Probability Basics

1. Definition of Probability

Probability is defined as the ratio of the number of favorable outcomes to the total number of possible outcomes. If S is the sample space, and A is an event, then:

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

2. Axioms of Probability

- **Non-negativity:**
$$P(A) \geq 0 \quad \forall A \subseteq S$$
- **Normalization:**
$$P(S) = 1$$
- **Additivity (for mutually exclusive events $A_1, A_2, \ldots$):**
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Expected Value

1. Definition

The expected value of a random variable X is the weighted average of all possible values that X can take, weighted by their probabilities.

- For a **discrete random variable**:
$$E[X] = \sum_{x \in S} x \cdot P(X = x)$$
- For a **continuous random variable**:
$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

2. Example

For a dice roll with outcomes $X = \{1, 2, 3, 4, 5, 6\}$, where each outcome has a probability of $\frac{1}{6}$:

$$E[X] = \sum_{x=1}^{6} x \cdot \frac{1}{6} = 3.5$$

Variance

1. Definition

Variance is a measure of the spread of a random variable around its mean. It is defined as:

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

Alternatively:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

2. Example

For the same dice roll example:

- First, calculate $E[X^2]$:

$$E[X^2] = \sum_{x=1}^{6} x^2 \cdot \frac{1}{6} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} = \frac{91}{6}$$

- Variance:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$

Probability of Events

1. Union of Events

If A and B are two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. Conditional Probability

If B is an event with $P(B) > 0$, then the conditional probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

3. Independence

Events A and B are independent if:

$$P(A \cap B) = P(A) \cdot P(B)$$

Example Problem

1. Problem

What is the probability of rolling an even number on a six-sided die?

2. Solution

- Sample space:
$$S = \{1, 2, 3, 4, 5, 6\}$$
- Favorable outcomes (even numbers):
$$A = \{2, 4, 6\}$$
- Probability:
$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total outcomes}} = \frac{3}{6} = 0.5$$

Detailed Explanation of Probability

Probability is the branch of mathematics that deals with measuring the likelihood of events. It is fundamental for understanding randomness and uncertainty, which are central to many applications in statistics, machine learning, and data science.

Key Definitions

1. **Sample Space (S)**:

- The set of all possible outcomes of a random experiment.
- **Example**: Rolling a die, $S = \{1, 2, 3, 4, 5, 6\}$.

2. **Event (A)**:

- A subset of the sample space.
- **Example**: Event $A = \{\text{rolling an even number}\} = \{2, 4, 6\}$.

3. **Probability ($P(A)$)**:

- A measure of how likely an event is to occur.
- **Formula**:

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

4. **Axioms of Probability**:

- $0 \leq P(A) \leq 1$
- $P(S) = 1$
- If $A_1, A_2, \ldots$ are mutually exclusive events:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Types of Probability

1. **Classical Probability**:

- Assumes all outcomes are equally likely.
- **Example**: Tossing a fair coin, $P(\text{Heads}) = \frac{1}{2}$.

2. **Empirical Probability**:

- Based on observation or experiment.
- **Example**: If a die is rolled 100 times and the number "3" appears 15 times, $P(3) = \frac{15}{100} = 0.15$.

3. **Subjective Probability**:

- Based on personal judgment or belief.
- **Example**: A weather forecaster estimating a 70% chance of rain.

Conditional Probability

The probability of an event A occurring given that another event B has occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Example: Probability of drawing a red card (A) given that the card is a heart (B):

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{13/52}{1/4} = 1$$

Bayes' Theorem

A fundamental result in probability theory:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \quad P(B) > 0$$

Example: In a medical test for a disease:

- $P(D)$: Probability of having the disease.
- $P(T|D)$: Probability of a positive test given the disease.
- $P(D|T)$: Probability of having the disease given a positive test.

Random Variables

1. **Definition**:

- A variable that takes on values based on the outcome of a random experiment.
- **Example**: Tossing a coin twice, $X =$ Number of heads, possible values: $X = \{0, 1, 2\}$.

2. **Probability Mass Function (PMF)**:

- For a discrete random variable X:

$$P(X = x) = f_X(x)$$

3. **Probability Density Function (PDF)**:

- For a continuous random variable X:

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$

Expectation and Variance

1. **Expected Value**:

- The mean or average of a random variable.
- Discrete:

$$E[X] = \sum_x x \cdot P(X = x)$$

- Continuous:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

Example: For a fair die, $X = \{1, 2, 3, 4, 5, 6\}$, $P(X = x) = \frac{1}{6}$:

$$E[X] = \sum_{x=1}^{6} x \cdot \frac{1}{6} = 3.5$$

2. **Variance**:

- A measure of how spread out a random variable is:

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Python Examples for Probability

Calculating Probability

```
# Example: Rolling a die
favorable_outcomes = 3  # {2, 4, 6}
total_outcomes = 6
probability = favorable_outcomes / total_outcomes
print(f"Probability of rolling an even number: {probability}")
```

Bayes' Theorem in Python

```
# Bayes' Theorem Example: Medical Test
P_D = 0.01  # Probability of having the disease
P_T_given_D = 0.99  # Probability of testing positive given disease
P_T_given_not_D = 0.05  # Probability of testing positive without disease
P_not_D = 1 - P_D

# Total probability of testing positive
P_T = P_T_given_D * P_D + P_T_given_not_D * P_not_D

# Probability of having the disease given a positive test
P_D_given_T = (P_T_given_D * P_D) / P_T
print(f"Probability of disease given a positive test: {P_D_given_T}")
```

Expected Value and Variance

```
# Dice roll example
outcomes = [1, 2, 3, 4, 5, 6]
probabilities = [1/6] * 6  # Uniform distribution

# Expected Value
expected_value = sum(x * p for x, p in zip(outcomes, probabilities))
print(f"Expected Value: {expected_value}")

# Variance
variance = sum((x - expected_value)**2 * p for x, p in zip(outcomes, probabilities))
print(f"Variance: {variance}")
```