

Leibniz Integral Rule Proof

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In this document we will consider the following:

- $(\Omega, \mathcal{F}, \mu)$ is a measure space.
- (E, d) is a metric space.
- $\mathfrak{F}(X, Y) = Y^X$ is the set of all the functions $f: X \rightarrow Y$, where X and Y are sets.
- $\mathbb{N}_0 := \{0, 1, \dots\}$ and $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$.
- $[0, \infty] := [0, \infty] \cup \{\infty\}$.
- $\mathcal{Y} := \{\mathbb{R}, [0, \infty]\}$.
- $\mathcal{I} := \{]a, b[: (a, b) \in \mathbb{R}^2 \text{ and } a < b\}$ is the set of all open intervals of \mathbb{R} .
- $B(x_0, r) := \{x \in E : d(x, x_0) < r\}$ where $x_0 \in E$ and $r > 0$.
- $m \in \mathbb{N}$ and $p \in \mathbb{N}$ with $p \geq 2$.
- $\llbracket 1, p \rrbracket := [1, p] \cap \mathbb{N}$.
- $\delta: \llbracket 1, p \rrbracket^2 \rightarrow \mathbb{R}$ is the function defined by:

$$\forall (i, j) \in \llbracket 1, p \rrbracket^2, \delta(i, j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For $i \in \llbracket 1, p \rrbracket$:

- $e_i := (\delta_{i1}, \dots, \delta_{ip})$.
- $\mathcal{I}_i(x, A) := \{I \in \mathcal{I} : 0 \in I \text{ and } \forall t \in I, x + te_i \in A\}$ where $x \in A$ and $A \subset \mathbb{R}^p$.

For $Y \in \mathcal{Y}$:

- $\{f < t\} := \{\omega \in \Omega : f(\omega) < t\}$ where $f: \Omega \rightarrow Y$ is a function and $t \in \mathbb{R}$.
- $\mathcal{L}(\Omega, \mathcal{F}, Y) := \{f: \Omega \rightarrow Y : \forall t \in \mathbb{R}, \{f < t\} \in \mathcal{F}\}$.
- $\mathcal{L}^1(\Omega, \mathcal{F}, \mu, Y) := \{f: \Omega \rightarrow Y : f \in \mathcal{L}(\Omega, \mathcal{F}, Y) \text{ and } \int |f| d\mu < \infty\}$.

For $A \subset E$:

- $A^\circ := \{x \in E : \exists r > 0, B(x, r) \subset A\}$.
- $A' := \{x \in E : \forall r > 0, [B(x, r) \cap A] \setminus \{x\} \neq \emptyset\}$.
- $\mathcal{T}_p := \{\mathcal{U} \subset \mathbb{R}^p : \mathcal{U} = \mathcal{U}^\circ\}$ is the set of all open subsets of \mathbb{R}^p .

Definition 1

Let X, Y and Z be sets, $f: X \times Y \rightarrow Z$ is a function.

- $\forall x \in X, f(x, \cdot): Y \rightarrow Z$ is the function defined by:

$$\forall y \in Y, f(x, \cdot)(y) = f(x, y).$$

- $\forall y \in Y, f(\cdot, y): X \rightarrow Z$ is the function defined by:

$$\forall x \in X, f(\cdot, y)(x) = f(x, y).$$

Definition 2

Let $I \in \mathcal{I}$, $f: I \times \Omega \rightarrow \mathbb{R}$ is a function, $t_0 \in I$ and $\omega \in \Omega$.

If the function $f(\cdot, \omega)$ has a derivative at t_0 , we define:

$$\partial_1 f(t_0, \omega) = \frac{\partial f}{\partial t}(t_0, \omega) := \lim_{t \rightarrow t_0} \frac{f(t, \omega) - f(t_0, \omega)}{t - t_0}.$$

Definition 3

Let $\mathcal{U} \in \mathcal{T}_p$, $f: \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ is a function, $i \in \llbracket 1, p \rrbracket$, $x_0 \in \mathcal{U}$ and $\omega \in \Omega$.

If the function $f(\cdot, \omega)$ has an i th partial derivative at x_0 , we define:

$$\partial_i f(x_0, \omega) = \frac{\partial f}{\partial x_i}(x_0, \omega) := \lim_{t \rightarrow 0} \frac{f(x_0 + te_i, \omega) - f(x_0, \omega)}{t}.$$

Definition 4

Let A and X be sets. An indexed family $(x_\alpha)_{\alpha \in A} \in X^A$ is a function $x: A \rightarrow X$.

We have the following:

- A is called the index set.
- $\forall \alpha \in A, x_\alpha := x(\alpha)$.
- $\{x_\alpha\}_{\alpha \in A} := \{x_\alpha : \alpha \in A\}$.
- if $A \in \{\mathbb{N}, \mathbb{N}_0\}$, the indexed family $(x_\alpha)_{\alpha \in A} \in X^A$ is called a sequence.

Theorem 1

Let $f: E \rightarrow \mathbb{R}$ be a function and $x_0 \in E$. The following are equivalent:

- $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ (f is continuous at x_0).
- $\forall (x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

Theorem 2

Let $A \subset \mathbb{R}^m$, $f: A \rightarrow \mathbb{R}$ is a function, $x_0 \in A'$ and $\ell \in \mathbb{R}$. The following are equivalent:

- $\lim_{x \rightarrow x_0} f(x) = \ell$.
- $\forall (x_n)_{n \in \mathbb{N}} \in (A \setminus \{x_0\})^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = \ell$.

Theorem 3

Let $(f, g) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2$ and $\alpha \in \mathbb{R}$. Then:

$$(\alpha f, f + g) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2.$$

Theorem 4

Let $f \in \mathfrak{F}(\Omega, \mathbb{R})$ and $(f_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$ such that:

- $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$;
- $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$.

Then:

$$f \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R}).$$

(DCT)

Let $g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$, $f \in \mathfrak{F}(\Omega, \mathbb{R})$ and $(f_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$ such that:

- $f \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$;
- $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$;
- $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$;
- $\forall \omega \in \Omega, \forall n \in \mathbb{N}, |f_n(\omega)| \leq g(\omega)$.

Then:

- $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$.

Theorem a

Let $x_0 \in E$, $f: E \times \Omega \rightarrow \mathbb{R}$ is a function such that:

- $(a_1) \forall x \in E, f(x, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $(a_2) \forall \omega \in \Omega, f(\cdot, \omega)$ is continuous at x_0 ;
- $(a_3) \exists g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$ such that:

$$\forall (x, \omega) \in E \times \Omega, |f(x, \omega)| \leq g(\omega).$$

Then:

- The function $F: E \rightarrow \mathbb{R}$ defined by:

$$\forall x \in E, F(x) := \int_{\Omega} f(x, \omega) d\mu(\omega)$$

is continuous at x_0 .

Proof:

Let $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$, assume that $\lim_{n \rightarrow \infty} x_n = x_0$.

Define $\varphi: \Omega \rightarrow \mathbb{R}$ by: $\varphi := f(x_0, \cdot)$.

(a_1) gives $\varphi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ so: $\varphi \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$.

Define $(\varphi_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$ by: $\forall n \in \mathbb{N}, \varphi_n := f(x_n, \cdot)$.

(a_1) gives $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ so: $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$.

From (a_2) we get:

$$\forall \omega \in \Omega, \lim_{x \rightarrow x_0} f(x, \omega) = f(x_0, \omega).$$

And by **Theorem 1** we have:

$$\forall \omega \in \Omega, \lim_{n \rightarrow \infty} f(x_n, \omega) = f(x_0, \omega).$$

Which is the same as: $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} \varphi_n(\omega) = \varphi(\omega)$.

We know that: $\forall n \in \mathbb{N}, x_n \in E$.

So from (a_3) we get:

$$\forall \omega \in \Omega, \forall n \in \mathbb{N}, |f(x_n, \omega)| \leq g(\omega).$$

Which is the same as: $\forall \omega \in \Omega, \forall n \in \mathbb{N}, |\varphi_n(\omega)| \leq g(\omega)$.

Then, we can apply (\mathfrak{DCT}) to get:

$$\lim_{n \rightarrow \infty} \int \varphi_n d\mu = \int \varphi d\mu.$$

With:

$$\forall n \in \mathbb{N}, \int \varphi_n d\mu = \int_{\Omega} f(x_n, \omega) d\mu(\omega) = F(x_n);$$

and:

$$\int \varphi d\mu = \int_{\Omega} f(x_0, \omega) d\mu(\omega) = F(x_0).$$

Since $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ is arbitrary, we have proven that:

$$\forall (x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} F(x_n) = F(x_0).$$

Thus by **Theorem 1**: $\lim_{x \rightarrow x_0} F(x) = F(x_0)$, Therefore:

- F is continuous at x_0 .

□

Theorem b

Let $I \in \mathcal{I}$ and $t_0 \in I$, $f: I \times \Omega \rightarrow \mathbb{R}$ is a function such that:

- $(b_1) \forall t \in I, f(t, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $(b_2) \forall \omega \in \Omega, f(\cdot, \omega)$ has a derivative at t_0 ;
- $(b_3) \exists g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$ such that:

$$\forall t \in I \setminus \{t_0\}, \forall \omega \in \Omega, \left| \frac{f(t, \omega) - f(t_0, \omega)}{t - t_0} \right| \leq g(\omega).$$

Then:

- The function $F: I \rightarrow \mathbb{R}$ defined by:

$$\forall t \in I, F(t) := \int_{\Omega} f(t, \omega) d\mu(\omega)$$

has a derivative at t_0 ;

- $\partial_1 f(t_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $\frac{dF}{dt}(t_0) = \int_{\Omega} \frac{\partial f}{\partial t}(t_0, \omega) d\mu(\omega).$

Proof:

Let $(t_n)_{n \in \mathbb{N}} \in (I \setminus \{t_0\})^{\mathbb{N}}$, assume that $\lim_{n \rightarrow \infty} t_n = t_0$.

Define $(\xi_{\omega})_{\omega \in \Omega} \in \mathfrak{F}(I \setminus \{t_0\}, \mathbb{R})^{\Omega}$ by:

$$\forall t \in I \setminus \{t_0\}, \forall \omega \in \Omega, \xi_{\omega}(t) := \frac{f(t, \omega) - f(t_0, \omega)}{t - t_0}.$$

We also define $\zeta: I \setminus \{t_0\} \rightarrow \mathbb{R}$ by:

$$\forall t \in I \setminus \{t_0\}, \zeta(t) := \frac{F(t) - F(t_0)}{t - t_0}.$$

In addition, we define $(\Psi_n)_{n \in \mathbb{N}} \in \mathfrak{F}(\Omega, \mathbb{R})^{\mathbb{N}}$ by:

$$\forall \omega \in \Omega, \forall n \in \mathbb{N}, \Psi_n(\omega) := \xi_{\omega}(t_n).$$

(b_2) gives: $\forall \omega \in \Omega, \lim_{t \rightarrow t_0} \xi_{\omega}(t) = \frac{\partial f}{\partial t}(t_0, \omega) \in \mathbb{R}.$

So we can define $\Psi: \Omega \rightarrow \mathbb{R}$ by: $\Psi := \partial_1 f(t_0, \cdot).$

It is not hard to see that: $t_0 \in (\mathbb{I} \setminus \{t_0\})'$ and $(\mathbb{I} \setminus \{t_0\}) \setminus \{t_0\} = \mathbb{I} \setminus \{t_0\}$.
So by **Theorem 2**:

$$\forall \omega \in \Omega, \lim_{n \rightarrow \infty} \xi_\omega(t_n) = \partial_1 f(t_0, \omega).$$

Which is the same as: $\forall \omega \in \Omega, \lim_{n \rightarrow \infty} \Psi_n(\omega) = \Psi(\omega)$.

(b_1) gives: $\forall n \in \mathbb{N}, (f(t_n, \cdot), f(t_0, \cdot)) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2$.

Thus by **Theorem 3**:

$$\forall n \in \mathbb{N}, \left(\frac{f(t_n, \cdot)}{t_n - t_0}, -\frac{f(t_0, \cdot)}{t_n - t_0} \right) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})^2.$$

Therefore by **Theorem 3** again:

$$\forall n \in \mathbb{N}, \Psi_n = \frac{f(t_n, \cdot) - f(t_0, \cdot)}{t_n - t_0} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R}).$$

In other words: $\{\Psi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$ so: $\{\Psi_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$.

So by **Theorem 4**: $\Psi \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$.

We know that: $\forall n \in \mathbb{N}, t_n \in \mathbb{I} \setminus \{t_0\}$.

So from (b_3) we get:

$$\forall \omega \in \Omega, \forall n \in \mathbb{N}, |\xi_\omega(t_n)| = \left| \frac{f(t_n, \omega) - f(t_0, \omega)}{t_n - t_0} \right| \leq g(\omega).$$

Which is the same as: $\forall \omega \in \Omega, \forall n \in \mathbb{N}, |\Psi_n(\omega)| \leq g(\omega)$.

Then, we can apply $(\mathfrak{D}\mathfrak{C}\mathfrak{T})$ to get:

$$\Psi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R}) \text{ and } \lim_{n \rightarrow \infty} \int \Psi_n d\mu = \int \Psi d\mu \in \mathbb{R}.$$

With:

$$\begin{aligned} \Psi &= \partial_1 f(t_0, \cdot); \\ \forall n \in \mathbb{N}, \int \Psi_n d\mu &= \int_{\Omega} \frac{f(t_n, \omega) - f(t_0, \omega)}{t_n - t_0} d\mu(\omega) = \frac{F(t_n) - F(t_0)}{t_n - t_0} = \zeta(t_n); \end{aligned}$$

and:

$$\int \Psi d\mu = \int_{\Omega} \frac{\partial f}{\partial t}(t_0, \omega) d\mu(\omega).$$

Since $(t_n)_{n \in \mathbb{N}} \in (\mathbb{I} \setminus \{t_0\})^{\mathbb{N}}$ is arbitrary, we have proven that:

$$\forall (t_n)_{n \in \mathbb{N}} \in (\mathbb{I} \setminus \{t_0\})^{\mathbb{N}}, \lim_{n \rightarrow \infty} t_n = t_0 \implies \lim_{n \rightarrow \infty} \zeta(t_n) = \int \Psi d\mu.$$

Thus by **Theorem 2**:

$$\lim_{t \rightarrow t_0} \zeta(t) = \frac{dF}{dt}(t_0) = \int \Psi d\mu.$$

Therefore:

- F has a derivative at t_0 ;
- $\partial_1 f(t_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $\frac{dF}{dt}(t_0) = \int_{\Omega} \frac{\partial f}{\partial t}(t_0, \omega) d\mu(\omega)$.

□

Theorem c

Let $\mathcal{U} \in \mathcal{T}_p$, $i \in \llbracket 1, p \rrbracket$ and $x_0 \in \mathcal{U}$, $f: \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ is a function such that:

- $(c_1) \forall x \in \mathcal{U}, f(x, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $(c_2) \forall \omega \in \Omega, f(\cdot, \omega)$ has an i th partial derivative at x_0 ;
- $(c_3) \exists I \in \mathcal{I}_i(x_0, \mathcal{U}), \exists g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, [0, \infty])$ such that:

$$\forall t \in I \setminus \{0\}, \forall \omega \in \Omega, \left| \frac{f(x_0 + te_i, \omega) - f(x_0, \omega)}{t} \right| \leq g(\omega).$$

Then:

- The function $F: \mathcal{U} \rightarrow \mathbb{R}$ defined by:

$$\forall x \in \mathcal{U}, F(x) := \int_{\Omega} f(x, \omega) d\mu(\omega)$$

has an i th partial derivative at x_0 ;

- $\partial_i f(x_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $\frac{\partial F}{\partial x_i}(x_0) = \int_{\Omega} \frac{\partial f}{\partial x_i}(x_0, \omega) d\mu(\omega).$

Proof:

Define $\lambda: I \times \Omega \rightarrow \mathbb{R}$ by: $\forall (t, \omega) \in I \times \Omega, \lambda(t, \omega) := f(x_0 + te_i, \omega).$

We know from (c_3) that: $\forall t \in I, x_0 + te_i \in \mathcal{U}.$

So (c_1) gives: $\forall t \in I, f(x_0 + te_i, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R}).$

Which is the same as: $\forall t \in I, \lambda(t, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R}).$

We have also:

$$\forall \omega \in \Omega, \forall t \in I \setminus \{0\}, \frac{\lambda(t, \omega) - \lambda(0, \omega)}{t - 0} = \frac{f(x_0 + te_i, \omega) - f(x_0, \omega)}{t}.$$

So (c_2) gives:

$$\forall \omega \in \Omega, \lim_{t \rightarrow 0} \frac{\lambda(t, \omega) - \lambda(0, \omega)}{t - 0} = \frac{\partial f}{\partial x_i}(x_0, \omega) \in \mathbb{R}.$$

Thus: $\forall \omega \in \Omega, \lambda(\cdot, \omega)$ has a derivative at 0.

And: $\forall \omega \in \Omega, \frac{\partial \lambda}{\partial t}(0, \omega) = \frac{\partial f}{\partial x_i}(x_0, \omega).$

So: $\partial_1 \lambda(0, \cdot) = \partial_i f(x_0, \cdot).$

Also (c_3) gives: $\forall t \in I \setminus \{0\}, \forall \omega \in \Omega, \left| \frac{\lambda(t, \omega) - \lambda(0, \omega)}{t - 0} \right| \leq g(\omega).$

Then, we can apply **Theorem b** to get:

- The function $\Lambda: I \rightarrow \mathbb{R}$ defined by: $\forall t \in I, \Lambda(t) := \int_{\Omega} \lambda(t, \omega) d\mu(\omega)$ has a derivative at 0;

- $\partial_1 \lambda(0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $\frac{d\Lambda}{dt}(0) = \int_{\Omega} \frac{\partial \lambda}{\partial t}(0, \omega) d\mu(\omega) \in \mathbb{R}$.

We have also $\forall t \in \mathbb{I}$, $\Lambda(t) = \int_{\Omega} f(x_0 + te_i, \omega) d\mu(\omega) = F(x_0 + te_i)$ so:

$$\forall t \in \mathbb{I} \setminus \{0\}, \frac{\Lambda(t) - \Lambda(0)}{t - 0} = \frac{F(x_0 + te_i) - F(x_0)}{t}.$$

Thus:

$$\lim_{t \rightarrow 0} \frac{F(x_0 + te_i) - F(x_0)}{t} = \frac{\partial F}{\partial x_i}(x_0) = \frac{d\Lambda}{dt}(0).$$

Therefore:

- F has an i th partial derivative at x_0 ;
- $\partial_i f(x_0, \cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathbb{R})$;
- $\frac{\partial F}{\partial x_i}(x_0) = \int_{\Omega} \frac{\partial f}{\partial x_i}(x_0, \omega) d\mu(\omega) .$

□