Mobolaji Williams (mwilliams@physics.harvard.edu➡) First Version: July 28, 2016

Schwinger-Dyson Equation(s)

In these notes we derive the Schwinger-Dyson equation, a functional differential equation whose solution encodes (variously) all the connected and disconnected vacuum-to-vacuum transition amplitudes of a theory, the connected vacuum-to-vacuum transition amplitudes, and the quantum/effective action. It provides a functional differential equation analog to the functional integral definition of QFT, and thus leads an equivalent way to calculate standard QFT results like perturbative correlation functions or the Coleman Weinberg potential.

– **What questions does this topic answer?** What is a functional integral differential equation which can be employed to study QFT? This topic establishes quantum field theory as arising from a functional differential equation which is distinct from (but consistent with) the formulations of quantum field theory based on the Hamiltonian interaction picture or Lagrangians and functional integrals.

– **Why are these questions important?** Having multiple ways to formulate a theory (for example, having both Wave *and* Matrix Mechanics, or both Newton's equations *and* the principle of least action) leads to a better understanding of the theory itself. Also in comparison to the other formulations of QFT, a functional differential equation formulation better elucidates the complementarity between the connected-correlation function generating functional $W[j]$ and the quantum action $\Gamma[\phi_{\text{cl}}]$.

– **How does the topic answer these questions?** The equation (in its many manifestations) extends both from the functional integral form of the fundamental theorem of calculus and the various Legendre transforms which relate effective actions to generating functionals. Feynman diagrams are also employed as usual to organize perturbative corrections to correlation functions.

– **What are applications to systems/problems?** The Schwinger-Dyson equations provide an alternative way to compute all of the standard perturbative correlation functions of QFT. It thus can be seen as existing on the same epistemic level as the Hamiltonian and Lagrangian formulations of QFT. It also can be used to derive the Coleman-Weinberg potential [1].

1 Derivation

The Schwinger Dyson equation is a result of the Fundamental Theorem of Functional Calculus (or at least the functional calculus analog to the Fundamental Theorem of Calculus). We begin with the the path integral definition of a field theory, and we compute the expectation value of the functional derivative of a functional $F[\phi]$:

$$\begin{aligned}\left\langle \frac{\delta}{\delta\phi(x)}F[\phi]\right\rangle_J \equiv \left\langle 0\middle| \frac{\delta}{\delta\phi(x)}F[\phi]\middle|0\right\rangle_J &= \int D\phi\, \frac{\delta}{\delta\phi(x)}F[\phi]\exp\left(iS[\phi]+i\int d^4x\,J(x)\phi(x)\right)\\ &= -\int D\phi\, F[\phi]\frac{\delta}{\delta\phi(x)}\exp\left(iS[\phi]+i\int d^4x\,J(x)\phi(x)\right)\\ &= -\int D\phi\, F[\phi]\left(i\frac{\delta}{\delta\phi(x)}S[\phi]+iJ(x)\right)\exp\left(iS[\phi]+i\int d^4x\,J(x)\phi(x)\right)\\ &= -i\left\langle F[\phi]\left(\frac{\delta}{\delta\phi(x)}S[\phi]+J(x)\right)\right\rangle_J ,\end{aligned} \tag{1}$$

where we used the functional integral analog to the fundamental theorem of calculus:

$$\int D\phi \, \frac{\delta}{\delta\phi(x)} G[\phi] = 0, \tag{2}$$

for an appropriately chosen functional $G[\phi]$[1]. Our general Schwinger Dyson equation is then

$$\left\langle \frac{\delta}{\delta\phi(x)} F[\phi] \right\rangle_J = -i \left\langle F[\phi] \left(\frac{\delta}{\delta\phi(x)} S[\phi] + J(x) \right) \right\rangle_J . \tag{3}$$

The idea is we choose $F[\phi]$ so as to yield what ever correlation function we wish to compute, and then we apply this equation to find a differential equation for that correlation function. As a first step, we often take $F[\phi] = 1$, so that Eq.(3) becomes

$$0 = -i \left\langle \frac{\delta}{\delta\phi(x)} S[\phi] + J(x) \right\rangle_J . \tag{4}$$

Eq.(4) can be written another way by making use of the fact that $Z[J]$ is the generating functional for correlation functions. Namely,

$$\frac{\delta}{\delta J(x_n)} \cdots \frac{\delta}{\delta J(x_1)} Z[J] = i^n \langle \phi(x_1) \cdots \phi(x_n) \rangle_J Z[J], \tag{5}$$

where the inner product is time-ordered[2]. Thus multiplying Eq.(4) by $Z[J]$, we can write the Schwinger-Dyson equation as

$$\boxed{\frac{\delta}{\delta\phi(x)} S\left[\phi = \frac{1}{i}\frac{\delta}{\delta J(x)}\right] Z[J] + J(x) Z[J] = 0.} \tag{6}$$

This is the form of the equation as it is expressed on Wikipedia's article for the equation.

To obtain the form of the equation presented in [2], we take various partial derivatives of Eq.(6). Specifically, we note

$$\frac{1}{i^n} \frac{\delta}{\delta J(x_n)} \cdots \frac{\delta}{\delta J(x_1)} J(x) Z[J] = \frac{1}{i} \sum_{i=1}^{n} \langle \phi(x_1) \cdots \delta^4(x - x_i) \cdots \phi(x_n) \rangle_J . \tag{7}$$

Similarly, we have

$$\frac{1}{i^n} \frac{\delta}{\delta J(x_n)} \cdots \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta\phi(x)} S\left[\phi = \frac{1}{i}\frac{\delta}{\delta J(x)}\right] Z[J] = \left\langle \frac{\delta}{\delta\phi(x)} S[\phi] \, \phi(x_1) \cdots \phi(x_n) \right\rangle_J . \tag{8}$$

With these two results Eq.(6) becomes

$$\boxed{\left\langle \frac{\delta}{\delta\phi(x)} S[\phi] \, \phi(x_1) \cdots \phi(x_n) \right\rangle_J + \frac{1}{i} \sum_{i=1}^{n} \langle \phi(x_1) \cdots \delta^4(x - x_i) \cdots \phi(x_n) \rangle_J = 0} \tag{9}$$

which is the expression found in Srednicki.

[1] I'm not absolutely sure about the mathematical correctness of this result

[2] It is important to note the time-ordered ness of this result because propagation from "here-to-there" in some quantum field theories is not equivalent to propagation from "there-to-here"

2 Applications

The Schwinger-Dyson equation, like any mathematical formulation of quantum field theory, is used mainly to compute correlation functions. We will show it yields correlation functions which are consistent with results grounded in the perturbative expansion of the functional integral.

We will use scalar ϕ^3 theory as our example field theory. The action is

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \right), \tag{10}$$

which yields the Schwinger-Dyson equation

$$\left(\partial_x^2 - m^2\right) \langle\phi(x)\rangle_J Z[J] - \frac{g}{2}\langle\phi^2(x)\rangle_J Z[J] + J(x)Z[J] = 0. \tag{11}$$

Functionally differentiating this expression with respect to various $J(x_i)$ and alternately setting $J = 0$ gives us a ladder of coupled differential equations for the correlation functions of this theory. The first four differential equations are

$$\begin{aligned}
(\partial_x^2 - m^2)\langle\phi(x)\rangle_0 &= \frac{g}{2}\langle\phi^2(x)\rangle_0 \\
(\partial_x^2 - m^2)\langle\phi(x)\phi(x_1)\rangle_0 &= \frac{g}{2}\langle\phi^2(x)\phi(x_1)\rangle_0 - \frac{1}{i}\delta^4(x-x_1) \\
(\partial_x^2 - m^2)\langle\phi(x)\phi(x_1)\phi(x_2)\rangle_0 &= \frac{g}{2}\langle\phi^2(x)\phi(x_1)\phi(x_2)\rangle_0 - \frac{1}{i}\delta^4(x-x_1)\langle\phi(x_1)\rangle_0 \\
&\quad - \frac{1}{i}\delta^4(x-x_2)\langle\phi(x_1)\rangle_0 \\
(\partial_x^2 - m^2)\langle\phi(x)\phi(x_1)\phi(x_2)\phi(x_3)\rangle_0 &= \frac{g}{2}\langle\phi^2(x)\phi(x_1)\phi(x_2)\phi(x_3)\rangle_0 - \frac{1}{i}\delta^4(x-x_3)\langle\phi(x_1)\phi(x_2)\rangle_0 \\
&\quad - \frac{1}{i}\delta^4(x-x_1)\langle\phi(x_2)\phi(x_3)\rangle_0 - \frac{1}{i}\delta^4(x-x_2)\langle\phi(x_1)\phi(x_3)\rangle_0
\end{aligned} \tag{12}$$

where $\langle\mathcal{O}\rangle_0 \equiv \langle\mathcal{O}\rangle_{J=0}$. Each of these equations can be solved by the theory of Green's functions. Namely, if we have

$$\left(\partial_x^2 - m^2\right) F(x) = H(x), \tag{13}$$

Then the particular solution to $F(x)$ is

$$F(x) = \frac{1}{i}\int d^4x'\, \Delta_0(x-x')\, H(x') \tag{14}$$

where

$$\Delta_0(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}. \tag{15}$$

Therefore, for the first equation, we have the solution

$$\langle\phi(x)\rangle_0 = \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\, \langle\phi^2(y)\rangle_0. \tag{16}$$

For the second equation we have the solution

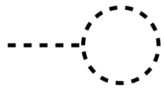
$$\langle\phi(x)\phi(x_1)\rangle_0 = \frac{1}{i}\int d^4y \left[-\frac{1}{i}\delta^4(x-x_1) + \frac{g}{2}\langle\phi(y)^2\phi(x_1)\rangle \right] \Delta_0(x-y)$$

$$= \Delta(x-x_1) + \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\, \langle \phi(y)^2\phi(x_1)\rangle_0. \tag{17}$$

Which implies that Eq.(16) is

$$\langle \phi(x)\rangle_0 = \frac{g}{2i}\int d^4y \Delta_0(x-y)\Delta_0(y-y) + \mathcal{O}(g^2)$$

$$= \quad + \mathcal{O}(g^2), \tag{18}$$



as expected.

By our list of differential equations, the three-point correlation function is defined as

$$\begin{aligned}\big\langle \phi(x)\phi(x_1)\phi(x_2)\big\rangle_0 &= \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\big\langle \phi^2(y)\phi(x_1)\phi(x_2)\big\rangle_0 \\ &\quad + \big\langle \phi(x_2)\big\rangle_0 \Delta_0(x-x_1) + \big\langle \phi(x_1)\big\rangle_0 \Delta_0(x-x_2).\end{aligned} \tag{19}$$

To compute this quantity perturbatively, we must first compute the four-point correlation function (again perturbatively). We have

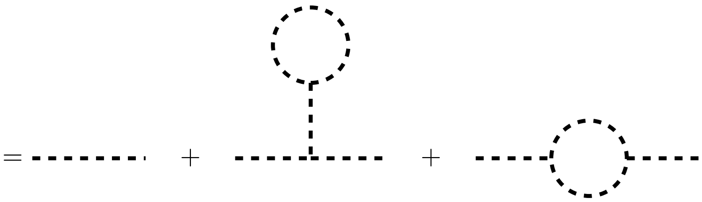
$$\begin{aligned}\big\langle \phi(x)\phi(x_1)\phi(x_2)\phi(x_3)\big\rangle_0 &= \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\big\langle \phi^2(y)\phi(x_1)\phi(x_2)\phi(x_3)\big\rangle_0 + \big\langle \phi(x_2)\phi(x_3)\big\rangle_0 \Delta_0(x-x_1) \\ &\quad + \big\langle \phi(x_1)\phi(x_3)\big\rangle_0 \Delta_0(x-x_2) + \big\langle \phi(x_1)\phi(x_2)\big\rangle_0 \Delta_0(x-x_3) \\ &= \Delta_0(x_2-x_3)\Delta_0(x-x_1) + \Delta_0(x_1-x_3)\Delta_0(x-x_2) \\ &\quad + \Delta_0(x_1-x_2)\Delta_0(x-x_3) + \mathcal{O}(g).\end{aligned} \tag{20}$$

Therefore, the three-point correlation function becomes

$$\begin{aligned}\big\langle \phi(x)\phi(x_1)\phi(x_2)\big\rangle_0 &= \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\big\langle \phi^2(y)\phi(x_1)\phi(x_2)\big\rangle_0 + \mathcal{O}(g) \\ &= \frac{g}{2i}\int d^4z\, \Delta_0(y-z)\Delta_0(x_1-x_2)\Delta_0(y-y) \\ &\quad + \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\Delta_0(y-x_2)\Delta_0(y-x_1) \\ &\quad + \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\Delta_0(y-x_1)\Delta_0(y-x_2) + \mathcal{O}(g) \\ &= \frac{g}{2i}\int d^4y\, \Delta_0(x-y)\Delta_0(x_1-x_2)\Delta_0(y-y) \\ &\quad + \frac{g}{i}\int d^4y\, \Delta_0(x-y)\Delta_0(y-x_2)\Delta_0(y-x_1) + \mathcal{O}(g).\end{aligned} \tag{21}$$

We can then return to Eq.(17), to find that its perturbative expansion yields

$$\begin{aligned}\langle \phi(x)\phi(x_1)\rangle_0 &= \frac{1}{i}\int d^4z\, \left[-\frac{1}{i}\delta^4(x-x_1) + \frac{g}{2}\langle \phi(z)^2\phi(x_1)\rangle\right]\Delta_0(x-z) \\ &= \Delta(x-x_1) + \frac{g^2}{4i^2}\int d^4z\, d^4y\, \Delta_0(z-y)\Delta_0(z-x_1)\Delta_0(y-y)\Delta_0(x-z) \\ &\quad + \frac{g^2}{2i^2}\int d^4z\, d^4y\, \Delta_0(z-y)\Delta_0(y-x_1)\Delta_0(y-z)\Delta_0(x-z) + \mathcal{O}(g^2)\end{aligned}$$

$=$  $+ \mathcal{O}(g^4), \qquad (22)$

which is the expected result. So, the Schwinger-Dyson equation appropriately reproduces the results from the perturbative expansion of the functional integral.

3 Alternative Forms

As in all areas of physics, having alternative ways to express a mathematical formalism affords us great flexibility in applying the formalism to problems. The Schwinger-Dyson equation Eq.(6) has a number of mathematical expressions each of which extends from the way the vacuum-to-vacuum transition amplitude $Z[J]$ is connected to other functionals in quantum field theory.

For instance, we can define the vacuum-to-vacuum transition amplitude $Z[J]$ in terms of the generating functional for connected diagrams $W[J]$ as

$$Z[J] = \exp(iW[J]). \tag{23}$$

Thus Eq.(6) can be written as

$$\begin{aligned} J(x) &= -Z[J]^{-1} \frac{\delta}{\delta\phi(x)} S\left[\phi = \frac{1}{i}\frac{\delta}{\delta J(x)}\right] Z[J] \\ &= e^{-iW[J]} \frac{\delta}{\delta\phi(x)} S\left[\phi = \frac{1}{i}\frac{\delta}{\delta J(x)}\right] e^{iW[J]} \\ &= \frac{\delta}{\delta\phi(x)} S\left[\phi = e^{-iW[J]}\frac{1}{i}\frac{\delta}{\delta J(x)} e^{iW[J]}\right] \\ &= \frac{\delta}{\delta\phi(x)} S\left[\phi = \frac{\delta W[J]}{\delta J(x)} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right], \end{aligned} \tag{24}$$

and so the Schwinger-Dyson equation becomes

$$\boxed{\frac{\delta}{\delta\phi(x)} S\left[\phi = \frac{\delta W[J]}{\delta J(x)} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right] \cdot 1 + J(x) = 0,} \tag{25}$$

where we wrote a 1 to the right of the functional derivatives to make precise that terms with a derivative and no other terms yield zero. The third equality above is established as a functional derivative analog to the calculation

$$\begin{aligned} e^{g(x)} \sum_{k=0}^{n} b_k \left(\frac{d}{dx}\right)^k e^{-g(x)} &= e^{g(x)} \sum_{k=0}^{n} b_k \left(\frac{d}{dx} \cdots \frac{d}{dx}\right) e^{-g(x)} \\ &= \sum_{k=0}^{n} b_k \left(e^{g(x)} \frac{d}{dx} \cdots \frac{d}{dx} e^{-g(x)}\right) \\ &= \sum_{k=0}^{n} b_k \left(e^{g(x)} \frac{d}{dx} e^{-g(x)}\right) \cdots \left(e^{g(x)} \frac{d}{dx} e^{-g(x)}\right) \end{aligned}$$

$$= \sum_{k=0}^{n} b_k \left(e^{g(x)} \frac{d}{dx} e^{-g(x)} \right)^k . \tag{26}$$

As an example of how the derivatives in Eq.(25) are applied, consider the function $\mathcal{G}(\phi) = \phi^2$. Taking $\delta S[\phi]/\delta\phi(x) = \mathcal{G}(\phi)$, we find

$$\begin{aligned} Z[J]^{-1}\mathcal{G}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) Z[J] &= -e^{-iW[J]}\frac{\delta^2}{\delta J(x)^2}e^{iW[J]} \\ &= -ie^{-iW[J]}\frac{\delta}{\delta J(x)}\left(e^{iW[J]}\frac{\delta W[J]}{\delta J}\right) \\ &= -i\frac{\delta^2 W[J]}{\delta J(x)^2} + \left(\frac{\delta W[J]}{\delta J(x)}\right)^2 , \end{aligned} \tag{27}$$

or

$$\begin{aligned} \mathcal{G}\left(\frac{\delta W[J]}{\delta J(x)} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right) &= \left(\frac{\delta W[J]}{\delta J(x)} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{\delta W[J]}{\delta J(x)} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right)\cdot 1 \\ &= \left(\frac{\delta W[J]}{\delta J(x)}\right)^2 + \frac{1}{i}\frac{\delta^2 W[J]}{\delta J(x)^2}. \end{aligned} \tag{28}$$

For the final useful form of the Schwinger-Dyson equations we use the relationship between the effective action and the generating functional for connected diagrams:

$$W[J] = \Gamma[\phi_J] + \int d^4x \, J(x)\phi_J(x), \tag{29}$$

where $\phi_J(x) \equiv \langle 0|\phi(x)|0\rangle_J$. From the discussion in Chapter 21 of [2] we have

$$\frac{\delta W[J]}{\delta J(x)} = \phi_J(x), \tag{30}$$

and

$$\frac{\delta\Gamma[\phi]}{\delta\phi_J(x)} = -J(x). \tag{31}$$

And from the definition of the functional derivative chain rule (which can be checked by applying $J(z)$ to both sides)

$$\begin{aligned} \frac{\delta}{\delta J(x)} &= \int d^4y \, \frac{\delta\phi_J(y)}{\delta J(x)}\frac{\delta}{\delta\phi_J(y)} \\ &= \int d^4y \, \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\frac{\delta}{\delta\phi_J(y)}. \end{aligned} \tag{32}$$

The second functional derivative of $W[J]$ is the exact (sourced) propagator for our field theory. Since $W[J]$ is a functional of J we expect this propagator to be a function of J, but with Eq.(31) we can express $J(x)$ as a function of ϕ_J, so we define this exact propagator (as a function of ϕ_J) as

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \equiv \Delta(x, y|\phi_J) \tag{33}$$

Assembling Eq.(31), Eq.(32), Eq.(33), Eq.(25) becomes

$$\boxed{\frac{\delta}{\delta\phi(x)}S\left[\phi=\phi_J(x)+\frac{1}{i}\int d^4z\,\Delta(x,z|\phi_J)\frac{\delta}{\delta\phi_J(x)}\right]\cdot 1-\frac{\delta\Gamma[\phi]}{\delta\phi_J(x)}=0,} \tag{34}$$

the effective action version of the Schwinger-Dyson equation.

4 Computation of Effective Potential

Following the derivation in [1], we can use Eq.(34) to compute the Coleman-Weinberg potential for a quantum field theory. Our example quantum field theory will be ϕ^4 theory. We begin with the action

$$S[\phi]=\int d^4x\left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4\right). \tag{35}$$

Computing the first term in Eq.(34), and noting that the derivatives are acting from the left, we find

$$\frac{\delta}{\delta\phi(x)}S\left[\phi=\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right]\cdot 1=(\partial^2-m^2)\phi_J(x)-\frac{\lambda}{3!}\left(\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3, \tag{36}$$

where we replaced the functional derivative with respect to $\phi_J(x)$ with a functional derivative with respect to $J(x)$ in anticipation of a future simplification. Expanding the final term, we obtain

$$\begin{aligned}\left(\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3&=\left(\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\\&=\left(\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\phi_J(x)^2+\frac{1}{i}\frac{\delta\phi_J(x)}{\delta J(x)}\right)\\&=\phi_J(x)^3+\frac{3}{i}\phi_J(x)\frac{\delta\phi_J(x)}{\delta J(x)}-\frac{\delta^2\phi_J(x)}{\delta J(x)^2}.\end{aligned} \tag{37}$$

Using Eq.(30) and Eq.(33) we can then write this as

$$\left(\phi_J(x)+\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3=\phi_J(x)^3+\frac{3}{i}\phi_J(x)\Delta(x,x|\phi_J)-\frac{\delta^3W[J]}{\delta J(x)^3}. \tag{38}$$

Focusing on the final term in Eq.(38) and applying the chain rule, we obtain

$$\begin{aligned}\frac{\delta^3W[J]}{\delta J(x)^3}&=\int d^4u_1\,d^4u_2\,d^4u_3\,\frac{\delta\phi_J(u_1)}{\delta J(x)}\frac{\delta\phi_J(u_2)}{\delta J(x)}\frac{\delta\phi_J(u_3)}{\delta J(x)}\frac{\delta^3W[J]}{\delta\phi_J(u_1)\delta\phi_J(u_2)\delta\phi_J(u_3)}\\&=\int d^4u_1\,d^4u_2\,d^4u_3\,\Delta(x,u_1|\phi_J)\Delta(x,u_2|\phi_J)\Delta(x,u_3|\phi_J)\frac{\delta^3\Gamma[\phi]}{\delta\phi_J(u_1)\delta\phi_J(u_2)\delta\phi_J(u_3)},\end{aligned} \tag{39}$$

where we used Eq.(33) and Eq.(31). In all then, we find that the Schwinger-Dyson equation, for the effective-action of ϕ^4 theory becomes

$$\begin{aligned}\frac{\delta\Gamma[\phi_J]}{\delta\phi_J(x)}=&\frac{\delta S\left[\phi_J\right]}{\delta\phi(x)}-\frac{\lambda}{2i}\phi_J(x)\Delta(x,x|\phi_J)\\&+\frac{\lambda}{3!}\int d^4u_1\,d^4u_2\,d^4u_3\,\Delta(x,u_1|\phi_J)\Delta(x,u_2|\phi_J)\Delta(x,u_3|\phi_J)\frac{\delta^3\Gamma[\phi_J]}{\delta\phi_J(u_1)\delta\phi_J(u_2)\delta\phi_J(u_3)}.\end{aligned} \tag{40}$$

In what follows it will prove useful to review some tree-level results of ϕ^4 theory. First we note that the inverse classical propagator for the theory is defined by the second order functional derivative of $S[\phi]$:

$$\begin{aligned}\Delta_0^{-1}(x,x'|\phi) &= -\frac{\delta^2}{\delta\phi(x)\delta\phi(x')}S[\phi] \\ &= \left(-\partial_x^2 + m^2 + \frac{\lambda}{2}\phi(x)^2\right)\delta^4(x-x'). \end{aligned} \tag{41}$$

The three and four-leg vertices are defined similarly:

$$\begin{aligned}\mathcal{V}_3(x_1,x_2,x_3|\phi) &= \frac{\delta^3 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)} \\ &= -\lambda\phi(x_1)\delta^4(x_1-x_2)\delta^4(x_2-x_3) \end{aligned} \tag{42}$$

$$\begin{aligned}\mathcal{V}_4(x_1,x_2,x_3,x_4) &= \frac{\delta^4 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)\delta\phi(x_4)} \\ &= -\lambda\delta^4(x_1-x_4)\delta^4(x_1-x_2)\delta^4(x_2-x_3). \end{aligned} \tag{43}$$

Our larger goal is to compute the Coleman-Weinberg potential for this theory. To do so, we take the Salam-Stradhee *ansatz* for the lowest order effective action. Defining the lowest order effective action as $\Gamma_0[\phi]$, our *ansatz* is

$$\Gamma_0[\phi] = S[\phi] - \int d^4x\, V\left(\phi(x)\right), \tag{44}$$

where V is the effective potential of this QFT. Computing a single functional derivative of this expression gives

$$\frac{\delta\Gamma_0[\phi]}{\delta\phi(x)} = \frac{\delta S[\phi]}{\delta\phi(x)} - \frac{d}{d\phi}V\left(\phi(x)\right), \tag{45}$$

and computing successive functional derivatives of this expression yields

$$\begin{aligned}\frac{\delta^3\Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)} &= \frac{\delta^3 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)} - \frac{d^3}{d\phi^3}V\left(\phi(x)\right)\delta^4(x_3-x_2)\delta^4(x_2-x_1) \\ &= -\left(\lambda\phi(x_1) + \frac{d^3}{d\phi^3}V\left(\phi(x)\right)\right)\delta^4(x_3-x_2)\delta^4(x_2-x_1). \end{aligned} \tag{46}$$

Substituting Eq.(45) and Eq.(46) into Eq.(40) for $\Gamma[\phi] = \Gamma_0[\phi]$, we find the integro-differential equation

$$\frac{d}{d\phi}V(\phi(x)) - \frac{\lambda}{2i}\phi(x)\Delta(x,x|\phi) - \frac{\lambda}{3!}\int d^4u_1\,\Delta(x,u_1|\phi)^3\left(\lambda\phi(u_1) + \frac{d^3}{d\phi^3}V\left(\phi(u_1)\right)\right) = 0. \tag{47}$$

To compute the effective potential V under the same assumptions/approximations as Coleman-Weinberg's original derivation we will take ϕ to be a constant field. This allows us to factor the integrand in Eq.(47) and makes its implicit x dependence spurious. The exact propagator $\Delta(x,x|\phi)$ can now be defined according to the effective action in a way analogous to Eq.(41):

$$\Delta^{-1}(x,y|\phi) = -\frac{\delta^2\Gamma_0[\phi]}{\delta\phi(x)\delta\phi(y)} = \left(-\partial_x^2 + m^2 + \frac{\lambda}{2}\phi^2 + \frac{d^2}{d\phi^2}V\right)\delta^4(x-y), \tag{48}$$

which corresponds to the free field propagator with effective mass

$$m_{\text{eff}}^2 = m^2 + \frac{\lambda\phi^2}{2} + \frac{d^2}{d\phi^2}V. \tag{49}$$

We can now consider Eq.(47) to lowest order in λ and derivatives of V. Doing so gives us the effective potential differential equation

$$\frac{dV(\phi)}{d\phi} - \frac{\lambda}{2i}\phi\Delta(x,x|\phi) + \cdots . \tag{50}$$

Solving this equation to this order we have

$$\begin{aligned} V(\phi) - V(\phi = 0) &= \frac{\lambda}{2i}\int_0^{\phi} d\phi'\, \phi'\Delta(x,x|\phi') + \cdots \\ &= \frac{\lambda}{2i}\int_0^{\phi} d\phi'\, \phi' \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2+\frac{1}{2}\lambda\phi'^2} + \cdots \\ &= \frac{1}{2i}\int \frac{d^4k}{(2\pi)^4} \ln\left(\frac{k^2+m^2+\frac{1}{2}\lambda\phi^2}{k^2+m^2}\right) + \cdots \\ &= \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4} \ln\left(\frac{k_E^2+m^2+\frac{1}{2}\lambda\phi^2}{k_E^2+m^2}\right) + \cdots , \end{aligned} \tag{51}$$

where in the last line we performed a Wick rotation to take $k^0 \to ik_E^0$. This is the standard result for the Coleman-Weinberg potential of this problem.

5 Concept Map

6 Yang-Mills Theory

We complete these notes by extending this formalism to Yang-Mills Theory. Namely, we will derive the theory's corresponding Schwinger-Dyson Equations. The result is not very informative, but we write it anyway. We begin with the Lagrangian of Yang-Mills Theory:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\mathrm{Tr}(F^2) = -\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \\ &= -\frac{1}{2}\mathrm{Tr}(T^aT^b)F^{a\mu\nu}F^b_{\mu\nu} \\ &= -\frac{1}{2}\left(\frac{1}{2}\delta^{ab}\right)F^{a\mu\nu}F^b_{\mu\nu} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}, \end{aligned} \tag{52}$$

where we used the non-abelian gauge theory generator definition $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. The field strength for Yang-Mills Theory is defined (in component form) as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu, \tag{53}$$

or in matrix form as $F^{\mu\nu} \equiv F^{a\mu\nu}T^a$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{i}[A_\mu, A_\nu]. \tag{54}$$

For the matrix form, the corresponding field equation is

$$(D_\mu F^{\mu\nu})^a + J^{a\mu} = 0. \tag{55}$$

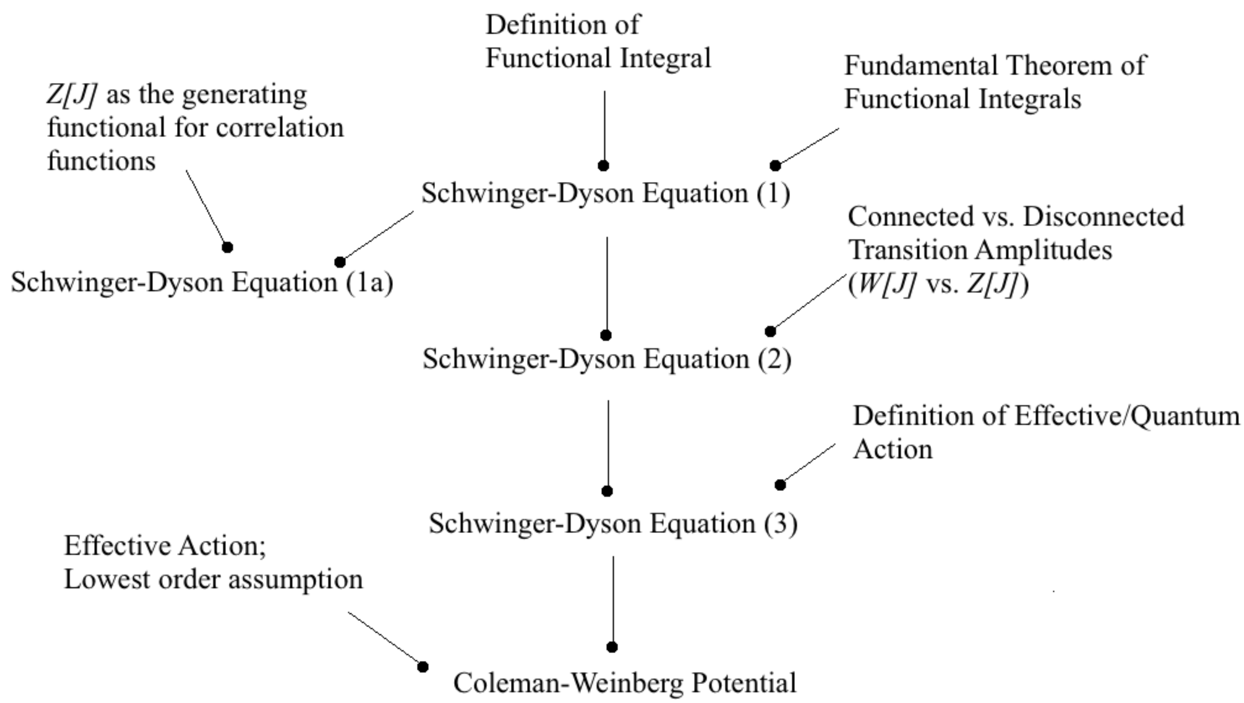


Figure 1: Concept map showing the relationships between various manifestations of the Schwinger-Dyson equations and how they lead to a derivation of the Coleman-Weinberg potential. (1) refers to Eq.(6); (1a) to Eq.(9); (2) to Eq.(25); (3) to Eq.(34).

Or writing out in component form

$$\begin{aligned}(D_\mu F^{\mu\nu})^a &= (\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}])^a \\ &= \left(\partial_\mu F^{a\mu\nu}T^a - i[T^b, T^c]A^b_\mu F^{c\mu\nu}\right)^a \\ &= \left(\partial_\mu F^{a\mu\nu}T^a + f^{abc}A^b_\mu F^{c\mu\nu}T^a\right)^a \\ &= \partial_\mu F^{a\mu\nu} + f^{abc}A^b_\mu F^{c\mu\nu}, \end{aligned} \tag{56}$$

so that we have the dynamical equation

$$\partial_\mu F^{a\mu\nu} + f^{abc}A^b_\mu F^{c\mu\nu} + J^{a\mu}, \tag{57}$$

or written in its full glory in terms of $A^{a\mu}$

$$\partial_\mu\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right) + f^{abc}\partial_\mu\left(A^{b\mu}A^{c\nu}\right) + f^{abc}A^b_\mu\left(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu}\right) + f^{abc}f^{cde}A^b_\mu A^{d\mu}A^{e\nu} + J^{a\nu} = 0. \tag{58}$$

Heuristically, to derive the Schwinger-Dyson equation from an equation of motion, we replace every field variable by a functional derivative with respect to a source. Doing so above we find the equation

$$\partial_\mu\left(\partial^\mu \frac{\delta Z[J^\sigma]}{\delta J^a_\nu(x)} - \partial^\nu \frac{\delta Z[J^\sigma]}{\delta J^a_\mu(x)}\right) + f^{abc}\partial_\mu\left(\frac{\delta^2 Z[J^\sigma]}{\delta J^b_\mu(x)\delta J^c_\nu(x)}\right)$$

$$+ f^{abc}\frac{\delta}{\delta J^b_\mu(x)}\left(\partial^\mu\frac{\delta Z[J^\sigma]}{\delta J^c_\nu(x)} - \partial^\nu\frac{\delta Z[J^\sigma]}{\delta J^c_\mu(x)}\right) + f^{abc}f^{cde}\frac{\delta^3 Z[J^\sigma]}{\delta J^{b\mu}(x)\delta J^d_\nu(x)\delta J^e_\nu(x)} + J^{a\mu}Z[J^\sigma] = 0, \tag{59}$$

which is not very informative. As a final question it is worth seeing if this equation simplifies when we express it in our various equivalent forms, in particular the form for the effective action.

References

[1] A. Salam and J. Strathdee, "Comment on the computation of effective potentials," *Physical Review D*, vol. 9, no. 4, p. 1129, 1974.

[2] M. Srednicki, *Quantum field theory*. Cambridge University Press, 2007.