

Representation theory of finite groups

Formalization project

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Outline

- 1 Representation Theory
- 2 Finite abelian groups
- 3 Formalization
- 4 Mathlib
- 5 Future work

Representations of finite groups

Definition

For a group G and a field k , a **representation** of G over k is a pair (V, ρ) where V is a vector space over k and $\rho : G \rightarrow \mathrm{GL}(V)$ is an action of G on V .

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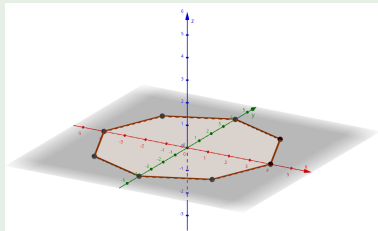
Convention: V has finite dimension, unless explicitly stated otherwise.

Definition

$\dim(V)$ is the **dimension** or **degree** of (V, ρ) .

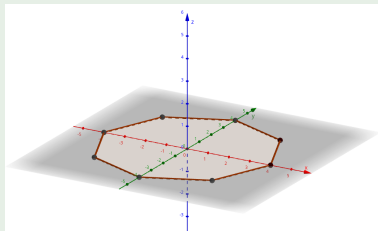
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Representation $\rho : D_{2n} \rightarrow GL(\mathbb{R}^3)$ with

- $\rho(a)$ as rotation about the Z-axis
- $\rho(b)$ as a rotation about a suitable axis in the XY-plane

Invariant subspaces, Irreducibility

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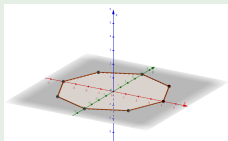
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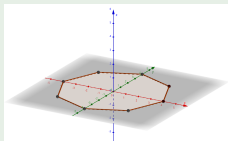


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The XY -Plane is an invariant subspace.

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- $\text{Im}(\theta)$ and $\text{Ker}(\theta)$ are invariant subspaces
- If V and W are irreducible, then $\theta : V \rightarrow W$ is 0 or bijective.

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Lemma

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Lemma

Every Representation Endomorphism is given by multiplication with a scalar.

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Now, we can use proof by contradiction:

- 1 Assume $\dim(V) > 1$.
- 2 Then, V has a proper subspace with dimension 1.
- 3 So V has a proper invariant subspace.
- 4 This is a contradiction to the irreducibility of V .

Representations in Mathlib

Representations are already defined in Mathlib:

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  G →* V →1[k] V  
  
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Apart from that, the definitions introduced in the beginning of this presentation are missing in Mathlib.

Definitions

```
/-- A predicate for a subspace being invariant -/  
def IsInvariantSubspace {k G V : Type*} [CommSemiring k]  
  [Monoid G] [AddCommMonoid V] [Module k V]  
  (U : Submodule k V) (ρ : Representation k G V) :=  
    ∀ g : G, ∀ u : U, ρ g u ∈ U  
  
/-- defines degree of a representation as rank of its  
    module -/  
def degree {k G V : Type*} [CommSemiring k] [Monoid G]  
  [AddCommMonoid V] [Module k V]  
  (ρ : Representation k G V) : Cardinal := (Module.rank k  
    V)
```

Representation Homomorphisms

```
/-- Definition of Homomorphisms between Representations -/  
@[ext] class RepresentationHom {k G V W : Type*}  
  [CommSemiring k] [Monoid G] [AddCommMonoid V] [Module  
    k V] [AddCommMonoid W] [Module k W]  
  ( $\rho$  : Representation k G V) ( $\psi$  : Representation k G W)  
  extends LinearMap (RingHom.id k) V W where  
  reprStructure :  $\forall$  g : G,  $\forall$  v : V, toLinearMap ( $\rho$  g v) =  
     $\psi$  g (toLinearMap v)
```

Coercions

```
/-- Coercions of RepresentationHom to Function and Linear
Map -/
instance {k G V W : Type*} [CommSemiring k] [Monoid G]
  [AddCommMonoid V] [Module k V] [AddCommMonoid W]
  [Module k W] {ρ : Representation k G V} {ψ :
    Representation k G W} : CoeFun (RepresentationHom ρ ψ
  ) (fun _ ↦ V →1 [k] W) where
coe := by
  intro θ
  use ⟨θ.toFun, ?_⟩
  simp; intro x y; simp
```

Registering Instances

Lemma

For all $g \in G$, $\rho(g)$ is a Representation Endomorphism.

```
instance repr_yields_reprHom_commMonoid {k G V : Type*}
  [CommSemiring k] [CommMonoid G] [AddCommMonoid V]
  [Module k V]
  (ρ : Representation k G V) (g : G) : (RepresentationHom
    ρ ρ) where
toFun := ρ g
map_add' := by intro x y; simp
map_smul' := by intro m x; simp
reprStructure := sorry
```

Representation Theory in Mathlib

Things already contained in Mathlib:

- Definition of Representations, basic properties, duality
- Category Theory
- Characters

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Things not contained in Mathlib:

- Subrepresentations, Homomorphisms, Irreducibility
- Direct sums, reducibility, Maschke for Representations
- ...

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Take a k -module with basis $\{g | g \in G\}$ and multiplication

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with action

$$kG \times V \rightarrow V, \left(\sum_{g \in G} \lambda_g g, v \right) \mapsto \sum_{g \in G} \lambda_g (gv)$$

on V .

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- `Representation.asModule` translates `Representation` to algebra
- `Representation.ofModule` translates algebra to `Representation`

Future work

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- Connect Algebras and Representations: Irreducibility, ...
- Formulate theorem of Maschke for Representations
- Add additional theorems about Representation Theory of finite groups
- ...

Questions?