STA447/2006: Midterm Exam #1

Instructor: Wenlong Mou

Feb 5th, 2025

Student name: ________________

Student ID: ________________

Student signature: ________________

This exam contains 10 pages.

Total marks: 100 pts

Time Allowed: 110 minutes

Question 1. [30 points, 3 points for each question] Mark each of the following statements with T (true) or F (false). *No justification is required.* Your grade will be solely based on your true-or-false choices.

(1) Let i, j be a pair of states of a Markov chain P. If $f_{ij} < 1$ and j is recurrent, then i is transient.

(2) There exists an irreducible and transient Markov chain P, such that $f_{ij} < 1$ for any pair $i, j \in S$.

(3) Let i, j be a pair of states of a Markov chain P. If $i \leftrightarrow j$, and j is null recurrent, then i is also null recurrent.

(4) If $i \to k$ and $\ell \to j$. When $\sum_{n\geq 0} p_{kl}^{(n)} < +\infty$, we have $\sum_{n\geq 0} p_{ij}^{(n)} < +\infty$.

(5) There exists a Markov chain that has exactly two stationary distributions.

(6) Let P be an irreducible and recurrent Markov chain. If $(X_k)_{k\geq 0}$ and $(X'_k)_{k\geq 0}$ are two independent Markov chains following P. Then the joint chain $Y_k = (X_k, X'_k)$ is also irreducible and recurrent.

(7) Let i, j be a pair of states of an irreducible Markov chain P. If $f_{ij} = f_{ji} = 1$. Then P is recurrent.

(8) Let i, j be a pair of states of a Markov chain P. If i has period 3 and $i \to j$. Then j also has period 3.

(9) Let P be an irreducible Markov chain, and let i be a state. If $\mathbb{E}_i[T_i] < +\infty$ (T_i is the first return time to i), then for any j, the limit

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} p_{ij}^{(n)}$$

exists, and is strictly larger than 0.

(10) Let P be an irreducible Markov chain on a finite set S. Then for any $i \in S$, we have $\mathbb{E}_i[T_i^2] < +\infty$, where T_i is the first return time to i.

Question 2. [25 pts] Consider a Markov chain on a finite state space $S = \{1, 2, 3, 4, 5\}$, with the transition matrix given by

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(1) [5 pts]. Which states are recurrent? Which states are transient? Please explain your reasoning.

(2) [10 pts]. Compute f_{15} and $\mathbb{E}_1[N(3)]$, where $N(i)$ is the number of visits to the state i. Explain your reasoning.

(3) [10 pts]. Let $X_0 = 1$, compute the probability that state 2 and state 5 are both visited in the trajectory $(X_k)_{k=0,1,2,\cdots}$.

Question 3. [20 pts, 10 pts each] Prove the following statements.

(1). Let P be an irreducible Markov chain in the state space S. If there exist $i, j \in S$, such that $\mathbb{E}_i[T_j] < +\infty$ and $\mathbb{E}_j[T_i] < +\infty$, then P is positive recurrent.

(2). There exists a Markov chain P, such that P has a stationary distribution π, but there exists $i \in S$, such that for any state j in the support of π (i.e. a state j with $\pi(j) > 0$), we have

$$i \to j, \quad \text{but} \quad \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} p_{ij}^{(n)} \neq \pi_j$$

Question 4. [10 pts] Let $S = \{0, 1, 2, \cdots\}$, and consider the following Markov chain

$$P_{0i} = \frac{1}{i(i+1)}, \quad \text{and} \quad P_{i(i-1)} = 1,$$

for every $i = 1, 2, \cdots$. Find a stationary measure of P, and determine whether the Markov chain has a stationary distribution.

Question 5. [15 pts] Let P be a reversible Markov chain over a finite state space S ($|S| < +\infty$). Let π be its stationary distribution. Suppose that P is irreducible and aperiodic. From the class, we know that the Markov chain converges to its stationary distribution. Through this question, we will quantify how fast it converges.

(1) [6 pts]. Show that the matrix P is diagonalizable and all its eigenvalues are real.

(2) [9 pts]. Show that there exists a pair of constants $c, \lambda > 0$ depending on P, such that

$$\sum_{j \in S} \left| p_{ij}^{(n)} - \pi_j \right| \leq c e^{-\lambda n},$$

for every $i \in S$.
[Hint: you can use the result of part (1) as given, even if you have not proven it.]