

# Leibniz Integral Rule Proof

Moussa El Moussaoui

In this document we will consider the following:

- $(\Omega, \mathcal{F}, \mu)$  is a measure space.
- $(E, d)$  is a metric space.
- $\mathfrak{F}(X, Y) = Y^X$  is the set of all the functions  $f: X \rightarrow Y$ , where  $X$  and  $Y$  are sets.
- $\mathbb{N}_0 := \{0, 1, \dots\}$  and  $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ .
- $[0, \infty] := [0, \infty] \cup \{\infty\}$ .
- $\mathcal{Y} := \{\mathbb{R}, [0, \infty]\}$ .
- $\mathcal{I} := \{[a, b] : (a, b) \in \mathbb{R}^2 \text{ and } a < b\}$  is the set of all open intervals of  $\mathbb{R}$ .
- $B(x_0, r) := \{x \in E : d(x, x_0) < r\}$  where  $x_0 \in E$  and  $r > 0$ .
- $m \in \mathbb{N}$  and  $p \in \mathbb{N}$  with  $p \geq 2$ .
- $\llbracket 1, p \rrbracket := [1, p] \cap \mathbb{N}$ .
- $\delta: \llbracket 1, p \rrbracket^2 \rightarrow \mathbb{R}$  is the function defined by:

$$\forall (i, j) \in \llbracket 1, p \rrbracket^2, \delta(i, j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For  $i \in \llbracket 1, p \rrbracket$ :

- $e_i := (\delta_{i1}, \dots, \delta_{ip})$ .
- $\mathcal{I}_i(x, A) := \{\mathbb{I} \in \mathcal{I} : 0 \in \mathbb{I} \text{ and } \forall t \in \mathbb{I}, x + te_i \in A\}$  where  $x \in A$  and  $A \subset \mathbb{R}^p$ .

For  $Y \in \mathcal{Y}$ :

- $\{f < t\} := \{\omega \in \Omega : f(\omega) < t\}$  where  $f: \Omega \rightarrow Y$  is a function and  $t \in \mathbb{R}$ .
- $\mathcal{L}(\Omega, \mathcal{F}, Y) := \{f: \Omega \rightarrow Y : \forall t \in \mathbb{R}, \{f < t\} \in \mathcal{F}\}$ .
- $\mathcal{L}^1(\Omega, \mathcal{F}, \mu, Y) := \{f: \Omega \rightarrow Y : f \in \mathcal{L}(\Omega, \mathcal{F}, Y) \text{ and } \int |f| d\mu < \infty\}$ .

For  $A \subset E$ :

- $A^\circ := \{x \in E : \exists r > 0, B(x, r) \subset A\}$ .
- $A' := \{x \in E : \forall r > 0, [B(x, r) \cap A] \setminus \{x\} \neq \emptyset\}$ .
- $\mathcal{T}_p := \{\mathcal{U} \subset \mathbb{R}^p : \mathcal{U} = \mathcal{U}^\circ\}$  is the set of all open subsets of  $\mathbb{R}^p$ .

### Definition 1

Let  $X, Y$  and  $Z$  be sets,  $f: X \times Y \rightarrow Z$  is a function.

- $\forall x \in X, f(x, \cdot): Y \rightarrow Z$  is the function defined by:

$$\forall y \in Y, f(x, \cdot)(y) = f(x, y).$$

- $\forall y \in Y, f(\cdot, y): X \rightarrow Z$  is the function defined by:

$$\forall x \in X, f(\cdot, y)(x) = f(x, y).$$

### Definition 2

Let  $I \in \mathcal{I}, f: I \times \Omega \rightarrow \mathbb{R}$  is a function,  $t_0 \in I$  and  $\omega \in \Omega$ .  
If the function  $f(\cdot, \omega)$  has a derivative at  $t_0$ , we define:

$$\partial_1 f(t_0, \omega) = \frac{\partial f}{\partial t}(t_0, \omega) := \lim_{t \rightarrow t_0} \frac{f(t, \omega) - f(t_0, \omega)}{t - t_0}.$$

### Definition 3

Let  $\mathcal{U} \in \mathcal{T}_p, f: \mathcal{U} \times \Omega \rightarrow \mathbb{R}$  is a function,  $i \in [1, p]$ ,  $x_0 \in \mathcal{U}$  and  $\omega \in \Omega$ .  
If the function  $f(\cdot, \omega)$  has an  $i$ th partial derivative at  $x_0$ , we define:

$$\partial_i f(x_0, \omega) = \frac{\partial f}{\partial x_i}(x_0, \omega) := \lim_{t \rightarrow 0} \frac{f(x_0 + te_i, \omega) - f(x_0, \omega)}{t}.$$

### Definition 4

Let  $A$  and  $X$  be sets. An indexed family  $(x_\alpha)_{\alpha \in A} \in X^A$  is a function  $x: A \rightarrow X$ .  
We have the following:

- $A$  is called the index set.
- $\forall \alpha \in A, x_\alpha := x(\alpha)$ .
- $\{x_\alpha\}_{\alpha \in A} := \{x_\alpha : \alpha \in A\}$ .
- if  $A \in \{\mathbb{N}, \mathbb{N}_0\}$ , the indexed family  $(x_\alpha)_{\alpha \in A} \in X^A$  is called a sequence.

### Theorem 1

Let  $f: E \rightarrow \mathbb{R}$  be a function and  $x_0 \in E$ . The following are equivalent:

- $\lim_{x \rightarrow x_0} f(x) = f(x_0)$  ( $f$  is continuous at  $x_0$ ).
- $\forall (x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .

### Theorem 2

Let  $A \subset \mathbb{R}^m$ ,  $f: A \rightarrow \mathbb{R}$  is a function,  $x_0 \in A'$  and  $\ell \in \mathbb{R}$ . The following are equivalent:

- $\lim_{x \rightarrow x_0} f(x) = \ell$ .
- $\forall (x_n)_{n \in \mathbb{N}} \in (A \setminus \{x_0\})^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = \ell$ .

### Theorem 3

Let  $(f, g) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2$  and  $\alpha \in \mathbb{R}$ . Then:

$$(\alpha f, f + g) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2.$$

### Theorem 4

Let  $f \in \mathfrak{F}(\Omega, \mathbb{R})$  and  $(f_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$  such that:

- $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ ;
- $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$ .

Then:

$$f \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R}).$$

### (Доказательство)

Let  $g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$ ,  $f \in \mathfrak{F}(\Omega, \mathbb{R})$  and  $(f_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$  such that:

- $f \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ ;
- $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ ;
- $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$ ;
- $\forall \omega \in \Omega, \forall n \in \mathbb{N}, |f_n(\omega)| \leq g(\omega)$ .

Then:

- $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$ .

### Theorem a

Let  $x_0 \in E$ ,  $f: E \times \Omega \rightarrow \mathbb{R}$  is a function such that:

- $(a_1) \forall x \in E, f(x, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $(a_2) \forall \omega \in \Omega, f(\cdot, \omega)$  is continuous at  $x_0$ ;
- $(a_3) \exists g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$  such that:

$$\forall (x, \omega) \in E \times \Omega, |f(x, \omega)| \leq g(\omega).$$

Then:

- The function  $F: E \rightarrow \mathbb{R}$  defined by:

$$\forall x \in E, F(x) := \int_{\Omega} f(x, \omega) d\mu(\omega)$$

is continuous at  $x_0$ .

#### Proof:

Let  $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ , assume that  $\lim_{n \rightarrow \infty} x_n = x_0$ .

Define  $\varphi: \Omega \rightarrow \mathbb{R}$  by:  $\varphi := f(x_0, \cdot)$ .

$(a_1)$  gives  $\varphi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$  so:  $\varphi \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ .

Define  $(\varphi_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$  by:  $\forall n \in \mathbb{N}, \varphi_n := f(x_n, \cdot)$ .

$(a_1)$  gives  $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$  so:  $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ .

From  $(a_2)$  we get:

$$\forall \omega \in \Omega, \lim_{x \rightarrow x_0} f(x, \omega) = f(x_0, \omega).$$

And by **Theorem 1** we have:

$$\forall \omega \in \Omega, \lim_{n \rightarrow \infty} f(x_n, \omega) = f(x_0, \omega).$$

Which is the same as:  $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} \varphi_n(\omega) = \varphi(\omega)$ .

We know that:  $\forall n \in \mathbb{N}, x_n \in E$ .

So from  $(a_3)$  we get:

$$\forall \omega \in \Omega, \forall n \in \mathbb{N}, |f(x_n, \omega)| \leq g(\omega).$$

Which is the same as:  $\forall \omega \in \Omega, \forall n \in \mathbb{N}, |\varphi_n(\omega)| \leq g(\omega)$ .

Then, we can apply  $(\mathfrak{DET})$  to get:

$$\lim_{n \rightarrow \infty} \int \varphi_n d\mu = \int \varphi d\mu.$$

With:

$$\forall n \in \mathbb{N}, \int \varphi_n d\mu = \int_{\Omega} f(x_n, \omega) d\mu(\omega) = F(x_n);$$

and:

$$\int \varphi d\mu = \int_{\Omega} f(x_0, \omega) d\mu(\omega) = F(x_0).$$

Since  $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$  is arbitrary, we have proven that:

$$\forall (x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} F(x_n) = F(x_0).$$

Thus by **Theorem 1**:  $\lim_{x \rightarrow x_0} F(x) = F(x_0)$ , Therefore:

- $F$  is continuous at  $x_0$ .

□

### Theorem b

Let  $I \in \mathcal{I}$  and  $t_0 \in I$ ,  $f: I \times \Omega \rightarrow \mathbb{R}$  is a function such that:

- $(b_1) \forall t \in I, f(t, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $(b_2) \forall \omega \in \Omega, f(\cdot, \omega)$  has a derivative at  $t_0$ ;
- $(b_3) \exists g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$  such that:

$$\forall t \in I \setminus \{t_0\}, \forall \omega \in \Omega, \left| \frac{f(t, \omega) - f(t_0, \omega)}{t - t_0} \right| \leq g(\omega).$$

Then:

- The function  $F: I \rightarrow \mathbb{R}$  defined by:

$$\forall t \in I, F(t) := \int_{\Omega} f(t, \omega) d\mu(\omega)$$

has a derivative at  $t_0$ ;

- $\partial_1 f(t_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $\frac{dF}{dt}(t_0) = \int_{\Omega} \frac{\partial f}{\partial t}(t_0, \omega) d\mu(\omega)$ .

### Proof:

Let  $(t_n)_{n \in \mathbb{N}} \in (I \setminus \{t_0\})^{\mathbb{N}}$ , assume that  $\lim_{n \rightarrow \infty} t_n = t_0$ .

Define  $(\xi_{\omega})_{\omega \in \Omega} \in \mathfrak{F}(I \setminus \{t_0\}, \mathbb{R})^{\Omega}$  by:

$$\forall t \in I \setminus \{t_0\}, \forall \omega \in \Omega, \xi_{\omega}(t) := \frac{f(t, \omega) - f(t_0, \omega)}{t - t_0}.$$

We also define  $\zeta: I \setminus \{t_0\} \rightarrow \mathbb{R}$  by:

$$\forall t \in I \setminus \{t_0\}, \zeta(t) := \frac{F(t) - F(t_0)}{t - t_0}.$$

In addition, we define  $(\Psi_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$  by:

$$\forall \omega \in \Omega, \forall n \in \mathbb{N}, \Psi_n(\omega) := \xi_{\omega}(t_n).$$

$(b_2)$  gives:  $\forall \omega \in \Omega, \lim_{t \rightarrow t_0} \xi_{\omega}(t) = \frac{\partial f}{\partial t}(t_0, \omega) \in \mathbb{R}$ .

So we can define  $\Psi: \Omega \rightarrow \mathbb{R}$  by:  $\Psi := \partial_1 f(t_0, \cdot)$ .

It is not hard to see that:  $t_0 \in (\mathbb{I} \setminus \{t_0\})'$  and  $(\mathbb{I} \setminus \{t_0\}) \setminus \{t_0\} = \mathbb{I} \setminus \{t_0\}$ .  
So by **Theorem 2**:

$$\forall \omega \in \Omega, \lim_{n \rightarrow \infty} \xi_\omega(t_n) = \partial_1 f(t_0, \omega).$$

Which is the same as:  $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} \Psi_n(\omega) = \Psi(\omega)$ .

(b<sub>1</sub>) gives:  $\forall n \in \mathbb{N}, (f(t_n, \cdot), f(t_0, \cdot)) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2$ .

Thus by **Theorem 3**:

$$\forall n \in \mathbb{N}, \left( \frac{f(t_n, \cdot)}{t_n - t_0}, -\frac{f(t_0, \cdot)}{t_n - t_0} \right) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2.$$

Therefore by **Theorem 3** again:

$$\forall n \in \mathbb{N}, \Psi_n = \frac{f(t_n, \cdot) - f(t_0, \cdot)}{t_n - t_0} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R}).$$

In other words:  $\{\Psi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$  so:  $\{\Psi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ .

So by **Theorem 4**:  $\Psi \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ .

We know that:  $\forall n \in \mathbb{N}, t_n \in \mathbb{I} \setminus \{t_0\}$ .

So from (b<sub>3</sub>) we get:

$$\forall \omega \in \Omega, \forall n \in \mathbb{N}, |\xi_\omega(t_n)| = \left| \frac{f(t_n, \omega) - f(t_0, \omega)}{t_n - t_0} \right| \leq g(\omega).$$

Which is the same as:  $\forall \omega \in \Omega, \forall n \in \mathbb{N}, |\Psi_n(\omega)| \leq g(\omega)$ .

Then, we can apply (DCT) to get:

$$\Psi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R}) \text{ and } \lim_{n \rightarrow \infty} \int \Psi_n d\mu = \int \Psi d\mu \in \mathbb{R}.$$

With:

$$\Psi = \partial_1 f(t_0, \cdot);$$

$$\forall n \in \mathbb{N}, \int \Psi_n d\mu = \int_{\Omega} \frac{f(t_n, \omega) - f(t_0, \omega)}{t_n - t_0} d\mu(\omega) = \frac{F(t_n) - F(t_0)}{t_n - t_0} = \zeta(t_n);$$

and:

$$\int \Psi d\mu = \int_{\Omega} \frac{\partial f}{\partial t}(t_0, \omega) d\mu(\omega).$$

Since  $(t_n)_{n \in \mathbb{N}} \in (\mathbb{I} \setminus \{t_0\})^{\mathbb{N}}$  is arbitrary, we have proven that:

$$\forall (t_n)_{n \in \mathbb{N}} \in (\mathbb{I} \setminus \{t_0\})^{\mathbb{N}}, \lim_{n \rightarrow \infty} t_n = t_0 \implies \lim_{n \rightarrow \infty} \zeta(t_n) = \int \Psi d\mu.$$

Thus by **Theorem 2**:

$$\lim_{t \rightarrow t_0} \zeta(t) = \frac{dF}{dt}(t_0) = \int \Psi d\mu.$$

Therefore:

- $F$  has a derivative at  $t_0$ ;
- $\partial_1 f(t_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $\frac{dF}{dt}(t_0) = \int_{\Omega} \frac{\partial f}{\partial t}(t_0, \omega) d\mu(\omega)$ .

□

### Theorem c

Let  $\mathcal{U} \in \mathcal{T}_p$ ,  $i \in \llbracket 1, p \rrbracket$  and  $x_0 \in \mathcal{U}$ ,  $f: \mathcal{U} \times \Omega \rightarrow \mathbb{R}$  is a function such that:

- $(c_1) \forall x \in \mathcal{U}, f(x, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $(c_2) \forall \omega \in \Omega, f(\cdot, \omega)$  has an  $i$ th partial derivative at  $x_0$ ;
- $(c_3) \exists I \in \mathcal{I}_i(x_0, \mathcal{U}), \exists g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$  such that:

$$\forall t \in I \setminus \{0\}, \forall \omega \in \Omega, \left| \frac{f(x_0 + te_i, \omega) - f(x_0, \omega)}{t} \right| \leq g(\omega).$$

Then:

- The function  $F: \mathcal{U} \rightarrow \mathbb{R}$  defined by:

$$\forall x \in \mathcal{U}, F(x) := \int_{\Omega} f(x, \omega) d\mu(\omega)$$

has an  $i$ th partial derivative at  $x_0$ ;

- $\partial_i f(x_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $\frac{\partial F}{\partial x_i}(x_0) = \int_{\Omega} \frac{\partial f}{\partial x_i}(x_0, \omega) d\mu(\omega)$ .

### Proof:

Define  $\lambda: I \times \Omega \rightarrow \mathbb{R}$  by:  $\forall (t, \omega) \in I \times \Omega, \lambda(t, \omega) := f(x_0 + te_i, \omega)$ .

We know from  $(c_3)$  that:  $\forall t \in I, x_0 + te_i \in \mathcal{U}$ .

So  $(c_1)$  gives:  $\forall t \in I, f(x_0 + te_i, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ .

Which is the same as:  $\forall t \in I, \lambda(t, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ .

We have also:

$$\forall \omega \in \Omega, \forall t \in I \setminus \{0\}, \frac{\lambda(t, \omega) - \lambda(0, \omega)}{t - 0} = \frac{f(x_0 + te_i, \omega) - f(x_0, \omega)}{t}.$$

So  $(c_2)$  gives:

$$\forall \omega \in \Omega, \lim_{t \rightarrow 0} \frac{\lambda(t, \omega) - \lambda(0, \omega)}{t - 0} = \frac{\partial f}{\partial x_i}(x_0, \omega) \in \mathbb{R}.$$

Thus:  $\forall \omega \in \Omega, \lambda(\cdot, \omega)$  has a derivative at 0.

And:  $\forall \omega \in \Omega, \frac{\partial \lambda}{\partial t}(0, \omega) = \frac{\partial f}{\partial x_i}(x_0, \omega)$ .

So:  $\partial_1 \lambda(0, \cdot) = \partial_i f(x_0, \cdot)$ .

Also  $(c_3)$  gives:  $\forall t \in I \setminus \{0\}, \forall \omega \in \Omega, \left| \frac{\lambda(t, \omega) - \lambda(0, \omega)}{t - 0} \right| \leq g(\omega)$ .

Then, we can apply **Theorem b** to get:

- The function  $\Lambda: I \rightarrow \mathbb{R}$  defined by:  $\forall t \in I, \Lambda(t) := \int_{\Omega} \lambda(t, \omega) d\mu(\omega)$  has a derivative at 0;

- $\partial_1 \lambda(0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ ;
- $\frac{d\Lambda}{dt}(0) = \int_{\Omega} \frac{\partial \lambda}{\partial t}(0, \omega) d\mu(\omega) \in \mathbb{R}$ .

We have also  $\forall t \in \mathbb{I}$ ,  $\Lambda(t) = \int_{\Omega} f(x_0 + te_i, \omega) d\mu(\omega) = F(x_0 + te_i)$  so:

$$\forall t \in \mathbb{I} \setminus \{0\}, \frac{\Lambda(t) - \Lambda(0)}{t - 0} = \frac{F(x_0 + te_i) - F(x_0)}{t}.$$

Thus:

$$\lim_{t \rightarrow 0} \frac{F(x_0 + te_i) - F(x_0)}{t} = \frac{\partial F}{\partial x_i}(x_0) = \frac{d\Lambda}{dt}(0).$$

Therefore:

- $F$  has an  $i$ th partial derivative at  $x_0$  ;
- $\partial_i f(x_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$  ;
- $\frac{\partial F}{\partial x_i}(x_0) = \int_{\Omega} \frac{\partial f}{\partial x_i}(x_0, \omega) d\mu(\omega)$  .

□