Practice Questions

April 16, 2024

Question 1. Consider a Markov chain with state space $\{1, 2, 3, 4, 5\}$, with transition matrix given by

$$P = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.1 & 0 & 0.6 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Compute f_{32}.

Question 2. Consider a Markov chain on the state space $S = \{0, 1, 2, \cdots\}$. For any $i \geq 1$, we define the transition from the state i as

$$p_{i,i+1} = \frac{i}{2i+1}, \quad \text{and} \quad p_{i,i-1} = \frac{i+1}{2i+1},$$

and $p_{i,j} = 0$ for $j \notin \{i-1, i+1\}$. Show that the Markov chain is null recurrent.

Question 3. Let $(B_t : t \geq 0)$ be a standard Brownian motion.

- If the process $M_t := \sin(tB_t) - \int_0^t f(B_s)ds$ is a martingale. Write down the function form of f, and express M_t in the form of an Itô integral.
- Find the probability $\mathbb{P}\big(B_1 > -1 \text{ and } \max_{0 \leq t \leq 1} M_t > 1\big)$.
- Apply Itô's formula to the process $\left(e^{\lambda B_t - \lambda^2 t/2}\right)_{t \geq 0}$, and use it to compute the moment generating function of τ, where $\tau := \inf\left\{t > 0 : |B_t| = 1\right\}$.

Question 4. Let $(X_t)_{t \geq 0}$ be a recurrent Markov chain on the state space S, and let $V : S \to \mathbb{R}$ be a real-valued function, such that

$$\sum_{j \in S} p_{i,j} V(j) = V(i), \quad \text{for } i \in S.$$

- If V is uniformly bounded in $[0, 1]$, show that V is a constant for all states.
- Let the Markov chain be simple symmetric random walk on $\mathbb{Z}$. Find a non-constant and unbounded function V such that the above equation is true.