

# Representation theory of finite groups

## Formalization project

Raphael Gaedtke, Paul Neumann

University of Bonn

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# Outline

- 1 Representation Theory
- 2 Finite abelian groups
- 3 Formalization
- 4 Mathlib
- 5 Future work

# Representations of finite groups

## Definition

For a group  $G$  and a field  $k$ , a **representation** of  $G$  over  $k$  is a pair  $(V, \rho)$  where  $V$  is a vector space over  $k$  and  $\rho : G \rightarrow \mathrm{GL}(V)$  is an action of  $G$  on  $V$ .

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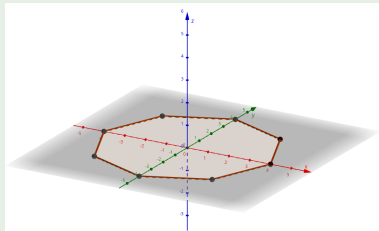
Convention:  $V$  has finite dimension, unless explicitly stated otherwise.

## Definition

$\dim(V)$  is the **dimension** or **degree** of  $(V, \rho)$ .

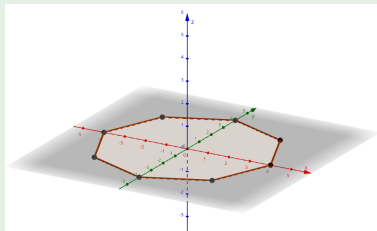
## Example

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Representation  $\rho : D_{2n} \rightarrow GL(\mathbb{R}^3)$  with

- $\rho(a)$  as rotation about the Z-axis
- $\rho(b)$  as a rotation about a suitable axis in the XY-plane

# Invariant subspaces, Irreducibility

## Definition

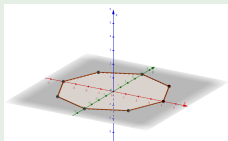
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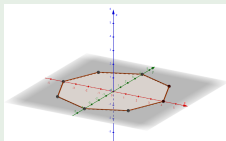


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## Example



The  $XY$ -Plane is an invariant subspace.

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- $\text{Im}(\theta)$  and  $\text{Ker}(\theta)$  are invariant subspaces
- If  $V$  and  $W$  are irreducible, then  $\theta : V \rightarrow W$  is  $0$  or bijective.

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*Let  $G$  be a finite abelian group.*

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*Then  $\rho$  is irreducible if and only if  $\dim_k(V) = 1$ .*

# Proof of the theorem

## Theorem

*$\rho$  is irreducible if and only if  $\dim_k(V) = 1$ .*

“ $\Leftarrow$ ” is trivial.

For “ $\Rightarrow$ ”, we use the following two lemmas:

## Lemma

*For all  $g \in G$ ,  $\rho(g)$  is a Representation Endomorphism.*

## Lemma

*Every Representation Endomorphism is given by multiplication with a scalar.*



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- 1 Assume  $\dim(V) > 1$ .
- 2 Then,  $V$  has a proper subspace with dimension 1.
- 3 So  $V$  has a proper invariant subspace.
- 4 This is a contradiction to the irreducibility of  $V$ .

# Representations in Mathlib

Representations are already defined in Mathlib:

```
abbrev Representation :=  
  G →* V →1[k] V  
  
end
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Apart from that, the definitions introduced in the beginning of this presentation are missing in Mathlib.



# Definitions

```
/-- A predicate for a subspace being invariant -/
def IsInvariantSubspace {k G V : Type*} [CommSemiring k]
  [Monoid G] [AddCommMonoid V] [Module k V]
  (U : Submodule k V) (ρ : Representation k G V) :=
  ∀ g : G, ∀ u : U, ρ g u ∈ U

/-- defines degree of a representation as rank of its
    module -/
def degree {k G V : Type*} [CommSemiring k] [Monoid G]
  [AddCommMonoid V] [Module k V]
  (ρ : Representation k G V) : Cardinal := (Module.rank k
    V)
```

# Representation Homomorphisms

```
-- Definition of Homomorphisms between Representations -/  
@[ext] class RepresentationHom {k G V W : Type*}  
  [CommSemiring k] [Monoid G] [AddCommMonoid V] [Module  
    k V] [AddCommMonoid W] [Module k W]  
  (ρ : Representation k G V) (ψ : Representation k G W)  
  extends LinearMap (RingHom.id k) V W where  
  reprStructure : ∀ g : G, ∀ v : V, toLinearMap (ρ g v) =  
    ψ g (toLinearMap v)
```

# Coercions

```
/-- Coercions of RepresentationHom to Function and Linear
Map -/
instance {k G V W : Type*} [CommSemiring k] [Monoid G]
  [AddCommMonoid V] [Module k V] [AddCommMonoid W]
  [Module k W] {ρ : Representation k G V} {ψ :
    Representation k G W} : CoeFun (RepresentationHom ρ ψ
  ) (fun _ ↦ V →1 [k] W) where
coe := by
  intro θ
  use ⟨θ.toFun, ?_⟩
  simp; intro x y; simp
```

# Registering Instances

## Lemma

*For all  $g \in G$ ,  $\rho(g)$  is a Representation Endomorphism.*

```
instance repr_yields_reprHom_commMonoid {k G V : Type*}
  [CommSemiring k] [CommMonoid G] [AddCommMonoid V]
  [Module k V]
  (ρ : Representation k G V) (g : G) : (RepresentationHom
    ρ ρ) where
toFun := ρ g
map_add' := by intro x y; simp
map_smul' := by intro m x; simp
reprStructure := sorry
```

# Representation Theory in Mathlib

Things already contained in Mathlib:

- Definition of Representations, basic properties, duality
- Category Theory
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Things not contained in Mathlib:

- Subrepresentations, Homomorphisms, Irreducibility
- Direct sums, reducibility, Maschke for Representations
- ...

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with action

$$kG \times V \rightarrow V, \left( \sum_{g \in G} \lambda_g g, v \right) \mapsto \sum_{g \in G} \lambda_g (gv)$$

on  $V$ .

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The theorem `asAlgebraHomdef` and the definition `Representation.asModule` translate a `Representation` to the corresponding algebra, `Representation.ofModule` is for the other direction.

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Things like the theorem of Maschke are not formulated for `Representations`, but for algebras.  
Irreducibility of representations translates to irreducibility of modules.

# Future work

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- Formulate theorem of Maschke for Representations
- Add additional theorems about Representation Theory of finite groups
- ...



# Questions?