H 1

$$H = U n_{0\uparrow} n_{0\downarrow} + t \sum_{\sigma} \left(c^{\dagger}_{0\sigma} c_{1\sigma} + c^{\dagger}_{1\sigma} c_{0\sigma} \right) + \varepsilon \sum_{\sigma} n_{1\sigma}$$

where $n_{i\sigma} = c^{\dagger}_{i\sigma} c_{i\sigma}$

$U \bullet \overset{t}{\text{———}} \circ\, \varepsilon$

site : 0 1

possible states: $|0\rangle_0 \quad |\uparrow\rangle_0 \quad |\downarrow\rangle_0 \quad |\uparrow\downarrow\rangle_0$

$|0\rangle_1 \quad |\uparrow\rangle_1 \quad |\downarrow\rangle_1 \quad |\uparrow\downarrow\rangle_1$

$0e^-$: $\underset{00}{|0\rangle_0} \underset{00}{|0\rangle_1} = |vac\rangle = \underset{0}{|0\rangle}$

$1e^-$: $\underset{10}{|\uparrow\rangle_0} \underset{00}{|0\rangle_1} = \underset{1}{|1\rangle}$, $\underset{00}{|0\rangle_0} \underset{10}{|\uparrow\rangle_1} = \underset{4}{|2\rangle}$,

$\underset{01}{|\downarrow\rangle_0} \underset{00}{|0\rangle_1} = \underset{2}{|3\rangle}$, $\underset{00}{|0\rangle_0} \underset{01}{|\downarrow\rangle_1} = \underset{8}{|4\rangle}$

$2e^-$: $\underset{10}{|\uparrow\rangle_0} \underset{10}{|\uparrow\rangle_1} = \underset{5}{|5\rangle}$, $\underset{01}{|\downarrow\rangle_0} \underset{01}{|\downarrow\rangle_1} = \underset{10}{|6\rangle}$,

$\underset{10}{|\uparrow\rangle_0} \underset{01}{|\downarrow\rangle_1} = \underset{9}{|7\rangle}$, $\underset{01}{|\downarrow\rangle_0} \underset{10}{|\uparrow\rangle_1} = \underset{6}{|8\rangle}$,

$\underset{11}{|\uparrow\downarrow\rangle_0} \underset{00}{|0\rangle_1} = \underset{3}{|9\rangle}$, $\underset{00}{|0\rangle_0} \underset{11}{|\uparrow\downarrow\rangle_1} = \underset{12}{|10\rangle}$

$3e^-$: $\underset{11}{|\uparrow\downarrow\rangle_0} \underset{10}{|\uparrow\rangle_1} = \underset{7}{|11\rangle}$, $\underset{10}{|\uparrow\rangle_0} \underset{11}{|\uparrow\downarrow\rangle_1} = \underset{13}{|12\rangle}$,

$\underset{11}{|\uparrow\downarrow\rangle_0} \underset{01}{|\downarrow\rangle_1} = \underset{11}{|13\rangle}$, $\underset{01}{|\downarrow\rangle_0} \underset{11}{|\uparrow\downarrow\rangle_1} = \underset{14}{|14\rangle}$

$4e^-$: $\underset{11}{|\uparrow\downarrow\rangle_0} \underset{11}{|\uparrow\downarrow\rangle_1} = \underset{15}{|15\rangle}$

written in pink: "native" state representation of computer: see code

$\hat{=} \; |n_{0\uparrow}\, n_{0\downarrow}\, n_{1\uparrow}\, n_{1\downarrow}\rangle$ where $n_{i\sigma} = 0, 1$

$\Rightarrow 4^2$ states

Symmetries of H: does not change # particles and doesn't have ~~has no~~ spin flip terms.

$\Rightarrow$ Hamiltonian conserves n_{tot}, S^2_{tot}, S_z^{tot}

$\Rightarrow$ these are good quantum numbers, $\exists$ a common eigenbasis

$\Rightarrow$ Hamiltonian is block-diagonal in $(n, S^2_{tot}, S^z_{tot})$

Symmetrization of $|\Psi_i\rangle$: is encoded in operator algebra

$$\boxed{\{c_i, c_j\} = \{c_j, c_i\} = 0, \quad \{c_i, c_j^{\dagger}\} = \delta_{ij}}$$

Convention: read state from right to left and write creation operators in that order, starting from $|vac\rangle$

f.e.: $\cdot\ |\uparrow\rangle_0 |\downarrow\rangle_1 = c^\dagger_{0\uparrow} c^\dagger_{1\downarrow} |vac\rangle$

$\cdot\ |\uparrow\rangle_0 |\uparrow\downarrow\rangle_1 = c^\dagger_{0\uparrow} c^\dagger_{1\uparrow} c^\dagger_{1\downarrow} |vac\rangle$

note: $(|\uparrow\rangle_0 |\downarrow\rangle_1)^\dagger = \langle\downarrow_1| \langle\uparrow_0| = c_{\downarrow 1} c_{\uparrow 0}$

Convention needs to be fixed bc as one can see

for $n=2$: $|t\rangle = \frac{1}{\sqrt{2}}\left(|\phi_0\rangle|\phi_1\rangle - |\phi_1\rangle|\phi_0\rangle\right) =$

$\neq \frac{1}{\sqrt{2}}\left(|\phi_1\rangle|\phi_0\rangle - |\phi_0\rangle|\phi_1\rangle\right) = -|t\rangle$

Hamiltonian Matrix in blocks of (n, S, S_z):

$n=0$: $H|vac\rangle = 0|vac\rangle$

$n=1$: considering the off-diagonals, only $\underset{=\langle 1|H|2\rangle}{\langle 2|H|1\rangle}$ and $\underset{=\langle 3|H|4\rangle}{\langle 4|H|3\rangle}$ ~~(and their conjugates)~~ can be $\neq 0$, bc

$|1\rangle, |2\rangle$: $S^z_{tot} = +\frac{1}{2}$; $|3\rangle, |4\rangle$: $S^z_{tot} = -\frac{1}{2}$

$\langle 2|H|1\rangle = \langle\uparrow|_1 H |\uparrow\rangle_0 = \langle vac| c_{1\uparrow} H c^\dagger_{0\uparrow} |vac\rangle =$

$= t\langle vac| \underbrace{c_{1\uparrow} c^\dagger_{1\uparrow}}_{(1-c^\dagger c)} \underbrace{c_{0\uparrow} c^\dagger_{0\uparrow}}_{(1-c^\dagger c)} |vac\rangle = t$

$\langle 4|H|3\rangle = \langle\downarrow|_1 H |\downarrow\rangle_0 = t\langle vac| c_{1\downarrow} c^\dagger_{1\downarrow} c_{0\downarrow} c^\dagger_{0\downarrow} |vac\rangle = t$

since $\langle i|i\rangle = 1$: $\langle 1|H|1\rangle = 0$, $\langle 2|H|2\rangle = \varepsilon$

diagonals $\langle 3|H|3\rangle = 0$, $\langle 4|H|4\rangle = 0$

$$\begin{array}{c|cccc} & \overset{1}{|1\rangle} & \overset{4}{|2\rangle} & \overset{2}{|3\rangle} & \overset{8}{|4\rangle} \\ \hline \langle 1| & 0 & t & 0 & 0 \\ \langle 2| & t & \varepsilon & 0 & 0 \\ \langle 3| & 0 & 0 & 0 & t \\ \langle 4| & 0 & 0 & t & \varepsilon \end{array} = \underline{\underline{H}}_1$$

H3

$n=2$: $|5\rangle$ and $|6\rangle$ are eigenstates, bc there is no ~~other~~ term in H connecting them to any other state

$\Rightarrow$ $H|5\rangle = \varepsilon|5\rangle$ 5, $\quad H|6\rangle = \varepsilon|6\rangle$ 10

What is left: $|7\rangle, |8\rangle, |9\rangle, |10\rangle$:

$\langle 7|H|7\rangle = \varepsilon$, $\langle 8|H|8\rangle\ \varepsilon$,

$\langle 9|H|9\rangle = U$, $\langle 10|H|10\rangle = 2\varepsilon$

Off diagonals: both $|7\rangle$ and $|8\rangle$ can connect to $|9\rangle$ and $|10\rangle$, but $|7\rangle$ can't connect to $|8\rangle$, bc we only have "quadratic" t-terms

$$\langle 9|H|7\rangle = \langle 0|c_{0\downarrow}c_{0\uparrow} H c^\dagger_{0\uparrow} c^\dagger_{1\downarrow}|0\rangle =$$
$$= t\langle 0|c_{0\downarrow}c_{0\uparrow} c^\dagger_{0\downarrow} c_{1\downarrow} c^\dagger_{0\uparrow} c^\dagger_{1\downarrow}|0\rangle =$$
$$= (-1)^2\, t\langle 0|c_{0\downarrow}c^\dagger_{0\downarrow} c_{0\uparrow} c^\dagger_{0\uparrow} c_{1\downarrow} c^\dagger_{1\downarrow}|0\rangle = t$$

$$\langle 10|H|7\rangle = \langle 0|c_{1\downarrow}c_{1\uparrow} H c^\dagger_{0\uparrow} c^\dagger_{1\downarrow}|0\rangle =$$
$$= t\langle 0|c_{1\downarrow}c_{1\uparrow} c^\dagger_{1\uparrow} c_{0\uparrow} c^\dagger_{0\uparrow} c^\dagger_{1\downarrow}|0\rangle = t$$

$$\langle 9|H|8\rangle = \langle 0|c_{0\downarrow}c_{0\uparrow} H c^\dagger_{0\downarrow} c^\dagger_{1\uparrow}|0\rangle =$$
$$= t\langle 0|c_{0\downarrow}c_{0\uparrow} c^\dagger_{0\uparrow} c_{1\uparrow} c^\dagger_{0\downarrow} c^\dagger_{1\uparrow}|0\rangle =$$
$$= (-1)\, t\langle 0|c_{0\downarrow}c_{0\uparrow} c^\dagger_{0\uparrow} c^\dagger_{0\downarrow} c_{1\uparrow} c^\dagger_{1\uparrow}|0\rangle = -t$$

$$\langle 10|H|8\rangle = \langle 0|c_{1\downarrow}c_{1\uparrow} H c^\dagger_{0\downarrow} c^\dagger_{1\uparrow}|0\rangle =$$

5 10 $$= t\langle 0|c_{1\downarrow}c_{1\uparrow} c^\dagger_{1\downarrow} c_{0\downarrow} c^\dagger_{0\downarrow} c^\dagger_{1\uparrow}|0\rangle =$$

ε $$= (-1)\, t\langle 0|c_{1\downarrow}c^\dagger_{1\downarrow} c_{1\uparrow} c_{0\downarrow} c^\dagger_{0\downarrow} c^\dagger_{1\uparrow}|0\rangle = -t$$

ε 9 6 3 12

$$\begin{array}{c} \\ \langle 7| \\ \langle 8| \\ \langle 9| \\ \langle 10| \end{array}
\begin{array}{c}
\begin{array}{cccc} |7\rangle & |8\rangle & |9\rangle & |10\rangle \end{array} \\
\begin{pmatrix} \varepsilon & 0 & t & t \\ 0 & \varepsilon & -t & -t \\ t & -t & U & 0 \\ t & -t & 0 & 2\varepsilon \end{pmatrix}
\end{array} = \underline{\underline{H'_2}}$$

n=3: Particle with unpaired spin projection can change site. $|11\rangle \leftrightarrow |12\rangle$, $|13\rangle \leftrightarrow |14\rangle$

diagonals: $\langle 11|H|11\rangle = U + \varepsilon$, $\langle 12|H|12\rangle = 2\varepsilon$

$\langle 13|H|13\rangle = U + \varepsilon$, $\langle 14|H|14\rangle = 2\varepsilon$

off diagonals:

$$\langle 12|H|11\rangle = \langle 0|c_{1\downarrow}c_{1\uparrow}c_{0\uparrow}\, H\, c^\dagger_{0\uparrow}c^\dagger_{0\downarrow}c^\dagger_{1\uparrow}|0\rangle =$$
$$= t\,\langle 0|c_{1\downarrow}c_{1\uparrow}c_{0\uparrow}\, c^\dagger_{1\downarrow}c_{0\downarrow}\, c^\dagger_{0\uparrow}c^\dagger_{0\downarrow}c^\dagger_{1\uparrow}|0\rangle =$$
$$= t(-1)^2\langle 0|c_{1\downarrow}c_{1\uparrow}c^\dagger_{1\downarrow}\, c_{0\uparrow}c_{0\downarrow}c^\dagger_{0\downarrow}c^\dagger_{0\uparrow}c^\dagger_{1\uparrow}|0\rangle = (-1)^3 t = -t$$

$$\langle 14|H|13\rangle = \langle 0|c_{1\downarrow}c_{1\uparrow}c_{0\downarrow}\, H\, c^\dagger_{0\uparrow}c^\dagger_{0\downarrow}c^\dagger_{1\downarrow}|0\rangle =$$
$$= \langle 0|c_{1\downarrow}c_{1\uparrow}c_{0\downarrow}\, c^\dagger_{1\uparrow}c_{0\uparrow}\, c^\dagger_{0\uparrow}c^\dagger_{0\downarrow}c^\dagger_{1\downarrow}|0\rangle =$$
$$= (-1)t\langle 0|c_{1\downarrow}c_{1\uparrow}c^\dagger_{1\uparrow}c_{0\downarrow}c_{0\uparrow}c^\dagger_{0\uparrow}c^\dagger_{0\downarrow}c^\dagger_{1\downarrow}|0\rangle = -t$$

$$\begin{array}{c|c} & \begin{array}{cccc} \overset{7}{|11\rangle} & \overset{13}{|12\rangle} & \overset{11}{|13\rangle} & \overset{14}{|14\rangle} \end{array} \\ \begin{array}{c} \langle 11| \\ \langle 12| \\ \langle 13| \\ \langle 14| \end{array} & \left(\begin{array}{cc|cc} U+\varepsilon & -t & 0 & 0 \\ -t & 2\varepsilon & 0 & 0 \\ \hline 0 & 0 & U+\varepsilon & -t \\ 0 & 0 & -t & 2\varepsilon \end{array}\right) \end{array} = \underline{\underline{H}}_3$$

n=4: Only one state. $\overset{15}{}$

$$H|15\rangle = (U+2\varepsilon)|15\rangle$$

~~Solving the Hamiltonian for n=1,2,4~~

~~n=1:~~

H5 Solving the Hamiltonian:

n=0: $\boxed{H|\Psi_0\rangle = E_0|\Psi_0\rangle : \quad E_0 = 0, \quad |\Psi_0\rangle = |0\rangle}$

n=1: $\underline{\underline{H}}_1 - \lambda \underline{\underline{I}} =$

$$\begin{array}{ccc} & & |1\rangle & |2\rangle & |3\rangle & |4\rangle \\ 1 & \langle 1| & & & & \\ 4 & \langle 2| & & & & \\ 2 & \langle 3| & & & & \\ 8 & \langle 4| & & & & \end{array} = \left(\begin{array}{cc|cc} 0-\lambda & t & 0 & 0 \\ t & \varepsilon-\lambda & 0 & 0 \\ \hline 0 & 0 & 0-\lambda & t \\ 0 & 0 & t & \varepsilon-\lambda \end{array}\right)$$

same blocks because of ~~spin~~ ~~degeneracy~~ SU(2) symmetry

$$\tilde{\underline{\underline{H}}}_1 - \lambda \underline{\underline{I}} = \begin{pmatrix} -\lambda & t \\ t & \varepsilon-\lambda \end{pmatrix} \qquad \det\left(\underline{\underline{H}}_1 - \lambda \underline{\underline{I}}\right) = -\lambda(\varepsilon-\lambda) - t^2 \overset{!}{=} 0$$

$$\Leftrightarrow \lambda^2 - \varepsilon\lambda - t^2 = 0$$

$$\lambda_{1,2} = \frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} + t^2} \quad \left(\begin{array}{l} \nearrow \lambda_1 = \text{with } - \\ \searrow \lambda_2 = \text{with } + \end{array} \right)$$

$$= a \pm b$$

$\underline{\lambda_1 = a - b:}$

$$\begin{pmatrix} b-a & t \\ t & b+a \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0} : \quad \begin{array}{l} \mathrm{I}: v_1(b-a) + t v_2 = 0 \\ \mathrm{II}: v_1 t + (b+a) v_2 = 0 \end{array}$$

$$\mathrm{I}: \quad v_2 = \frac{(a-b)}{t} v_1 \quad \text{in } \mathrm{II}$$

$$v_1 \underbrace{\left(t + \frac{(a-b)(a+b)}{t} \right)}_{0} = 0$$

$$\Rightarrow t^2 + a^2 - b^2 = 0$$

$$\Leftrightarrow t^2 + \frac{\varepsilon^2}{4} - \frac{\varepsilon^2}{4} - t^2 = 0 \checkmark$$

$$v_1 := t \Rightarrow v_2 = (a-b) = \lambda_1 : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} t \\ \lambda_1 \end{pmatrix}$$

$$\Rightarrow \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_2 = \sqrt{t^2 + \lambda_1^2}$$

$\underline{\lambda_2 = a+b}$

$$\begin{pmatrix} -(a+b) & t \\ t & a-b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 : \quad \begin{matrix} \mathrm{I}: & -v_1(a+b) + v_2 t = 0 \\ \mathrm{II}: & v_1 t + v_2(a-b) = 0 \end{matrix}$$

$$\mathrm{I}: \quad v_2 = \frac{a+b}{t} v_1 \quad \text{in } \mathrm{II}$$

$$v_1 \Big(\underbrace{t + \frac{(a-b)(a+b)}{t}}_{=0\ \checkmark} \Big) = 0$$

$$v_1 := t \quad \Rightarrow \quad v_2 = a+b = \lambda_2$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} t \\ \lambda_2 \end{pmatrix}, \quad \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\| = \sqrt{t^2 + \lambda_2^2}$$

$\underline{n=1:}$

$\Rightarrow$

$$H|\Psi_1^{(1)}\rangle = E_1 |\Psi_1^{(1)}\rangle, \quad H|\Psi_1^{(2)}\rangle = E_1 |\Psi_1^{(2)}\rangle$$

$$E_1 = \frac{\varepsilon}{2} - \sqrt{\frac{\varepsilon^2}{4} + t^2}$$

$$|\Psi_1^{(1)}\rangle = \frac{1}{\sqrt{t^2 + E_1^2}} \Big(t|\underset{1}{1}\rangle + E_1|\underset{4}{2}\rangle \Big)$$

$$|\Psi_1^{(2)}\rangle = \frac{1}{\sqrt{t^2 + E_1^2}} \Big(t|\underset{2}{3}\rangle + E_1|\underset{8}{4}\rangle \Big)$$

degeneracy bc of SU(2) symmetry

$$H|\Psi_2^{(i)}\rangle = E_2 |\Psi_2^{(i)}\rangle, \quad i = 1, 2$$

$$E_2 = \frac{\varepsilon}{2} + \sqrt{\frac{\varepsilon^2}{4} + t^2}$$

$$|\Psi_2^{(1)}\rangle = \frac{1}{\sqrt{t^2 + E_2^2}} \Big(t|\underset{1}{1}\rangle + E_2|\underset{4}{2}\rangle \Big)$$

$$|\Psi_2^{(2)}\rangle = \frac{1}{\sqrt{t^2 + E_2^2}} \Big(t|\underset{2}{3}\rangle + E_2|\underset{8}{4}\rangle \Big)$$

H7 $\underline{n=2}$:

$$\boxed{\begin{array}{ll} H|\Psi_3\rangle = E_3|\Psi_3\rangle, & H|\Psi_4\rangle = E_4|\Psi_4\rangle \\ E_3 = \varepsilon & E_4 = \varepsilon \\ |\Psi_3\rangle = |\underset{5}{5}\rangle & |\Psi_4\rangle = |\underset{10}{6}\rangle \end{array}}$$

✓

$$\underline{\underline{H}}_2' - \lambda \underline{\underline{I}} =$$

$$\begin{array}{c|c} & |7\rangle \quad |8\rangle \quad |9\rangle \quad |10\rangle \\ \begin{array}{c} \langle 7|9 \\ \langle 8|6 = \\ \langle 9|3 \\ \langle 10|12 \end{array} & \begin{pmatrix} \varepsilon-\lambda & 0 & t & t \\ 0 & \varepsilon-\lambda & -t & -t \\ t & -t & U-\lambda & 0 \\ t & -t & 0 & 2\varepsilon-\lambda \end{pmatrix} \end{array} \xrightarrow{z_1+z_2} \begin{pmatrix} \varepsilon-\lambda & \varepsilon-\lambda & 0 & 0 \\ 0 & \varepsilon-\lambda & -t & -t \\ t & -t & U-\lambda & 0 \\ t & -t & 0 & 2\varepsilon-\lambda \end{pmatrix}$$

$$\xrightarrow{s_2-s_1} \begin{pmatrix} \varepsilon-\lambda & 0 & 0 & 0 \\ 0 & \varepsilon-\lambda & -t & -t \\ t & -2t & U-\lambda & 0 \\ t & -2t & 0 & 2\varepsilon-\lambda \end{pmatrix}$$

$$\det(\underline{\underline{H}}_2' - \lambda \underline{\underline{I}}) = (\varepsilon-\lambda)\cdot\det\begin{pmatrix} \varepsilon-\lambda & -t & -t \\ -2t & U-\lambda & 0 \\ -2t & 0 & 2\varepsilon-\lambda \end{pmatrix}$$

$$\xrightarrow{s_1+\frac{2t}{2\varepsilon-\lambda}s_3} \begin{pmatrix} \varepsilon-\lambda-\frac{2t^2}{2\varepsilon-\lambda} & -t & -t \\ -2t & U-\lambda & 0 \\ 0 & 0 & 2\varepsilon-\lambda \end{pmatrix}$$

$$\boxed{\lambda_1 = \varepsilon} \qquad \lambda_2 \neq 2\varepsilon \text{ (i.A.), bc of singularity in matrix}$$

$$\Rightarrow \det(\underline{\underline{H}}_2' - \lambda\underline{\underline{I}}) = (\varepsilon-\lambda)(2\varepsilon-\lambda)\cdot$$

$$\cdot\left(\underbrace{\left(\varepsilon-\lambda-\frac{2t^2}{2\varepsilon-\lambda}\right)(U-\lambda)-2t^2}_{\overset{!}{=}0}\right)$$

$$\big(\overbrace{(\varepsilon-\lambda)(2\varepsilon-\lambda)}^{2\varepsilon^2-3\lambda\varepsilon+\lambda^2}-2t^2\big)(U-\lambda)-2t^2(2\varepsilon-\lambda) = 0$$

$$\Leftrightarrow 2\varepsilon^2U - 3\lambda\varepsilon U + \lambda^2 U - 2t^2U - 2\varepsilon^2\lambda - 3\lambda^2\varepsilon - \lambda^3 + 2\lambda t^2 - 4t^2\varepsilon + 2t^2\lambda = 0$$

$$\Leftrightarrow \boxed{\lambda^3 + \lambda^2(-U-3\varepsilon) + \lambda(3\varepsilon U + 2\varepsilon^2 - 4t^2) + 2U(t^2-\varepsilon^2) + 4t^2\varepsilon = 0}$$

$\underline{\lambda_1 = \varepsilon:}$

$$\begin{pmatrix} 0 & 0 & t & t \\ 0 & 0 & -t & -t \\ t & -t & U-\varepsilon & 0 \\ t & -t & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \overset{!}{=} 0$$

$\mathrm{I}: \; v = -v$

$\mathrm{II}: \; v = -v$

$\mathrm{III}: \; t(v_1 - v_2) + v_3(U-\varepsilon) = 0$

$\mathrm{IV}: \; t(v_1 - v_2) + v_4 \varepsilon = 0$

$\mathrm{III}: \quad t(v_1 - v_2) + v_3(U-\varepsilon) = 0$

$\mathrm{IV}: \quad t(v_1 - v_2) - v_3 \varepsilon = 0 \qquad (v_4 = -v_3)$

$\ominus \qquad 0 \quad + \quad v_3 U = 0 \qquad \Rightarrow v_3 = 0 \Rightarrow v_4 = 0$

$\Rightarrow v_1 = v_2$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \|v\|_2 = \sqrt{2}$$

$$\boxed{H|\psi_5\rangle = E_5|\psi_5\rangle: \quad E_5 = \varepsilon, \quad |\psi_5\rangle = \frac{1}{\sqrt{2}}\left(|1\underset{9}{7}\rangle + |\underset{6}{8}\rangle\right)}$$

$\underline{\lambda_2, \lambda_3, \lambda_4: \; \ldots}$

n=2: $U = \pm 2\varepsilon$ "Half filling"

$$H_2 = \begin{pmatrix} \varepsilon & 0 & t & t \\ 0 & \varepsilon & -t & -t \\ t & -t & 2\varepsilon & 0 \\ t & -t & 0 & 2\varepsilon \end{pmatrix}$$

$$(H_2 - I\lambda) = \begin{pmatrix} \varepsilon-\lambda & 0 & t & t \\ 0 & \varepsilon-\lambda & -t & -t \\ t & -t & 2\varepsilon-\lambda & 0 \\ t & -t & 0 & 2\varepsilon-\lambda \end{pmatrix}$$

$$\xrightarrow{z_1+z_2} \begin{pmatrix} \varepsilon-\lambda & \varepsilon-\lambda & 0 & 0 \\ 0 & \varepsilon-\lambda & -t & -t \\ t & -t & 2\varepsilon-\lambda & 0 \\ t & -t & 0 & 2\varepsilon-\lambda \end{pmatrix}$$

$$\xrightarrow{s_2-s_1} \begin{pmatrix} \varepsilon-\lambda & 0 & 0 & 0 \\ 0 & \varepsilon-\lambda & -t & -t \\ t & -2t & 2\varepsilon-\lambda & 0 \\ t & -2t & 0 & 2\varepsilon-\lambda \end{pmatrix} \qquad \boxed{(\varepsilon-\lambda)\cdot \det\begin{pmatrix} \varepsilon-\lambda & -t & -t \\ -2t & 2\varepsilon-\lambda & 0 \\ -2t & 0 & 2\varepsilon-\lambda \end{pmatrix}}$$

$$\xrightarrow{z_2-z_3} \begin{pmatrix} \varepsilon-\lambda & -t & -t \\ 0 & 2\varepsilon-\lambda & -(2\varepsilon-\lambda) \\ -2t & 0 & 2\varepsilon-\lambda \end{pmatrix} \xrightarrow{s_2+s_3} \begin{pmatrix} \varepsilon-\lambda & -2t & -t \\ 0 & 0 & -(2\varepsilon-\lambda) \\ -2t & 2\varepsilon-\lambda & 2\varepsilon-\lambda \end{pmatrix}$$

$$\det\begin{pmatrix} \varepsilon-\lambda & -2t \\ -2t & 2\varepsilon-\lambda \end{pmatrix} \cdot \left[-(2\varepsilon-\lambda)\right](\varepsilon-\lambda) =$$

$$= (\varepsilon-\lambda)(\lambda-2\varepsilon)\Big(\underbrace{(\varepsilon-\lambda)(2\varepsilon-\lambda)}_{2\varepsilon^2-3\varepsilon\lambda+\lambda^2} - 4t^2\Big) =$$

$$= (\varepsilon-\lambda)(\lambda-2\varepsilon)(2\varepsilon^2-3\varepsilon\lambda+\lambda^2-4t^2) = 0$$

$\Rightarrow \lambda_1 = \varepsilon, \quad \lambda_2 = 2\varepsilon,$

$$\lambda^2 - 3\lambda\varepsilon + 2\varepsilon^2 - 4t^2 = 0$$

$$\rightarrow \lambda_{3,4} = \frac{3\varepsilon}{2} \pm \sqrt{\frac{9\varepsilon^2}{4} + 4t^2 - 2\varepsilon^2} =$$

$$= \frac{3\varepsilon}{2} \pm \sqrt{\frac{1}{4}\varepsilon^2 + 4t^2} \geq \forall t, \varepsilon$$

$\boxed{\lambda_1 = \varepsilon:}$

$$\begin{pmatrix} 0 & 0 & t & t \\ 0 & 0 & -t & -t \\ t & -t & \varepsilon & 0 \\ t & -t & 0 & \varepsilon \end{pmatrix} \vec{v} = \vec{0}$$

I: $v_3 = -v_4$, II: $v_3 = -v_4$

III: $(v_1 - v_2)t + v_3\varepsilon = 0$

IV: $(v_1 - v_2)t + v_4\varepsilon = 0$

III: $(v_1 - v_2)t + v_3\varepsilon = 0$

IV: $(v_1 - v_2)t - v_3\varepsilon = 0$ $\Big|\, +$

$\Rightarrow v_1 = v_2$ $\quad \Rightarrow v_3 = 0, \ v_4 = 0$

$$\rightarrow v^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \|v^{(1)}\| = \sqrt{2}$$

$\boxed{\lambda_2 = 2\varepsilon:}$

$$\begin{pmatrix} -\varepsilon & 0 & t & t \\ 0 & -\varepsilon & -t & -t \\ t & -t & 0 & 0 \\ t & -t & 0 & 0 \end{pmatrix} \vec{v} = \vec{0}$$

$$\boxed{v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \ \|v^{(2)}\| = \sqrt{2}}$$

$\Rightarrow$ I: $-\varepsilon v_1 + t(v_3 + v_4) = 0$

II: $-\varepsilon v_2 - t(v_3 + v_4) = 0$ $\Big\}$ I − II: $v_3 = -v_4$

$\Rightarrow v_1 = v_2 = 0$

III = IV: $t(v_1 - v_2) = 0 \rightarrow v_1 = v_2$

$$\boxed{\lambda_{3,4} = \frac{3\varepsilon}{2} \mp\pm \sqrt{\frac{1}{4}\varepsilon^2 + 4t^2} = \cancel{a \pm b}\ \frac{3\varepsilon}{2} \pm b}$$

$$\cancel{\Rightarrow \varepsilon = \frac{2}{3} a,\ 2\varepsilon = 4}$$

$$\left(\begin{array}{cccc|c} -\frac{\varepsilon}{2}\pm\mp b & 0 & t & t & 0 \\ 0 & -\frac{\varepsilon}{2}\pm\mp b & -t & -t & 0 \\ t & -t & \frac{\varepsilon}{2}\pm\mp b & 0 & 0 \\ t & -t & 0 & \frac{\varepsilon}{2}\pm\mp b & 0 \end{array}\right)$$

$$\begin{array}{c} z_1 + z_2 \\ \longrightarrow \\ z_3 - z_4 \\ \longrightarrow \end{array} \begin{pmatrix} -\frac{\varepsilon}{2}\pm\mp b & -\frac{\varepsilon}{2}\pm\mp b & 0 & 0 \\ 0 & -\frac{\varepsilon}{2}\pm\mp b & -t & -t \\ 0 & 0 & \frac{\varepsilon}{2}\pm\mp b & -\frac{\varepsilon}{2}\mp\pm b \\ t & -t & 0 & \frac{\varepsilon}{2}\pm\mp b \end{pmatrix}$$

$$\mathrm{I}: \left(-\frac{\varepsilon}{2}\pm\mp b\right)(v_1 + v_2) = 0 \quad \Rightarrow \quad v_2 = -v_1$$

$$\mathrm{III}: \left(\frac{\varepsilon}{2}\pm\mp b\right)(v_3 - v_4) = 0 \quad \Rightarrow \quad v_3 = v_4$$

$$\mathrm{II}: \left(-\frac{\varepsilon}{2}\pm\mp b\right) v_2 - t(2v_3) = 0$$

$$\Rightarrow v_3 = \frac{1}{2t}\left(-\frac{\varepsilon}{2}\pm\mp b\right) v_2$$

wähle $v_2 := 2t \Rightarrow v_1 = -2t$

$$\Rightarrow v_3 = -\frac{\varepsilon}{2}\pm\mp b = v_4 =$$

$$= \left(-\frac{3\varepsilon}{2}\pm\mp b\right) + \varepsilon = \varepsilon - \lambda_{3,4} \checkmark$$

$$\boxed{\Rightarrow v^{(3),(4)} = \begin{pmatrix} -2t \\ 2t \\ \varepsilon - \lambda_{3,4} \\ \varepsilon - \lambda_{3,4} \end{pmatrix} \checkmark \quad \varepsilon - \lambda_{3,4} \rightarrow -\frac{\varepsilon}{2} \mp \sqrt{\frac{1}{4}\varepsilon^2 + 4t^2} = \frac{1}{2}\left(-\varepsilon \mp \sqrt{\varepsilon^2 + 16t^2}\right)}$$

$$\boxed{\|v\|_2 = \sqrt{8t^2 + 2(\varepsilon - \lambda_{3,4})^2}}$$

$$\begin{pmatrix} -2t \\ 2t \\ \frac{1}{2}\left(-\varepsilon \mp \sqrt{\varepsilon^2+16t^2}\right) \\ \frac{1}{2}\left(-\varepsilon \mp \sqrt{\varepsilon^2+16t^2}\right) \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{4t}{-\varepsilon \mp \sqrt{\varepsilon^2+16t^2}} \\ \frac{4t}{-\varepsilon \mp \sqrt{\varepsilon^2+16t^2}} \\ 1 \\ 1 \end{pmatrix}$$

H9 $n=3$: $\underline{\underline{H}}_3 - \lambda \underline{\underline{\mathbb{I}}} =$

$$
\begin{array}{c|cccc}
 & |11\rangle & |12\rangle & |13\rangle & |14\rangle \\
\langle 11| \; 7 & & & & \\
\langle 12| \; 13 & & & & \\
\langle 13| \; 11 & & & & \\
\langle 14| \; 14 & & & &
\end{array}
= \begin{pmatrix} \boxed{\begin{matrix} U+\varepsilon-\lambda & -t \\ -t & 2\varepsilon \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} U+\varepsilon-\lambda & -t \\ -t & 2\varepsilon-\lambda \end{matrix}} \end{pmatrix}
$$

same blocks because of SU(2) symmetry

$$\tilde{\underline{\underline{H}}}_3 - \lambda \underline{\underline{\mathbb{I}}} = \begin{pmatrix} U+\varepsilon-\lambda & -t \\ -t & 2\varepsilon-\lambda \end{pmatrix} \qquad \det(\tilde{\underline{\underline{H}}}_3 - \lambda \underline{\underline{\mathbb{I}}}) = $$
$$= (U+\varepsilon-\lambda)(2\varepsilon-\lambda) - t^2$$

$$\det(\tilde{\underline{\underline{H}}}_3 - \lambda \underline{\underline{\mathbb{I}}}) = 0: \quad \lambda^2 - \lambda(U+3\varepsilon) + 2\varepsilon(U+\varepsilon) - t^2 = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{3\varepsilon+U}{2} \pm \sqrt{\underbrace{\frac{(3\varepsilon+U)^2}{4} - 2\varepsilon(U+\varepsilon) + t^2}_{D}}$$

$$D = \frac{9\varepsilon^2 + 6\varepsilon U + U^2 - 8\varepsilon U - 8\varepsilon^2 + 4t^2}{4} =$$

$$= \frac{1}{4}\left(\varepsilon^2 - 2\varepsilon U + U^2 + 4t^2\right) =$$

$$= \frac{1}{4}\left((\varepsilon - U)^2 + 4t^2\right) \geq 0 \quad \checkmark$$

$$\Rightarrow \lambda_{1,2} = \frac{3\varepsilon+U}{2} \pm \frac{1}{2}\sqrt{(\varepsilon-U)^2 + 4t^2} = \quad \begin{matrix} \nearrow \lambda_1 = \text{with} - \\ \searrow \lambda_2 = \text{with} + \end{matrix}$$

$$= a \pm b$$

$\underline{\lambda_1 = a - b}$: E_9

$$\begin{pmatrix} \frac{U-\varepsilon}{2} + b & -t \\ -t & \frac{\varepsilon - U}{2} + b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$$

$$\text{I}: \left(\frac{U-\varepsilon}{2} + b\right) v_1 - t v_2 = 0 \rightarrow v_2 = \frac{U-\varepsilon+2b}{2t} v_1$$

$$v_1 := t \Rightarrow v_2 = \boxed{\frac{U-\varepsilon}{2} + b} = \lambda_2 - 2\varepsilon \quad \checkmark$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} t \\ \lambda_2 - 2\varepsilon \end{pmatrix} \qquad \|v\| = \sqrt{t^2 + (\lambda_2 - 2\varepsilon)^2}$$

$\underline{\lambda_2 = \cancel{\quad} a + b}$: E_{10}

$$\Rightarrow \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} t \\ \lambda_1 - 2\varepsilon \end{pmatrix} \qquad \| v \|_2 = \sqrt{t^2 + (\lambda_1 - 2\varepsilon)^2}$$

$$H|\Psi_9\rangle = E_9 |\Psi\rangle$$

$$E_9 = \frac{3\varepsilon + U}{2} - \frac{1}{2}\sqrt{(\varepsilon - U)^2 + 4t^2}$$

$$|\Psi_9^{(1)}\rangle = \frac{1}{\sqrt{(E_{10} - 2\varepsilon)^2 + t^2}} \left(t \, |\underset{7}{11}\rangle + (E_{10} - 2\varepsilon) |\underset{13}{12}\rangle \right)$$

$$|\Psi_9^{(2)}\rangle = \text{——} \, '' \, \text{——} \left(\, '' \, |\underset{11}{13}\rangle \quad '' \quad |\underset{14}{14}\rangle \right)$$

$$H|\Psi_{10}\rangle = E_{10} |\Psi_{10}\rangle$$

$$E_{10} = \frac{3\varepsilon + U}{2} + \frac{1}{2}\sqrt{(\varepsilon - U)^2 + 4t^2}$$

$$|\Psi_{10}^{(1)}\rangle = \frac{1}{\sqrt{t^2 + (E_9 - 2\varepsilon)^2}} \left(t \, |\underset{7}{11}\rangle + (E_9 - 2\varepsilon) |\underset{13}{12}\rangle \right)$$

$$|\Psi_{10}^{(2)}\rangle = \quad '' \quad \left(\, '' \, |\underset{11}{13}\rangle \quad '' \quad |\underset{14}{14}\rangle \right)$$