Representation theory of finite groups

Formalization project

Raphael Gaedtke, Paul Neumann

University of Bonn

January 10, 2025

Outline

Representations of finite groups

Definition

For a group G and a field k, a **representation** of G over k is a pair (V, ρ) where V is a vector space over k and $\rho : G \to \mathrm{GL}(V)$ is an action of G on V.

Representations of finite groups

Definition

For a group G and a field k, a **representation** of G over k is a pair (V, ρ) where V is a vector space over k and $\rho : G \to \mathrm{GL}(V)$ is an action of G on V.

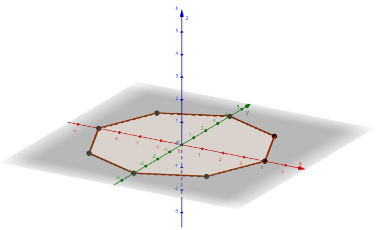
Convention: V has finite dimension, unless explicitly stated otherwise.

Definition

$\dim(V)$ is the **dimension** or **degree** of (V, ρ).

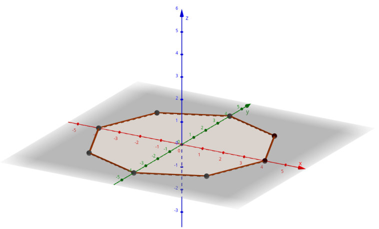
Example

$$D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$$



Example

$$D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$$



Representation $\rho : D_{2n} \to \mathrm{GL}(\mathbb{R}^3)$ with

- $\rho(a)$ as rotation about the Z-axis
- $\rho(b)$ as a rotation about a suitable axis in the XY-plane

Invariant subspaces, Irreducibility

Definition

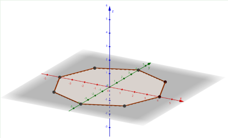
Let V be a representation and $U \subseteq V$ a subspace. U is an **invariant subspace** if $gu \in U$ for $\forall u \in U, g \in G$.

Invariant subspaces, Irreducibility

Definition

Let V be a representation and $U \subseteq V$ a subspace. U is an **invariant subspace** if $gu \in U$ for $\forall u \in U, g \in G$.

Example

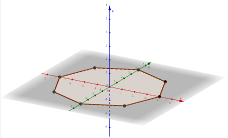


Invariant subspaces, Irreducibility

Definition

Let V be a representation and $U \subseteq V$ a subspace. U is an **invariant subspace** if $gu \in U$ for $\forall u \in U, g \in G$.

Example



The XY-Plane is an invariant subspace.

Irreducibility, Representation Homomorphisms

Definition

A representation V is **irreducible** provided $V \neq 0$ and the only invariant subspaces are 0 and V.

Irreducibility, Representation Homomorphisms

Definition

A representation V is **irreducible** provided $V \neq 0$ and the only invariant subspaces are 0 and V.

Definition

For representations V and W, a **homomorphism** is a linear map $\theta : V \to W$ with $\theta(gv) = g\theta(v)$ for $\forall g \in G, v \in V$.

Irreducibility, Representation Homomorphisms

Definition

A representation V is **irreducible** provided $V \neq 0$ and the only invariant subspaces are 0 and V.

Definition

For representations V and W, a **homomorphism** is a linear map $\theta : V \to W$ with $\theta(gv) = g\theta(v)$ for $\forall g \in G, v \in V$.

- $\mathrm{Im}(\theta)$ and $\mathrm{Ker}(\theta)$ are invariant subspaces
- If V and W are irreducible, then $\theta : V \to W$ is 0 or bijective.

Main theorem formalized in this project

This theorem is listed on the "Missing undergraduate mathematics in mathlib"-page.

Main theorem formalized in this project

This theorem is listed on the "Missing undergraduate mathematics in mathlib"-page.

Theorem

Let G be a finite abelian group.
Let V be a non-null vector space over an algebraically closed field k.
Let $\rho : G \to GL(V)$ be a representation.

Main theorem formalized in this project

This theorem is listed on the "Missing undergraduate mathematics in mathlib"-page.

Theorem

Let G be a finite abelian group.
Let V be a non-null vector space over an algebraically closed field k.
Let $\rho : G \to GL(V)$ be a representation.

Then ρ is irreducible if and only if $\dim_k(V) = 1$.

Proof of the theorem

Theorem

ρ is irreducible if and only if $\dim_k(V) = 1$.

"$\Leftarrow$" is trivial.
For "$\Rightarrow$", we use the following two lemmas:

Lemma

For all $g \in G$, $\rho(g)$ is a Representation Endomorphism.

Lemma

Every Representation Endomorphism is given by multiplication with a scalar.

Proof of the theorem

With these two lemmas, we can prove the following fact:

Lemma

Every one-dimensional subspace of V is an invariant subspace.

Proof of the theorem

With these two lemmas, we can prove the following fact:

Lemma

Every one-dimensional subspace of V is an invariant subspace.

Now, we can use proof by contradiction:

Proof of the theorem

With these two lemmas, we can prove the following fact:

Lemma

Every one-dimensional subspace of V is an invariant subspace.

Now, we can use proof by contradiction:

1. Assume $\dim(V) > 1$.

Proof of the theorem

With these two lemmas, we can prove the following fact:

Lemma

Every one-dimensional subspace of V is an invariant subspace.

Now, we can use proof by contradiction:

1. Assume $\dim(V) > 1$.
2. Then, V has a proper subspace with dimension 1.

Proof of the theorem

With these two lemmas, we can prove the following fact:

Lemma

Every one-dimensional subspace of V is an invariant subspace.

Now, we can use proof by contradiction:

1. Assume $\dim(V) > 1$.
2. Then, V has a proper subspace with dimension 1.
3. So V has a proper invariant subspace.

Proof of the theorem

With these two lemmas, we can prove the following fact:

Lemma

Every one-dimensional subspace of V is an invariant subspace.

Now, we can use proof by contradiction:

1. Assume $\dim(V) > 1$.
2. Then, V has a proper subspace with dimension 1.
3. So V has a proper invariant subspace.
4. This is a contradiction to the irreducibility of V.

Representations in Mathlib

Representations are already defined in Mathlib:

```
abbrev Representation :=
  G →* V →ₗ[k] V

end
```

Representations in Mathlib

Representations are already defined in Mathlib:

```
abbrev Representation :=
  G →* V →ₗ[k] V

end
```

Apart from that, the definitions introduced in the beginning of this presentation are missing in Mathlib.

Definitions

```
/-- A predicate for a subspace being invariant -/
def IsInvariantSubspace {k G V : Type*} [CommSemiring k]
    [Monoid G] [AddCommMonoid V] [Module k V]
  (U : Submodule k V) (ρ : Representation k G V) :=
  ∀ g : G, ∀ u : U, ρ g u ∈ U

/-- defines degree of a representation as rank of its
    module -/
def degree {k G V : Type*} [CommSemiring k] [Monoid G]
    [AddCommMonoid V] [Module k V]
  (ρ : Representation k G V) : Cardinal := (Module.rank k
    V)
```