Integral Calculus Primer

Moussa El Najmi

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1 Introduction

Here's an advanced primer on Integral Calculus, covering rules, equations, formulas, and techniques in detail.

2 Fundamentals of Integral Calculus

2.1 Definition of an Integral

2.1.1 Indefinite Integral (Antiderivative)

The indefinite integral of $f(x)$ is a function $F(x)$ such that:

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of integration.

2.1.2 Definite Integral

The definite integral of $f(x)$ over $[a, b]$ is:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

3 Fundamental Theorem of Calculus

3.1 Differentiation of an Integral:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

3.2 Evaluation of a Definite Integral:

$$\int_a^b f(x)dx = F(b) - F(a)$$

4 Basic Integration Rules

4.1 Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

4.2 Constant Rule

$$\int c\, dx = cx + C$$

4.3 Sum/Difference Rule

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

4.4 Constant Multiplication Rule

$$\int cf(x) dx = c \int f(x) dx$$

4.5 Integration by Substitution (Change of Variables)

If $u = g(x)$, then:

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

Example:

$$\int x e^{x^2} dx$$

Let $u = x^2$, so $du = 2x dx$.

$$\frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

5 Advanced Integration Techniques

5.1 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Choose: - u = a function that simplifies when differentiated. - dv = a function that is easy to integrate.

Example:

$$\int xe^x dx$$

Let $u = x$, $dv = e^x dx$, then:

$$du = dx, \quad v = e^x$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

5.2 Trigonometric Integrals

Integrate expressions involving sin, cos, tan, sec, csc, cot using identities.

Example:

$$\int \sin^3 x \cos x dx$$

Use $u = \sin x$, so $du = \cos x dx$:

$$\int u^3 du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

5.3 Trigonometric Substitution

Used for integrals involving: - $\sqrt{a^2 - x^2} \Rightarrow x = a\sin\theta$ - $\sqrt{a^2 + x^2} \Rightarrow x = a\tan\theta$ - $\sqrt{x^2 - a^2} \Rightarrow x = a\sec\theta$

Example:

$$\int \frac{dx}{\sqrt{9 - x^2}}$$

Let $x = 3\sin\theta$, then $dx = 3\cos\theta d\theta$.

$$\int \frac{3\cos\theta d\theta}{\sqrt{9 - 9\sin^2\theta}}$$

Since $\sqrt{9(1 - \sin^2\theta)} = 3\cos\theta$, it simplifies to:

$$\int d\theta = \theta + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$

5.4 Partial Fraction Decomposition

Used when integrating rational functions:

$$\int \frac{P(x)}{Q(x)} dx$$

where $Q(x)$ is factored.

Example:

$$\int \frac{3x+5}{(x+1)(x+2)} dx$$

Decompose:

$$\frac{3x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Solving for A, B, we integrate separately.

5.5 Improper Integrals

If $\int_a^\infty f(x)dx$ or $\int_{-\infty}^\infty f(x)dx$, take limits.

Example:

$$\int_1^\infty \frac{1}{x^2} dx$$

$$\lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$\lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1$$

6 Special Integrals

6.1 Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

6.2 Beta Function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

6.3 Gamma Function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, \quad \Gamma(n) = (n-1)!$$

6.4 Dirichlet Integrals

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

7 Applications of Integration

7.1 Area Under a Curve

$$A = \int_a^b f(x)dx$$

7.2 Arc Length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

7.3 Surface Area of Revolution

$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

7.4 Volume by Integration

- Disk Method:

$$V = \pi \int_a^b [f(x)]^2 dx$$

- Shell Method:

$$V = 2\pi \int_a^b x f(x) dx$$

This primer covers the rules, techniques, and advanced applications of Integral Calculus.