Differential Calculus Primer

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1 Introduction

Here's an advanced primer on Differential Calculus, covering rules, equations, formulas, and techniques comprehensively.

2 Foundations of Differential Calculus

2.1 Definition of a Derivative

For a function $f(x)$, the **derivative** is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This gives the **instantaneous rate of change** of $f(x)$ at x.

2.2 Alternative notations:

$$\frac{d}{dx}f(x), \quad Df(x), \quad \dot{f}(x) \text{ (Newton's notation for time derivatives)}$$

3 Basic Differentiation Rules

3.1 Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}, \quad \text{for } n \in R$$

3.2 Constant Rule

$$\frac{d}{dx}C = 0, \quad C \text{ is a constant}$$

3.3 Constant Multiple Rule

$$\frac{d}{dx}[cf(x)] = cf'(x), \quad c \text{ is a constant}$$

3.4 Sum and Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

3.5 Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

3.6 Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

3.7 Chain Rule (Composite Functions)

If $y = f(g(x))$, then:

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

4 Advanced Differentiation Techniques

4.1 Implicit Differentiation

Used when y is given implicitly in terms of x:

$$\frac{d}{dx}[F(x,y)=0] \Rightarrow F_x + F_y\frac{dy}{dx} = 0$$

Example: Differentiate $x^2 + y^2 = 1$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

4.2 Logarithmic Differentiation

Used for functions of the form $f(x)^{g(x)}$ or complicated products:

1. Take natural log:

$$y = f(x)^{g(x)} \Rightarrow \ln y = g(x)\ln f(x)$$

2. Differentiate both sides using the product rule.
3. Solve for $\frac{dy}{dx}$.

Example:

$$y = x^x \Rightarrow \ln y = x\ln x$$

Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

Multiply by y:

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

5 Higher-Order Derivatives

5.1 Second Derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

5.2 Nth Order Derivative

$$f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Example: If $f(x) = e^x$, then:

$$f^{(n)}(x) = e^x$$

6 Differentiation of Special Functions

6.1 Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\cot x = -\csc^2 x$$

6.2 Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

6.3 Exponential and Logarithmic Functions

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}a^x = a^x \ln a$$
$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

7 Applications of Differential Calculus

7.1 Critical Points & Extrema

Find local maxima/minima using: - First Derivative Test: $f'(x) = 0$ and sign analysis. - Second Derivative Test: - $f''(x) > 0 \rightarrow$ Local minimum - $f''(x) < 0$ $\rightarrow$ Local maximum

7.2 Concavity & Inflection Points

$$\text{If } f''(x) > 0, \text{ function is concave up (convex)}$$

$$\text{If } f''(x) < 0, \text{ function is concave down (concave)}$$

7.3 L'Hôpital's Rule (Indeterminate Forms)

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ results in $0/0$ or ∞/∞, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

7.3.1 Taylor and Maclaurin Series

Approximates a function using derivatives at a point a:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

Maclaurin Series (at $a = 0$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Example:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

8 Advanced Topics

8.1 Differentiation under the Integral Sign (Leibniz Rule)

If $I(a) = \int_a^b f(x, a)dx$, then:

$$\frac{d}{da}I(a) = \int_a^b \frac{\partial}{\partial a}f(x, a)dx$$

8.2 Functional Derivatives

For functionals $F[y]$, the Euler-Lagrange equation gives extremal functions:

$$\frac{\delta F}{\delta y} = 0$$

This covers core rules, techniques, and applications of Differential Calculus at an advanced level.