Representation theory of finite groups

Formalization project

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Outline

Representations of finite groups

Definition

For a group G and a field k, a **representation** of G over k is a pair (V, ρ) where V is a vector space over k and $\rho : G \to \mathrm{GL}(V)$ is an action of G on V.

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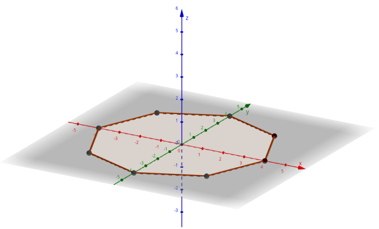
Convention: V has finite dimension, unless explicitly stated otherwise.

Definition

$\dim(V)$ is the **dimension** or **degree** of (V, ρ).

Example

$$D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$$



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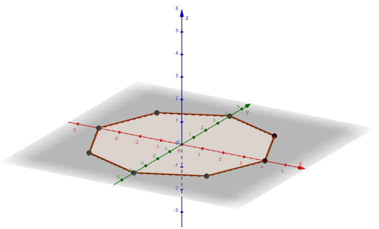

Representation $\rho : D_{2n} \to \mathrm{GL}(\mathbb{R}^3)$ with

- $\rho(a)$ as rotation about the Z-axis
- $\rho(b)$ as a rotation about a suitable axis in the XY-plane

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Let V be a representation and $U \subseteq V$ a subspace. U is an **invariant subspace** if $gu \in U$ for $\forall u \in U, g \in G$.

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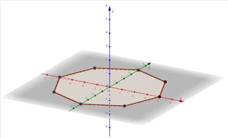

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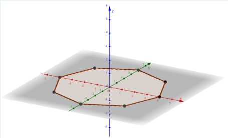

The XY-Plane is an invariant subspace.

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- $\mathrm{Im}(\theta)$ and $\mathrm{Ker}(\theta)$ are invariant subspaces
- If V and W are irreducible, then $\theta : V \to W$ is 0 or bijective.

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Lemma

Every Representation Endomorphism is given by multiplication with a scalar.

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Now, we can use proof by contradiction:

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2. Then, V has a proper subspace with dimension 1.
3. So V has a proper invariant subspace.
4. This is a contradiction to the irreducibility of V.

Representations in Mathlib

Representations are already defined in Mathlib:

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abbrev Representation :=
  G →* V →ₗ[k] V

end
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Apart from that, the definitions introduced in the beginning of this presentation are missing in Mathlib.

Definitions

```
/-- A predicate for a subspace being invariant -/
def IsInvariantSubspace {k G V : Type*} [CommSemiring k]
    [Monoid G] [AddCommMonoid V] [Module k V]
  (U : Submodule k V) (ρ : Representation k G V) :=
  ∀ g : G, ∀ u : U, ρ g u ∈ U

/-- defines degree of a representation as rank of its
    module -/
def degree {k G V : Type*} [CommSemiring k] [Monoid G]
    [AddCommMonoid V] [Module k V]
  (ρ : Representation k G V) : Cardinal := (Module.rank k
    V)
```

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```
/-- Definition of Homomorhpisms between Representations -/
@[ext] class RepresentationHom {k G V W : Type*}
    [CommSemiring k] [Monoid G] [AddCommMonoid V] [Module
    k V] [AddCommMonoid W] [Module k W]
  (ρ : Representation k G V) (ψ : Representation k G W)
    extends LinearMap (RingHom.id k) V W where
  reprStructure : ∀ g : G, ∀ v : V, toLinearMap (ρ g v) =
    ψ g (toLinearMap v)
```

Coercions

```
/-- Coercions of RepresentationHom to Function and Linear
    Map-/
instance {k G V W : Type*} [CommSemiring k] [Monoid G]
    [AddCommMonoid V] [Module k V] [AddCommMonoid W]
    [Module k W] {ρ : Representation k G V} {ψ :
    Representation k G W} : CoeFun (RepresentationHom ρ ψ
    ) (fun _ ↦ V →ₗ[k] W) where
  coe := by
    intro θ
    use ⟨θ.toFun, ?_⟩
    simp; intro x y; simp
```

Registering Instances

Lemma

For all $g \in G$, $\rho(g)$ is a Representation Endomorphism.

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```
instance repr_yields_reprHom_commMonoid {k G V : Type*}
    [CommSemiring k] [CommMonoid G] [AddCommMonoid V]
    [Module k V]
  (ρ : Representation k G V) (g : G) : (RepresentationHom
    ρ ρ) where
  toFun := ρ g
  map_add' := by intro x y; simp
  map_smul' := by intro m x; simp
  reprStructure := sorry
```

Representation Theory in Mathlib

Things already contained in Mathlib:

- Definition of Representations, basic properties, duality
- Category Theory
- Characters

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Things not contained in Mathlib:

- Subrepresentations, Homomorphisms, Irreducibility
- Direct sums, reducibility, Maschke for Representations
- ...

Two views on Representations

A representation (V, ρ) can be “translated” to a module over a k-algebra kG:

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Take a k-module with basis $\{g | g \in G\}$ and multiplication

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with action

$$kG \times V \to V, \left(\sum_{g \in G} \lambda_g g, v\right) \mapsto \sum_{g \in G} \lambda_g (gv).$$

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Irreducibility of representations translates to simplicity of modules.

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- `Representation.asModule` translates `Representation` to algebra
- `Representation.ofModule` translates algebra to `Representation`

Future work

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- Connect Algebras and Representations: Irreducibility, ...
- Formulate theorem of Maschke for Representations
- Add additional theorems about Representation Theory of finite groups
- ...

Questions?