Backpropagation

1. Gradient Descent & Chain Rule

Network parameters $\theta = \{w_1, w_2, \cdots, b_1, b_2, \cdots\}$

Starting Parameters $\theta^0 \longrightarrow \theta^1 \longrightarrow \theta^2 \longrightarrow \cdots\cdots$

$$\nabla L(\theta) = \begin{bmatrix} \partial L(\theta)/\partial w_1 \\ \partial L(\theta)/\partial w_2 \\ \vdots \\ \partial L(\theta)/\partial b_1 \\ \partial L(\theta)/\partial b_2 \\ \vdots \end{bmatrix}$$

$Compute\ \nabla L(\theta^0)$ $\quad \theta^1 = \theta^0 - \eta \nabla L(\theta^0)$

$Compute\ \nabla L(\theta^1)$ $\quad \theta^2 = \theta^1 - \eta \nabla L(\theta^1)$

Millions of parameters ……

To compute the gradients efficiently, we use ***backpropagation***.

在Neural Network用Gradient Descent的时候与在Logistic Regression、Liner Regression里使用是没有太多差别的，最大的差别是在Neural Network里有特别多的参数。

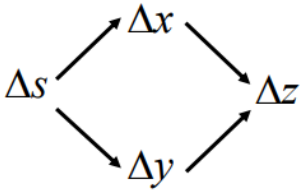
Backpropagation是一种Gradient Descent，是比较有效率的演算法，能更有效率的计算gradient。

Case 1 $\quad y = g(x) \quad z = h(y)$

$\Delta x \rightarrow \Delta y \rightarrow \Delta z$ $\qquad \dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx}$

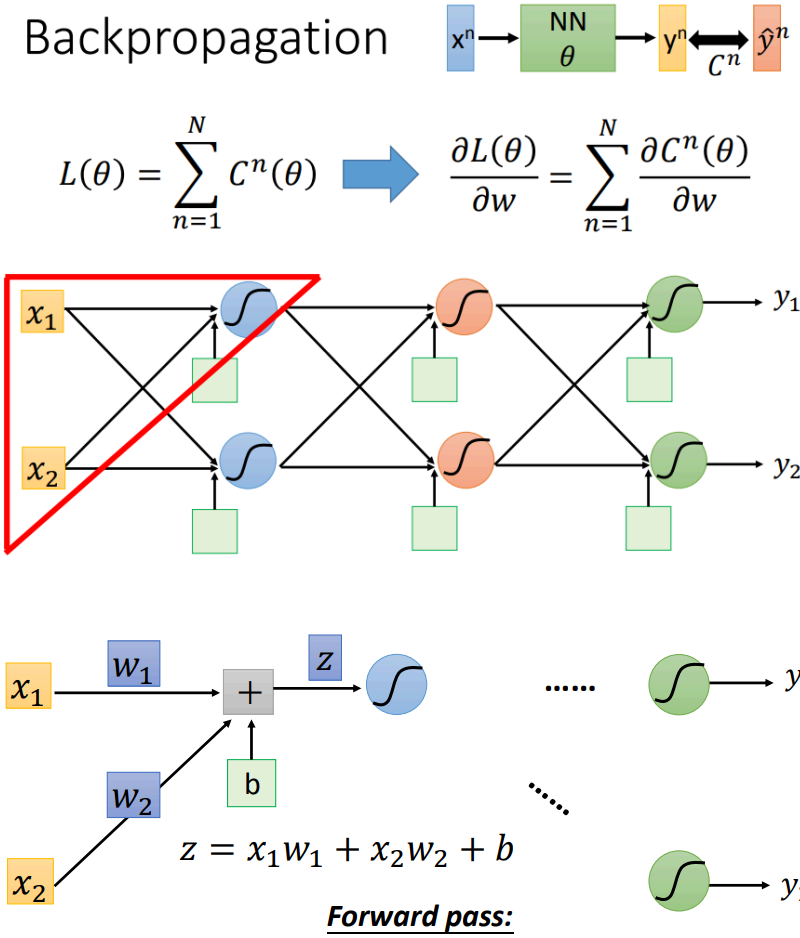
Case 2

$x = g(s) \quad y = h(s) \quad z = k(x, y)$



$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

2. Backpropagation



Forward pass:

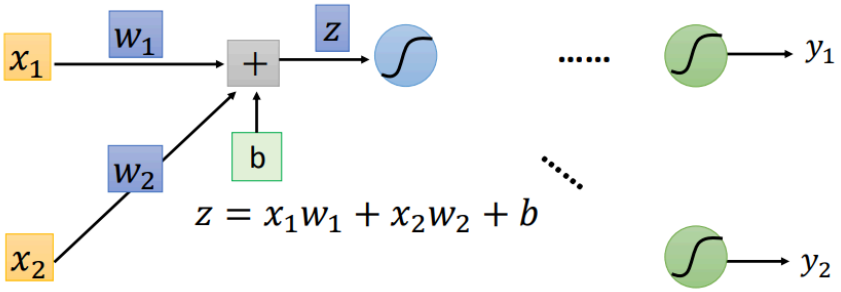
Compute $\partial z/\partial w$ for all parameters

Backward pass:

Compute $\partial C/\partial z$ for all activation function inputs z

2.1 Backpropagation – Forward pass（计算 $\frac{\partial z}{\partial w}$）

Compute $\partial z/\partial w$ for all parameters

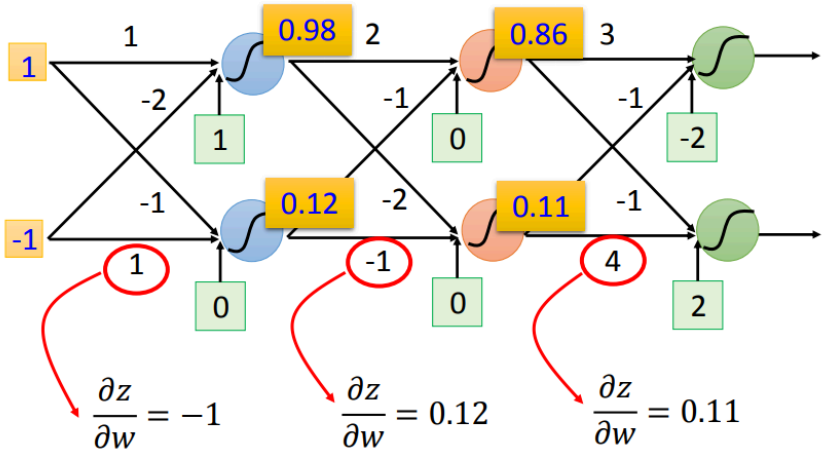


$\partial z/\partial w_1 =?\ x_1$

$\partial z/\partial w_2 =?\ x_2$

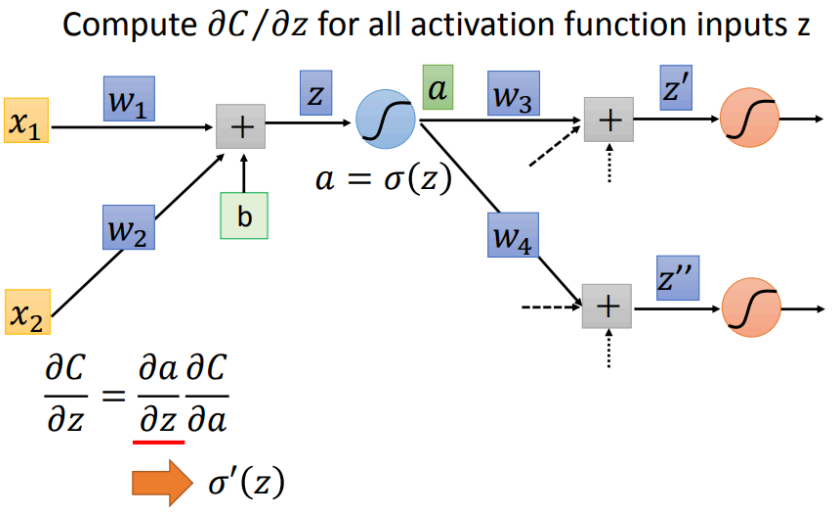
The value of the input connected by the weight

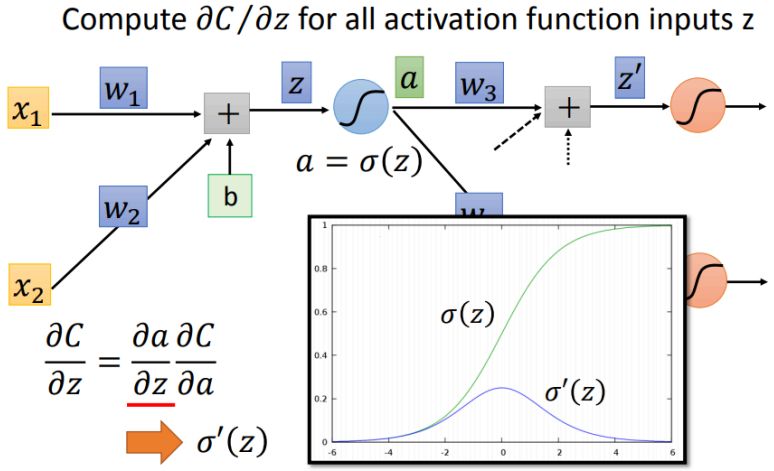
Compute $\partial z/\partial w$ for all parameters



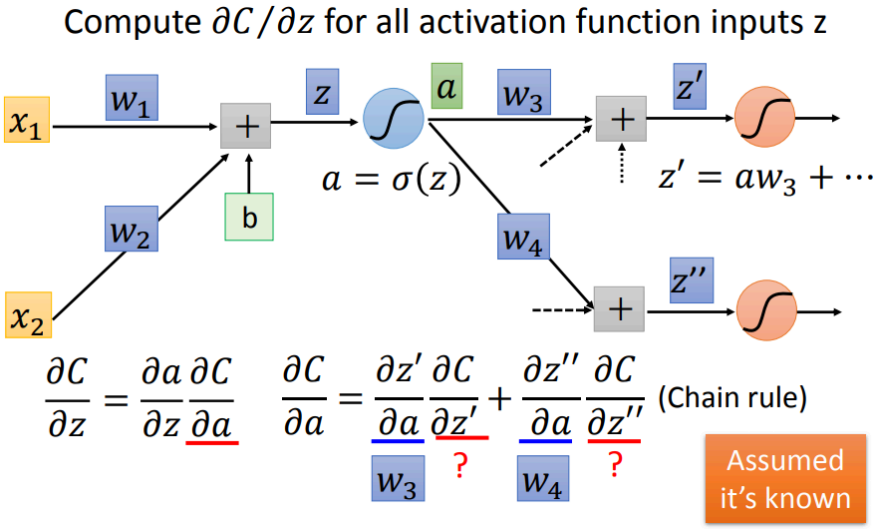
想要计算出Neural Network中每一个**w对activation function input的偏微分**，只需要把input丢进去，计算每一个neural的output

2.2 Backpropagation – pass（计算$\frac{\partial C}{\partial z}$）

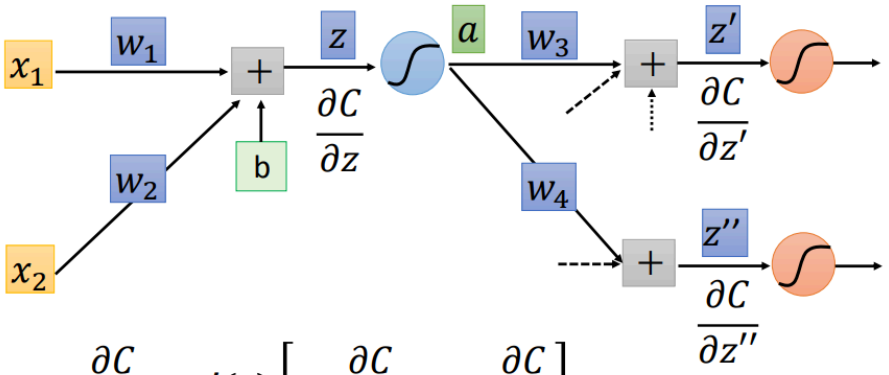




$\frac{\partial C}{\partial z}$可以拆分为$\frac{\partial a}{\partial z}\frac{\partial C}{\partial a}$，而$\frac{\partial a}{\partial z}$就是求激活函数的偏微分（右图），可以很容易的求出，所以现在的问题就是如何求$\frac{\partial C}{\partial a}$



Compute $\partial C/\partial z$ for all activation function inputs z \



$$\frac{\partial C}{\partial z} = \sigma'(z)\left[w_3\frac{\partial C}{\partial z'} + w_4\frac{\partial C}{\partial z''}\right]$$

$\frac{\partial C}{\partial a}$根据链式法则可以拆分，损失 C **不是直接由** a **决定的**，而是：$a \to z'$ ，$a \to z''$

所以：$C = C(z'(a), z''(a))$

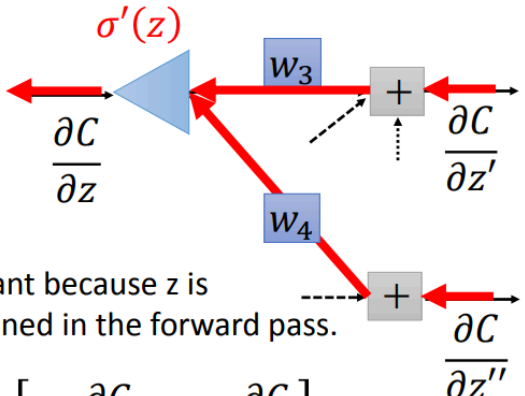
$$\boxed{\frac{\partial C}{\partial a} = \frac{\partial C}{\partial z'}\frac{\partial z'}{\partial a} + \frac{\partial C}{\partial z''}\frac{\partial z''}{\partial a}}$$

$$z'' = w_4 a + \ldots \quad \Rightarrow \quad \frac{\partial z''}{\partial a} = w_4$$

$$z' = w_3 a + \ldots \quad \Rightarrow \quad \frac{\partial z'}{\partial a} = w_3$$

$$\begin{aligned}\frac{\partial C}{\partial z} &= \frac{\partial a}{\partial z}\frac{\partial C}{\partial a} \\ &= \frac{\partial a}{\partial z}\left(\frac{\partial C}{\partial z'}\frac{\partial z'}{\partial a} + \frac{\partial C}{\partial z''}\frac{\partial z''}{\partial a}\right) \\ &= \sigma'(z)\left(w_3\frac{\partial C}{\partial z'} + w_4\frac{\partial C}{\partial z''}\right)\end{aligned}$$

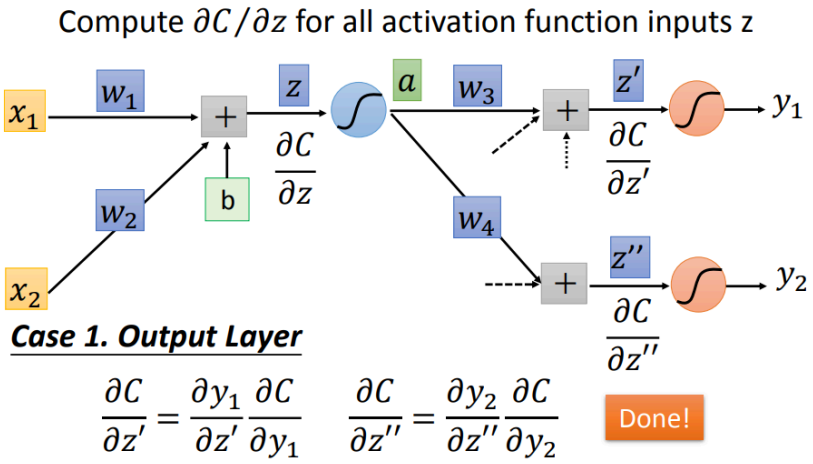
所以，相较于最开始的神经网络，我们需要依靠后面层级的数据来计算$\frac{\partial C}{\partial a}$：

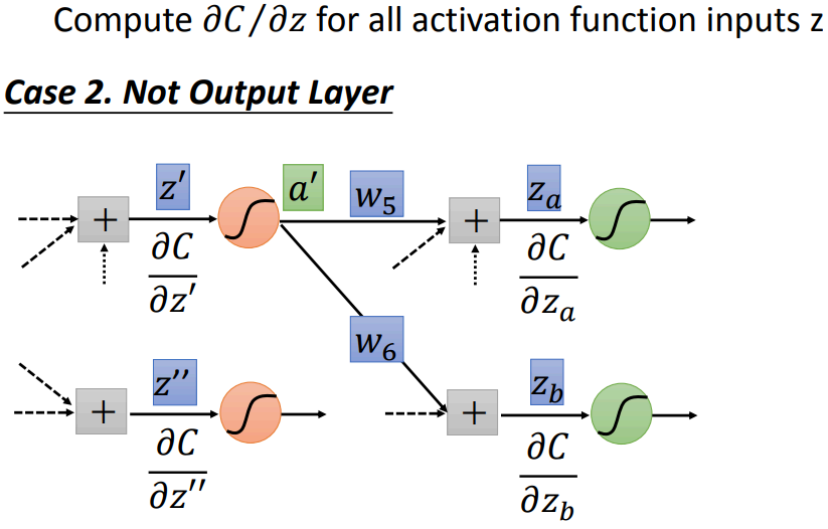


$\sigma'(z)$ is a constant because z is already determined in the forward pass.

$$\frac{\partial C}{\partial z} = \sigma'(z)\left[w_3\frac{\partial C}{\partial z'} + w_4\frac{\partial C}{\partial z''}\right]$$

后续层级的情况有两种：最终输出层、不是最终输出层

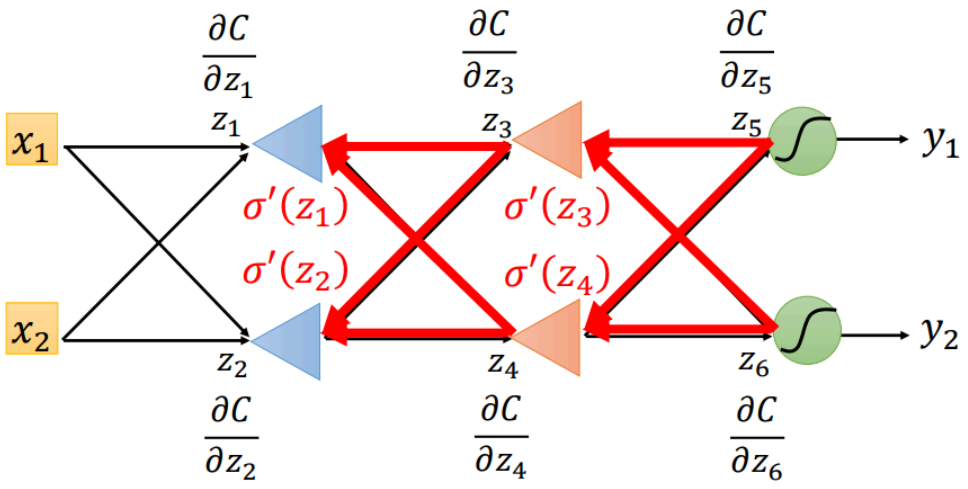




- 若后续层级是最终输出层，则该层的梯度可以直接由损失函数对输出的导数得到。
- 若后续层级不是最终输出层，则该层的梯度需要由后续层级的梯度通过链式法则反向传递得到，因此反向传播为高效计算梯度提供了统一机制

Compute $\partial C/\partial z$ for all activation function inputs z

Compute $\partial C/\partial z$ from the output layer



2.3 Backpropagation – Summary