Representation theory of finite groups

Formalization project

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Outline

Representations of finite groups

Definition

For a group G and a field k, a **representation** of G over k is a pair (V, ρ) where V is a vector space over k and $\rho : G \to \mathrm{GL}(V)$ is an action of G on V.

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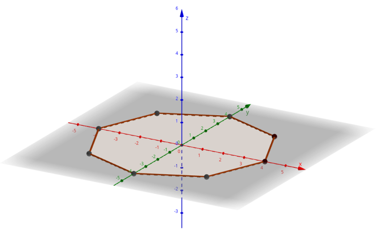
Convention: V has finite dimension, unless explicitly stated otherwise.

Definition

$\dim(V)$ is the **dimension** or **degree** of (V, ρ).

Example

$$D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$$



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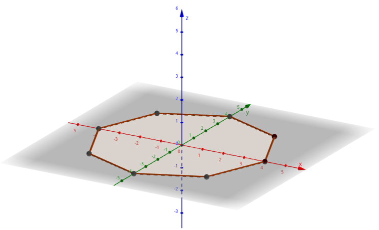

Representation $\rho : D_{2n} \to \mathrm{GL}(\mathbb{R}^3)$ with

- $\rho(a)$ as rotation about the Z-axis
- $\rho(b)$ as a rotation about a suitable axis in the XY-plane

Invariant subspaces, Irreducibility

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Let V be a representation and $U \subseteq V$ a subspace. U is an **invariant subspace** if $gu \in U$ for $\forall u \in U, g \in G$.

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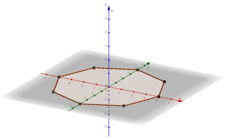

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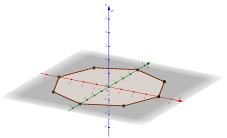

The XY-Plane is an invariant subspace.

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- $\mathrm{Im}(\theta)$ and $\mathrm{Ker}(\theta)$ are invariant subspaces
- If V and W are irreducible, then $\theta : V \to W$ is 0 or bijective.

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Let G be a finite abelian group.
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Proof of the theorem

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"$\Leftarrow$" is trivial.
For "$\Rightarrow$", we use the following two lemmas:

Lemma

For all $g \in G$, $\rho(g)$ is a Representation Endomorphism.

Lemma

Every Representation Endomorphism is given by multiplication with a scalar.

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1. Assume $\dim(V) > 1$.
2. Then, V has a proper subspace with dimension 1.
3. So V has a proper invariant subspace.
4. This is a contradiction to the irreducibility of V.