Some additional exercise questions of stochastic calculus

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Throughout the exercise questions, we use $(B_t)_{t\geq 0}$ to denote a standard Brownian motion.

Week 9.

- If the process $M_t := B_t^3 - \int_0^t f(B_s)ds$ is a martingale. Write down the function form of f, and express M_t in the form of an Itô integral.

Solution: $f(x) = 3x$, and

$$M_t = 3\int_0^t B_s^2 dB_s.$$

- If $f : \mathbb{R}_+ \to \mathbb{R}$ is a deterministic continuous function. Show that $\int_0^t f(s)dB_s$ follows a normal distribution, and compute its mean and variance.

Solution: Let $Z = \int_0^t f(s)dB_s$. We have $\mathbb{E}[Z] = 0$ and $\mathrm{var}(Z) = \int_0^t f^2(s)ds$.

Note that

$$Z_n = \sum_{j=0}^{n-1} f\big(jt/n\big) \cdot \Big\{B_{(j+1)t/n} - B_{jt/n}\Big\} \xrightarrow{\mathbb{L}^2} Z.$$

Each Z_n is zero-mean Gaussian, and their variances converges to $\mathrm{var}(Z)$. So we can use the fact that $\mathbb{L}^2$ convergence implies convergence in distribution to show that the CDF of SZ is also Gaussian.

Week 10.

- If the process $M_t := t^2 B_t^2 - \int_0^t f(s, B_s) ds$ is a martingale. Write down the function form of f, and express M_t in the form of an Itô integral.

Solution: $f(t, x) = 2tx^2 + 2t^2$, and

$$M_t = 2 \int_0^t s^2 B_s dB_s.$$

- Let $Y_t = B_t \cdot \int_0^t B_s dB_s$. Compute dY_t.

Solution: by product rule

$$\begin{aligned} dY_t &= \big(\int_0^t B_s dB_s \big) \cdot dB_t + B_t \cdot B_t dB_t + d\langle B, \int_0^\bullet B_s dB_s \rangle_t \\ &= \frac{1}{2}\big(B_t^2 - t\big) dB_t + B_t^2 dB_t + B_t dt. \end{aligned}$$