Practice Questions

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Question 1. Consider a Markov chain with state space $\{1, 2, 3, 4, 5\}$, with transition matrix given by

$$P = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.1 & 0 & 0.6 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Compute f_{32}.

By f-expansion, we have

$$f_{32} = 0.6f_{32} + 0.1f_{12} + 0.2f_{42} + 0.1f_{52},$$
$$f_{12} = 0.7 + 0.3f_{12}.$$

4 and 5 are absorbing states. So we have $f_{42} = f_{52} = 0$. Solving the equation, we get $f_{32} = 0.25$

Question 2. Consider a Markov chain on the state space $S = \{0, 1, 2, \cdots\}$. For any $i \geq 1$, we define the transition from the state i as

$$p_{i,i+1} = \frac{i}{2i+1}, \quad \text{and} \quad p_{i,i-1} = \frac{i+1}{2i+1},$$

and $p_{i,j} = 0$ for $j \notin \{i-1, i+1\}$. We further let $p_{0,1} = 1$ Show that the Markov chain is null recurrent.

Clearly the Markov chain is irreducible. At each $i \neq 0$, the probability of moving to the left is larger than that of SRW. So $f_{i0}(\text{this chain}) \geq f_{i0}(\text{SRW}) = 1$. The chain is recurrent.

We define

$$\mu_i := \begin{cases} \dfrac{2(2i+1)}{3i(i+1)} & i \geq 1 \\ \dfrac{3}{2} & i = 0. \end{cases}$$

It is easy to verify that $\mu_i p_{i,i+1} = \mu_{i+1} p_{i+1,i}$ for each $i \geq 0$. So μ is a stationary measure. However, we note that

$$\sum_{i \in S} \mu_i \geq \sum_{i=1}^{+\infty} \frac{2}{3i} = +\infty.$$

So the stationary distribution does not exist, and therefore null recurrent.

Question 3. Let $(B_t : t \geq 0)$ be a standard Brownian motion.

- If the process $M_t := \sin(tB_t) - \int_0^t f(s, B_s)ds$ is a martingale. Write down the function form of f, and express M_t in the form of an Itô integral.

$$dM_t = \cos(tB_t) + \cos(tB_t)dB_t - \frac{1}{2}\sin(tB_t).$$

  So we let $f(t,x) = \cos(tx) - \frac{1}{2}\sin(tx)$, and the martingale is $M_t = \int_0^t \cos(sB_s)dB_s$.

- Find the probability $\mathbb{P}\big(B_1 > -1 \text{ and } \max_{0\leq t\leq 1} B_t > 1\big)$.

  We decompose

$$\mathbb{P}\big(B_1 > -1 \text{ and } \max_{0\leq t\leq 1} B_t > 1\big) = \mathbb{P}\big(\max_{0\leq t\leq 1} B_t > 1\big) - \mathbb{P}\big(B_1 \leq -1 \text{ and } \max_{0\leq t\leq 1} B_t > 1\big).$$

  Using reflection principle, we can derive

$$\mathbb{P}\big(\max_{0\leq t\leq 1} B_t > 1\big) = 2\mathbb{P}\big(B_1 \geq 1\big), \quad \text{and}$$
$$\mathbb{P}\big(B_1 \leq -1 \text{ and } \max_{0\leq t\leq 1} B_t > 1\big) = \mathbb{P}\big(B_1 \geq 3\big).$$

  So the answer is

$$\mathbb{P}\big(B_1 > -1 \text{ and } \max_{0\leq t\leq 1} B_t > 1\big) = \frac{2}{\sqrt{2\pi}}\int_1^{+\infty} e^{-x^2/2}dx - \frac{1}{\sqrt{2\pi}}\int_3^{+\infty} e^{-x^2/2}dx.$$

- Apply Itô's formula to the process $\big(e^{\lambda B_t - \lambda^2 t/2}\big)_{t\geq 0}$, and use it to compute the moment generating function of τ, where $\tau := \inf\big\{t > 0 : |B_t| = 1\big\}$.

$$d\big(e^{\lambda B_t - \lambda^2 t/2}\big) = -\frac{\lambda^2}{2}e^{\lambda B_t - \lambda^2 t/2}dt + \lambda e^{\lambda B_t - \lambda^2 t/2}dB_t + \frac{\lambda^2}{2}e^{\lambda B_t - \lambda^2 t/2}dt = \lambda e^{\lambda B_t - \lambda^2 t/2}dB_t.$$

  So the process is a martingale. The martingale is bounded up to time τ. So by OST,

$$\mathbb{E}\big[e^{\lambda B_\tau - \lambda^2\tau/2}\big] = 1.$$

  By symmetry, B_τ and τ are independent. So we have

$$\mathbb{E}\big[e^{-\lambda^2\tau/2}\big] = 1/\mathbb{E}\big[e^{\lambda B_\tau}\big] = \frac{2}{e^\lambda + e^{-\lambda}}.$$

Question 4. Let $(X_t)_{t\geq 0}$ be a recurrent Markov chain on the state space S, and let $V : S \to \mathbb{R}$ be a real-valued function, such that

$$\sum_{j\in S} p_{i,j}V(j) = V(i), \quad \text{for } i \in S.$$

- If V is uniformly bounded in $[0,1]$, show that V is a constant for all states.
- Let the Markov chain be simple symmetric random walk on $\mathbb{Z}$. Find a non-constant and unbounded function V such that the above equation is true.

For the chain starting from i, the process $\big(M_n := V(X_n)\big)_{n\geq 0}$ is a martingale. Let τ_j be the first hitting time of the state j. By recurrence $\mathbb{P}(\tau_j < +\infty) = 1$. Since the martingale is uniformly bounded, by OST, we have

$$V(i) = M_0 = \mathbb{E}[M_{\tau_j}] = V(j).$$

So V is a constant function.

For SRW, we let $V(x) = x$.