Representations and characters of groups

Jan Schröer

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ABSTRACT. This is a script of a lecture course I gave in Leeds for 4th year students. I'm using it now in Bonn for a student seminar for 1st year students (2nd semester). The course is only based on 1st semester Linear Algebra. Sections II. 12,13,14 were recently added by Maurizio Martino.

CHAPTER 1

Representations: Definitions, examples, basic theory

1. Introduction

A typical situation where groups occur:

Take a geometrical object, i.e. a cube or a regular n-gon, and then take its 'symmetry group'.

For example, the **dihedral group** D_{2n} of order $2n$ is just the 'symmetry group' of the regular n-gon. We have

$$D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle.$$

Here a is a rotation of the n-gon by $2\pi/n$ degrees, and b is one of the n reflections. All other rotations and reflections are generated by a and b.

As another example, let V be a vector space over a field k. In this case, the 'symmetry group' is the **general linear group** $\mathrm{GL}(V)$ of V, which consists of all bijective k-linear maps $V \to V$.

In representation theory, we go somehow the other way round: We fix a group G, and then we want to know how G 'acts' (or 'operates') on gemetrical objects, vector spaces, etc

Besides dihedral groups and the general linear group of a vector space, let us mention some other important groups:

The **cyclic group** C_n of order n is

$$C_n = \langle a \mid a^n = 1 \rangle.$$

By S_n we denote the **symmetric group** of permutations of n elements. Thus S_n is a group of order $n!$.

By $\mathrm{GL}_n(k)$ we denote the group of invertible $n \times n$ matrices with entries in the field k.

Some conventions:

We write maps always on the left side. Thus

$$(\theta\phi)(v) = \theta(\phi(v)).$$

When applying a map θ to an element v we also sometimes write θv instead of $\theta(v)$.

Be careful to distinguish '$=$' and '$\simeq$'. The sign '$\simeq$' means 'is isomorphic to', but this does usually not imply that something is equal. For example, if $V = \mathrm{M}_2(k)$ is the vector space of 2×2 matrices with entries in k, then we get $\mathrm{GL}(V) \simeq \mathrm{GL}(k^4)$, but of course $\mathrm{GL}(V) \neq \mathrm{GL}(k^4)$.

Often we just say 'linear map' instead of 'k-linear map' if there is no danger of misunderstandings. Also we just talk about 'vector spaces' instead of 'vector spaces over k' if it's clear which field k is meant.

2. Actions

An **action** of a group G on a vector space V is a group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

Thus

$$\rho(gh) = \rho(g)\rho(h)$$

for all $g, h \in G$. This is just the definition of a group homomorphism.

Recall that this implies $\rho(1) = 1$ ($\rho(1) = \rho(11) = \rho(1)\rho(1)$, then multiply on both sides with $\rho(1)^{-1}$ and get $1 = \rho(1)$) and $\rho(g^{-1}) = \rho(g)^{-1}$ for all g ($1 = \rho(1) = \rho(gg^{-1}) = \rho(g)\rho(g^{-1})$).

LEMMA 2.1. *To give an action $\rho : G \to \mathrm{GL}(V)$ is the same as giving a map*

$$G \times V \to V, (g, v) \mapsto gv$$

such that the following hold:

(1) $g(\lambda v + \mu w) = \lambda(gv) + \mu(gw)$ *for all* $g \in G$, $v, w \in V$, $\lambda, \mu \in k$;
(2) $1v = v$ *for all* $v \in V$;
(3) $g(hv) = (gh)v$ *for all* $g, h \in G$, $v \in V$.

Proof. Given $\rho : G \to \mathrm{GL}(V)$. Define $\eta : G \times V \to V$ with $\eta(g, v) = gv := \rho(g)(v)$. One easily checks that η satisfies (1), (2) and (3):

(1): $g(\lambda v + \mu w) = \rho(g)(\lambda v + \mu w) = \lambda(\rho(g)(v)) + \mu(\rho(g)(w)) = \lambda(gv) + \mu(gw)$, since $\rho(g)$ is a linear map.

(2): $1v = \rho(1)(v) = v$, since ρ is a group homomorphism, thus $\rho(1) = 1$.

(3): $g(hv) = g(\rho(h)(v)) = \rho(g)(\rho(h)(v)) = (\rho(g)\rho(h))(v) = \rho(gh)(v) = (gh)v$, since ρ is a group homomorphism.

Vice versa: Given a map $\eta : G \times V \to V$ which satisfies (1), (2) and (3). Let $g \in G$. Define a map $\rho(g) : V \to V$ with $\rho(g)(v) = \eta(g, v) = gv$. This map is linear by (1). It is invertible since it has an inverse, namely $\rho(g^{-1})$ (to check that $\rho(g^{-1})$ is the inverse, we use (2) and (3)). Thus we get $\rho(g) \in \mathrm{GL}(V)$. Furthermore, it follows from (3) that $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$. Thus $\rho : G \to \mathrm{GL}(V)$, $g \mapsto \rho(g)$ is a group homomorphism. $\square$

Remark:

A more general setting would look as follows: If X is any mathematical object, then let $\mathrm{Aut}(X)$ be the set of bijections $X \to X$ which preserve the structure of X. An **action** of G on X is then a homomorphism $G \to \mathrm{Aut}(X)$.

For example, if X is a set, then $\mathrm{Aut}(X)$ is the group of permutations of X.

Or if X is a topological space, then $\mathrm{Aut}(X)$ is the set of homeomorphisms $X \to X$.

Or X is a complex inner product space, then $\mathrm{Aut}(X)$ is the set of invertible linear maps $\theta : X \to X$ with $(\theta(x), \theta(y)) = (x, y)$ for all $x, y \in X$.

etc

3. Representations

A **representation** of G over k is a pair (V, ρ) where V is a vector space over k and $\rho : G \to \mathrm{GL}(V)$ is an action of G on V.

Convention: V is finite-dimensional, unless explicitely stated otherwise.

$\dim(V)$ is the **dimension** or the **degree** of the representation (V, ρ).

A **real** or **complex** representation means $k = \mathbb{R}$ or $k = \mathbb{C}$, respectively.

Sometimes we just say 'a representation $\rho : G \to \mathrm{GL}(V)$' or 'a representation V' (here $\rho : G \to \mathrm{GL}(V)$ is not named but could be denoted by ρ_V).

Representations V and W are **isomorphic** (or **equivalent**) if there exists an invertible linear map

$$\theta : V \to W$$

such that

$$\theta(gv) = g\theta(v)$$

for all $g \in G$, $v \in V$.

4. Matrix representations

A **matrix representation** of G over k is a group homomorphism

$$A : G \to \mathrm{GL}_n(k).$$

It has **degree** n.

Two matrix representations $A, B : G \to \mathrm{GL}_n(k)$ are **conjugate** (or **equivalent**) if there exists some $C \in \mathrm{GL}_n(k)$ with

$$B(g) = CA(g)C^{-1}$$

for all $g \in G$.

Matrix representations give representations as follows:

Let k^n be the vector space of column vectors. Then $\mathrm{GL}(k^n)$ may be identified with $\mathrm{GL}_n(k)$ via $\theta \mapsto A$ where $\theta(v) = Av$.

Thus a matrix representation $A : G \to \mathrm{GL}_n(k)$ gives a representation $\rho_A : G \to \mathrm{GL}(k^n)$.

Vice versa, representations give matrix representations as follows:

Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Choose a basis $e_1, \cdots, e_n$ of V. Then $\mathrm{GL}(V)$ can be identified with $\mathrm{GL}_n(k)$ via $\theta \mapsto A$ where

$$\theta(e_i) = \sum_{j=1}^{n} A_{ji} e_j.$$

Here A_{ji} is the entry of A which is in the jth row and the ith column.

Thus a representation $\rho : G \to \mathrm{GL}(V)$ gives a matrix representation $A : G \to \mathrm{GL}_n(k)$ where

$$\rho(g)(e_i) = \sum_{j=1}^{n} A(g)_{ji} e_j$$

where $A(g)_{ji}$ is the jith entry of the matrix $A(g)$.

Example

Let $G = C_2 = \langle g \mid g^2 = 1 \rangle$, and let $\rho : G \to \mathrm{GL}(\mathbb{R}^2)$ be the representation with

$$g \mapsto \left(\rho(g) : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ -b \end{pmatrix} \right).$$

Now choose a basis of $\mathbb{R}^2$, for example $e_1 = \binom{2}{0}$ and $e_2 = \binom{1}{1}$. The matrix representation corresponding to ρ with respect to the basis e_1, e_2 is given by

$$G \to \mathrm{GL}_2(\mathbb{R}), \mathrm{g} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

It is easy to show that the following hold:

(1) A and B are conjugate if and only if ρ_A and ρ_B are isomorphic;
(2) As $e_1, \cdots, e_n$ runs through all bases of a representation V, the corresponding matrix representations run through a conjugacy class;
(3) Two representations are isomorphic if and only if they have the same dimension and have bases giving equal matrix representations.

Thus there is a 1-1 correspondence between conjugacy classes of matrix reprsentations $G \to \mathrm{GL}_n(k)$ and isomorphism classes of n-dimensional representations of G over k.

5. Some remarks

1) Since this course is for more advanced and mature students, we will from now on mix the 3 concepts of actions, representations and matrix representations freely.

All 3 points of view contain the same information, and it shouldn't cause any difficulties to jump from one to the other.

Often we will just talk about 'representations' and mean any of the 3 mentioned concepts.

2) There are many different 'representation theories' in mathematics. For example, representation theory of

groups, semigroups, algebras, partially ordered sets, Lie groups, Lie algebras, quivers, etc

Many of these can be translated into each other:

representation theory of groups '=' representation theory of group algebras

or

representation theory of Lie algebras '=' representation theory of universal enveloping algebras

etc

3) We will study 'only' the classical case: Complex representations of finite groups. All information is contained in the 'character table' which is just a square of numbers. The theory of characters will be developed in the second part of the course.

Classical representation theory of groups was invented by Frobenius, Burnside, Schur and others (1895 - 1910).

Representations are used for the classification of finite simple groups, one of the biggest projects in modern mathematics (1960 - 1985). For example the Fischer monster group has around 10^{54} elements, and is the biggest finite simple group. In 1973 it was conjectured that it has a 'faithful' representation of dimension 196883. In 1980 the Fischer monster was constructed as a subgroup of $\mathrm{GL}_{196883}(\mathbb{C})$.

Representations occur in ALL branches of mathematics (maybe other representation theories (see above), maybe infinite groups, or infinite dimensional vector spaces, or other fields than $\mathbb{C}$). But: The theory ALWAYS is build on the classical case.

There are applications of representation theory in

Quantum Mechanics, Dynamical Systems, Lie Theory, Number Theory (Understanding representations

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL_n}(\mathbb{F}_\mathrm{p})$$

is central for Wile's proof of Fermat's Last Theorem), Chemistry, etc

6. Examples

Let G be a group.

1) The **zero representation** of G: Take $V = 0$. Thus $\mathrm{GL}(V) = \{1\}$. The corresponding homomorphism is

$$G \to \mathrm{GL}(V), g \mapsto 1$$

which corresponds to the map

$$G \times V \to V, (g, 0) \mapsto 0.$$

2) The **trivial representation** of G over k: $V = k$ (so $\dim(V) = 1$) and maps

$$G \to \mathrm{GL}(V), g \mapsto 1$$

or

$$G \times V \to V, (g, \lambda) \mapsto \lambda$$

or (as a matrix representation)

$$G \to \mathrm{GL}_1(k), g \mapsto (1)$$

where (1) is the 1×1 matrix with entry 1.

3) The **regular representation** of G over k: Let V be a vector space over k with basis $\{e_g \mid g \in G\}$. Define

$$\rho : G \to \mathrm{GL}(V), g \mapsto \left(\rho(g) : \sum_{h \in G} \lambda_h e_h \mapsto \sum_{h \in G} \lambda_h e_{gh} \right)$$

or

$$G \times V \to V, (g, \sum_{h \in G} \lambda_h e_h) \mapsto \sum_{h \in G} \lambda_h e_{gh}$$

where $\lambda_h \in k$ for all $h \in G$, and all but finitely many λ_h are 0 (G might be an infinite group and there are no infinite sums in mathematics). The regular representation of G over k is denoted by kG.

4) **Permutation representations**: Suppose G acts on a set X, i.e. we have a homomorphism

$$\pi : G \to \mathrm{Symm}(X)$$

where $\mathrm{Symm}(X)$ is the group of permutations of X. Let V be a vector space with basis $\{e_x \mid x \in X\}$. It becomes a representation via

$$ge_x = e_{\pi(g)(x)}.$$

Note that this short notation contains all information. It is enough to know how G acts on the basis vectors of V, then we extend this action linearly. A more precise (but also more lengthy) notation is

$$\rho : G \to \mathrm{GL}(V), g \mapsto \left(\rho(g) : \sum_{x \in X} \lambda_x e_x \mapsto \sum_{x \in X} \lambda_x e_{\pi(g)(x)} \right).$$

If $G = X$ with G acting on itself by left multiplication, then this is the regular representation.

5) A permutation $\pi \in S_n$ has a signiture $\epsilon(\pi) \in \{-1, 1\}$ (where $\epsilon(\pi) = 1$ if π is a product of an even number of transpositions (thus elements of the form $(i, i+1)$), and $\epsilon(\pi) = -1$, else). This gives a matrix representation

$$\epsilon : S_n \to \mathrm{GL}_1(k), \pi \mapsto (\epsilon(\pi)).$$

6) The symmetry group of a cube is S_4. Let $X \subset \mathbb{R}^3$ be a cube centred at the origin. Number the long diagonals $\{1, \cdots, 4\}$. We get a real representation

$$\rho : S_4 \to \mathrm{GL}(\mathbb{R}^3)$$

with $\rho(\pi)$ the unique rotation with $\rho(\pi)(X) \subseteq X$ which acts as the permutation π on the diagonals.

7) The symmetry group of a regular n-gon is

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle.$$

Let $X \subset \mathbb{R}^3$ be a regular n-gon at the origin in the XY-plane. We get a real representation

$$\rho : D_{2n} \to \mathrm{GL}(\mathbb{R}^3)$$

with $\rho(a)$ a rotation by $2\pi/n$ about the Z-axis, and $\rho(b)$ is a rotation by π about a suitable axis in the XY-plane.

8) Example of 2 conjugate matrix representations: Let $G = D_8 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$. Define

$$\rho : G \to \mathrm{GL}_2(\mathbb{C}),$$

$$a \mapsto A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b \mapsto B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, $a^i b^j \mapsto A^i B^j$ where $0 \leq i \leq 3$ and $0 \leq j \leq 1$.

Now let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

This matrix is invertible with inverse

$$T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

and we get

$$TAT^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, TBT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This define a new representation

$$\sigma : G \to \mathrm{GL}_2(\mathbb{C}),$$

$$a \mapsto C = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, b \mapsto D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and we know that ρ and σ are conjugate matrix representations. Note that it is enough to specify what the image of the generators a and b are.

Of course, we should still check whether ρ is really a matrix representation of G. Thus we have to check that $A^4 = B^2 = I$ and $BAB = A^{-1}$.

9) Let $G = \langle a \mid a^2 = 1 \rangle$. Then

$$\rho : G \to \mathrm{GL}_2(\mathbb{C}), \mathrm{a} \mapsto \mathrm{A} = \begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix}$$

is a matrix representation. We have to check that $A^2 = I$ to see that ρ is a representation. One can show that ρ is conjugate to the matrix representation

$$\sigma : G \to \mathrm{GL}_2(\mathbb{C}), \mathrm{a} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

7. Invariant subspaces

Let V be a representation. A subspace $U \subseteq V$ is an **invariant subspace** (or **sub-representation**) if $gu \in U$ for all $u \in U$, $g \in G$.

In this case, the following hold:

(1) U becomes a representation via restriction

$$G \times U \to U.$$

(2) V/U becomes a representation, the **quotient representation**, via
$$g(v+U) = gv + U$$
Note that this map is well defined: Namely, $v+U = v'+U \Rightarrow v - v' \in U \Rightarrow gv - gv' = g(v-v') \in U \Rightarrow gv + U = gv' + U$.

(3) If U and W are invariant subspaces, then $U \cap W$ and $U + W$ are invariant subspaces.

Examples:

1) Take the real representation $D_{2n} \to \mathrm{GL}(\mathrm{V} = \mathbb{R}^3)$ of symmetries of the regular n-gon (as in Section 6).

The XY-plane is an invariant subspace U.

The quotient representation $D_{2n} \to \mathrm{GL(V/U)}$ sends the rotations of the n-gon to 1, and the reflections to -1.

Also the Z-axis is an invariant subspace.

2) Let G act on a set X, and let V be the permutation representation with basis $\{e_x \mid x \in X\}$.

Then the set $\{e_x - e_y \mid x, y \in X\}$ spans an invariant subspace.

3) If V is any representation, then
$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$$
is an invariant subspace.

Matrix version of invariant subspaces:

Let V be a representation, and let U be a subspace of V.

Choose a basis $e_1, \cdots, e_r$ of U, and extend it to a basis $e_1, \cdots, e_r$, $e_{r+1}, \cdots, e_n$ of V. Let
$$A : G \to \mathrm{GL_n(k)}$$
be the corresponding matrix represenation. Then U is an invariant subspace of V if and only if for all $g \in G$ we have
$$A(g) = \begin{pmatrix} B(g) & D(g) \\ 0 & C(g) \end{pmatrix}.$$
The representation U corresponds to $B : G \to \mathrm{GL_r(k)}$ via the basis $e_1, \cdots, e_r$.

The quotient representation V/U corresponds to $C : G \to \mathrm{GL_{n-r}(k)}$ via the basis $e_{r+1} + U, \cdots, e_n + U$ of V/U.

A representation V is **irreducible** (or **simple**) provided $V \neq 0$ and the only invariant subspaces are 0 and V.

Examples:

1) Any 1-dimensional representation is irreducible.

2) The real representation $S_4 \to \mathrm{GL}(\mathbb{R}^3)$ of symmetries of a cube is irreducible, since no line or plane of $\mathbb{R}^3$ is invariant.

8. Homomorphisms

If V and W are representations, then a **homomorphism** from V to W is a linear map $\theta : V \to W$ such that $\theta(gv) = g\theta(v)$ for all $g \in G$, $v \in V$.

The set of all homomorphisms from V to W is denoted by $\mathrm{Hom}_G(V, W)$. This is a subspace of the vector space $\mathrm{Hom}_k(V, W)$ of linear maps $V \to W$.

An **isomorphism** is an invertible homomorphism. In this case, the inverse is again a homomorphism.

An **endomorphism** of V is a homomorphism $V \to V$. The set of all endomorphisms of V is denoted by $\mathrm{End}_G(V)$. It is a ring with multiplication given by composition of endomorphisms.

Reminder: A **ring** $R = (R, +, \cdots)$ consists of a set R and two maps $+ : R \times R \to R$, $(r, s) \mapsto r + s$ and $\cdot : R \times R \to rs$ such that the following hold:

(1) $(R, +)$ is an abelian group. We denote the identity element by 0;
(2) $a(bc) = (ab)c$ for all $a, b, c \in R$;
(3) There exists an element $1 \in R$ such that $1 \neq 0$ and $1r = r1 = r$ for all $r \in R$;
(4) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Compare the definition of a ring with the definition of a field: Take the definition of a field and delete the conditions on the commutativity of the multiplication and the existence of an inverse with respect to the multiplication. Then we get the definition of a ring.

If $\theta : V \to W$ is a homomorphism, then $\mathrm{Im}(\theta)$ is an invariant subspace of W, and $\mathrm{Ker}(\theta)$ is an invariant subspace of V, and θ induces an isomorphism $V/\mathrm{Ker}(\theta) \to \mathrm{Im}(\theta)$.

Lemma 8.1. *If V and W are irreducible, then any homomorphism $\theta : V \to W$ is either 0 or an isomorphism.*

Proof. If $\theta \neq 0$, then $\mathrm{Im}(\theta) = W$ and $\mathrm{Ker}(\theta) = 0$. □

Corollary 8.2 (Schur's Lemma (Version 1)). *The endomorphism ring of an irreducible representation is a division ring, i.e. every non-zero endomorphism has an inverse.*

Corollary 8.3 (Schur's Lemma (Version 2)). *Every endomorphism of an irreducible complex representation V is given by multiplication by a scalar. Thus $\mathrm{End}_G(V) \simeq \mathbb{C}$.*

Proof. Let θ be an endomorphism of V. Now θ has an eigenvalue $\lambda \in \mathbb{C}$ with eigenvector v. The endomorphism $\theta - \lambda 1$ of V is not an isomorphism, since $(\theta - \lambda 1)(v) = 0$. Thus $\theta - \lambda 1 = 0$ by the above lemma. Thus $\theta = \lambda 1$. □

Matrix version of homomorphisms:

If $A : G \to \mathrm{GL_n(k)}$ and $B : G \to \mathrm{GL_m(k)}$ are matrix representations corresponding to V and W, then $\mathrm{Hom}_G(V, W)$ corresponds to

$$\{\theta \in \mathrm{M}_{m\times n}(k) \mid B(g)\theta = \theta A(g) \text{ for all } g \in G\}.$$

9. Direct sums

If U and W are representations, then their **direct sum** $U \oplus W$ becomes a representation via $g(u, w) = (gu, gw)$.

One checks easily that the following hold:

(1) If V is a representation with invariant subspaces U and W such that $U \oplus W = V$, then V is isomorphic to the direct sum of the representations U and W;
(2) $U \simeq (U \oplus W)/W$;

(3) If $U_1, \cdots, U_n$ and $V_1, \cdots, V_m$ are representations, then

$$\mathrm{Hom}_G\left(\bigoplus_{i=1}^{n} U_i, \bigoplus_{j=1}^{m} V_j\right) \simeq \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} \mathrm{Hom}_G(U_i, V_j).$$

The isomorphism is given by $\theta \mapsto (\theta_{ij})_{ij}$ where

$$\theta(u_1, \cdots, u_n) = (\theta_{11}(u_1) + \cdots + \theta_{1n}(u_n), \cdots, \theta_{m1}(u_1) + \cdots + \theta_{mn}(u_n)).$$

Matrix version of direct sums:

Suppose U corresponds to the matrix representation $B : G \to \mathrm{GL_r(k)}$ with respect to a basis $e_1, \cdots, e_r$ of U. Suppose W corresponds to the matrix representation $C : G \to \mathrm{GL_{n-r}(k)}$ with respect to a basis $e_{r+1}, \cdots, e_n$ of W. Then $U \oplus W$ corresponds to the matrix representation $A : G \to \mathrm{GL_n(k)}$ where

$$A(g) = \begin{pmatrix} B(g) & 0 \\ 0 & C(g) \end{pmatrix}$$

with respect to the basis $e_1, \cdots, e_n$ of $U \oplus W$.

10. Complete reducibility

A representation V is **completely reducible** (or **semisimple**) if any invariant subspace $U \subseteq V$ has an invariant complement W, i.e. $U \oplus W = V$.

Lemma 10.1. *Every sub-representation and quotient representation of a completely reducible representation is completely reducible.*

Proof. Sub-representation: If $U \subseteq V$ is a sub-representation, and $X \subseteq U$ is an invariant subspace, then X has an invariant complement W in V. We have $U = X \oplus (W \cap U)$. (If $u = x + w$ with $x \in X$ and $w \in W$, then $u - x \in W \cap U$). Note that $W \cap U$ is invariant.

Quotient representation: Say $U \subseteq V$ is invariant. Then it has an invariant complement W. By the first part, W is also completely reducible. Now $V/U \simeq W$, so V/U is completely reducible. □

Theorem 10.2. *The following are equivalent:*

(1) V *is completely reducible;*
(2) V *is a direct sum* $V_1 \oplus \cdots V_r$ *of irreducible sub-representations;*
(3) V *is the sum of all its irreducible sub-representations.*

Proof. (1) $\Rightarrow$ (2): Proof by induction on $\dim(V)$. If V is irreducible, then there is nothing to do. If not, then $V = U \oplus W$ with $U, W \neq 0$. By induction U and W are direct sums of irreducible representations, hence so is V.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let U be an invariant subspace of V. Let W be an invariant subspace with $U \cap W = 0$ and W of maximal dimension with this property. Suppose for a contradiction that $U + W \neq V$. Then there is an irreducible sub-representation Z of V which is not contained in $U + W$. Now $Z + W \neq W$, and we have $U \cap (Z + W) = 0$. (We have $Z \cap (U + W) = 0$, since this is a proper invariant subspace of Z and Z is irreducible. If $u \in U \cap (Z + W)$, then $u = z + w$ with $z \in Z$ and $w \in W$. So $z = u - w \in Z \cap (U + W) = 0$. Thus $u = w \in U \cap W = 0$.) A contradiction to the maximality of W. □

Corollary 10.3. *A direct sum of completely reducible representations is completely reducible.*

Lemma 10.4. *If $V \simeq V_1 \oplus \cdots \oplus V_r$ with V_i irreducible for all i, and U irreducible, then the following numbers are equal:*

1. *The number of V_i isomorphic to U;*
2. $\dim \mathrm{Hom}_G(U, V)/\dim \mathrm{End}_G(U)$*;*
3. $\dim \mathrm{Hom}_G(V, U)/\dim \mathrm{End}_G(U)$*.*

Proof. We have

$$\dim \mathrm{Hom}_G(U, V) = \sum_{i=1}^{r} \dim \mathrm{Hom}_G(U, V_i)$$

and $\mathrm{Hom}_G(U, V_i) = \mathrm{End}_G(U)$ if $U \simeq V_i$, and $\mathrm{Hom}_G(U, V_i) = 0$, else. Similarly

$$\dim \mathrm{Hom}_G(V, U) = \sum_{i=1}^{r} \dim \mathrm{Hom}_G(V_i, U).$$

□

If U is irreducible and V is completely reducible, then the **multiplicity** $[V : U]$ of U in V is the number of V_i isomorphic to U in any decomposition $V \simeq V_1 \oplus \cdots \oplus V_r$ with V_i irreducible for all i. By the above lemma, this does not depend on the decomposition.

If V is a representation and $n \geq 0$, then let V^n be the direct sum of n copies of V. In particular $V^0 = 0$.

By a **complete set of irreducible complex representations** of a finite group G, we mean a set U_i, $i \in I$, of irreducible complex representations, which are pairwise non-isomorphic and any other irreducible complex representation is isomorphic to some U_i.

COROLLARY 10.5. *Let U_i, $i \in I$, be a complete set of irreducible complex representations. If V is completely reducible, then*

$$V \simeq \bigoplus_{i \in I} U_i^{[V:U_i]}.$$

COROLLARY 10.6. *Let U_i, $i \in I$, be a complete set of irreducible complex representations. Let V and W be completely reducible. Then*

$$\dim \operatorname{Hom}_G(V, W) = \sum_{i \in I} [V : U_i][W : U_i] \dim \operatorname{End}_G(U_i).$$

11. Maschke's Theorem

THEOREM 11.1 (Maschke). *Let G be a finite group, and let k be a field with* $\operatorname{char}(k) = 0$. *Then any representation of G over k is completely reducible.*

Proof. Let U be an invariant subspace. Write $V = U \oplus W$ with W not necessarily invariant. Let $\theta : V \to V$ be the projection onto U. So $\theta(u) = u$ and $\theta(w) = 0$ for all $u \in U$, $w \in W$. Define $f : V \to V$ via

$$f(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \theta(gv).$$

We claim that $f \in \operatorname{End}_G(V)$: Clearly, f is a linear map. For $h \in G$ we have

$$f(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \theta(ghv).$$

Set $x = gh$. Thus $g^{-1} = hx^{-1}$. So we get

$$f(hv) = \frac{1}{|G|} \sum_{g \in G} hx^{-1} \theta(xv) = hf(v).$$

Thus f is an endomorphism of V. We have $\mathrm{Im}(f) = U$: Namely, $\mathrm{Im}(f) \subseteq U$ since each term in the sum is in U. If $u \in U$, then

$$f(u) = \frac{1}{|G|}\sum_{g\in G} g^{-1}\theta(gu) = \frac{1}{|G|}\sum_{g\in G} g^{-1}gu = \frac{1}{|G|}\sum_{g\in G} u = u.$$

Clearly, this implies $U \cap \mathrm{Ker}(f) = 0$: Namely, if $0 \neq u \in U \cap \mathrm{Ker}(f)$, then $f(u) = u = 0$, a contradiction.

We have $V/\mathrm{Ker}(f) \simeq \mathrm{Im}(f) = U$. Thus $\dim \mathrm{Ker}(f) + \dim \mathrm{Im}(f) = \dim(V)$. This implies

$$V = \mathrm{Im}(f) \oplus \mathrm{Ker}(f) = U \oplus \mathrm{Ker}(f),$$

and $\mathrm{Ker}(f)$ is an invariant subspace. Now choose U an irreducible invariant subspace. We get $V = U \oplus \mathrm{Ker}(f)$. By induction on $\dim(V)$, $\mathrm{Ker}(f)$ is completely reducible. Thus V is completely reducible. $\square$

Remark: The proof of Maschke's Theorem works also if the characteristic of k is non-zero and does not divide the order of G.

12. Decomposition of the regular representation over $\mathbb{C}$

In this section, let G be a finite group, and let $k = \mathbb{C}$. Recall that we denoted the regular representation of G by $\mathbb{C}G$. By Maschke's Theorem we know that $\mathbb{C}G$ is completely reducible. What are the multiplicities of the irreducible direct summands of $\mathbb{C}G$?

Lemma 12.1. *If V is any representation, then there is an isomorphism of vector spaces $\mathrm{Hom}_G(\mathbb{C}G, V) \simeq V$ with $v \in V$ corresponding to the homomorphism $\theta_v : \mathbb{C}G \to V$ defined by $\theta_v(g) = gv$ (and then extend linearly).*

Proof. For any $v \in V$, θ_v is clearly a homomorphism. Conversely, if θ is any homomorphism, let $v = \theta(1)$. To obtain the isomorphism we need to check that these are inverses:

$$V \to \mathrm{Hom}_G(\mathbb{C}G, V) \to V, v \mapsto \theta_v \mapsto \theta_v(1) = 1v = v$$

and

$$\mathrm{Hom}_G(\mathbb{C}G, V) \to V \to \mathrm{Hom}_G(\mathbb{C}G, V), \theta \mapsto \theta(1) \mapsto \theta_{\theta(1)}$$

and we have $\theta_{\theta(1)}(g) = g\theta(1) = \theta(g)$. This finishes the proof. $\square$

Theorem 12.2. *If U is irreducible, then $[\mathbb{C}G : U] = \dim(U) \neq 0$.*

Proof. We have $\dim \mathrm{Hom}_G(\mathbb{C}G, U) = \dim(U)$ by the previous lemma, and we have $\dim \mathrm{End}_G(U) = 1$ by Schur's Lemma. Now we use the lemma preceeding the definition of $[V : U]$. □

COROLLARY 12.3. *Up to isomorphism there are only finitely many irreducible complex representations of G.*

Proof. Write $\mathbb{C}G = V_1 \oplus \cdots \oplus V_r$ as a direct sum of irreducibles. Any irreducible U is isomorphic to some V_i since $[\mathbb{C}G : U] \neq 0$. □

Remark: The previous corollary holds for any field k, but one needs a different proof for the general case.

COROLLARY 12.4. *Let $U_1, \cdots, U_r$ be a complete set of irreducible complex representations of G. Then*

$$|G| = \sum_{i=1}^{r} (\dim(U_i))^2$$

Proof. We have

$$\mathbb{C}G \simeq \bigoplus_{i=1}^{r} U_i^{[\mathbb{C}G:U_i]}.$$

Then use $[\mathbb{C}G : U_i] = \dim(U_i)$ for all i. □

13. Group algebras and Wedderburn's Theorem

Let G be a group, and let kG be the vector space over k with basis $\{g \mid g \in G\}$. Then kG becomes a ring via

$$(\sum_g \lambda_g g)(\sum_h \mu_h h) = \sum_{g,h} \lambda_g \mu_h gh.$$

We call kG the **group ring** of G over k. In fact, kG is even a k-algebra. So often kG is also called the **group algebra** of G over k.

If V is a representation of G, then the group ring kG 'acts' also on V via

$$kG \times V \to V, \left(x = \sum_g \lambda_g g, v \right) \mapsto xv = \sum_g \lambda_g (gv).$$

In other words, V becomes a 'kG-module' (see books for a precise definition).

By $M_n(k)$ we denote the ring of $n \times n$ matrices with entries in k. If V is a vector space over k, then $\mathrm{End}_k(V)$, the set of linear maps $V \to V$ is a ring, where the multiplication is given by composition of maps.

If $\dim(V) = n$, then $\mathrm{End}_k(V) \simeq M_n(k)$ as a ring.

If R and S are rings, then so is $R \oplus S = \{(r, s) \mid r \in R, s \in S\}$ via $(r, s)(r', s') = (rr', ss')$. The identity element of $R \oplus S$ is $(1, 1)$.

THEOREM 13.1 (Wedderburn). *If G is a finite group, then*

$$\mathbb{C}G \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

as a ring, and the irreducible complex representations of G have dimensions $n_1, \cdots, n_r$. More precisely, if $U_1, \cdots, U_r$ is a complete set of irreducible complex representations of G, then the map

$$F : \mathbb{C}G \to \mathrm{End}_{\mathbb{C}}(U_1) \oplus \cdots \oplus \mathrm{End}_{\mathbb{C}}(U_r)$$

with

$$x \mapsto (\eta_1, \cdots, \eta_r)$$

where $\eta_i(u) = xu$ for all $u \in U_i$ is an isomorphism of rings.

Proof. Clearly, F is a linear map. First, we show that F is injective: Suppose that $F(x) = 0$. Then $xu = 0$ for all $u \in U_i$ and for all i. By Maschke's Theorem, $\mathbb{C}G$ is isomorphic to a direct sum of copies of the U_i. So $xy = 0$ for all $y \in \mathbb{C}G$. In particular, for $y = 1$, we get $x1 = 0$. Thus $x = 0$.

Next, we claim that F is invertible: This follows by comparing dimensions. Namely,

$$\mathbb{C}G \simeq \bigoplus_{i=1}^{r} U_i^{\dim(U_i)}$$

Thus

$$\dim(\mathbb{C}G) = \sum_{i=1}^{r} (\dim(U_i))^2.$$

On the other hand, we have

$$\dim \mathrm{End}_{\mathbb{C}}(U_i) = (\dim(U_i))^2.$$

Using this and the injectivity of F, we get that F is invertible.

Finally, one easily checks that F is an isomorphism of rings: Check that $F(1) = 1$, $F(x + x') = F(x) + F(x')$ and $F(xx') = F(x)F(x')$ for all $x, x' \in \mathbb{C}G$. This finishes the proof. $\square$

If R is a ring, then its **centre** is the ring $Z(R) = \{r \in R \mid rs = sr \text{ for all } s \in R\}$.

LEMMA 13.2. *We have*

$$Z(\mathbb{C}G) = \{\sum_g \lambda_g g \in \mathbb{C}G \mid g \textit{ conjugate to } h \Rightarrow \lambda_g = \lambda_h\}.$$

Thus a basis of $Z(\mathbb{C}G)$ is given by the 'class sums', i.e. sums of the form $g_1 + \cdots + g_n$ where $\{g_1, \cdots, g_n\}$ is a conjugacy class of G.

Proof. We have $\sum \lambda_g g \in Z(\mathbb{C}G)$ if and only if $\sum \lambda_g g = x^{-1}(\sum \lambda_g g)x$ for all $x \in G$. If $h \in G$, then the coefficient of h in the LHS of this equation is λ_h, and the coefficient in the RHS is $\lambda_{xhx^{-1}}$. This implies the result. $\square$

LEMMA 13.3. *$Z(R \oplus S) = Z(R) \oplus Z(S)$.*

LEMMA 13.4. *We have $Z(M_n(\mathbb{C})) = \mathbb{C}I$. If V is a vector space over $\mathbb{C}$, then $Z(\mathrm{End}_{\mathbb{C}}(V)) = \mathbb{C}1$.*

Proof. The multiples of the identity matrix are central elements. If A is central, then use the fact that $AE_{ij} = E_{ij}A$ for all i, j to prove that $A \in \mathbb{C}I$. Here E_{ij} is the matrix with ijth entry 1, and 0 else. $\square$

THEOREM 13.5. *If $U_1, \cdots, U_r$ is a complete set of irreducible complex representations of G, then $Z(\mathbb{C}G) \simeq \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (r copies).*

Proof. We have $\mathbb{C}G \simeq \mathrm{End}_C(U_1) \oplus \cdots \oplus \mathrm{End}_{\mathbb{C}}(U_r)$ by the Wedderburn Theorem. Then use the previous 2 lemmas. $\square$

COROLLARY 13.6. *If G is a finite group, then the following numbers are equal:*

(1) $\dim Z(\mathbb{C}G)$;
(2) *The number of isomorphism classes of irreducible complex representations of G;*
(3) *The number of conjugacy classes of G.*

14. Representations of finite abelian groups

Reminder: If G and H are groups, then $G \times H = \{(g, h) \mid g \in G, h \in H\}$ is again a group via $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1, h_2)$.

We state the following theorem without proof:

THEOREM 14.1. *Let G be a finite abelian group, then*

$$G \simeq C_{n_1} \times \cdots \times C_{n_l}$$

for some $n_i \geq 1$.

THEOREM 14.2. *A group G is abelian if and only if every irreducible complex representation is 1-dimensional.*

Proof. Let $U_1, \cdots, U_r$ be a complete set of irreducible complex representations of G.

By the Wedderburn Theorem we have

$$\mathbb{C}G \simeq \mathrm{End}_{\mathbb{C}}(U_1) \oplus \cdots \oplus \mathrm{End}_{\mathbb{C}}(U_r).$$

Thus the ring $\mathbb{C}G$ is commutative if and only if $\dim(U_i) = 1$ for all i. But G is commutative if and only if $\mathbb{C}G$ is commutative. □

An alternative proof is the following:

Proof. G is abelian if and only if $r = |G|$ (since r is the number of conjugacy classes of G). But $r = |G|$ if and only if $\dim(U_i) = 1$ for all i (since $|G| = \sum_{i=1}^{r}(\dim(U_i))^2$). □

Now let $G = C_{n_1} \times \cdots \times C_{n_l}$ and let $g_i = (1, \cdots, 1, c_i, 1, \cdots, 1)$ with c_i a generator of C_{n_i} being the ith entry. Then

$$G = \langle g_1, \cdots, g_l \mid g_i^{n_i} = 1, g_i g_j = g_j g_i \text{ for all } i, j \rangle.$$

Let $\rho : G \to \mathrm{GL_n}(\mathbb{C})$ be an irreducible representation. Thus after the previous theorem we have $n = 1$, and $\rho(g_i) = (\lambda_i)$ where (λ_i) is a 1×1 matrix for all i.

The order of g_i is n_i. So $\rho(1) = \rho(g_i^{n_i}) = \rho(g_i)^{n_i} = (\lambda_i^{n_i}) = (1)$. Thus $\lambda_i^{n_i} = 1$. So λ_i is an n_ith root of 1. Also observe that the values $\lambda_1, \cdots, \lambda_l$ determine ρ. Since for $g = g_1^{i_1} \cdots g_l^{i_l}$ we have $\rho(g) = (\lambda_1^{i_1} \cdots \lambda_l^{i_l})$. We denote such a representation by $\rho_{\lambda_1 \cdots \lambda_l}$.

Vice versa, for any λ_i an n_ith root of 1, $1 \leq i \leq l$, the function $g_1^{i_1} \cdots g_l^{i_l} \mapsto (\lambda_1^{i_1} \cdots \lambda_l^{i_l})$ is an irreducible representation of G. There are $n_1 n_2 \cdots n_l$ such representations. One can show easily that no two of these are equivalent.

15. Diagonalization

Theorem 15.1. *Let G be a finite group, let $g \in G$, and let $\rho : G \to \mathrm{GL(V)}$ be an n-dimensional representation of G over $\mathbb{C}$. Then there exists a basis of V such that $A(g)$ is a diagonal matrix, where $A : G \to \mathrm{GL_n}(\mathbb{C})$ is the corresponding matrix representation with respect to this basis.*

Proof. Let $H = \{g^i \mid i \geq 0\}$. This is a finite abelian subgroup of G. If we restrict the homomorphism ρ to the subgroup H, then we get a homomorphism $\rho' : H \to \mathrm{GL(V)}$. So we can regard V also as a representation of H over $\mathbb{C}$. By Maschke's Theorem we can decompose V as a direct sum, say $V = U_1 \oplus \cdots \oplus U_n$, with U_i irreducible complex representations of H for all i. By the previous section, we know that $\dim(U_i) = 1$ for all i (since H is abelian). Let u_i be a basis of U_i (just take any non-zero vector in U_i). Then $u_1, \cdots, u_n$ form a basis of V, and the corresponding matrix representation $A' : H \to \mathrm{GL_n}(\mathbb{C})$ has the property that $A'(g)$ is a diagonal matrix. But A' is only the restriction to H of the matrix representation $A : G \to \mathrm{GL_n}(\mathbb{C})$ corresponding to V with respect to the basis $u_1, \cdots, u_n$. So $A(g)$ is a diagonal matrix. $\square$

In the above theorem and its proof, note that the entries of the diagonal matrix $A(g)$ are lth roots of 1, where l is the order of H (or equivalently the order of g).

Warning: The previous theorem does NOT imply that we can always construct a matrix representation $A : G \to \mathrm{GL_n}(\mathbb{C})$ corresponding to ρ such that $A(g)$ is a diagonal matrix for ALL $g \in G$.

CHAPTER 2

Characters of groups

In this chapter, we always assume that G is a finite group, and that $k = \mathbb{C}$ is the field of complex numbers. All representations will be complex representations.

1. Trace of a matrix

Let $A = (a_{ij}) \in M_n(k)$ be a matrix. Then the **trace** of A is

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

LEMMA 1.1. *Let $A, B \in M_n(k)$. Then the following hold:*

(1) $\mathrm{tr}(I_n) = n$.
(2) $\mathrm{tr}(\lambda A + \mu B) = \lambda \mathrm{tr}(A) + \mu \mathrm{tr}(B)$ *for all* $\lambda, \mu \in k$.
(3) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
(4) *If* $T \in \mathrm{GL_n(k)}$, *then* $\mathrm{tr}(TAT^{-1}) = \mathrm{tr}(A)$.

Warning: In general we have $\mathrm{tr}(AB) \neq \mathrm{tr}(A)\mathrm{tr}(B)$.

If V is a vector space over k and $\theta \in \mathrm{End}_k(V)$, then $\mathrm{tr}(\theta)$ is the trace of the matrix of θ with respect to any basis of V. This does not depend on the chosen basis.

2. Characters

Let $\rho : G \to \mathrm{GL(V)}$ be a representation. Then the **character of** V is the function $\chi_V : G \to \mathbb{C}$ given by the composition

$$G \xrightarrow{\rho} \mathrm{GL(V)} \xrightarrow{\mathrm{tr}} \mathbb{C}.$$

A function $\chi : G \to \mathbb{C}$ is a **character of** G if it is the character of some representation of G. A character is called **irreducible** if it is the character of an irreducible representation.

LEMMA 2.1. *Let V and W be representations of G. Then the following hold:*

(1) *If $V \simeq W$, then $\chi_V = \chi_W$.*
(2) *$\chi_{V \oplus W} = \chi_V + \chi_W$.*
(3) *$\chi_V(gh) = \chi_V(hg)$ for all $g, h \in G$.*
(4) *$\chi_V(g) = \chi_V(hgh^{-1})$ for all $g, h \in G$, i.e. χ_V is constant on conjugacy classes of G (so χ_V is a 'class function').*

Proof. (1): By Lemma 1.1, (4).

(2): By Lemma 1.1, (2).

(3): By Lemma 1.1, (3).

(4): By Lemma 1.1, (4). □

The character of the trivial representation of G is called the **trivial character** of G, denoted by 1_G. Thus $1_G(g) = 1$ for all $g \in G$.

Example:

Let $G = D_8 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$, and let

$$\rho : G \to \mathrm{GL}_2(\mathbb{C}), \mathrm{a} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathrm{b} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we get

g	1	a	a^2	a^3	b	ab	a^2b	a^3b
$\chi(g)$	2	0	-2	0	0	0	0	0

3. The values of characters

For a complex number $z = a+bi$ let $\overline{z}$ be its conjugate. Thus $\overline{z} = a-bi$.

LEMMA 3.1. *Let χ be the character of a representation V of G. Suppose $g \in G$ is an element of order m. Then the following hold:*

(1) *$\chi(1) = \dim(V)$;*
(2) *$\chi(g)$ is a sum of mth roots of 1;*
(3) *$\chi(g^{-1}) = \overline{\chi(g)}$;*
(4) *If $g = hg^{-1}h^{-1}$ for some $h \in G$, then $\chi(g) \in \mathbb{R}$.*

Proof. (1): Let $n = \dim(V)$. Clearly with respect to any basis the identity $1 : V \to V$ corresponds to the $n \times n$ identity matrix. Thus $\chi(1) = n$.

(2): Let $\rho : G \to \mathrm{GL(V)}$ be a representation such that χ is the character of ρ. Now we use Theorem 15.1: Namely for $g \in G$ choose a basis of V such that the corresponding matrix representation $A : G \to \mathrm{GL_n}(\mathbb{C})$ has the property that $A(g)$ is a diagonal matrix. Since g has order m, we get $A(g)^m = 0$. Thus the diagonal entries of $A(g)$ are mth roots of 1. So the trace of $A(g)$ is a sum of mth root of 1. But this trace does not depend on the chosen basis.

(3): Let $g \in G$. Again choose a basis such that $A(g)$ is a diagonal matrix, where $A : G \to \mathrm{GL_n}(\mathbb{C})$ is the matrix representation corresponding to V with respect to this basis. Let $\lambda_1, \cdots, \lambda_n$ be the diagonal entries of $A(g)$. Thus $A(g)^{-1}$ has diagonal entries $\lambda_1^{-1}, \cdots, \lambda_n^{-1}$. Thus we get $\chi(g^{-1}) = \sum_{i=1}^n \lambda_i^{-1}$. But every complex mth root of 1, say ω, satisfies $\omega^{-1} = \overline{\omega}$ (since for all real θ we have $(e^{i\theta})^{-1} = e^{-i\theta} = \overline{e^{i\theta}}$. Therefore $\chi(g^{-1}) = \sum_{i=1}^n \overline{\lambda_i} = \overline{\chi(g)}$.

(4): If $g = hg^{-1}h^{-1}$, then $\chi(g) = \chi(hg^{-1}h^{-1}) = \chi(g^{-1})$. This implies $\chi(g) = \overline{\chi(g)}$. Thus $\chi(g)$ is real. □

If χ is a character of a representation V, then the **degree** of χ is the dimension of V. Characters of degree 1 are called **linear characters**. It follows immediately that linear characters are irreducible.

By $\mathbb{C}^*$ we denote the (multiplicative) group of non-zero complex numbers.

Corollary 3.2. *Every linear character defines a homomorphism $G \to \mathbb{C}^*$. Vice versa, every character, which defines a homomorphism $G \to \mathbb{C}^*$ is linear.***??**

Proof. If a character χ of a representation ρ is linear, then $\chi(g) = \lambda$ with λ a root of 1. In particular $\chi(g) \neq 0$. Thus we can regard χ as a map $G \to \mathbb{C}^*$. Now $\chi(1) = \mathrm{tr}(1) = 1$ and $\chi(gh) = \mathrm{tr}(\rho(gh)) = \mathrm{tr}(\rho(g)\rho(h)) = \mathrm{tr}(\rho(g))\mathrm{tr}(\rho(h)) = \chi(g)\chi(h)$. Note that the equality $\mathrm{tr}(\rho(g)\rho(h)) = \mathrm{tr}(\rho(g))\mathrm{tr}(\rho(h))$ does not hold in general, but for linear characters it does.

On the other hand, assume χ is a character of a representation V with $\chi(g) \neq 0$ for all $g \in G$ and $\chi : G \to \mathbb{C}^*$ a homomorphism, then we get $\chi(1) = \dim(V) = 1$. Thus χ is linear. □

THEOREM 3.3. *Let $\rho : G \to \mathrm{GL_n}(\mathbb{C})$ be a representation of G, and let χ be the character of ρ. Then the following hold:*

(1) *For $g \in G$ we have $|\chi(g)| = \chi(1)$ if and only if $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$.*
(2) $\ker(\rho) = \{g \in G \mid \chi(g) = \chi(1)\}$.

Proof. (1): Let $g \in G$. Suppose g has order m. If $\rho(g) = \lambda I_n$ with $\lambda \in \mathbb{C}$, then λ is an mth root of 1 and $\chi(g) = n\lambda$. So $|\chi(g)| = n = \chi(1)$.

Conversely, suppose $|\chi(g)| = \chi(1)$. By Theorem 15.1 there is a basis $e_1, \cdots, e_n$ of $\mathbb{C}^n$ such that $A(g)$ is a diagonal matrix with entries $\omega_1, \cdots, \omega_n$ where A is the matrix representation corresponding to ρ with respect to the basis $e_1, \cdots, e_n$, and the ω_i are mth roots of 1. This implies $|\chi(g)| = |\omega_1 + \cdots + \omega_n| = \chi(1) = n$. For any complex numbers $z_1, \cdots, z_n$ we have $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$ with $=$ if and only if $\arg z_1 = \cdots = \arg z_n$. Since $|\omega_i| = 1$ for all i we get $\omega_i = \omega_j$ for all i, j. Thus $\rho(g)$ is conjugate to the matrix $\omega_1 I_n$. But this implies $\rho(g) = \omega_1 I_n$.

(2): If $g \in \ker(\rho)$, then $\rho(g) = I_n$. Thus $\chi(g) = n = \chi(1)$.

Conversely, suppose $\chi(g) = \chi(1)$. Then by (1) we have $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. This implies $\chi(g) = \lambda\chi(1)$. Thus we get $\lambda = 1$. So $\rho(g) = I_n$ which means $g \in \ker(\rho)$. □

Let χ be a character of G. The **kernel** of χ is

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}.$$

Thus by the above theorem, if $\rho : G \to \mathrm{GL_n}(\mathbb{C})$ is a representation with character χ, then $\ker(\rho) = \ker(\chi)$. In particular, $\ker(\chi)$ is a normal subgroup of G.

The representation ρ is called **faithful** if $\ker(\rho) = \{1\}$, and χ is **faithful** if $\ker(\chi) = \{1\}$.

Example:

The irreducible characters of $D_6 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1}\rangle$ are the following:

g	1	a	a^2	b	ab	a^2b
$\chi_1(g)$	1	1	1	1	1	1
$\chi_2(g)$	1	1	1	-1	-1	-1
$\chi_3(g)$	2	-1	-1	0	0	0

Now one checks easily that $\ker(\chi_1) = G$, $\ker(\chi_2) = \{1, a, a^2\}$ and $\ker(\chi_3) = \{1\}$. Thus χ_3 is faithful.

For a character χ of G define $\overline{\chi} : G \to \mathbb{C}$ by $\overline{\chi}(g) = \overline{\chi(g)}$.

LEMMA 3.4. *If χ is a character of G, then $\overline{\chi}$ is also a character of G. If χ is irreducible, then $\overline{\chi}$ is irreducible.*

Proof. Suppose χ is a character of a representation $\rho : G \to \mathrm{GL_n}(\mathbb{C})$. So $\chi(g) = \mathrm{tr}(\rho(g))$ for all $g \in G$. If $A = (a_{ij})_{ij} \in M_n(\mathbb{C})$, then let $\overline{A} = (\overline{a_{ij}})_{ij}$. For matrices $A, B \in M_n(\mathbb{C})$ we have $\overline{AB} = \overline{A}\,\overline{B}$. It follows that $\overline{\rho} : G \to \mathrm{GL_n}(\mathbb{C})$ where $\overline{\rho}(g) = \overline{\rho(g)}$ is a representation of G. Since $\mathrm{tr}(\overline{\rho}(g)) = \mathrm{tr}(\overline{\rho(g)}) = \overline{\mathrm{tr}(\rho(g))} = \overline{\chi(g)}$, the character of $\overline{\rho}$ is $\overline{\chi}$. Clearly, if ρ is completely reducible, then $\overline{\rho}$ is completely reducible. Hence χ is irreducible if and only if $\overline{\chi}$ is irreducible. $\square$

4. The regular character

The **regular character** of G is the character of the regular representation of G, and is denoted by χ_{reg}.

PROPOSITION 4.1. *Let $V_1, \cdots, V_r$ be a complete set of irreducible representations of G, and let $\chi_i : G \to \mathbb{C}$ be the character of V_i and $d_i = \chi_i(1)$, $1 \le i \le r$. Then*

$$\chi_{\mathrm{reg}} = \sum_{i=1}^{r} d_i \chi_i.$$

Proof. We have $\mathbb{C}G \simeq V_1^{d_1} \oplus \cdots \oplus V_r^{d_r}$. Then we use Lemma 2.1, (2). $\square$

LEMMA 4.2. *If χ_{reg} is the regular character of G, then $\chi_{\mathrm{reg}}(1) = |G|$ and $\chi_{\mathrm{reg}}(g) = 0$ for all $g \neq 1$.*

Proof. Let $g_1, \cdots, g_n$ be the elements of G, let $\mathcal{B} = \{g_1, \cdots, g_n\}$ be the usual basis of the regular representation $\mathbb{C}G$, and let $\rho_{\mathrm{reg}} : G \to \mathrm{GL_n}(\mathbb{C})$ be the corresponding matrix representation. Now $\chi_{\mathrm{reg}}(1) = \dim(\mathbb{C}G) = |G|$ by Lemma 3.1, (1).

Let $g \in G$ with $g \neq 1$. Then for $1 \leq i \leq n$ we get $gg_i = g_j$ for some j with $j \neq i$. Thus the ith column of the matrix $\rho_{\text{reg}}(g)$ has entry everywhere 0, except in the jth row. Especially the iith entry is 0. It follows that $\chi_{\text{reg}}(g) = \text{tr}(\rho_{\text{reg}}(g)) = 0$. □

Example:

Again let $G = D_6 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$. The irreducible characters χ_1, χ_2, χ_3 of G look as follows:

g	1	a	a^2	b	ab	a^2b
$\chi_1(g)$	1	1	1	1	1	1
$\chi_2(g)$	1	1	1	-1	-1	-1
$\chi_3(g)$	2	-1	-1	0	0	0

Thus $\chi_{\text{reg}} = \chi_1 + \chi_2 + 2\chi_3$. Thus $\chi_{\text{reg}}(1) = 6 = |G|$.

5. Inner products

The characters of a finite group are certain functions $G \to \mathbb{C}$. The set of ALL functions $G \to \mathbb{C}$ form a vector space over $\mathbb{C}$:

Let $\psi, \theta : G \to \mathbb{C}$ be arbitrary functions, and let $\lambda \in \mathbb{C}$. Then define $\psi + \theta : G \to \mathbb{C}$ via $(\psi + \theta)(g) = \psi(g) + \theta(g)$ and $(\lambda\psi)(g) = \lambda\psi(g)$ for all $g \in G$.

It is easy to check that this defines a vector space structure on the set of all functions $G \to \mathbb{C}$.

Now let V be a vector space over $\mathbb{C}$. An **inner product** on V is a map $\langle -, - \rangle : V \times V \to \mathbb{C}$ such that the following hold:

(1) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$;
(2) $\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $v_1, v_2, w \in V$;
(3) $\langle v, v \rangle > 0$ for all $0 \neq v \in V$.

Remarks: Condition (1) implies that $\langle v, v \rangle$ is real for all $v \in V$. Furthermore, (1) and (2) imply

$$\langle v, \lambda_1 w_1 + \lambda_2 w_2 \rangle = \overline{\lambda_1} \langle v, w_1 \rangle + \overline{\lambda_2} \langle v, w_2 \rangle$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $v, w_1, w_2 \in V$.

We now introduce an inner product on the vector space of all functions $G \to \mathbb{C}$. This will be very important for the study of characters.

Let ψ, θ be functions $G \to \mathbb{C}$. Define

$$\langle \psi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g) \overline{\theta(g)}.$$

As an exercise, check that (1), (2) and (3) in the above definition hold.

Example:

For $G = C_3 = \langle a \mid a^3 = 1 \rangle$, let ψ and θ be defined as follows:

g	1	a	a^2
$\psi(g)$	2	i	-1
$\theta(g)$	1	1	1

Then we get

$$\begin{aligned} \langle \psi, \theta \rangle &= \frac{1}{3}(2 + i - 1) = \frac{1}{3}(1 + i), \\ \langle \psi, \psi \rangle &= \frac{1}{3}(4 + i\bar{i} + (-1)(-1)) = 2, \\ \langle \theta, \theta \rangle &= \frac{1}{3}(1 + 1 + 1) = 1. \end{aligned}$$

6. Inner products of characters

Reminder on conjugacy classes: Let G be a finite group, and let $x \in G$. The **conjugacy class** of G containing x is defined as

$$x^G = \{gxg^{-1} \mid g \in G\}.$$

The **centralizer** of x in G is

$$C_G(x) = \{g \in G \mid gxg^{-1} = x\}.$$

We know from elementary group theory that

$$|x^G| = \frac{|G|}{|C_G(x)|}.$$

Furthermore, given conjugacy classes x^G and y^G, then either $x^G \cap y^G = \emptyset$ or $x^G = y^G$.

LEMMA 6.1. *Assume G has exactly l conjugacy classes. Choose representatives $g_1, \cdots, g_l$. Let χ and ψ be characters of G. Then the following hold:*

(1)
$$\langle \chi, \psi \rangle = \langle \psi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1})$$
and this is a real number.

(2)
$$\langle \chi, \psi \rangle = \sum_{i=1}^{l} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}.$$

Proof. (1): By Lemma 3.1, (3) we know that $\psi(g^{-1}) = \overline{\psi(g)}$. Therefore
$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$
Since $\{g^{-1} \mid g \in G\} = G$ we get
$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})\psi(g) = \langle \psi, \chi \rangle.$$
Since $\langle \psi, \chi \rangle = \overline{\langle \chi, \psi \rangle}$. It follows that $\langle \chi, \psi \rangle$ is real.

(2): Let $g_i^G = \{g g_i g^{-1} \mid g \in G\}$ be the conjugacy class of G which contains g_i. Characters are constant on conjugacy classes. So we get
$$\sum_{g \in g_i^G} \chi(g)\overline{\psi(g)} = |g_i^G| \chi(g_i)\overline{\psi(g_i)}.$$
Now
$$G = \bigcup_{i=1}^{l} g_i^G$$
and
$$|g_i^G| = \frac{|G|}{|C_G(g_i)|}.$$
Hence we get
$$\begin{aligned} \langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} &= \frac{1}{|G|} \sum_{i=1}^{l} \sum_{g \in g_i^G} \chi(g)\overline{\psi(g)} \\ &= \sum_{i=1}^{l} \frac{|g_i^G|}{|G|} \chi(g_i)\overline{\psi(g_i)} = \sum_{i=1}^{l} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}. \end{aligned}$$
□

Example:

A_4 has 4 conjugacy classes with representatives $g_1 = 1$, $g_2 = (12)(34)$, $g_3 = (123)$ and $g_4 = (132)$. Let $\omega = e^{2\pi i/3}$. Then there exist characters χ and ψ which look as follows:

g_i	g_1	g_2	g_3	g_4
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ	1	1	ω	ω^2
ψ	4	0	ω^2	ω

Thus we get

$$\langle \chi, \psi \rangle = \frac{4}{12} + \frac{0}{4} + \frac{\omega\overline{\omega}^2}{3} + \frac{\omega^2\overline{\omega}}{3} = 0$$

and

$$\langle \psi, \psi \rangle = \frac{16}{12} + \frac{0}{4} + \frac{\omega^2\overline{\omega}^2}{3} + \frac{\omega\overline{\omega}}{3} = 2.$$

Exercise: Check that $\langle \chi, \chi \rangle = 1$.

Reminder: We have $\mathbb{C}G = U_1 \oplus \cdots \oplus U_r$ with U_i irreducible representations for all i, and every irreducible representation of G is isomorphic to one of the U_i. There are several ways of decomposing $\mathbb{C}G = W_1 \oplus W_2$ such that $\mathrm{Hom}_G(W_1, W_2) = 0$ (this implies immediately $\mathrm{Hom}_G(W_2, W_1) = 0$).

For example, take W_1 to be the sum of the U_i which are isomorphic to U_1, and let W_2 be the sum of the remaining U_i.

In the following we work with the following assumption: Let $\mathbb{C}G = W_1 \oplus W_2$ where W_1 and W_2 are representations with $\mathrm{Hom}_G(W_1, W_2) = 0$, and let $e_1 \in W_1$ and $e_2 \in W_2$ such that $1 = e_1 + e_2$.

LEMMA 6.2. *For all $w_1 \in W_1$ and $w_2 \in W_2$ we have $e_1 w_1 = w_1$, $e_1 w_2 = 0$, $e_2 w_1 = 0$ and $e_2 w_2 = w_2$.*

Proof. The function $W_2 \to W_1$, $w_2 \mapsto w_2 w_1$ is a homomorphism. But $\mathrm{Hom}_G(W_2, W_1) = 0$. Thus $w_2 w_1 = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. Similarly, $w_1 w_2 = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. In particular, $e_2 w_1 = e_1 w_2 = 0$. Now $w_1 = 1w_1 = (e_1 + e_2)w_1 = e_1 w_1$ and $w_2 = 1w_2 = (e_1 + e_2)w_2 = e_2 w_2$. □

COROLLARY 6.3. *We have $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 e_2 = e_2 e_1 = 0$.*

Proof. In the previous lemma take $w_1 = e_1$ and $w_2 = e_2$. □

LEMMA 6.4. *Let χ be the character of the representation W_1. Then*

$$e_1 = \frac{1}{|G|}\sum_{g\in G}\chi(g)g^{-1}.$$

Proof. Let $x \in G$. The function $\theta : \mathbb{C}G \to \mathbb{C}G$, $w \mapsto xe_1w$ is a linear map. For $w_1 \in W_1$ and $w_2 \in W_2$ we have $\theta(w_1) = xe_1w_1 = xw_1$ and $\theta(w_2) = xe_1w_2 = 0$. Thus we get the following restriction map: $\theta_x : W_1 \to W_1$, $w_1 \mapsto xw_1$. The trace of this linear map is $\chi(x)$, and the restriction $W_2 \to W_2$, $w_2 \mapsto 0$ is a linear map with trace 0. Therefore $\mathrm{tr}(\theta) = \chi(x)$.

Secondly, $e_1 \in \mathbb{C}G$. So $e_1 = \sum_{g\in G}\lambda_g g$ for some $\lambda_g \in \mathbb{C}$. Now we use Lemma 4.2: The linear map $\mathbb{C}G \to \mathbb{C}G$, $w \mapsto xgw$ has trace 0 if $g \neq x^{-1}$, and it has trace $|G|$ if $g = x^{-1}$. This follows from the analysis of the regular representation of G.

Hence as $\theta(w) = x(\sum_{g\in G}\lambda_g g)w$, we get $\mathrm{tr}(\theta) = \lambda_{x^{-1}}|G|$. Thus $\mathrm{tr}(\theta) = \chi(x) = \lambda_{x^{-1}}|G|$. So

$$\lambda_{x^{-1}} = \frac{\chi(x)}{|G|}.$$

Remember that x was chosen arbitrary. Therefore we get

$$e_1 = \frac{1}{|G|}\sum_{g\in G}\chi(g)g^{-1}.$$

□

COROLLARY 6.5. *Let χ be the character of W_1. Then $\langle\chi,\chi\rangle = \chi(1)$.*

Proof. The coefficient of 1 in e_1^2 is

$$\frac{1}{|G|^2}\sum_{g\in G}\chi(g)\chi(g^{-1}) = \frac{1}{|G|}\langle\chi,\chi\rangle.$$

On the other hand, we know that $e_1^2 = e_1$, and the coefficient of 1 in e_1 is $\frac{\chi(1)}{|G|}$. Hence we get $\chi(1) = \langle\chi,\chi\rangle$. □

Now we can prove the main theorem concerning inner products:

THEOREM 6.6. *Let U, V be non-isomorphic irreducible representations with characters χ, ψ, respectively. Then $\langle\chi,\chi\rangle = 1$ and $\langle\chi,\psi\rangle = 0$.*

Proof. Recall that $\mathbb{C}G$ can be written as a direct sum of irreducible sub-representations. Say $\mathbb{C}G = U_1 \oplus \cdots \oplus U_r$ where the number of U_i

which are isomorphic to U is $\dim(U)$. Let $m = \dim(U)$, and let W be the sum of the U_i which are isomorphic to U, and let X be the sum of the remaining U_i. Thus $\mathbb{C}G = W \oplus X$ and $\mathrm{Hom}_G(W, X) = 0$.

The character of W is $m\chi$ since $W \simeq U^m$, and χ is the character of U. Now we apply the previous corollary and get $\langle m\chi, m\chi\rangle = (m\chi)(1) = m\chi(1)$. As $\chi(1) = \dim(U) = m$, and since $\langle m\chi, m\chi\rangle = m^2\langle\chi,\chi\rangle$, we get $\langle\chi,\chi\rangle = 1$.

Next, let Y be the sum of U_i which are isomorphic to U or to V, and let Z be the sum of the remaining U_i. Thus $\mathbb{C}G = Y \oplus Z$ and $\mathrm{Hom}_G(Y, Z) = 0$. The character of Y is $m\chi + n\psi$ where $n = \dim(V)$. By the previous corollary we get

$$\begin{aligned}(m\chi + n\psi)(1) &= \langle m\chi + n\psi, m\chi + n\psi\rangle \\ &= m^2\langle\chi,\chi\rangle + n^2\langle\psi,\psi\rangle + mn(\langle\chi,\psi\rangle + \langle\psi,\chi\rangle).\end{aligned}$$

Now $\langle\chi,\chi\rangle = \langle\psi,\psi\rangle = 1$, as proved above, and $\chi(1) = m$ and $\psi(1) = n$. Therefore

$$m^2 + n^2 = m^2 + n^2 + mn(\langle\chi,\psi\rangle + \langle\psi,\chi\rangle).$$

This implies $0 = \langle\chi,\psi\rangle + \langle\psi,\chi\rangle$. By Lemma 6.1, (1) we have $\langle\chi,\psi\rangle = \langle\psi,\chi\rangle$. Thus $\langle\chi,\psi\rangle = \langle\psi,\chi\rangle = 0$. □

7. Applications

In this section, let G be a finite group, and let $V_1, \cdots, V_r$ be a complete set of irreducible representations of G. Let χ_i be the character of V_i, $1 \le i \le r$. Then by Theorem 6.6 we have $\langle\chi_i,\chi_j\rangle = \delta_{ij}$ for all i, j. In particular, this implies that the irreducible characters are all distinct. Now let V be any representation of G. Thus $V \simeq V_1^{d_1} \oplus \cdots \oplus V_r^{d_r}$ for some $d_i \ge 0$. Therefore the character of V is $\psi = \sum_{i=1}^r d_i\chi_i$. Thus we get

$$\langle\psi,\chi_i\rangle = \langle\chi_i,\psi\rangle = d_i$$

for $1 \le i \le r$, and

$$\langle\psi,\psi\rangle = \sum_{i=1}^r d_i^2.$$

Summarizing we get the following:

THEOREM 7.1. *Let $\chi_1, \cdots, \chi_r$ be the irreducible characters of G. If ψ is any character of G, then $\psi = \sum_{i=1}^r d_i\chi_i$ for some $d_i \ge 0$, $d_i = \langle\psi,\chi_i\rangle$ and $\langle\psi,\psi\rangle = \sum_{i=1}^r d_i^2$.*

Example:

Let $G = S_3$. There are 3 irreducible characters χ_1, χ_2, χ_3 which look as follows:

g_i	1	(12)	(123)
$\lvert C_G(g_i)\rvert$	6	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Let ψ be the character of the 3-dimensional permutation representation V. Thus V has basis $\{e_1, e_2, e_3\}$ and S_3 acts as follows on V:

$$S_3 \times V \to V, \left(\pi, \sum_{i=1}^{3} \lambda_i e_i\right) \mapsto \sum_{i=1}^{3} \lambda_i e_{\pi(i)}.$$

Thus as a matrix representation we get

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus ψ looks as follows:

g_i	1	(12)	(123)
ψ	3	1	0

Therefore by Lemma 6.1 we get

$$\langle \psi, \chi_1 \rangle = \frac{3}{6} + \frac{1}{2} + \frac{0}{3} = 1.$$

Similarly we get $\langle \psi, \chi_2 \rangle = 0$ and $\langle \psi, \chi_3 \rangle = 1$. Thus by Theorem 7.1 we get $\psi = \chi_1 + \chi_3$.

THEOREM 7.2. *Let V be a representation of G with character ψ. Then V is irreducible if and only if $\langle \psi, \psi \rangle = 1$.*

Proof. If V is irreducible, then $\langle \psi, \psi \rangle = 1$ by Theorem 6.6. Conversely, assume that $\langle \psi, \psi \rangle = 1$. We have $\psi = \sum_{i=1}^{r} d_i \chi_i$ for some $d_i \geq 0$, and by Theorem 7.1 we get

$$1 = \langle \psi, \psi \rangle = \sum_{i=1}^{r} d_i^2.$$

It follows that one d_i is 1, and the others must be 0. Thus V is irreducible. $\square$

THEOREM 7.3. *Let V, W be representations of G with characters χ, ψ, respectively. Then $V \simeq W$ if and only if $\chi = \psi$.*

Proof. In Lemma 2.1 we proved that $V \simeq W$ implies $\chi = \psi$. Thus suppose that $\chi = \psi$. There are $c_i, d_i \geq 0$ such that $V \simeq V_1^{c_1} \oplus \cdots \oplus V_r^{c_r}$ and $W \simeq V_1^{d_1} \oplus \cdots \oplus V_r^{d_r}$ where $V_1, \cdots, V_r$ is a complete set of irreducible representations of G. Let χ_i be the character of V_i, $1 \leq i \leq r$. By Theorem 7.1 we have $c_i = \langle \chi, \chi_i \rangle$ and $d_i = \langle \psi, \chi_i \rangle$ for $1 \leq i \leq r$. Since $\chi = \psi$, we get $c_i = d_i$ for all i. Thus $V \simeq W$. □

Example:

Let $\omega = e^{2\pi i/3}$. Let $G = C_3 = \langle a \mid a^3 = 1 \rangle$. The following define representations of G:

$$\rho_1 : a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix},$$

$$\rho_2 : a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},$$

$$\rho_3 : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$\rho_4 : a \mapsto \begin{pmatrix} 1 & \omega^{-1} \\ 0 & \omega \end{pmatrix}.$$

To check that the above really define representations we just have to check that $(\rho_i(a))^3 = I$ for all i. The character ψ_i of ρ_i, $1 \leq i \leq 4$, looks as follows:

g_i	1	a	a^2
ψ_1	2	2ω	$2\omega^2$
ψ_2	2	-1	-1
ψ_3	2	-1	-1
ψ_4	2	$1+\omega$	$1+\omega^2$

Thus we have $\rho_2 \simeq \rho_3$, and no other isomorphisms between the ρ_i by Theorem 7.3.

LEMMA 7.4. *Let $\chi_1, \cdots, \chi_r$ be the irreducible characters of G. Then $\chi_1, \cdots, \chi_r$ are linearly independent vectors in the vector space of all functions $G \to \mathbb{C}$.*

Proof. Assume $\sum_{i=1}^{r} \lambda_i \chi_i = 0$ with $\lambda_i \in \mathbb{C}$. Then

$$0 = \langle 0, \chi_i \rangle = \langle \sum_{i=1}^{r} \lambda_i \chi_i, \chi_i \rangle = \lambda_i \langle \chi_i, \chi_i \rangle = \lambda_i.$$

Therefore $\lambda_i = 0$ for all i. Thus the χ_i are linearly independent. □

THEOREM 7.5. *Let V, W be representations of G with characters χ, ψ, respectively. Then*

$$\dim \mathrm{Hom}_G(V, W) = \langle \chi, \psi \rangle.$$

Proof. Let $V_1, \cdots, V_r$ be a complete set of irreducible representations of G. We have $V \simeq V_1^{c_1} \oplus \cdots \oplus V_r^{c_r}$ and $W \simeq V_1^{d_1} \oplus \cdots \oplus V_r^{d_r}$ for some $c_i, d_i \geq 0$. We also know that $\dim \mathrm{Hom}_G(V_i, V_j) = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$, else. Hence we get

$$\dim \mathrm{Hom}_G(V, W) = \sum_{i=1}^{r} c_i d_i.$$

On the other hand, $\chi = \sum_{i=1}^{r} c_i \chi_i$ and $\psi = \sum_{i=1}^{r} d_i \chi_i$ where χ_i is the character of V_i, $1 \leq i \leq r$. Thus we get

$$\langle \chi, \psi \rangle = \sum_{i=1}^{r} c_i d_i$$

by Theorem 6.6. □

8. Class functions

A **class function** on G is a function $\psi : G \to \mathbb{C}$ such that $\psi(x) = \psi(y)$ whenever x and y are conjugate elements of G, i.e. there exists some $g \in G$ such that $gxg^{-1} = y$.

For example, characters are class functions by Lemma 2.1, (4). The set C of all class functions on G is a subspace of the vector space of all functions $G \to \mathbb{C}$.

A basis is given by those functions which take value 1 on precisely one conjugacy class of G and 0 on all other conjugacy classes. Thus if l is the number of conjugacy classes of G, then $\dim(C) = l$. The following theorem follows from the results on group algebras and from Wedderburn's Theorem, see I, Section 13:

THEOREM 8.1. *The following numbers coincide:*

(1) *The number of irreducible characters of* G*;*
(2) *The number of conjugacy classes of* G*;*
(3) *The number of isomorphism classes of irreducible representations of* G*;*
(4) $\dim Z(\mathbb{C}G)$*;*
(5) $\dim(C)$.

Corollary 8.2. *The irreducible characters* $\chi_1, \cdots, \chi_r$ *of* G *form a basis of the vector space* C *of all class functions* $G \to \mathbb{C}$*.*

Proof. The functions $\chi_1, \cdots, \chi_r$ are linearly independent by Lemma 7.4, and they are class functions. Thus they are in C. Then use the previous theorem. □

Corollary 8.3. *Again let* $\chi_1, \cdots, \chi_r$ *be the irreducible character of* G*. If* $\psi : G \to \mathbb{C}$ *is a class function, then* $\psi = \sum_{i=1}^{r} \lambda_i \chi_i$ *for some* $\lambda_i \in \mathbb{C}$*, and we have* $\langle \psi, \chi_i \rangle = \lambda_i$*.*

Corollary 8.4. *Suppose* $g, h \in G$*. Then* g *is conjugate to* h *if and only if* $\psi(g) = \psi(h)$ *for all characters* ψ *of* G*.*

Proof. If g is conjugate to h then $\psi(g) = \psi(h)$ for all characters ψ, since characters are class functions. Conversely, suppose $\psi(g) = \psi(h)$ for all characters ψ of G. Then by Corollary 8.2 we get $\psi(g) = \psi(h)$ for all class functions ψ of G. In particular, if ψ takes value 1 on the conjugacy class of g, and takes 0 elsewhere, then we get $\psi(g) = \psi(h) = 1$. Thus g is conjugate to h. □

9. Character tables

Let $\chi_1, \cdots, \chi_r$ be the irreducible characters of G, and let $g_1, \cdots, g_r$ be representatives of the conjugacy classes of G. The $r \times r$ matrix whose ijth entry is $\chi_i(g_j)$, $1 \leq i, j \leq r$ is called the **character table** of G.

Convention: Let $\chi_1 = 1_G$ always be the trivial character, and let $g_1 = 1$ always be the identity element of G.

Thus the rows of the character table of G are indexed by the irreducible characters of G, and the columns are indexed by the conjugacy classes of G.

Lemma 9.1. *The character table of* G *is an invertible matrix.*

Proof. The irreducible characters are linearly independent by Lemma 7.4. Thus the rows of the character table are linearly independent. □

Examples:

1) Let $G = D_6 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1}\rangle$. Then the character table of G looks as follows:

	1	a	b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

2) Let $G = C_2 = \langle a \mid a^2 = 1\rangle$. Then the character table of G is the following:

	1	1
χ_1	1	1
χ_2	1	-1

3) $G = C_3 = \langle a \mid a^3 = 1\rangle$:

	1	a	a^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where $\omega = e^{2\pi i/3}$.

4) $G = D_8 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1}\rangle$:

	1	a	a^2	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0

10. Othogonality relations

THEOREM 10.1. *Let $\chi_1, \cdots, \chi_k$ be the irreducible characters of G, and let $g_1, \cdots, g_k$ be representatives of the conjugacy classes of G. Then for $r, s \in \{1, \cdots, k\}$ the following hold:*

(1) *Row orthogonality relation:*

$$\langle \chi_r, \chi_s \rangle = \sum_{i=1}^{k} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \delta_{rs}$$

(2) *Column orthogonality relation:*

$$\sum_{i=1}^{k} \frac{\chi_i(g_r)\overline{\chi_i(g_s)}}{|C_G(g_s)|} = \delta_{rs}$$

Proof. Part (1) follows from Lemma 6.1,(2) combined with Theorem 6.6.

For $1 \le s \le k$ let ψ_s be the class function with $\psi_s(g_r) = \delta_{rs}$, $1 \le r \le k$. By Corollary 8.2 the class function ψ_s is a linear combination of $\chi_1, \cdots, \chi_k$, say $\psi_s = \sum_{i=1}^{k} \lambda_i \chi_i$ with $\lambda_i \in \mathbb{C}$. We know that $\langle \chi_i, \chi_j \rangle = \delta_{ij}$, so

$$\lambda_i = \langle \psi_s, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \psi_s(g)\overline{\chi_i(g)}.$$

Now $\psi_s(g) = 1$ if g is conjugate to g_s and $\psi_s(g) = 0$, otherwise. Also there are $\frac{|G|}{|C_G(g_s)|}$ elements of G which are conjugate to g_s. Hence we get

$$\lambda_i = \frac{1}{|G|} \sum_{g \in g_s^G} \psi_s(g)\overline{\chi_i(g)} = \frac{\overline{\chi_i(g_s)}}{|C_G(g_s)|}.$$

Thus

$$\delta_{rs} = \psi_s(g_r) = \sum_{i=1}^{k} \lambda_i \chi_i(g_r) = \sum_{i=1}^{k} \frac{\chi_i(g_r)\overline{\chi_i(g_s)}}{|C_G(g_s)|}.$$

□

Example: Suppose we are given the following part of the character table of a group G of order 12 which has exactly 4 conjugacy classes:

g_i	g_1	g_2	g_3	g_4
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	x_1	x_2	x_3	x_4

where $\omega = e^{2\pi i/3}$. The entries of the first column of the character table are the degrees of the irreducible characters, so they are positive integers. By the column orthogonality relation with $r = s = 1$, the sum of the squares is 12. So $1^2 + 1^2 + 1^2 + x_1^2 = 12$. So $x_1 = 3$. Next, we have

$$\sum_{i=1}^{4} \chi_i(g_1)\overline{\chi_i(g_2)} = \delta_{12}|C_G(g_2)| = 0.$$

So $1 \cdot \overline{1} + 1 \cdot \overline{1} + 1 \cdot \overline{1} + 3 \cdot \overline{x_2} = 0$. Thus $x_2 = -1$. In the same way, we get $x_3 = x_4 = 0$.

Exercise: Verify the other orthogonality relations.

For example

$$\sum_{i=1}^{4} \chi_i(g_3)\overline{\chi_i(g_3)} = 1 \cdot \overline{1} + \omega\overline{\omega} + \omega^2\overline{\omega}^2 + 0 \cdot \overline{0} = \delta_{33}|C_G(g_3)| = 3.$$

Later we prove: The above is the character table of A_4.

Compare the following (which we get from the column orthogonality relations) with Proposition 4.1:

$$\sum_{i=1}^{k} \chi_i(1)\overline{\chi_i(g)} = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{else.} \end{cases}$$

11. Lifted characters

Proposition 11.1. *Assume $N \trianglelefteq G$, and let $\tilde{\chi}$ be a character of G/N. Define $\chi : G \to \mathbb{C}$ by*

$$\chi(g) = \tilde{\chi}(gN)$$

for $g \in G$. Then χ is a character of G, and χ and $\tilde{\chi}$ have the same degree.

Proof. Let $\tilde{\rho} : G/N \to \mathrm{Gl}_n(\mathbb{C})$ be a representation of G/N with character $\tilde{\chi}$. The function $\rho : G \to \mathrm{Gl}_n(\mathbb{C})$ which is given by the composition

$$g \mapsto gN \mapsto \tilde{\rho}(gN)$$

is a homomorphism from G to $\mathrm{GL}_n(\mathbb{C})$. Thus ρ is a representation of G. The character χ of ρ satisfies

$$\chi(g) = \mathrm{tr}(\rho(g)) = \mathrm{tr}(\tilde{\rho}(gN)) = \tilde{\chi}(gN)$$

for all $g \in G$. Moreover $\chi(1) = \tilde{\chi}(N)$. So χ and $\tilde{\chi}$ have the same degree. $\square$

If $N \trianglelefteq G$ and $\tilde{\chi}$ is a character of G/N, then the character χ of G which is given by $\chi(g) = \tilde{\chi}(gN)$ is called the **lift** of $\tilde{\chi}$ to G.

THEOREM 11.2. *Assume $N \trianglelefteq G$. By associating every character of G/N with its lift to G, we get a 1-1 correspondence*

$$\{\textit{characters of } G/N\} \to \{\textit{characters } \chi \textit{ of } G \textit{ with } N \leq \ker(\chi)\}.$$

Irreducible characters of G/N correspond to irreducible characters of G which have N in their kernel.

Proof. If $\tilde{\chi}$ is a character of G/N, and χ is the lift of $\tilde{\chi}$ to G, then $\tilde{\chi}(N) = \chi(1)$. Also, if $g \in N$, then $\chi(g) = \tilde{\chi}(gN) = \tilde{\chi}(N) = \chi(1)$. So $N \leq \ker(\chi)$. Now let χ be a character of G with $N \leq \ker(\chi)$. Suppose that $\rho : G \to \mathrm{Gl}_n(\mathbb{C})$ is a representation of G with character χ. If $g_1, g_2 \in G$ and $g_1N = g_2N$, then $g_1g_2^{-1} \in N$. So $\rho(g_1g_2^{-1}) = I_n$. Hence $\rho(g_1) = \rho(g_2)$. We may therefore define a function $\tilde{\rho} : G/N \to \mathrm{Gl}_n(\mathbb{C})$ by $\tilde{\rho}(gN) = \rho(g)$ for all $g \in G$. Then for all $g, h \in G$ we have

$$\tilde{\rho}(gNhN) = \tilde{\rho}(ghN) = \rho(gh) = \rho(g)\rho(h) = \tilde{\rho}(gN)\tilde{\rho}(hN).$$

So $\tilde{\rho}$ is a representation of G/N. If $\tilde{\chi}$ is the character of $\tilde{\rho}$, then $\tilde{\chi}(gN) = \chi(g)$ for all $g \in G$. Thus χ is the lift of $\tilde{\chi}$ to G. Thus the function which sends each character of G/N to its lift to G is a bijection

$$\{\text{characters of } G/N\} \to \{\text{characters } \chi \text{ of } G \text{ with } N \leq \ker(\chi)\}.$$

It remains to show that irreducible characters correspond to irreducible characters. Let U be a subspace of $\mathbb{C}^n$. Note that $\rho(g)(u) \in U$ for all $u \in U$ if and only if $\tilde{\rho}(gN)(u) \in U$ for all $u \in U$. So U is an invariant subspace of the representation $\mathbb{C}^n$ of G/N if and only if it is an invariant subspace of the representation $\mathbb{C}^n$ over G. So ρ is irreducible if and only if $\tilde{\rho}$ is irreducible. So χ is irreducible if and only if $\tilde{\chi}$ is irreducible. $\square$

Example:

Let $G = S_4$ and $N = V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$. So $N \triangleleft G$. Let $a = (123)N$ and $b = (12)N$. Then $G/N = \langle a, b \mid a^3 = b^2 = N, bab^{-1} = a^{-1}\rangle$. So $G/N \simeq D_6$. The character table of D_6 is

	N	$(12)N$	$(123)N$
$\tilde{\chi}_1$	1	1	1
$\tilde{\chi}_2$	1	-1	1
$\tilde{\chi}_3$	2	0	-1

To calculate the lift χ of a character $\tilde{\chi}$ of G/N, note that $\chi((12)(34)) = \tilde{\chi}(N)$ since $(12)(34) \in N$, and $\chi((1234)) = \tilde{\chi}((13)N)$ since $(1234)N = (13)N$. Hence the lifts are

	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0

The characters $\chi_1, \cdots, \chi_3$ are irreducible since $\tilde{\chi_1}, \cdots, \tilde{\chi_3}$ are irreducible.

12. The character table of S_4

Let $G = S_n$. Recall that there is a permutation module V of G with basis $v_1, \cdots, v_n$ given by the homomorphism

$$\eta : G \to \mathrm{GL}(V)$$

$$g \mapsto (\eta(g) : v_i \mapsto v_{gi}).$$

The character π of V is

$$\pi(g) = |\{1 \leq i \leq n \mid gi = i\}|.$$

One calls π the **permutation character** of G. Define

$$\mathrm{fix}(g) = \{1 \leq i \leq n \mid gi = i\}.$$

LEMMA 12.1. *Let $H \leq S_n$ be a subgroup. Then the map $\nu : H \to \mathbb{C}$ defined by $\nu(g) = |\mathrm{fix}(g)| - 1$ for all $g \in H$ is a character of H.*

Proof. Let $v_1, \cdots, v_n$ be a basis of the permutation module V of H, and as above let π be the character of V. Let $u = v_1 + \cdots + v_n$ and let $U =< u >$ be the 1-dimensional subspace generated by u. So $gu = u$ for all $g \in H$. Then U is the trivial representation of H. By Maschke's Theorem there is a representation W of H such that $V = U \oplus W$. Let ν be the character of W. Then $\pi = 1_H + \nu$. So $|\mathrm{fix}(g)| = 1 + \nu(g)$. So $\nu(g) = |\mathrm{fix}(g)| - 1$. □

Example. Let $G = S_4$. There are 5 conjugacy classes in G. As seen in the example in Section 11, there is a surjective homomorphism $S_4 \to S_3$, with kernel given by the subgroup

$$\{1, (12)(34), (13)(24), (14)(23)\}.$$

The irreducible characters of S_3 lift to S_4, and the remaining two characters are of degree 3 (since the only way that $24-6=18$ can be written as a sum of two squares is via $9+9=18$). So we have:

g_i	1	(12)	(123)	(12)(34)	(1234)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	x_1	x_2	x_3	x_4
χ_5	3	y_1	y_2	y_3	y_4

Recall the character ν of S_4: $\nu(g)=|\mathrm{Fix}(g)|-1$.

g_i	1	(12)	(123)	(12)(34)	(1234)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
ν	3	1	0	-1	-1

Notice that

$$\langle \nu, \nu\rangle = \frac{1}{24}3\cdot 3+\frac{1}{4}1\cdot 1+0+\frac{1}{8}(-1)^2+\frac{1}{4}(-1)^2=1.$$

Therefore, ν is an irreducible character of degree 3. Let us call it χ_4. We can use the column orthogonality relations, or the equation:

$$\chi_{reg}=\chi_1+\chi_2+2\chi_3+3\chi_4+3\chi_5,$$

to calculate χ_5.

Therefore the character table of S_4 is:

g_i	1	(12)	(123)	(12)(34)	(1234)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

13. The character table of A_4

We now want to construct the character table of A_4.

Consider the group A_4 consisting of all permutations $\sigma \in S_4$ such that $\epsilon(\sigma) = 1$. The elements of A_4 consist of all permutations of the form

$$1, (**)(**), (***),$$

where the $*$ entries denote distinct elements of the set $\{1, 2, 3, 4\}$. The conjugacy classes and sizes of centralisers in A_4 are given by:

g_i	1	(12)(34)	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2				
χ_3				
χ_4				

Thus there are 4 irreducible characters $\chi_1, \ldots, \chi_4$.

The character ν of A_4 from Lemma 12.1 is:

g_i	1	(12)(34)	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
ν	3	-1	0	0

We see that

$$\langle \nu, \nu \rangle = \frac{1}{12} 3^2 + \frac{1}{4} 1 + 0 + 0 = 1,$$

so that this character is irreducible. Let us call it χ_4. The remaining two irreducible characters χ_2, χ_3 of A_4 have degree 1, so we have:

g_i	1	(12)(34)	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	x_1	x_2	x_3
χ_3	1	y_1	y_2	y_3
χ_4	3	-1	0	0

By using the equation $\chi_{reg} = \chi_1 + \chi_2 + \chi_3 + 3\chi_4$ we deduce that

$$x_1 + y_1 = 2, x_2 + y_2 = -1, x_3 + y_3 = -1.$$

Notice that (12)(34) is its own inverse, so that $\chi_i((12)(34)) \in \mathbb{R}$ for all i, by Lemma 3.1 (4). In particular, $x_1, y_1 \in \mathbb{R}$.

Now take the column orthogonality rule for column 2 of our character table:

$$\frac{1}{4}(1 + x_1^2 + y_1^2 + 1) = 1.$$

Rearranging, and using the relation $y_1 = 2 - x_1$:

$$x_1^2 + (2 - x_1)^2 = 4 - 2 = 2,$$

or equivalently,

$$x_1^2 - 2x_1 + 1 = 0.$$

Thus $x_1 = 1 = y_1$.

Now take the row orthogonality relation for rows 1 and 2:

$$\frac{1}{12} 1 \cdot \overline{1} + \frac{1}{4} 1 \cdot \overline{1} + \frac{1}{3} x_2 \cdot \overline{1} + \frac{1}{3} x_3 \cdot \overline{1} = 0.$$

Thus $x_3 = -1 - x_2$, and a similar calculation for rows 1 and 3 yields $y_3 = -1 - y_2$. So our character table is:

g_i	1	(12)(34)	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	x_2	$-1 - x_2$
χ_3	1	1	$-1 - x_2$	x_2
χ_4	3	-1	0	0

To find the value of x_2 take column orthogonality for row 3 with itself:

$$\frac{1}{3}(1 + x_2\overline{x_2} + (-1 - x_2)(-1 - \overline{x_2})) = 1,$$

that is,

$$-1 + x_2 + \overline{x_2} + 2x_2\overline{x_2} = 0.$$

But χ_2 is a linear character, so $x_2 \in \mathbb{C}^*$ is a root of unity, see Corollary 3.2. In particular, $\overline{x_2} = x_2^{-1}$, so that

$$-1 + x_2 + x_2^{-1} + 2 = 0,$$

that is,

$$x_2^2 + x_2 + 1 = 0.$$

Thus $x_2 = \omega$ or $\overline{\omega}$, where $\omega = e^{\frac{2\pi i}{3}}$. Without loss of generality, we can assume $x_2 = \omega$ (since $-1 - \omega = \overline{\omega}$). Thus the character table of A_4 is:

g_i	1	(12)(34)	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

14. Dihedral groups

Let $G = D_{2n}$ with $n \geq 3$ be the dihedral group. We have $G = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$.

We want to calculate the character table of G. There are two cases.

(i) Suppose that n is odd. The conjugacy classes of D_{2n} are:

$$\{1\}, \{a^r, a^{-r}\}(1 \leq r \leq \frac{(n-1)}{2}), \{a^s b \mid 0 \leq s \leq n-1\}.$$

There are $\frac{n+3}{2}$ of these. There are one dimensional representations are given by:

$$\chi_1, \chi_2 : D_{2n} \to \mathbb{C}^*; \chi_1(a) = \chi_2(a) = 1, \chi_i(b) = (-1)^{i+1}.$$

Now there are $\frac{n-1}{2}$ 2-dimensional representations given by:

$$\rho_j : a \mapsto \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $1 \leq j < \frac{n}{2}$ and $\omega = e^{\frac{2\pi i}{n}}$. These are irreducible and pairwise nonisomorphic.

We have thus found all irreducible representations of G, since

$$2 \times 1 + \frac{n-1}{2} \times 4 = 2n = |G|.$$

Let ψ_j denote the character of ρ_j. Then the character table of D_{2n} is:

g_i	1	a^r $(1 \leq r \leq \frac{n-1}{2})$	b
$\lvert C_G(g_i)\rvert$	$2n$	n	2
χ_1	1	1	1
χ_2	1	1	-1
ψ_j	2	$\omega^{rj} + \omega^{-rj}$	$0.$

(ii) Suppose now that n is even, and let $m = \frac{n}{2}$. Then the conjugacy classes of D_{2n} are:

$$\{1\}, \{a^m\}, \{a^r, a^{-r}\}(1 \leq r \leq m-1), \{a^s b \mid s \text{ even}\}, \{a^s b \mid s \text{ odd}\}.$$

There are $m+3$ of them. Arguing as before there are 4 1-dimensional characters, and $m-1$ 2-dimenional ones. The 2-dimensional ones are given by the ρ_j from part (i) (and these are still irreducible and pairwise nonisiomorphic), so the character table of D_{2n} in this case is:

g_i	1	a^m	a^r $(1 \leq r \leq m-1)$	b	ab
$\lvert C_G(g_i)\rvert$	$2n$	$2n$	n	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_4	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	$0.$

15. Finding normal subgroups

Recall that we can compute the kernel of an irreducible character χ from the character table, since $\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}$. Also $\ker(\chi) \trianglelefteq G$. Of course, any subgroup which is an intersection of the kernels of irreducible characters is a normal subgroup, too (just take the kernel of the sum of these characters). The converse is also true:

PROPOSITION 15.1. *If $N \trianglelefteq G$, then there exist irreducible characters $\chi_1, \cdots, \chi_s$ of G such that*

$$N = \bigcap_{i=1}^{s} \ker(\chi_i).$$

Proof. If g belongs to the kernel of each irreducible character of G, then $\chi(g) = \chi(1)$ for all characters χ. This implies $g = 1$ by Corollary 8.4. Thus the intersection of all kernels of all irreducible characters is $\{1\}$. Now let $\tilde{\chi_1}, \cdots, \tilde{\chi_s}$ be irreducible characters of G/N where $N \trianglelefteq G$. By the above observation we get

$$\bigcap_{i=1}^{s} \ker(\tilde{\chi_i}) = \{N\}.$$

For $1 \leq i \leq s$ let χ_i be the lift of $\tilde{\chi_i}$ to G. If $g \in \ker(\chi_i)$, then $\tilde{\chi_i}(N) = \chi_i(1) = \chi_i(g) = \tilde{\chi_i}(gN)$. Thus $gN \in \ker(\tilde{\chi_i})$. Therefore, if $g \in \bigcap_{i=1}^{s} \ker(\chi_i)$, then $gN \in \bigcap_{i=1}^{s} \ker(\tilde{\chi_i}) = \{N\}$. So $g \in N$. Hence $N = \bigcap_{i=1}^{s} \ker(\chi_i)$, □

LEMMA 15.2. *A group G is NOT simple if and only if $\chi(g) = \chi(1)$ for some non-trivial irreducible character χ of G, and some non-identity element g of G.*

Proof. Suppose there is a non-trivial irreducible character χ such that $\chi(g) = \chi(1)$ with $1 \neq g \in G$. Then $g \in \mathrm{Ker}(\chi)$, so $\ker(\chi) \neq \{1\}$. If

ρ is a representation of G with character χ, then $\ker(\chi) = \ker(\rho)$ by Theorem 3.3, (2).

Since χ is non-trivial and irreducible, we get $\ker(\rho) \neq G$. Hence $\ker(\chi) \neq G$. Thus $\ker(\chi)$ is a normal subgroup of G which is not equal to $\{1\}$ or G. So G is not simple.

Conversely, suppose G is not simple. Thus there is a normal subgroup N of G which is not $\{1\}$ or G. Then by Proposition 15.1 there is an irreducible character χ of G such that $\ker(\chi)$ is not $\{1\}$ or G. As $\ker(\chi) \neq G$, χ is non-trivial, and taking $1 \neq g \in \ker(\chi)$ we have $\chi(g) = \chi(1)$. □

16. Linear characters

Recall that a **linear character** of a group is a character of degree 1. We want to find all linear characters of a group G.

If G is a group, then let G' be the subgroup generated by $\{g^{-1}h^{-1}gh \mid g, h \in G\}$. One calls G' the **derived subgroup** of G.

LEMMA 16.1. *If χ is a linear character of G, then $G' \leq \ker(\chi)$.*

Proof. Let χ be a linear character of G. Then χ is a homomorphism $G \to \mathbb{C}^*$. Therefore for all $g, h \in G$ we have $\chi(g^{-1}h^{-1}gh) = \chi(g)^{-1}\chi(h)^{-1}\chi(g)\chi(h) = 1$. Hence $G' \leq \ker(\chi)$. □

LEMMA 16.2. *Assume $N \trianglelefteq G$. Then the folllowing hold:*

1. *$G' \trianglelefteq G$.*
2. *$G' \leq N$ if and only if G/N is abelian. In particular G/G' is abelian.*

Proof. (1): Note that for all $a, b \in G$ we have

$$x^{-1}(ab)x = (x^{-1}ax)(x^{-1}bx)$$

and

$$x^{-1}a^{-1}x = (x^{-1}ax)^{-1}.$$

Now G' consits of elements of the form $g^{-1}h^{-1}gh$ and their inverses. Thus, to prove that $G' \trianglelefteq G$ it is sufficient to prove that

$$x^{-1}(g^{-1}h^{-1}gh)x \in G'$$

for all $g, h, x \in G$. We have

$$\begin{aligned} x^{-1}(g^{-1}h^{-1}gh)x &= (x^{-1}g^{-1}x)(x^{-1}h^{-1}x)(x^{-1}gx)(x^{-1}hx) \\ &= (x^{-1}gx)^{-1}(x^{-1}hx)^{-1}(x^{-1}gx)(x^{-1}hx) \end{aligned}$$

which is in G'.

(2): Let $g, h \in G$. We have $ghg^{-1}h^{-1} \in N$ if and only if $ghN = hgN$ if and only if $gNhN = hNgN$. Hence $G' \leq N$ if and only if G/N is abelian. Since we proved that $G' \trianglelefteq G$ we deduce that G/G' is abelian. □

It follows from Lemma 16.2 that G' is the smallest normal subgroup of G with abelian factor group.

Theorem 16.3. *The linear characters of G are precisely the lifts of the irreducible characters of G/G' to G. In particular, the number of distinct linear characters of G is equal to $|G/G'|$, so it divides $|G|$.*

Proof. Let $m = |G/G'|$. Since G/G' is abelian, we know that G/G' has exactly m irreducible characters, say $\tilde{\chi_1}, \cdots, \tilde{\chi_m}$. All of them have degree 1. The lifts $\chi_1, \cdots, \chi_m$ of these characters to G also have degree 1, thus by Theorem 11.2 they are precisely the irreducible characters χ of G such that $G' \leq \ker(\chi)$. Thus it follows from Lemma 16.1 that the χ_i are therefore all the linear characters of G. □

CHAPTER 3

Tensor products of representations

1. Tensor products

Let V, W be vector spaces over $\mathbb{C}$ with bases $v_1, \cdots, v_m$ and $w_1, \cdots, w_n$, respectively. For each i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$ we introduce a symbol $v_i \otimes w_j$. The **tensor product space** $V \otimes W$ is defined to be the mn-dimensional vector space over $\mathbb{C}$ with basis given by

$$\{v_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq m\}.$$

So $V \otimes W$ consists of all expressions

$$\sum_{i,j} \lambda_{ij}(v_i \otimes w_j)$$

with $\lambda_{ij} \in \mathbb{C}$. For $v \in V$ and $w \in W$ with $v = \sum_{i=1}^m \lambda_i v_i$ and $w = \sum_{j=1}^n \mu_j w_j$ define $v \otimes w \in V \otimes W$ by

$$v \otimes w = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j).$$

Example:

$$(2v_1 - v_2) \otimes (w_1 + w_2) = 2v_1 \otimes w_1 + 2v_1 \otimes w_2 - v_2 \otimes w_1 - v_2 \otimes w_2$$

Warning: Not every element of $V \otimes W$ is of the form $v \otimes w$ for some $v \in V$ and some $w \in W$. For example, $v_1 \otimes w_1 + v_2 \otimes w_2$ is not of this form.

PROPOSITION 1.1. *The following hold:*

(1) *If $v \in V$, $w \in W$ and $\lambda \in \mathbb{C}$, then*

$$v \otimes (\lambda w) = (\lambda v) \otimes w = \lambda(v \otimes w);$$

(2) *If $x_1, \cdots, x_a \in V$ and $y_1, \cdots, y_b \in W$, then*

$$\left(\sum_{i=1}^a x_i\right) \otimes \left(\sum_{j=1}^b y_j\right) = \sum_{i,j} x_i \otimes y_j.$$

Proof: Exercise or see [**JL**, Proposition 19.1].

PROPOSITION 1.2. *Let $e_1, \cdots, e_m$ be a basis of V, and $f_1, \cdots, f_n$ a basis of W. Then*

$$\{e_i \otimes f_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

is a basis of $V \otimes W$.

Proof: Exercise or see [**JL**, Proposition 19.2].

2. Tensor product modules

Let G be a finite group, and let V and W be representations of G. Let $v_1, \cdots, v_m$ be a basis of V, and $w_1, \cdots, w_n$ a basis of W. So by Proposition 1.2 we know that $\{v_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq i \leq n\}$ is a basis of $V \otimes W$. For $g \in G$ and all i, j define

$$\eta : G \to \mathrm{GL}(V \otimes W)$$

$$g \mapsto (\eta(g) : v_i \otimes w_j \mapsto gv_i \otimes gv_j)\,,$$

or more precisely

$$\eta(g) : \sum_{i,j} \lambda_{ij}(v_i \otimes w_j) \mapsto \sum_{i,j} \lambda_{ij}(gv_i \otimes gw_j).$$

By Proposition 1.1 we have $\eta(g)(v \otimes w) = gv \otimes gw$ for all $v \in V$, $w \in W$ and $g \in G$.

PROPOSITION 2.1. *In the way described above, $V \otimes W$ becomes a representation of G.*

Proof. For $g, h \in G$ we have

$$g(v_i \otimes w_j) = gv_i \otimes gw_j \in V \otimes W,$$

$$\begin{aligned}(gh)(v_i \otimes w_j) = (gh)v_i \otimes (gh)w_j &= g(hv_i) \otimes g(hw_j) \\ &= g(hv_i \otimes hw_j) = g(h(v_i \otimes w_j))\end{aligned}$$

and

$$1(v_i \otimes w_j) = v_i \otimes w_j.$$

□

THEOREM 2.2. *Let V and W be representations of G with characters χ and ψ, repsectively. Then the character of the representation $V \otimes W$ of G is the* **product** *$\chi\psi$ of the characters χ and ψ, where*

$$\chi\psi(g) = \chi(g)\psi(g)$$

for all $g \in G$.

Proof. Let $g \in G$. We can choose a basis $e_1, \cdots, e_m$ of V and a basis $f_1, \cdots, f_n$ of W such that $ge_i = \lambda_i e_i$, $1 \leq i \leq m$ and $gf_j = \mu_j f_j$, $1 \leq j \leq n$ for some $\lambda_i, \mu_j \in \mathbb{C}$. Then $\chi(g) = \sum_{i=1}^m \lambda_i$ and $\psi(g) = \sum_{j=1}^n \mu_j$. For $1 \leq i \leq m$ and $1 \leq j \leq n$ we have

$$g(e_i \otimes f_j) = ge_i \otimes gf_j = \lambda_i\mu_j(e_i \otimes f_j),$$

and by Proposition 1.2, the vectors $e_i \otimes f_j$ form a basis of $V \otimes W$. Hence if ϕ is the character of $V \otimes W$, then

$$\phi(g) = \sum_{i,j} \lambda_i\mu_j = \left(\sum_{i=1}^m \lambda_i\right)\left(\sum_{j=1}^n \mu_j\right) = \chi(g)\psi(g).$$

This finishes the proof. $\square$

COROLLARY 2.3. *The product of two characters of G is again a character of G.*

Example:

The character table of S_4 looks as follows:

g_i	1	(12)	(123)	(12)(34)	(1234)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

We have

g_i	1	(12)	(123)	(12)(34)	(1234)
$\chi_3\chi_4$	6	0	0	-2	0
$\chi_4\chi_4$	9	1	0	1	1

Recall that

$$\langle \chi, \psi \rangle = \sum_{i=1}^5 \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$$

for all characters χ and ψ of S_4. We get

$$\chi_3\chi_4 = \chi_4 + \chi_5$$

and

$$\chi_4\chi_4 = \chi_1 + \chi_3 + \chi_4 + \chi_5.$$

PROPOSITION 2.4. *Suppose that χ is a character of G and λ is a linear character of G. Then the product $\lambda\chi$ (defined by $(\lambda\chi)(g) = \lambda(g)\chi(g)$, $g \in G$) is a character of G. Moreover, if χ is irreducible then $\lambda\chi$ is irreducible.*

Proof. The first part follows from Theorem 2.2.

Now for all $g \in G$, the complex number $\lambda(g)$ is a root of unity, so $\lambda(g)\overline{\lambda(g)} = 1$. Therefore

$$\langle \lambda\chi, \lambda\chi \rangle = \frac{1}{|G|} \sum_{g \in G} \lambda(g)\chi(g)\overline{\lambda(g)\chi(g)} = \sum_{g \in G} \chi(g)\overline{\chi(g)} = \langle \chi, \chi \rangle.$$

By Theorem 7.2 it follows that $\lambda\chi$ is irreducible if and only if χ is irreducible. $\square$

3. Decomposing χ^2

Let V be a representation of G with character χ, and let $v_1, \cdots, v_n$ be a basis of V. Define a linear map by

$$T : V \otimes V \to V \otimes V$$

$$v_i \otimes v_j \mapsto v_j \otimes v_i$$

for all $1 \leq i, j \leq n$ and extend this linearly.

It follows that $T(v \otimes w) = w \otimes v$ for all $v, w \in V$. Define

$$S(V \otimes V) = \{x \in V \otimes V \mid Tx = x\},$$

the **symmetric part** of $V \otimes V$, and let

$$A(V \otimes V) = \{x \in V \otimes V \mid Tx = -x\},$$

the **antisymmetric part** of $V \otimes V$. Since T is linear, it follows that $S(V \otimes V)$ and $A(V \otimes V)$ are subspaces of $V \otimes V$ (they are both eigenspaces).

LEMMA 3.1. *$S(V \otimes V)$ and $A(V \otimes V)$ are invariant subspaces of $V \otimes V$ and*

$$V \otimes V = S(V \otimes V) \oplus A(V \otimes V).$$

Proof. For all $\lambda_{ij} \in \mathbb{C}$ and $g \in G$ we have

$$gT\left(\sum_{i,j} \lambda_{ij}(v_i \otimes v_j)\right) = \sum_{i,j} \lambda_{ij}(gv_j \otimes gv_i) = \sum_{i,j} \lambda_{ij} T(gv_i \otimes gv_j)$$
$$= Tg\left(\sum_{i,j} \lambda_{ij}(v_i \otimes v_j)\right).$$

Thus T is a homomorphism. For $x \in S(V \otimes V)$, $y \in A(V \otimes V)$ and $g \in G$ we have

$$T(gx) = g(Tx) = gx$$

and

$$T(gy) = g(Ty) = -gy.$$

So $gx \in S(V \otimes V)$ and $gy \in A(V \otimes V)$. So $S(V \otimes V)$ and $A(V \otimes V)$ are invariant subspaces. If $x \in S(V \otimes V) \cap A(V \otimes V)$, then $Tx = x = -x$, thus $x = 0$. Furthermore, for all $x \in V$ we have

$$x = \frac{1}{2}(x + Tx) + \frac{1}{2}(x - Tx).$$

Since T^2 is the identity, we have

$$\frac{1}{2}(x + Tx) \in S(V \otimes V)$$

and

$$\frac{1}{2}(x - Tx) \in A(V \otimes V).$$

Therefore,

$$V \otimes V = S(V \otimes V) \oplus A(V \otimes V).$$

$\square$

LEMMA 3.2. *Let $v_1, \cdots, v_n$ be a basis of V. Then the following hold:*

(1) $\{v_i \otimes v_j + v_j \otimes v_i \mid 1 \leq i \leq j \leq n\}$ *is a basis of* $S(V \otimes V)$. *In particular,*

$$\dim S(V \otimes V) = \frac{n(n+1)}{2}.$$

(2) $\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \leq i < j \leq n\}$ *is a basis of* $A(V \otimes V)$. *In particular,*

$$\dim A(V \otimes V) = \frac{n(n-1)}{2}.$$

Proof. Check that $v_i \otimes v_j + v_j \otimes v_i$, $1 \leq i \leq j \leq n$, are linearly independent in $S(V \otimes V)$ and that $v_i \otimes v_j - v_j \otimes v_i$, $1 \leq i < j \leq n$, are linearly independent in $A(V \otimes V)$. This implies

$$\dim S(V \otimes V) \geq \frac{n(n+1)}{2}.$$

and

$$\dim A(V \otimes V) \geq \frac{n(n-1)}{2}.$$

By Lemma 3.1, we have

$$\dim S(V \otimes V) + \dim A(V \otimes V) = \dim V \otimes V = n^2.$$

This finishes the proof. □

Let χ_S be the character of $S(V \otimes V)$ and let χ_A be the character of $A(V \otimes V)$. So

$$\chi^2 = \chi_S + \chi_A.$$

LEMMA 3.3. *Let $g \in G$. Then*

$$\chi_S(g) = \frac{1}{2}\left(\chi^2(g) + \chi(g^2)\right)$$

and

$$\chi_A(g) = \frac{1}{2}\left(\chi^2(g) - \chi(g^2)\right).$$

Proof. We can choose a basis $e_1, \cdots, e_n$ of V such that $ge_i = \lambda_i e_i$, $1 \leq i \leq n$ for some $\lambda_i \in \mathbb{C}$. Then

$$g(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i)$$

for $1 \leq i < j \leq n$. Thus by Lemma 3.2,(2) we get $\chi_A(g) = \sum_{i<j} \lambda_i \lambda_j$. Now $g^2 e_i = \lambda_i^2 e_i$. So $\chi(g) = \sum_i \lambda_i$ and $\chi(g^2) = \sum_i \lambda_i^2$. Therefore,

$$\chi^2(g) = (\chi(g))^2 = \sum_i \lambda_i^2 + 2\sum_{i<j} \lambda_i \lambda_j = \chi(g^2) + 2\chi_A(g).$$

Hence

$$\chi_A(g) = \frac{1}{2}\left(\chi^2(g) - \chi(g^2)\right).$$

Also $\chi^2 = \chi_S + \chi_A$. So

$$\chi_S(g) = \chi^2(g) - \chi_A(g) = \frac{1}{2}\left(\chi^2(g) + \chi(g^2)\right).$$

□

To illustrate this: [**JL**, Example 19.15].

4. The character table of S_5

Let $G = S_5$. Here some data on the conjugacy classes of G:

g_i	1	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
$\lvert g_i^G \rvert$	1	10	20	15	30	20	24
$\lvert C_G(g_i) \rvert$	120	12	6	8	4	6	5

So there are exactly 7 irreducible characters of S_5.

(a) Linear characters:

We have two linear characters: the trivial character and the signature character. Call these χ_1 and χ_2. We get

$$\chi_1 = 1_G$$

and

$$\chi_2(g) = \begin{cases} 1 & \text{if } g \text{ is an even permutation,} \\ -1 & \text{else.} \end{cases}$$

(b) The permutation character: By Lemma 12.1, we get a character $\nu : G \to \mathbb{C}$ with $\nu(g) = |\mathrm{fix}(g)| - 1$. We get

$$\langle \nu, \nu \rangle = \frac{4^2}{120} + \frac{2^2}{12} + \frac{1^2}{6} + \frac{(-1)^2}{6} + \frac{(-1)^2}{5} = 1.$$

Hence $\chi_3 = \nu$ is irreducible.

Define $\chi_4 = \chi_3\chi_2$. Since χ_3 is irreducible and χ_2 is linear, we know that χ_4 is irreducible. Thus, up to this point we get

g_i	1	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1

(c) Tensor products:

Write $\chi = \chi_3$. Compute χ_S and χ_A:

g_i	1	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_S	10	4	1	2	0	1	0
χ_A	6	0	0	-2	0	0	1

We have

$$\langle \chi_A, \chi_A \rangle = \frac{36}{120} + \frac{4}{8} + \frac{1}{5} = 1.$$

Set $\chi_5 = \chi_A$. Next,

$$\langle \chi_S, \chi_1 \rangle = \frac{10}{120} + \frac{4}{12} + \frac{1}{6} + \frac{2}{8} + \frac{1}{6} = 1,$$
$$\langle \chi_S, \chi_3 \rangle = \frac{40}{120} + \frac{8}{12} + \frac{1}{6} - \frac{1}{6} = 1,$$
$$\langle \chi_S, \chi_S \rangle = \frac{100}{120} + \frac{16}{12} + \frac{1}{6} + \frac{4}{8} + \frac{1}{6} = 3.$$

Note that $\chi_S(1) = 10$. We have $\chi_S = \sum_{i=1}^{7} n_i \chi_i$ and $\langle \chi_S, \chi_S \rangle = \sum_{i=1}^{7} n_i^2 = 3$. We also know that $n_1 = n_2 = 1$. So

$$\chi_S = \chi_1 + \chi_3 + \psi$$

where ψ is irreducible of degree 5.

Set $\chi_6 = \psi$. So $\chi_6 = \chi_S - \chi_1 - \chi_3$. Define $\chi_7 = \chi_6 \chi_2$ which is again a different irreducible character of degree 5.

Altogether we get the complete character table of S_5:

g_i	1	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0

CHAPTER 4

Burnside's theorem

1. Algebraic integers

A complex number λ is an **algebraic integer** if and only if λ is an eigenvalue of some matrix which has only integer entries. Thus for λ an algebraic integer we require that $\det(A - \lambda I) = 0$ for some matrix A with integer entries.

It is easy to check that some complex number λ is an algebraic integer if and only if λ is a root of a polynomial in $\mathbb{Z}[X]$, the ring of integer polynomials.

THEOREM 1.1. *If λ and μ are algebraic integers, then $\lambda\mu$ and $\lambda + \mu$ are also algebraic integers.*

Proof: [**JL**, Theorem 22.3].

COROLLARY 1.2. *If χ is a character of G and $g \in G$, then $\chi(g)$ is an algebraic integer.*

Proof. We know that $\chi(g)$ is a sum of mth roots of 1 for some m. Each root of 1 is an algebraic integer, since it is a root of the polynomial $\sum_{i=0}^{n-1} X^i$. Now we use the previous theorem. $\square$

PROPOSITION 1.3. *If λ is both a rational number and an algebraic integer, then λ is an integer.*

Proof. Suppose that λ is a rational number which is not an integer. We want to show that λ is not an algebraic integer. Write $\lambda = \frac{r}{s}$ where r and s are coprime integers and $s \neq 1, -1$. Let p be a prime which divides s. For every $n \times n$-matrix A of integers, the entries of $sA - rI$ which are not on the diagonal are divisible by s, and hence also by p. Therefore

$$\det(sA - rI) = (-r)^n + mp$$

for some integer m. As p does not divide r (since r and s are coprime), we deduce that $\det(sA - rI) \neq 0$. Thus

$$\det(A - \lambda I) = \left(\frac{1}{s}\right)^n \det(sA - rI) \neq 0,$$

hence λ is not an algebraic integer. $\square$

The next corollary follows directly from Proposition 1.3 and Corollary 1.2.

COROLLARY 1.4. *Let χ be a character of G and let $g \in G$. If $\chi(g)$ is a rational number, then $\chi(g)$ is an integer.*

2. The degree of every irreducible character divides $|G|$

Using the group algebra $\mathbb{C}G$ and the description of its centre, one can show the following:

LEMMA 2.1. *If χ is an irreducible character of G and $g \in G$, then*

$$\lambda = \frac{|G|}{|C_G(g)|} \frac{\chi(g)}{\chi(1)}$$

is an algebraic integer.

THEOREM 2.2. *If χ is an irreducible character of G, then $\chi(1)$ divides $|G|$.*

Proof. Let $g_1, \cdots, g_k$ be representatives of the conjugacy classes of G. Then for all i, both

$$\frac{|G|}{|C_G(g_i)|} \frac{\chi(g_i)}{\chi(1)}$$

and $\chi(g_i)$ are algebraic integers by Lemma 2.1 and Corollary 1.2. Hence by Theorem 1.1 we get that

$$\sum_{i=1}^{k} \frac{|G|}{|C_G(g_i)|} \frac{\chi(g_i)\overline{\chi(g_i)}}{\chi(1)}$$

is an algebraic integer. By the row orthogonality relations, this algebraic integer equals

$$\frac{|G|}{\chi(1)}.$$

As this is a rational number, we get from Proposition 1.3 that it is an integer. Thus $\chi(1)$ divides $|G|$. $\square$

LEMMA 2.3. *Let χ be a character of a finite group G, and let $g \in G$. Then $|\chi(g)/\chi(1)| \leq 1$, and if $0 < |\chi(g)/\chi(1)| < 1$ then $\chi(g)/\chi(1)$ is not an algebraic integer.*

Proof. Let $\chi(1) = d$. We know that $\chi(g) = \omega_1 + \cdots + \omega_d$ where the ω_i are roots of 1. Thus

$$\frac{\chi(g)}{\chi(1)} = \frac{(\omega_1 + \cdots + \omega_d)}{d}.$$

Since $|\chi(g)| = |\omega_1 + \cdots + \omega_d| \leq |\omega_1| + \cdots + |\omega_d| = d$, it follows that $|\frac{\chi(g)}{\chi(1)}| \leq 1$. Now suppose that $\frac{\chi(g)}{\chi(1)}$ is an algebraic integer and $|\frac{\chi(g)}{\chi(1)}| < 1$. We prove that $\chi(g) = 0$. Write $\gamma = \chi(g)/\chi(1)$, and let $p(x)$ be the minimum polynomial of γ, so that $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_i \in \mathbb{Z}$ for all i. Each root of $p(x)$ is of the form $\frac{(\omega_1' + \cdots + \omega_d')}{d}$ where ω_i' are roots of 1. This follows from a little Galois Theory. Thus each root has modulus at most 1. Let λ be the product of all roots of $p(x)$. Thus $|\lambda| < 1$. But the product of all roots of a polynomial is just the constant term a_0 or its negative. Since $a_0 \in \mathbb{Z}$ and $|\lambda| < 1$, it follows that $a_0 = 0$. As $p(x)$ is irreducible, this implies $p(x) = x$, which in turn forces $\gamma = 0$. Thus $\chi(g) = 0$. This finishes the proof. □

3. Burnside's theorem

THEOREM 3.1. *Let p be a prime number and let r be an integer with $r \geq 1$. Suppose that G is a finite group with a conjugacy class of size p^r. Then G is not simple.*

Proof. Let $g \in G$ with $|g^G| = p^r$. Since $p^r > 1$, G is not abelian and $g \neq 1$. As usual, denote the irreducible characters of G by $\chi_1, \cdots, \chi_k$ where χ_1 is the trivial character.

The column orthogonality relations in Theorem 10.1, (2) applied to the columns corresponding to 1 and g in the character table of G give

$$1 + \sum_{i=2}^{k} \chi_i(g)\chi_i(1) = 0.$$

Therefore

$$\sum_{i=2}^{k} \chi_i(g)\frac{\chi_i(1)}{p} = \frac{-1}{p}.$$

By Proposition 1.3, the number $\frac{-1}{p}$ is not an algebraic integer. Therefore for some $i \geq 2$,

$$\chi_i(g)\frac{\chi_i(1)}{p}$$

is not an algebraic integer. This follows from Theorem 1.1.

Since $\chi_i(g)$ is an algebraic integer by Corollary 1.2, it follows that $\frac{\chi_i(1)}{p}$ is not an algebraic integer. In other words, p does not divide $\chi_i(1)$. In particular $\chi_i(g) \neq 0$. As $|g^G| = p^r$, this means that $\chi_i(1)$ and $|g^G|$ are coprime integers. So there are integers a and b such that

$$a\frac{|G|}{|C_G(g)|} + b\chi_i(1) = 1.$$

Hence

$$a\frac{|G|\chi_i(g)}{|C_G(g)|\chi_i(1)} + b\chi_i(g) = \frac{\chi_i(g)}{\chi_i(1)}.$$

By Lemma 2.1 and Corollary 1.2 the left hand side of this equation is an algebraic integer, and since $\chi_i(g) \neq 0$, the left hand side is non-zero. Now Lemma 2.3 implies that $|\chi(g)/\chi(1)| = 1$.

Now let ρ be a representation of G with character χ_i. We know that there exists some $\lambda \in \mathbb{C}$ such that

$$\rho(g) = \lambda I$$

since $|\chi_i(g)| = \chi_i(1)$, see II, Theorem 3.3. Let $K = \ker(\rho)$. Thus $K \trianglelefteq G$. Since χ_i is not the trivial character ($i \neq 1$), we have $K \neq G$. If $K \neq \{1\}$, then G is not simple, as required.

So assume $K = \{1\}$. Thus ρ is a faithful representation. Since $\rho(g) = \lambda I$, the element $\rho(g)$ commutes with $\rho(h)$ for all $h \in G$. As ρ is faithful, it follows that g commutes with all $h \in G$. So $g \in Z(G)$. Therefore $Z(G) \neq 1$. But $Z(G) \trianglelefteq G$, and $Z(G) \neq G$, since G is not abelian. Thus G is not simple. This finishes the proof. $\square$

LEMMA 3.2. *Let G be a group of order p^n with $n \geq 1$ and p a pime. If $1 \neq H \trianglelefteq G$, then $H \cap Z(G) \neq 1$. In particular, $Z(G) \neq 1$.*

Proof. Since $H \trianglelefteq G$, H is a union of conjugacy classes of G, all of which have size a power of p. Furthermore, $H \cap Z(G)$ consists of those conjugacy classes in H which have size 1. So

$$|H| = |H \cap Z(G)| + mp$$

for some m. But also $|H| = m'p$ for some m', and $|H \cap Z(G)| \neq 0$, since $1 \in H \cap Z(G)$. So $H \cap Z(G) \neq 1$. $\square$

THEOREM 3.3 (Burnside's Theorem). *Let p, q be prime numbers, and let $a, b \geq 0$ be natural numbers with $a + b \geq 2$. If G is a group with $|G| = p^a q^b$, then G is not simple.*

Proof. First, suppose $a = 0$ or $b = 0$. Then $|G|$ is a prime power. So by Lemma 3.2 we have $Z(G) \neq 1$. Choose $g \in Z(G)$ of prime order. Then $< g > \trianglelefteq G$ with $1 \neq < g > \neq G$. So G is not simple. Next, assume $a > 0$ and $b > 0$. By Sylow's Theorem, G has a subgroup Q of order q^b. We have $Z(Q) \neq 1$ by Lemma 3.2. Let $g \in Z(Q)$ with $g \neq 1$. Then $Q \leq C_G(g)$, so

$$|g^G| = \frac{|G|}{|C_G(g)|} = p^r$$

for some r. If $p^r = 1$, then $g \in Z(G)$, so $Z(G) \neq 1$ and G is not simple as before. If $p^r > 1$, then G is not simple by Theorem 3.1. This finishes the proof. □

Bibliography

[JL] *Gordon James, Martin Liebeck*, Representations and Characters of Groups. Second edition. Cambridge University Press, New York 2001. viii+458pp.