STA 447 Lecture 2.

S, ν, P
$\uparrow$ states, $\uparrow$ initial, $\uparrow$ Transition

$X_0, X_1, X_2, \ldots$

$X_0 \sim \nu \quad X_{i+1} | X_i \sim P(X_i, \cdot)$.

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2) = \nu_{i_0} P_{i_0 i_1} P_{i_1 i_2}$$

$$\mathbb{P}(X_0 = i_0, X_2 = i_2) = \nu_{i_0} \sum_{i_1 \in S} P_{i_0 i_1} P_{i_1 i_2}$$

$$\mathbb{P}(X_2 = i_2) = \sum_{\substack{i_0 \in S \\ i_1 \in S}} \nu_{i_0} P_{i_0 i_1} P_{i_1 i_2}$$

$$\nu = [\nu_0, \nu_1, \ldots \nu_k]$$

$$\nu^{(2)}_i := \mathbb{P}(X_2 = i) \qquad \nu^{(2)} = \nu \cdot P^2.$$

$$\nu^{(3)} = \nu \cdot P^3 \quad \ldots \quad \nu^{(m)} = \nu \cdot P^m.$$

Def. $P^{(n)}_{ij} = \mathbb{P}(X_n = j \mid X_0 = i) \quad \forall i, j \in S$

$(X_0, X_n, X_{2n}, \ldots, X_{mn}, \ldots)$

$v_i = 1$ then $v \cdot P^m$ is the m-step
$v_l = 0$ for $l \neq i$ transition prob from the state i.

For the new chain, $\left(p_{ij}^{(n)}\right)_{i,j \in S} = P^n$

(each row is $[0 \cdots 1 0 \cdots 0] \cdot P^n$).

$$\left(p_{ij}^{(m+n)}\right)_{i,j \in S} = P^{m+n} = P^m \cdot P^n = \left(p_{ij}^{(m)}\right)_{i,j \in S} \cdot \left(p_{ij}^{(n)}\right)_{i,j \in S}$$

$$p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}.$$ (Chapman-Kolmogorov eq).

Recurrence and transience.

Def. $N(i) :=$ total number of times for MC to visit i

$$= \sum_{t=1}^{+\infty} \mathbb{1}\{X_t = i\}$$

$$f_{ij} := \mathbb{P}(N(j) \geq 1 \mid X_0 = i) = \mathbb{P}_i(N(j) \geq 1)$$

(Visit j at least once from i)

f_{ii}: prob of returning to i after leaving i.

Rem. $\mathbb{P}_i(N(i) \geq k) = (f_{ii})^k$

$$\mathbb{P}_i(N(i) \geq k) = \underbrace{\mathbb{P}_i(N(i) \geq k \mid N(i) \geq k-1)}_{\stackrel{?}{=} f_{ii}} \cdot \mathbb{P}(N(i) \geq k-1).$$

$\tau_i^{(k-1)} := \{$ time step that hits i for $(k-1)$ times $\}$

$$\begin{array}{cccccccccc} X_0, & X_1, & \cdots & X_{\tau_i^{(1)}}, & \cdots, & X_{\tau_i^{(k-1)}}, & X_{\tau_i^{(k-1)}+1}, & \cdots & X_{\tau_i^{(k)}} \\ \| & & & \| & \cdots & \| & & & \uparrow \\ i & & & i & & i & & & ? \end{array}$$

"strong Markov property"

Let τ be a hitting time for i

$$(X_\tau, X_{\tau+1}, \cdots) \stackrel{d}{=} (X_0, X_1, \cdots), \quad X_0 = i$$

$$\mathbb{P}_i(N(i) \geq k \mid N(i) \geq k-1) = \mathbb{P}_i(\tau_i^{(k)} < +\infty \mid \tau_i^{(k-1)} < +\infty)$$

$$= \mathbb{P}_i(\tau_i^{(1)} < +\infty) = f_{ii}$$

By induction $\mathbb{P}_i(N(i) \geq k) = f_{ii}^k$.

Corollary. $\mathbb{P}_i(N(j) \geq k) = f_{ij} \cdot (f_{jj})^{k-1}$

Proof. $\mathbb{P}_i(N(j) \geq k) = \underbrace{\mathbb{P}_i(\tau_j^{(1)} < +\infty)}_{= f_{ij}} \cdot \underbrace{\mathbb{P}_j(N(j) \geq k \mid \tau_j^{(1)} < +\infty)}_{= \mathbb{P}_j(N(j) \geq k-1)}$

Corollary: $\mathbb{E}_i[N(j)] = \sum_{k \geq 1}^{+\infty} \mathbb{P}(N(j) \geq k) = \begin{cases} \dfrac{f_{ij}}{1-f_{jj}} & (f_{jj} < 1) \\ +\infty & (f_{jj} = 1) \end{cases}$.

Def. A state i of MC is recurrent if $f_{ii} = 1$

transient if $f_{ii} < 1$.

Corollary: Recurrence $\Longleftrightarrow \mathbb{P}_i(N(i) = +\infty) = 1$.

Recurrent State Thm.

Recurrence $\Longleftrightarrow \sum_{n=1}^{+\infty} p_{ii}^{(n)} = +\infty$.

Proof. $\sum_{n=1}^{+\infty} p_{ii}^{n} = \sum_{n=1}^{+\infty} \mathbb{P}_i(X_n = i) = \sum_{n=1}^{+\infty} \mathbb{E}_i[\mathbb{1}_{X_n = i}]$

(Fubini-Tonelli) $\mathbb{E}_i\left[\sum_{n=1}^{+\infty} \mathbb{1}_{X_n = i}\right] = \mathbb{E}_i[N(i)]$.

$= \begin{cases} +\infty & f_{ii} = 1 \\ \dfrac{f_{ii}}{1-f_{ii}} & f_{ii} < 1 \end{cases}$

Extension: Borel–Cantelli Lemma. $(E_i)_{i=1}^{+\infty}$ sequence of events

if $\sum_{i=1}^{+\infty} \mathbb{P}(E_i) < +\infty$, then $\mathbb{P}\left((E)_{i=1}^{+\infty} \text{ happens finite times}\right) = 1$.

$$\sum_{i=1}^{+\infty} \mathbb{P}(E_i) = \mathbb{E}\left[\sum_{n=1}^{+\infty} \mathbb{1}_{E_n}\right] = \mathbb{E}\left[\# \text{ of } (E_i)_{i=1}^{+\infty} \text{ that happens}\right].$$

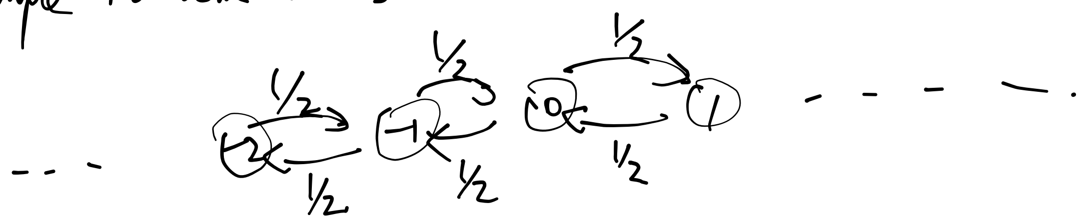
Detour: Exchange of infinite sums and expectations

$$\mathbb{E}\left[\sum_{n=1}^{+\infty} A_n\right] \overset{?}{=} \sum_{n=1}^{+\infty} \mathbb{E}[A_n].$$

Guideline: to check if $\sum_{n=1}^{+\infty} \mathbb{E}[|A_n|]$ is finite $\left(\text{or } \mathbb{E}\left[\sum_{n=1}^{+\infty} |A_n|\right]\right)$.

(Dominance Convergence Thm).

Simple random walks



Question $f_{00} \overset{?}{=} 1$

By recurrent state thm, need to check

$$\sum_{n=1}^{+\infty} p_{00}^{(n)}$$

For odd n : $p_{00}^{(n)} = 0$

For even n: $p_{00}^{(n)} = 2^{-n} \cdot \binom{n}{n/2}$

$$= 2^{-n} \cdot \frac{n!}{(n/2)!\,(n/2)!}$$

$\boxed{\begin{array}{l}\text{Stirling's approximation} \\ \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \le n! \le e\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\end{array}}$

$$p_{00}^{(n)} \geq 2^{-n} \cdot \frac{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n}{e^2 \cdot \sqrt{\pi n} \cdot \left(\frac{n}{2e}\right)^{n/2} \cdot \sqrt{\pi n} \cdot \left(\frac{n}{2e}\right)^{n/2}}$$

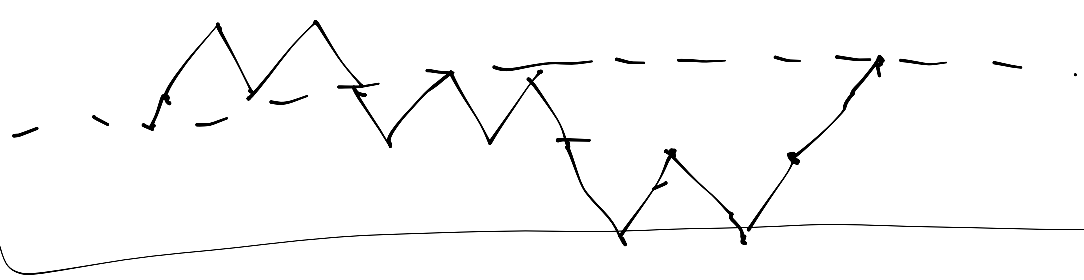
$$\geq \frac{\sqrt{2}}{e^2 \cdot \sqrt{\pi}} \cdot \frac{1}{\sqrt{n}}.$$

Similarly. $p_{00}^{(n)} \le \frac{e \cdot \sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}}$

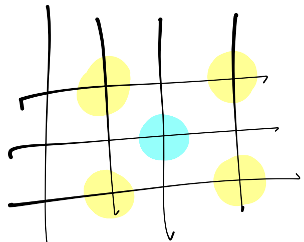
2^n paths

$n/2$ upward segments

$n/2$ downwards.



Multi-dim SRW: $S = \mathbb{Z}^d$



$$P_{(i_1,i_2,\dots,i_d)(j_1,j_2,\dots,j_d)} = \begin{cases} 2^{-d} & \text{if } |i_\ell - j_\ell| = 1 \text{ for any } \ell \\ 0 & \text{otherwise.} \end{cases}$$

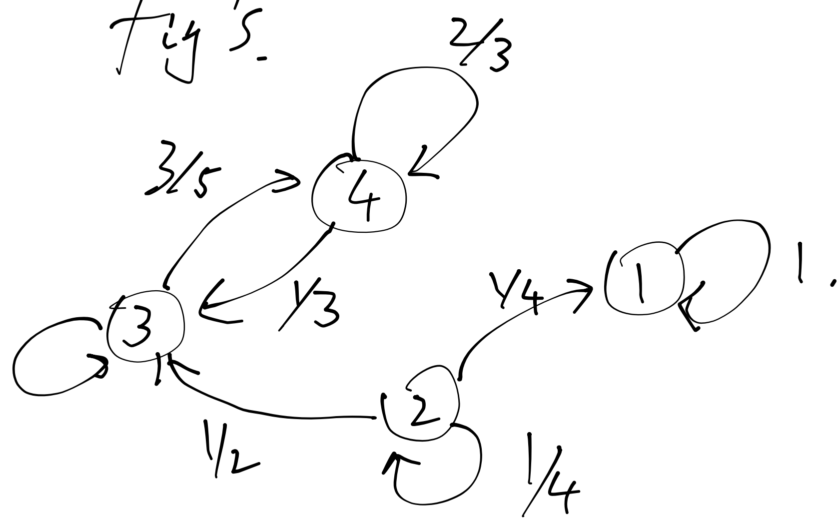
$$p_{00}^{(n)} = \left[2^{-n} \cdot \binom{n}{n/2} \right]^d \approx C_d \cdot n^{-d/2}$$

1d, 2d, $\sum_{n=1}^{+\infty} p_{00}^{(n)} = +\infty$

$d \geq 3$ $\sum_{n=1}^{+\infty} p_{00}^{(n)} < +\infty.$

Computing f_{ij}'s.

e.g.



$f_{11} = 1$

$f_{22} = 1/4.$

$f_{33} = 1$

$f_{44} = 1$

$\mathbb{P}_3\left(\begin{array}{l}\text{not coming back to } 3 \\ \text{after } n \text{ steps}\end{array}\right) = 3/5 \cdot (2/3)^{n-1}$

$$f_{12} = f_{13} = f_{14} = f_{32} = f_{31} = f_{42} = f_{41} = 0$$

$$f_{34} = f_{43} = 1.$$

How about f_{21}?

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f - expansion.

$$f_{ij} = \underbrace{P_{ij}}_{\text{First step transition is } j.} + \sum_{\substack{k \in S \\ k \neq j}} \underbrace{P_{ik}}_{\text{First step transition is } k.} f_{kj}.$$

$$\mathbb{P}(N(j) > 0 \mid X_0 = i) = \sum_{k \in S} \mathbb{P}(N(j) > 0 \mid X_0 = i, X_1 = k) \cdot \mathbb{P}(X_1 = k \mid X_0 = i)$$

$$f_{21} = \underbrace{P_{21}}_{1/4} + \underbrace{P_{22}}_{1/4} \cdot f_{21} + \underbrace{P_{23} f_{31}}_{f_{31} = 0}$$

$$f_{21} = 1/4 + 1/4 f_{21}$$

$$f_{21} = 1/3.$$