Bayesian Learning

Lecture 1 - Introduction and Bernoulli data: BDA ch. 1, 2.1-2.4

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Course overview

- All course material on Lisam.
- Teaching activities:
  - Lectures and mathematical exercises (Bertil Wegmann)
  - Computer labs (Bertil Wegmann, Amanda Olmin and Amirhossein Ahmadian)
- **Modules**:
  - Introduction to Bayesian inference: single- and multiparameter models
  - Regression and Classification models
  - Advanced models and Posterior Approximation methods
  - Model evaluation and comparison and Variable Selection
- **Examination**
  - Computer exam, 3 hp: distance-based
  - Lab reports, 3 hp: work in pairs, submit through Lisam.

Previous course evaluation and course modifications

- Course evaluation spring 2020 will be published on Lisam.
- Overall evaluation grade: 732A91(4.85)/TDDE07(5)
  Answer rate: 732A91(26 out of 56)/TDDE07(5 out of 35)
- The subject-specific content of the course gave me the opportunity to achieve the learning outcomes of the course. Grade: 4.8
- The various teaching and working methods of the course were relevant to the learning outcomes of the course. Grade: 4.8
- Lab assignments have been considered great, but somewhat heavy with 4 labs. The number of labs is decreased from 4 to 3.

Lecture overview

- The likelihood function
- Bayesian inference
- Bernoulli model

Likelihood function - Bernoulli trials

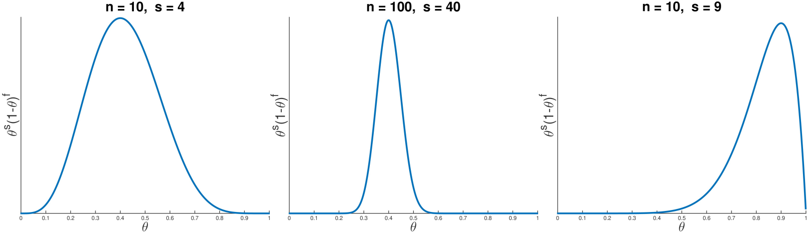
- **Bernoulli trials**:

$$X_1, ..., X_n|\theta \overset{iid}{\sim} Bern(\theta).$$

- **Likelihood** from $s = \sum_{i=1}^{n} x_i$ successes and $f = n - s$ failures.

$$p(x_1, ..., x_n|\theta) \quad = \quad p(x_1|\theta) \cdots p(x_n|\theta) = \theta^s (1-\theta)^f$$

- **Maximum likelihood estimator** $\hat{\theta}$ maximizes $p(x_1, ..., x_n|\theta)$.
- Given the data $x_1, ..., x_n$, plot $p(x_1, ..., x_n|\theta)$ as a function of θ.



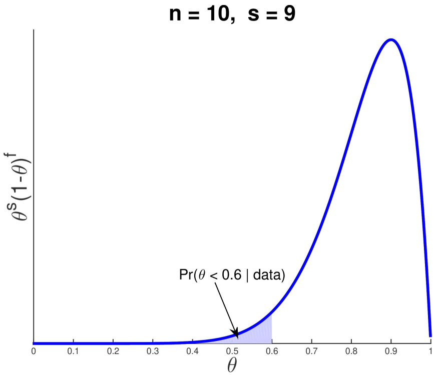
The likelihood function

- Say it out loud:

  The likelihood function is
  the probability of the observed data
  considered as a function of the parameter.

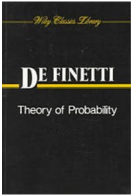
- The symbol $p(x_1, ..., x_n|\theta)$ plays two different roles:
- **Probability distribution** for the data.
  - The data $\mathbf{x} = (x_1, ..., x_n)$ are random.
  - θ is fixed.
- **Likelihood function** for the parameter
  - The data $\mathbf{x} = (x_1, ..., x_n)$ are fixed.
  - $p(x_1, ..., x_n|\theta)$ is a function of θ.

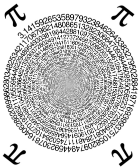
Probabilities from the likelihood?



Uncertainty and subjective probability

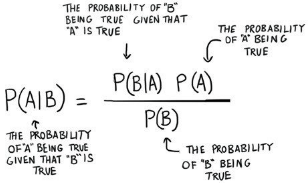
- $\Pr(\theta < 0.6|\text{data})$ only makes sense if θ is random.
- But θ may be a fixed natural constant?
- **Bayesian: doesn't matter if θ is fixed or random**.
- Do **You** know the value of θ or not?
- $p(\theta)$ reflects Your knowledge/**uncertainty** about θ.
- **Subjective probability**.
- The statement $\Pr(\text{10th decimal of } \pi = 9) = 0.1$ makes sense.





Bayesian learning

- **Bayesian learning** about a model parameter θ:
  - state your **prior** knowledge as a probability distribution $p(\theta)$.
  - collect **data** x and form the **likelihood** function $p(\mathrm{x}|\theta)$.
  - **combine** prior knowledge $p(\theta)$ with data information $p(\mathrm{x}|\theta)$.
- **How to combine** the two sources of information?
  Bayes' theorem



Learning from data - Bayes' theorem

- How to **update** from **prior** $p(\theta)$ to **posterior** $p(\theta|Data)$?
- **Bayes' theorem** for events A and B

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}.$$

- Bayes' Theorem for a model parameter θ

$$p(\theta|Data) = \frac{p(Data|\theta)p(\theta)}{p(Data)}.$$

- It is the prior $p(\theta)$ that takes us from $p(Data|\theta)$ to $p(\theta|Data)$.
- A probability distribution for θ is extremely useful.
  Predictions. **Decision making**.
- **No prior - no posterior - no useful inferences - no fun**.

Medical diagnosis

- A = {Very rare disease}, B ={Positive medical test}.
- $p(A) = 0.0001$. $p(B|A) = 0.9$. $p(B|A^c) = 0.05$.
- Probability of being sick when test is positive:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{p(B|A)p(A)}{p(B|A)p(A) + p(B|A^c)p(A^c)} \approx 0.0018.$$

- Probably not sick, but 18 times more probable now.

- Morale: If you want $p(A|B)$ then $p(B|A)$ does not tell the whole story. The prior probability $p(A)$ is also very important.

"You can't enjoy the Bayesian omelette without breaking the Bayesian eggs"
Leonard Jimmie Savage

The normalizing constant is not important

- Bayes theorem

$$p(\theta|Data) = \frac{p(Data|\theta)p(\theta)}{p(Data)} = \frac{p(Data|\theta)p(\theta)}{\int_\theta p(Data|\theta)p(\theta)d\theta}.$$

- Integral $p(Data) = \int_\theta p(Data|\theta)p(\theta)d\theta$ can be complex.
- $p(Data)$ **is just a constant** so that $p(\theta|Data)$ integrates to one.
- Example: $x \sim N(\mu, \sigma^2)$

$$p(x) = (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right].$$

- We may write

$$p(x) \propto \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right].$$

Bayes' theorem in a nutshell

- All you need to know:

$$p(\theta|Data) \propto p(Data|\theta)p(\theta)$$

or

$$\text{Posterior} \propto \text{Likelihood} \cdot \text{Prior}$$

- Thomas Bayes (1702-1761): English statistician, philosopher and Presbyterian minister.

Bernoulli trials - Beta prior

- **Model**

$$x_1, ..., x_n | \theta \overset{iid}{\sim} \mathrm{Bern}(\theta)$$

- **Prior**

$$\theta \sim \mathrm{Beta}(\alpha, \beta)$$

$$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} \quad \text{for } 0 \leq \theta \leq 1.$$

- **Posterior**

$$\begin{aligned} p(\theta | x_1, ..., x_n) &\propto p(x_1, ..., x_n | \theta) p(\theta) \\ &\propto \theta^s (1-\theta)^f \theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \theta^{s+\alpha-1}(1-\theta)^{f+\beta-1}. \end{aligned}$$

- Posterior is proportional to the $\mathrm{Beta}(\alpha+s, \beta+f)$ density.
- The prior-to-posterior mapping:

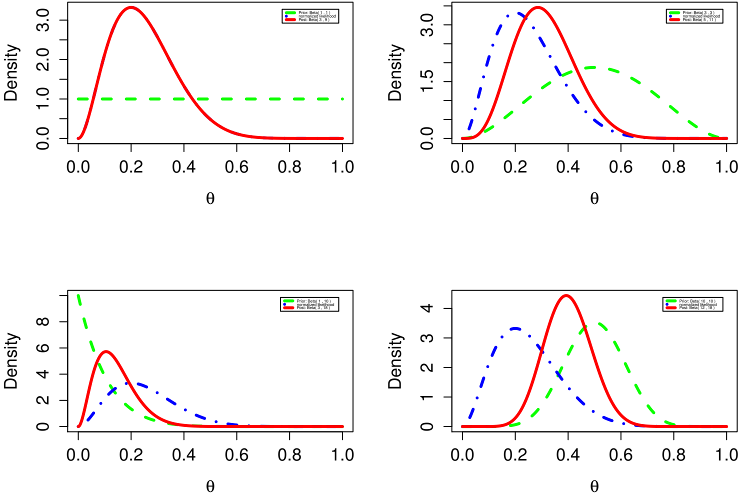
$$\theta \sim \mathrm{Beta}(\alpha, \beta) \overset{x_1, ..., x_n}{\Longrightarrow} \theta | x_1, ..., x_n \sim \mathrm{Beta}(\alpha+s, \beta+f)$$

Bernoulli example: spam emails

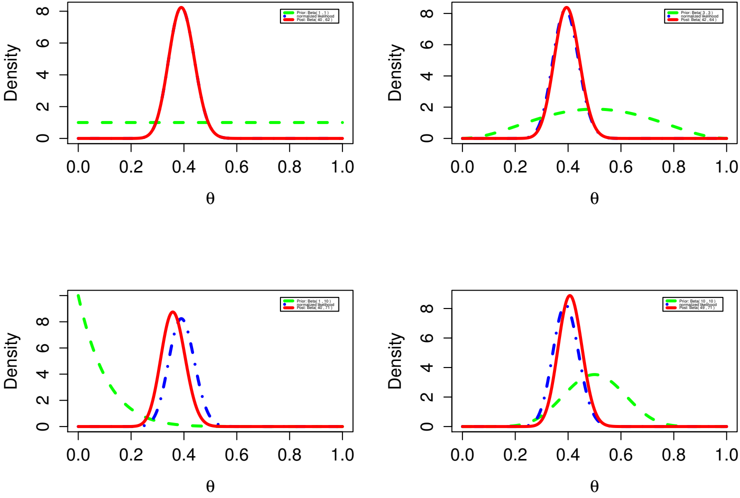
- George has gone through his collection of 4601 e-mails.
- He classified 1813 of them to be spam.
- Let $x_i = 1$ if i:th email is spam.
- **Model**: $x_i|\theta \overset{iid}{\sim} \mathrm{Bern}(\theta)$
- **Prior**: $\theta \sim \mathrm{Beta}(\alpha, \beta)$.
- **Posterior**

$$\theta|x \sim \mathrm{Beta}(\alpha + 1813, \beta + 2788)$$

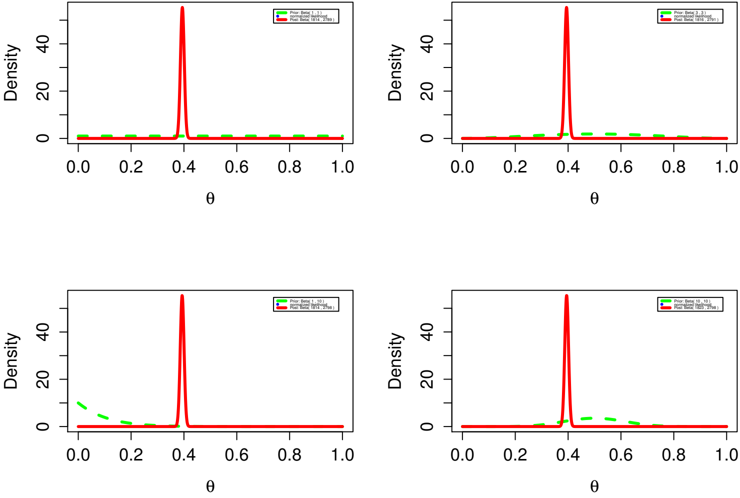
Spam data (n=10) - Prior is influential



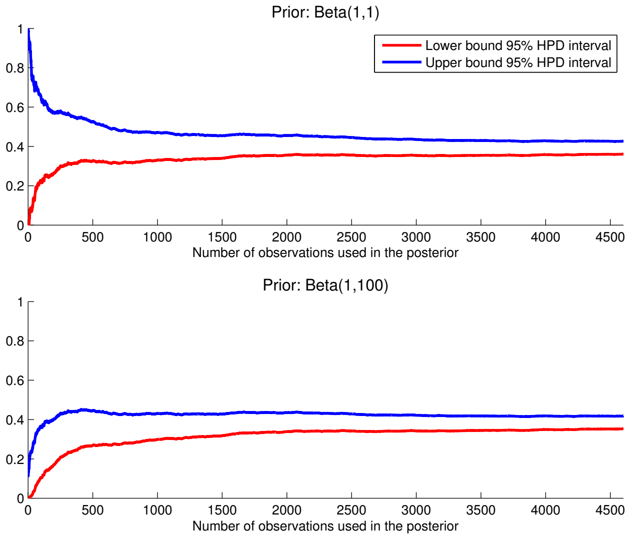
Spam data (n=100) - Prior is less influential



Spam data (n=4601) - Prior does not matter



Spam data - posterior convergence



Bayes respects the Likelihood Principle

- **Bernoulli trials with order**:
  $x_1 = 1, x_2 = 0, ..., x_4 = 1, ..., x_n = 1$

$$p(\mathrm{x}|\theta) = \theta^s (1-\theta)^f$$

- **Bernoulli trials without order**. n fixed, s random.

$$p(s|\theta) = \binom{n}{s} \theta^s (1-\theta)^f$$

- **Negative binomial sampling**: sample until you get s successes. s fixed, n random.

$$p(n|\theta) = \binom{n-1}{s-1} \theta^s (1-\theta)^f$$

- The **posterior distribution is the same** in all three cases.
- Bayesian inference respects the **likelihood principle**.