left(x + \frac{\pi}{2}\right)\end{aligned}$$

$$\sin(x + 2\pi) = \sin x$$

**Remark 1.1.** Special values which should be memorized are

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$$

**Definition 1.3.2.** If a function  $f$  has property  $f(x + a) = f(x)$ ,  $\forall x \in \text{dom}(f)$  it is called periodic. The period of  $f$  is the smallest possible  $a$  for which  $f(x + a) = f(x)$ ,  $\forall x \in \text{dom}(f)$ .

**Definition 1.3.3.** Other trigonometric functions can be written as a combination of sine and cosine:

$$\begin{aligned}\sec x &= \frac{1}{\cos x} \\ \operatorname{cosec} x &= \frac{1}{\sin x} \\ \tan x &= \frac{\sin x}{\cos x} \\ \cot x &= \frac{\cos x}{\sin x}\end{aligned}$$

## 1.4 Odd and Even Functions

**Definition 1.4.1.** A function  $f$  is even if

$$\forall x \in \text{dom}(f) : f(-x) = f(x)$$

A function  $f$  is odd if

$$\forall x \in \text{dom}(f) : f(-x) = -f(x)$$

**Remark 1.1.** These definitions assume that  $\text{dom}(f)$  is symmetric which means  $x \in \text{dom}(f) \Rightarrow -x \in \text{dom}(f)$

**Example 1.1.**  $\sin x$  is odd,  $\cos x$  is even.

**Property 1.1.** A function can be neither odd nor even. However, any function can be split into a sum of even and odd functions

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$$

The odd and even part of a function are unique.

**Example 1.2.**

$$e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})$$

## 1.5 Hyperbolic Functions

**Definition 1.5.1.**

$$\cosh x = \frac{1}{2} (e^x + e^{-x})$$

$$\sinh x = \frac{1}{2} (e^x - e^{-x})$$

**Property 1.1.**

- Addition theorem and derivatives:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

- The hyperbolic functions can also be expressed through power series:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

- Similarly to the trigonometrical Pythagoras the following equation holds:

$$\cosh^2 x - \sinh^2 x = 1$$

**Remark 1.1.** Origin of the name:

$$\begin{aligned} x &= \cosh t, & t &\in \mathbb{R} \\ y &= \sinh t \\ x^2 - y^2 &= 1 \end{aligned}$$

parametrizes a hyperbola.

## 1.6 Inverse Functions

**Definition 1.6.1.** The inverse function  $f^{-1}$ , if it exists, is a function  $f^{-1} : B \rightarrow A$  with the properties

$$\begin{aligned} f(f^{-1}(y)) &= y, & \forall y \in B \\ f^{-1}(f(x)) &= x, & \forall x \in A \end{aligned}$$

**Example 1.1.**

$$\begin{aligned} f(x) &= x^2 & A &= [0, \infty) = B \\ f^{-1}(y) &= \sqrt{y} \end{aligned}$$

**Remark 1.1.**

- A necessary condition for a function to be invertible is that  $f$  is injective (one-to-one).

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

or

$$f(x_1) \neq f(x_2) \quad \Leftarrow \quad x_1 \neq x_2$$

Graphical test:  $f$  is injective if its graph intersects any horizontal line at most once.

- The graph of the inverse  $f^{-1}$  is the set of the points of the graph of  $f$  with the  $x$  and  $y$  coordinates exchanged. The graph of  $f^{-1}$  can be obtained by reflecting the graph of  $f$  about the line  $y = x$ .
- If  $f$  is strictly increasing (decreasing), it is injective.

**Definition 1.6.2.**  $f$  is strictly increasing if

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

$f$  is strictly decreasing if

$$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$

**Example 2.1.** The exponential function is strictly increasing. (proof in problem sheet)

**Remark 2.1.**

- Any even function  $f$  is not injective if  $\text{dom } f \not\subseteq \{0\}$ .
- Any periodic function is not injective either.
- Therefore, the trigonometric functions

$$\sin, \cos, \tan$$

are not invertible.

In order to inverse the sin function, restrict the domain to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

In order to inverse the cos function, restrict the domain to  $[0, \pi]$ .

The inverse of the exponential function is called logarithm,  $\log x$ .

Analytic treatment is sometimes possible. Require the existence of  $f^{-1}(x)$  and 'Solve'  $y = f(x)$  to obtain  $x$  in terms of  $y$ ,  $x = f^{-1}(y)$ .

**Example 2.2.**

•

$$f(x) = e^{-\frac{1}{x}}$$

$$x = -\frac{1}{\log y}$$

- inverse hyperbolic functions

$$f(x) = \cosh x$$

$$f(x) = \frac{1}{2} (e^x + e^{-x})$$

$$e^{2x} - 2ye^x + 1 = 0$$

$$(e^x)^2 - 2ye^x + 1 = 0$$

$$e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$

$$e^x = y \pm \sqrt{y^2 - 1}$$

$$x = \log \left( y \pm \sqrt{y^2 - 1} \right)$$

Restrict domain of  $\cosh x$  to non-negative  $x$ .

$$x = \log(y + \sqrt{y^2 - 1})$$

$$\cosh^{-1} x = \log \left( x + \sqrt{x^2 - 1} \right)$$

$$\sinh^{-1} x = \log \left( x + \sqrt{1 + x^2} \right)$$

### 1.6.1 Derivatives of Inverse Functions

The slope of the inverse function is  $\frac{1}{f'(a)}$ , the reciprocal of slope of the original function.

**Example 2.3.**  $f(x) = e^x = y$  so that  $x = \log y$ .

$$\frac{dx}{dy} = \left( \frac{dy}{dx} \right)^{-1} = \frac{1}{e^x} = \frac{1}{y}$$

or

$$\frac{d}{dy} \log y = \frac{1}{y}$$

### 1.6.2 Inverse Trigonometrical functions

We are going to differentiate  $\sin^{-1}$ ,  $\tan^{-1}$ . We set  $y = \sin x$  so that  $x = \sin^{-1} y$ .

$$\begin{aligned} \frac{dx}{dy} &= \left( \frac{dy}{dx} \right)^{-1} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} \\ &= \frac{1}{\sqrt{1 - y^2}} \\ \frac{d}{dx} \sin^{-1} x &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

Similarly,

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

**Property 2.1.**

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

**Example 2.4.** Find the mistake in the following proof.

$$\begin{aligned} \sin \left( x + \frac{\pi}{2} \right) &= \cos x \\ x &= \cos^{-1} y \\ \sin \left( \cos^{-1} y + \frac{\pi}{2} \right) &= \cos \left( \cos^{-1} y \right) \\ \sin \left( \cos^{-1} y + \frac{\pi}{2} \right) &= y \\ \cos^{-1} y + \frac{\pi}{2} &= \sin^{-1} y \\ \sin^{-1} - \cos^{-1} y &= \frac{\pi}{2} \end{aligned}$$

## 2 Limits

**Definition 2.0.3.** The symbolic notation

$$L = \lim_{x \rightarrow a} f(x) \quad \text{or} \quad f(x) \xrightarrow{x \rightarrow a} f(x)$$

means:

$$\forall \epsilon > 0 \exists \delta > 0 : (|x - a| > \delta \wedge x \in \text{dom}(f)) \Rightarrow |f(x) - L| < \epsilon$$

$f(x)$  approaches  $L$  as  $x$  approaches  $a$ .

**Remark 3.1.** It is important that  $x$  approaches  $a$  from both the left and the right. A one sided limit is written as follows:

$$\lim_{x \rightarrow a^+} f(x)$$

or

$$\lim_{x \rightarrow a^-} f(x)$$

**Example 3.1.**

- Let

$$f(x) = \frac{x}{|x|}, \quad x \neq 0$$

Then

$$\lim_{x \rightarrow 0} f(x)$$

is undefined but one sided limits exist:

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= 1 \\ \lim_{x \rightarrow 0^-} f(x) &= -1 \end{aligned}$$

- Let

$$f(x) = x^2$$

Then

$$\lim_{x \rightarrow 2} f(x) = f(2) = 4$$

- Let

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$\lim_{x \rightarrow 1} f(x)$  is an indeterminate limit of the form  $\frac{0}{0}$ . However  $\frac{x^2-1}{x-1} = x+1$  if  $x \neq 1$ . Therefore  $\lim_{x \rightarrow 1} f(x) = 2$ .

**Remark 3.2.** Not all indeterminate limits are meaningful.

**Example 3.2.** The limit

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{(x - 1)^2}$$

of the form  $\frac{0}{0}$  does not exist.

## 2.1 Infinite Limits

**Example 0.3.**

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{x} &= 0 \\ \lim_{x \rightarrow -\infty} \tan^{-1} x &= -\frac{\pi}{2} \end{aligned}$$

$x \rightarrow \infty$  is the same as  $\frac{1}{x} \rightarrow 0^+$ .  
 $x \rightarrow -\infty$  is the same as  $\frac{1}{x} \rightarrow 0^-$ .

**Property 0.1.** Provided the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist we know the following rules:



- addition formula

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

- product rule

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

- quotient rule ( $\lim_{x \rightarrow a} g(x) \neq 0$ )

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

## 2.2 Computing Limits

- manipulate function so the limit is 'obvious'
- use power-series
- L'Hopital's rule

**Example 0.4.**

- 

$$\begin{aligned} L &= \lim_{x \rightarrow 1} \frac{\sqrt{2-x}-1}{1-x} \\ &= \lim_{x \rightarrow 1} \frac{\sqrt{2-x}-1}{1-x} \cdot \frac{\sqrt{2-x}+1}{\sqrt{2-x}+1} \\ &= \lim_{x \rightarrow 1} \frac{(2-x)-1}{(1-x)(\sqrt{2-x}+1)} \\ &= \lim_{x \rightarrow 1} \frac{1-x}{(1-x)(\sqrt{2-x}+1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{2-x}+1} \\ &= \frac{1}{2} \end{aligned}$$

Alternatively the power series can be used. Let  $s = 1 - x$ . Then

$$L = \lim_{s \rightarrow 0} \frac{\sqrt{1+s}-1}{s}$$

General Binomial expansion is

$$(1+s)^p = 1 + ps + \frac{p(p-1)}{2!} s^2 + \frac{p(p-1)(p-2)}{3!} s^3 + \dots$$

If  $p$  is a positive integer, the series terminates – and gives us the standard binomial theorem. If  $p$  is not a positive integer, the formula is valid for  $|s| < 1$ . Hence

$$(1+s)^{\frac{1}{2}} = 1 + \frac{1}{2}s + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} s^2 + \dots$$

can be inserted in our formula and we get:

$$\lim_{s \rightarrow 0} \frac{\sqrt{1+s}-1}{s} = \lim_{s \rightarrow 0} \frac{1 + \frac{1}{2}s + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} s^2 + \dots - 1}{s} = \frac{1}{2}$$

- Let us calculate the following well-known limit:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

Using a power series we get

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \left( 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right) = 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

Using limits for graph sketching.

$$f(x) = \frac{x}{e^x - 1} = \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots} \xrightarrow{x \rightarrow 0} 1$$

- Let

$$f(x) = \frac{\cos\left(\frac{\pi}{2}x\right)}{1 - x^2}$$

Now we can calculate  $\lim_{x \rightarrow 1} f(x)$ .

$$\begin{aligned} & \frac{\cos\left(\frac{\pi}{2}x\right)}{1 - x^2} \\ &= \frac{\cos\left(\frac{\pi}{2}(x-1) + \frac{\pi}{2}\right)}{(x-1)(x+1)} \\ &= \frac{\sin\left(\frac{\pi}{2}(x-1)\right)}{(x-1)(x+1)} \end{aligned}$$

Substituting  $s = x - 1$  that gives us

$$\begin{aligned} &= \frac{\sin\left(\frac{\pi}{2}s\right)}{s(2+s)} \\ &= \frac{\frac{\pi}{2}s - \frac{1}{3!}\left(\frac{\pi}{2}s\right)^3 + \frac{1}{5!}\left(\frac{\pi}{2}s\right)^5 + \dots}{s(2+s)} \\ &= \frac{\frac{\pi}{2} - \frac{1}{3!}\left(\frac{\pi}{2}s\right)^3 + \frac{1}{5!}\left(\frac{\pi}{2}s\right)^5 + \dots}{2+s} \\ &\xrightarrow{s \rightarrow 0} \frac{\pi}{4} \end{aligned}$$

- Using limits, the equivalence of the definitions for the exponential function can be proven.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

Derivation:

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x \\ &= \lim_{x \rightarrow \infty} \exp\left(\log\left(1 + \frac{a}{x}\right)^x\right) \\ &= \lim_{x \rightarrow \infty} \exp\left(x \log\left(1 + \frac{a}{x}\right)\right) \\ &= \exp\left(\lim_{x \rightarrow \infty} x \left(\frac{a}{x} - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} \dots\right)\right) \\ &= \exp(a) \end{aligned}$$