Bayesian Learning

Lecture 2 - Normal and Poisson data. Prior elicitation.

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Lecture overview

- The **Normal model** with known variance
- The **Poisson model**
- **Conjugate priors**
- **Prior elicitation**
- **Jeffreys' prior**

Normal data, known variance - uniform prior

- **Model**

$$x_1, ..., x_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2).$$

- **Prior**

$$p(\theta) \propto c \text{ (a constant)}$$

- **Likelihood**

$$\begin{aligned} p(x_1, ..., x_n | \theta, \sigma^2) &= \Pi_{i=1}^n (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}(x_i - \theta)^2\right] \\ &\propto \exp\left[-\frac{1}{2(\sigma^2/n)}(\theta - \bar{x})^2\right]. \end{aligned}$$

- **Posterior**

$$\theta | x_1, ..., x_n \sim N(\bar{x}, \sigma^2/n)$$

Normal data, known variance - normal prior

- **Prior**

$$\theta \sim N(\mu_0, \tau_0^2)$$

- **Posterior**

$$\begin{aligned} p(\theta|x_1, ..., x_n) &\propto p(x_1, ..., x_n|\theta, \sigma^2)p(\theta) \\ &\propto N(\theta|\mu_n, \tau_n^2), \end{aligned}$$

where

$$\frac{1}{\tau_n^2} = \frac{n}{\sigma^2} + \frac{1}{\tau_0^2},$$

$$\mu_n = w\bar{x} + (1 - w)\mu_0,$$

and

$$w = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}.$$

Normal data, known variance - normal prior

$$\theta \sim N(\mu_0, \tau_0^2) \overset{x_1,\ldots,x_n}{\Longrightarrow} \theta|x \sim N(\mu_n, \tau_n^2).$$

Posterior precision = Data precision + Prior precision

Posterior mean =

$$\frac{\text{Data precision}}{\text{Posterior precision}}(\text{Data mean}) + \frac{\text{Prior precision}}{\text{Posterior precision}}(\text{Prior mean})$$

Download speed

- **Data**: $x = (22.42, 34.01, 35.04, 38.74, 25.15)$ Mbit/sec.
- **Model**: $X_1, ..., X_5 | \theta, \sigma^2 \sim N(\theta, \sigma^2)$.
- Assume $\sigma = 5$ (measurements can vary $\pm$10MBit with 95% probability)
- My **prior**: $\theta \sim N(50, 5^2)$.

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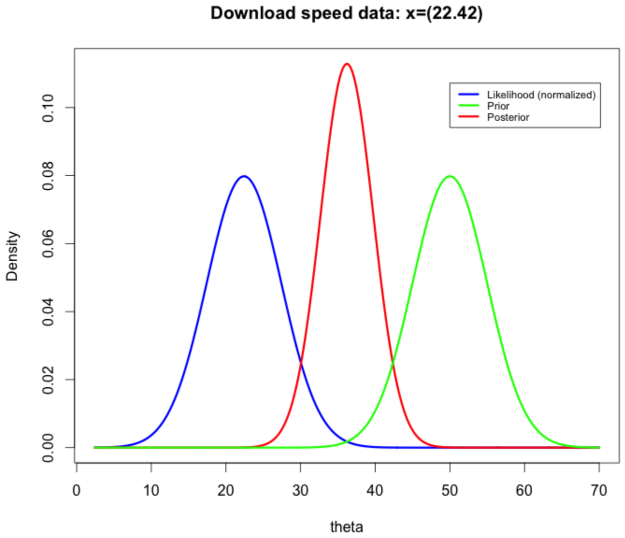

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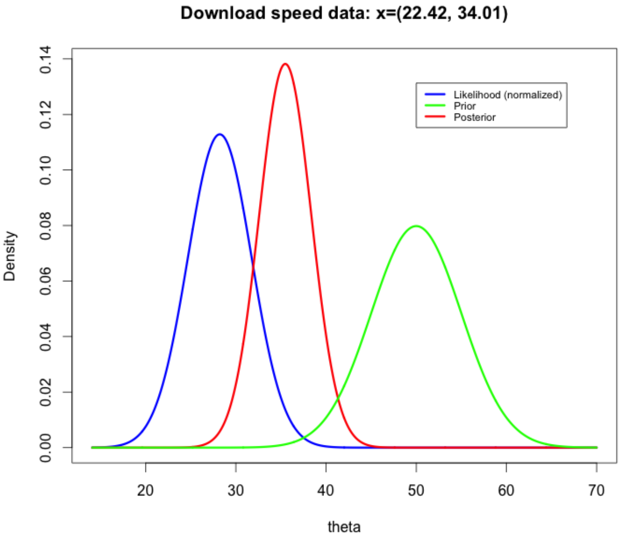

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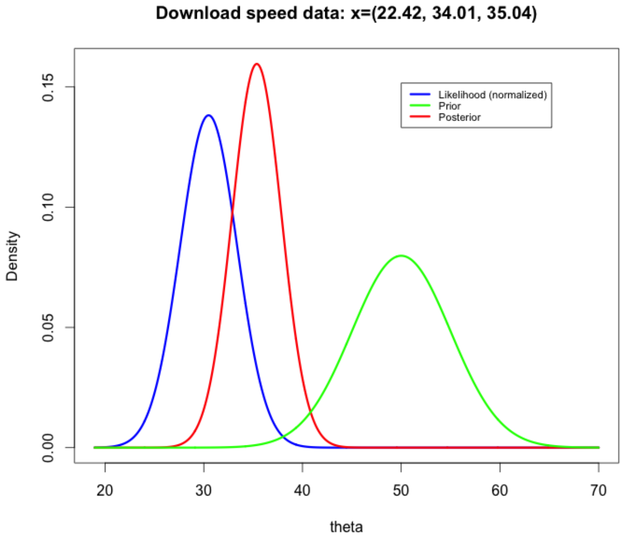

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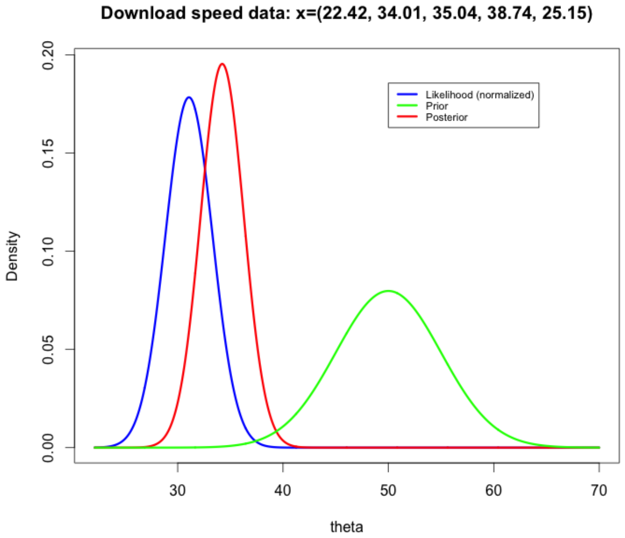
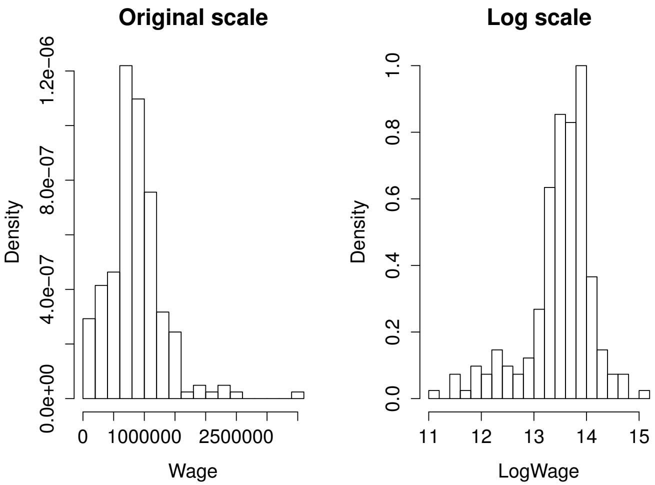


Canadian wages data

- Data on wages for 205 Canadian workers.



Canadian wages

- **Model**

$$X_1, ..., X_n | \theta \sim N(\theta, \sigma^2), \ \ \sigma^2 = 0.4$$

- **Prior**

$$\theta \sim N(\mu_0, \tau_0^2), \ \ \mu_0 = 12 \text{ and } \tau_0 = 10$$

- **Posterior**

$$\theta | x_1, ..., x_n \sim N\left(\mu_n, \tau_n^2\right),$$

where $\mu_n = w\bar{x} + (1 - w)\mu_0$.

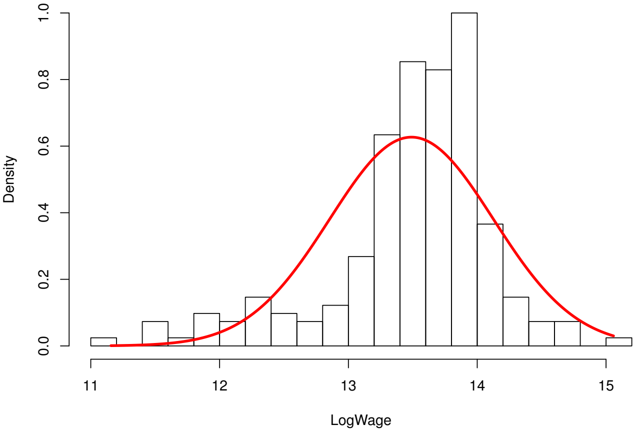
- For the Canadian wage data:

$$w = \frac{\sigma^{-2} n}{\sigma^{-2} n + \tau_0^{-2}} = \frac{2.5 \cdot 205}{2.5 \cdot 205 + 1/100} = 0.999.$$

$$\mu_n = w\bar{x} + (1 - w)\mu_0 = 0.999 \cdot 13.489 + (1 - 0.999) \cdot 12 \approx 13.489$$

$$\tau_n^2 = (2.5 \cdot 205 + 1/100)^{-1} = 0.00195$$

Canadian wages data - model fit



Poisson model

- **Model**

$$y_1, ..., y_n|\theta \overset{iid}{\sim} Pois(\theta)$$

- **Poisson distribution**

$$p(y_i|\theta) = \frac{\theta^{y_i} e^{-\theta}}{y_i!}, \; i = 1, \ldots, n$$

- **Likelihood** from iid Poisson sample $y = (y_1, ..., y_n)$

$$p(y|\theta) = \left[\prod_{i=1}^{n} p(y_i|\theta)\right] \propto \theta^{(\sum_{i=1}^{n} y_i)} \exp(-\theta n),$$

- **Prior**

$$p(\theta) \propto \theta^{\alpha-1} \exp(-\theta\beta) \propto Gamma(\alpha, \beta)$$

which contains the info: $\alpha - 1$ counts in β observations.

Poisson model, cont.

- **Posterior**

$$
\begin{aligned}
p(\theta|y_1,...,y_n) &\propto \left[\prod_{i=1}^n p(y_i|\theta)\right] p(\theta) \\
&\propto \theta^{\sum_{i=1}^n y_i} \exp(-\theta n)\theta^{\alpha-1}\exp(-\theta\beta) \\
&= \theta^{\alpha+\sum_{i=1}^n y_i - 1}\exp[-\theta(\beta+n)],
\end{aligned}
$$

which is proportional to the $Gamma(\alpha + \sum_{i=1}^n y_i, \beta + n)$ distribution.

- **Prior-to-Posterior mapping**

$$
\text{Model: } y_1,...,y_n|\theta \overset{iid}{\sim} Pois(\theta)
$$

$$
\text{Prior: } \theta \sim Gamma(\alpha,\beta)
$$

$$
\text{Posterior: } \theta|y_1,...,y_n \sim Gamma(\alpha + \sum_{i=1}^n y_i, \beta + n).
$$

Example - Number of bids in eBay auctions

- **Data**:
  - Number of placed bids in $n = 1000$ eBay coin auctions.
  - Sum of counts: $\sum_{i=1}^{n} y_i = 3635$.
  - Average number of bids per auction: $\bar{y} = 3635/1000 = 3.635$.
- **Prior**: $\alpha = 2$, $\beta = 1/2$.

$$E(\theta) = \frac{\alpha}{\beta} = 4$$

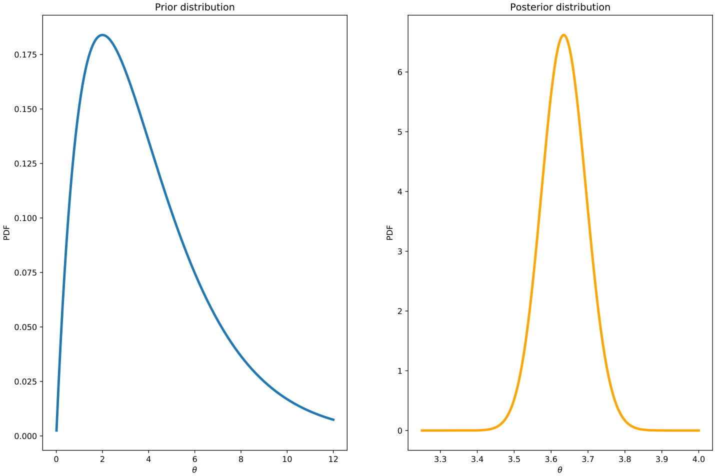
$$SD(\theta) = \left(\frac{\alpha}{\beta^2}\right)^{1/2} = 2.823$$

- **Posterior**

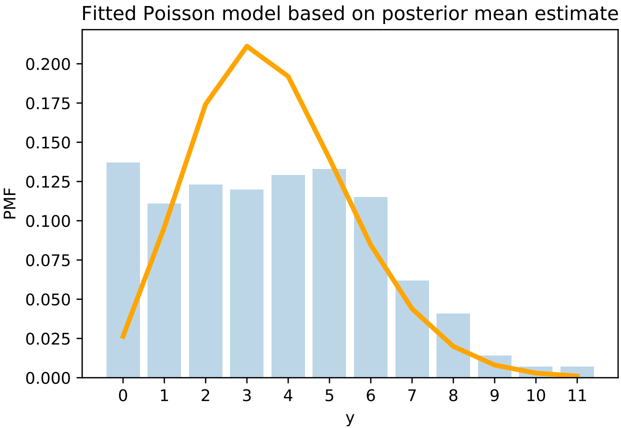
$$E(\theta|\mathbf{y}) = \frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n} = \frac{2 + 3635}{1/2 + 1000} \approx 3.635.$$

$$SD(\theta|\mathbf{y}) = \left(\frac{\alpha + \sum_{i=1}^{n} y_i}{(\beta + n)^2}\right)^{1/2} \approx 0.060.$$

eBay data - Prior and Posterior of θ



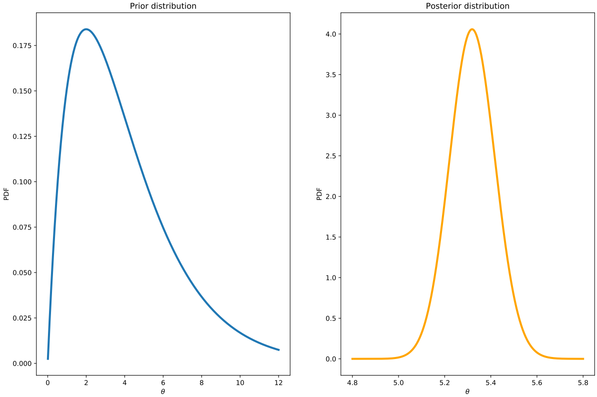
eBay data - Fit



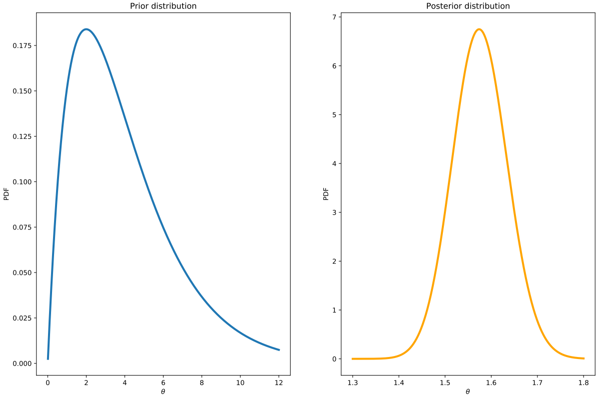
eBay - low/high seller's reservation price

- The data is very heterogenous. Some auctions start with very high reservations prices (lowest price accepted by the seller).
- Split the data into auctions with low/high reservation prices.
- **Low reservation price auctions**:
  - $n = 550$ eBay coin auctions.
  - Posterior mean: 5.321 bids.
- **High reservation price auctions**:
  - $n = 450$ eBay coin auctions.
  - Posterior mean: 1.576 bids.

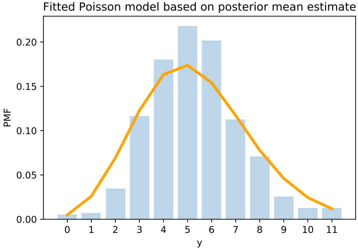
eBay - low seller's reservation price

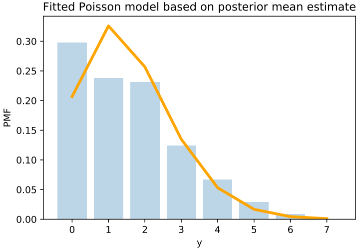


eBay - high seller's reservation price



eBay - fit low/high reservation prices





- Better fits, but still not good enough.
- Lab 3: Fit Poisson regression with reservation price as continuous covariate.

Posterior intervals

- **Bayesian 95% credible interval**: the probability that the unknown parameter θ lies in the interval is 0.95.

- Approximate 95% **credible interval** for θ

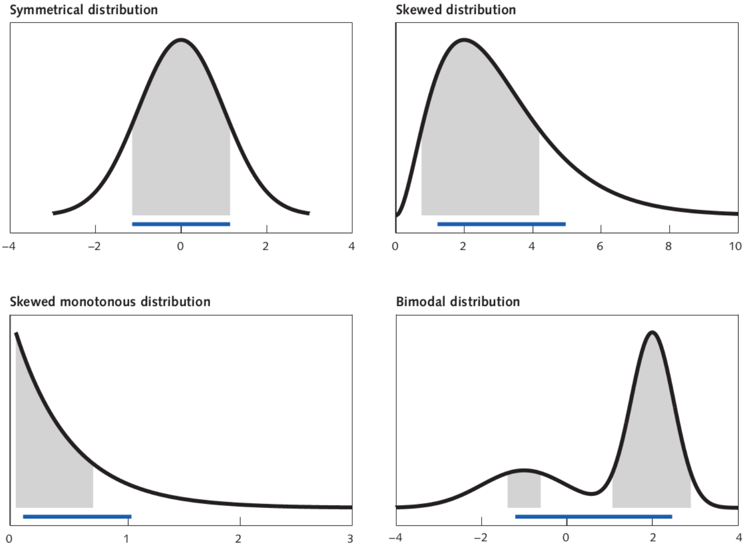
$$E(\theta|y) \pm 1.96 \cdot SD(\theta|y) = [3.517; 3.753]$$

- An exact 95% **equal-tail interval** is [3.518; 3.754]

- **Highest Posterior Density** (**HPD**) interval contains the θ values with highest pdf.

$$[3.518; 3.752]$$

Illustration of different interval types



Conjugate priors

- Normal likelihood: Normal prior→Normal posterior.
- Bernoulli likelihood: Beta prior→Beta posterior.
- Poisson likelihood: Gamma prior→Gamma posterior.

- **Conjugate priors**: A prior is conjugate to a model if the prior and posterior belong to the same distributional family.

- Formal definition: Let $\mathcal{F} = \{p(y|\theta), \theta \in \Theta\}$ be a class of sampling distributions. A family of distributions $\mathcal{P}$ is **conjugate** for $\mathcal{F}$ if

$$p(\theta) \in \mathcal{P} \Rightarrow p(\theta|x) \in \mathcal{P}$$

holds for all $p(y|\theta) \in \mathcal{F}$.

Prior elicitation

- The prior should be determined (elicited) by an **expert**. Typically, expert$\neq$statistician.
- Elicit the prior on a quantity that the expert knows well. Convert afterwards.
- Ask probabilistic questions to the expert:
  - $E(\theta) =?$
  - $SD(\theta) =?$
  - $Pr(\theta < c) =?$
  - $Pr(y > c) =?$
- Show some consequences of the elicitated prior to the expert.
- Beware of psychological effects, such as anchoring.

Prior elicitation - AR(p) example

- Autoregressive process of order p

$$y_t = \mu + \phi_1(y_{t-1} - \mu) + ... + \phi_p(y_{t-p} - \mu) + \varepsilon_t, \ \varepsilon_t \stackrel{iid}{\sim} N(0, \sigma^2)$$

- Informative prior on the unconditional mean: $\mu \sim N(\mu_0, \tau_0^2)$.
- "Noninformative" prior on σ^2: $p(\sigma^2) \propto 1/\sigma^2$
- Assume $\phi_i \sim N(\mu_i, \psi_i)$, $i = 1, ..., p$ are independent a priori.
- Prior on $\phi = (\phi_1, ..., \phi_p)$ centered on persistent AR(1) process: $\mu_1 = 0.8, \mu_2 = ... = \mu_p = 0$
- $Var(\phi_i) = \frac{c}{i^\lambda}$. "Longer" lags are more likely to be zero a priori.

Jeffreys' prior

- **Fisher information** (the amount of information that $\mathrm{x} = (x_1, \ldots, x_n)$ carries about θ):

$$I(\theta) = -E_{\mathrm{x}|\theta}\left(\frac{\partial^2 \ln p(\mathrm{x}|\theta)}{\partial\theta^2}\right)$$

- A common non-informative prior is **Jeffreys' prior**

$$p(\theta) = |I(\theta)|^{1/2}.$$

- **Invariant** to 1:1 transformations of θ.
- Often non-conjugate.
- Often problematic in multiparameter settings.

Jeffreys' prior for Bernoulli sampling

$$x_1, ..., x_n | \theta \overset{iid}{\sim} Bern(\theta).$$

$$\ln p(\mathrm{x}|\theta) = s \ln \theta + f \ln(1-\theta)$$

$$\frac{d \ln p(\mathrm{x}|\theta)}{d\theta} = \frac{s}{\theta} - \frac{f}{(1-\theta)}$$

$$\frac{d^2 \ln p(\mathrm{x}|\theta)}{d\theta^2} = -\frac{s}{\theta^2} - \frac{f}{(1-\theta)^2}$$

$$I(\theta) = \frac{E_{\mathrm{x}|\theta}(s)}{\theta^2} + \frac{E_{\mathrm{x}|\theta}(f)}{(1-\theta)^2} = \frac{n\theta}{\theta^2} + \frac{n(1-\theta)}{(1-\theta)^2} = \frac{n}{\theta(1-\theta)}$$

Thus, the Jeffreys' prior is

$$p(\theta) = |I(\theta)|^{1/2} \propto \theta^{-1/2}(1-\theta)^{-1/2} \propto Beta(1/2, 1/2).$$

Jeffreys' prior for negative binomial sampling

- Jeffreys' prior:

$$n|\theta \overset{iid}{\sim} NegBin(s, \theta).$$

$$\ln p(\mathrm{x}|\theta) = \ln \binom{n-1}{s-1} + s\ln\theta + f\ln(1-\theta)$$

$$\frac{d^2 \ln p(\mathrm{x}|\theta)}{d\theta^2} = -\frac{s}{\theta^2} - \frac{f}{(1-\theta)^2}$$

$$I(\theta) = \frac{s}{\theta^2} + \frac{E_{n|\theta}(n-s)}{(1-\theta)^2} = \frac{s}{\theta^2} + \frac{s/\theta - s}{(1-\theta)^2} = \frac{s}{\theta^2(1-\theta)}$$

- Thus, the Jeffreys' prior is

$$p(\theta) = |I(\theta)|^{1/2} \propto \theta^{-1}(1-\theta)^{-1/2} \propto Beta(\theta|0, 1/2).$$

- Jeffreys' prior is **improper**, but the posterior is proper: $\theta|n \sim \mathrm{Beta}(s, f + 1/2)$ is proper since $s \geq 1$.
- Jeffreys' prior **violates the likelihood principle** because $I(\theta)$ is sampling-based.

Different types of prior information

- Real **expert information**. Combo of previous studies and experience.
- **Vague prior** information.
- **Smoothness priors**. Regularization. Shrinkage. Big thing in modern statistics/machine learning.