# Dimensional Analysis as a Guide for Perturbation Theory

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Using dimensional analysis we estimate the discrepancy between the arc lengths of two particles in a gravitational field, one of which is subject to an air resistant drag force. We then calculate this value exactly and conclude that a consideration of the parameters of a problem provides a reliable estimate of the magnitude of correction terms in a perturbative analysis.

## I. INTRODUCTION

Approximations are essential in physics. Often a problem is intractable unless considered with certain simplifications. These simplifications often yield exact solutions which then provide an approximate answer to the real problem. Furthermore, many times these exact solutions can be slightly modified to provide better approximations to the true solution. The modification usually consists of including in the ideal case a small effect from the real situation in the form of numerically small parameters. The system is then solved in this quasi-real case to produce the ideal solution plus various correction terms which bring us closer to the answer we desire. This is the basis of perturbation theory; one solves the ideal case and then proceeds to implement the real situation through small parameters in the hopes that the new solution is more accurate. However, the actual procedure of implementing and solving for these corrections is usually quite difficult analytically. It would be beneficial if before these calculations were attempted one was able to obtain an estimate of how close the fictitious ideal situation was to the real case. My claim is that through dimensional analysis one could obtain a rough estimate of one's error in adopting the ideal solution and can therefore determine beforehand the necessity of perturbation theory.

## II. DIMENSIONAL ANALYSIS

We will use the calculation of the arc length of a particle as an archetypical example. Our ideal situation is the particle in a gravitational field and our real situation is the particle in a gravitational field in the presence of an air resistant drag force of the form

$$\mathbf{F}_{drag} = -m\alpha\mathbf{v} \tag{1}$$

where $\mathbf{v}$ is the velocity of the particle. We begin with the obvious but necessary statement that our quantity of interest has units of length. When we introduce air resistance of the form Eq (1) we bring another dimensional parameter $\alpha$ into our theory. If we were to obtain an exact analytical solution for the arc length in the air resistance case we should find that if we take $\alpha \to 0$ we recover the result of the case with no air resistance. Moreover, if we were to expand our result around $\alpha = 0$ we should find

$$L_R \approx L_0 + A\alpha \tag{2}$$

Where $L_R$ is the arc length in the presence of air resistance, $L_0$ is the arc length without this resistance, and $A$ is some quantity with the appropriate units. To determine the exact form of $A$ we need to consider the parameters in our original theory. Since $\alpha$ has units of 1/time, $A$ must have units of time×length. An obvious choice for $A$ must include $v_0$. But this cannot be the entire story; the correction term must also be dependent on the actual length of the trajectory otherwise some larger trajectories will have comparatively smaller corrections. Moreover, larger values of $v_0$ should correspond to longer trajectories and therefore larger corrections so $v_0$ must appear as a positive power. For similar reasons, $L_0$ should also appear as a positive power. Now, if we were to include $v_0$ and $L_0$ as positive powers, we would require some other quantity of dimension time$^2$/length to cancel the extraneous units. This quantity is $g$ of course and thus $A$ is fully determined

$$A = \frac{v_0 L_0}{g} \tag{3}$$

More specifically if we were to expand $L_R$ as some sort of perturbation series in these quantities we would have

$$L_R = \sum_{k=0}^{\infty} B_k \left(\frac{\alpha v_0}{g}\right)^k L_0 \tag{4}$$

where $B_k$ are undetermined constants. Less rigorously, if we were to approximate the percent error in adopting the ideal situation we would find

$$\begin{aligned}\frac{\Delta L}{L_R} &= \frac{L_R - L_0}{L_R} = \sum_{k=1}^{\infty} c_k \left(\frac{\alpha v_0}{g}\right)^k \frac{L_0}{L_R} \\ &\approx \left(\frac{\alpha v_0}{g}\right)\end{aligned}$$

This is merely an estimate: to obtain an exact error would require an honest calculation. Still, the utility of this result is clear. We were able to obtain, solely through dimensional analysis, an idea of the discrepancy between the ideal and real situations. However, to ensure the validity of this estimate we must do the calculation. As a sort of numerical experiment we will calculate $L_R$ and $L_0$ for various input parameters and ascertain how close we come to this predicted error.

## III. THEORETICAL FORMALISM

### A. No Air Resistance

We launch our particle and its trajectory may be described by a curve $\gamma$ parameterized by two coordinates ($x(t)$ and $y(t)$). . The curve is defined as

$$\gamma : t \longmapsto \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} \tag{5}$$

and the corresponding arc length

$$L \equiv \int_{t_0}^{t} |\gamma'(t)|\, dt = \int_{t_0}^{t_f} \sqrt{[x'(t)]^2 + [z'(t)]^2}\, dt \tag{6}$$

The functions $x(t)$ and $z(t)$ are determined by appealing to Newton's Second Law in the context of a constant gravitational force field

$$\mathbf{F} = m\mathbf{a} = m\mathbf{g}$$

which when seperated into it's vectorial components yields three equations

$$\begin{aligned} \ddot{x}(t) &= 0 \\ \ddot{y}(t) &= 0 \\ \ddot{z}(t) &= -g \end{aligned}$$

Based on the empirical fact that the trajectory is constrained to move in a plane, we may eliminate $y(t)$ (or $x(t)$) as a superfluous coordinate with no dynamics. This reduces us the necessary number of variables for our Eq(6). Solving the dynamical equations for their first derivatives, we obtain ( Ref [1])

$$\begin{aligned} \dot{x}(t) &= v_0 \cos\theta \\ \dot{z}(t) &= v_0 \sin\theta t - gt \end{aligned}$$

where we defined $\theta$ and $v_0$ as

$$\tan\theta = \frac{v_{0z}}{v_{0x}} \tag{7}$$

$$v_0 = \sqrt{v_{0z}^2 + v_{0x}^2} \tag{8}$$

substituting these values into Eq(6) we obtain

$$L = \int_0^T \sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2} \tag{9}$$

where $T = (2v_0/g)\sin\theta$ is the time it takes to complete the trajectory. To simplify the formula we employ two changes of variables. First we take

$$\begin{aligned} u &= v_0\sin\theta - gt \Longrightarrow du = -gdt \\ a &= v_0\cos\theta \end{aligned}$$

to obtain

$$\begin{aligned} L &= -\frac{1}{g}\int_{u_0}^{u_f} du\sqrt{a^2+u^2} \\ &= -\frac{a}{g}\int_{u_0}^{u_f} du\sqrt{1+(u/a)^2} \end{aligned}$$

and then we take

$$x = u/a \Longrightarrow a\, dx = du \tag{10}$$

so that finally

$$L = -\frac{a^2}{g}\int_{x_0}^{x_f} dx\sqrt{1+x^2} \tag{11}$$

and with with the aid of *Wolfram Integral* we obtain

$$L = -\frac{a^2}{g}\left[x\sqrt{1+x^2} + \ln(x+\sqrt{1+x^2})\right]_{x_0}^{x_f} \tag{12}$$

Now, we back substitute to relate this result to our original variables. The relations are

$$\begin{aligned} x &= u/a = \frac{v_0\sin\theta - gt}{v_0\cos\theta} \\ x_f &= x(T) = \frac{v_0\sin\theta - g(2v_0/g)\sin\theta}{v_0\cos\theta} = -\tan\theta \\ x_0 &= x(0) = \tan\theta \end{aligned}$$

so that $L$ is now

$$\begin{aligned} L &= -\frac{v_0^2\cos^2\theta}{2g}[-\tan\theta\sec\theta + \ln(-\tan\theta+\sec\theta) \\ &\quad -\tan\theta\sec\theta - \ln(\tan\theta+\sec\theta)] \\ &= \frac{v_0^2\cos^2\theta}{2g}\left[2\tan\theta\sec\theta + \ln\left(\frac{\tan\theta+\sec\theta}{-\tan\theta+\sec\theta}\right)\right] \end{aligned}$$

which, after further simplification, gives us our final result.

$$L_0 = \frac{v_0^2}{2g}\left[2\sin\theta + \cos^2\theta\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right]. \tag{13}$$

Later in this analysis the value of $\theta$ at which this arc length is maximum will prove useful. To determine this angle $\theta_0$ we differentiate $L$

$$\begin{aligned} L'(\theta) &= \frac{v_0^2}{2g}(2\cos\theta) - \frac{v_0^2}{g}\cos\theta\sin\theta\ln\left[\frac{1+\sin\theta}{1-\sin\theta}\right] \\ &\quad + \frac{v_0^2}{g}\cos^2\theta\left(\frac{\cos\theta}{1+\sin\theta} - \frac{-\cos\theta}{1-\sin\theta}\right) \\ &= \frac{v_0^2}{g}(2\cos\theta) - \frac{v_0^2}{g}\cos\theta\sin\theta\ln\left[\frac{1+\sin\theta}{1-\sin\theta}\right] \end{aligned} \tag{14}$$

When we set this result to zero we find that $\theta_0$ must satisfy the nonlinear equation

$$\sin\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right) = 2 \tag{15}$$

By using *Mathematica* to solve Eq.(15), we find that $\theta_0 \approx 56.46$. To ensure that this value produces a maximum (and not a minimum) arc length, we need to confirm $L''(\theta_0) < 0$. Computing $L''(\theta)$ from Eq (15) and plugging in condition 15, we obtain

$$\begin{aligned} L''(\theta_0) &= -\frac{2v_0^2}{g}\left(2\sin\theta + \frac{\cos 2\theta}{\sin\theta}\right) \\ &= -2.4\frac{v_0^2}{g} \end{aligned}$$

so the local maximum condition is satisfied. Laslty, when we substititue Eq (15) into Eq (13) we obtain a simplified arc length equation:

$$L_{max} = \frac{v_0^2}{g}(\sin\theta_0 + \cot\theta_0\cos\theta_0) \tag{16}$$

### B. Effects of Air Resistance

To implement the effects of air resistance we include an additional force proportional to velocity

$$\mathbf{F}_{drag} = -m\alpha\mathbf{v} \tag{17}$$

this force changes our dynamical equations to

$$\begin{aligned} \ddot{x}_d(t) &= -\alpha\dot{x} \\ \ddot{z}_d(t) &= -g - \alpha\dot{z} \end{aligned}$$

where $d$ denotes the fact that we are now considering drag forces. When the former initial conditions are applied we obtain the solution

$$\dot{x}_d(t) = v_0\cos\theta e^{-\alpha t} \tag{18}$$

$$\dot{z}_d(t) = \left(v_0\sin\theta + \frac{g}{\alpha}\right)e^{-\alpha t} - \frac{g}{\alpha} \tag{19}$$

In order to apply Eq (6) we must determine the time it takes the particle to complete its trajectory. This requires finding the time $T_f$ such that

$$z_d(T_f) = 0. \tag{20}$$

Integrating the $\dot{z}(t)$ equation, subject to the initial condition $z(t=0) = 0$, we find

$$z_d(t) = \frac{1}{\alpha}\left(v_0\sin\theta + \frac{g}{\alpha}\right)(1 - e^{-\alpha t}) - \frac{g}{\alpha}t \tag{21}$$

Using condition Eq (20) and simplifying we discover that $T_f$ must satisfy

$$e^{-\alpha T_f} + \frac{\alpha}{1 + \frac{\alpha v_0\sin\theta}{g}}T_f = 1 \tag{22}$$

we can ascertain the validity of this result by considering the limit of negligible air resistance ($\alpha \to 0$). Expanding the implicit equation to second order in $\alpha$

$$\begin{aligned} 1 &= 1 - \alpha T_f + \frac{(\alpha T_f)^2}{2} + \alpha T_f\left(1 - \frac{\alpha v_0\sin\theta}{g}\right) + O(\alpha^3) \\ 0 &= \frac{(\alpha T_f)^2}{2} - \alpha^2\frac{v_0\sin\theta}{g}T_f + O(\alpha^3) \\ 0 &= \frac{T_f}{2} - \frac{v_0\sin\theta}{g} \end{aligned}$$

where in the last line we divided by $\alpha^2$ and took the $\lim_{\alpha\to 0}$. So we recover the purely gravitational result and can be confident that our solution is valid for small $\alpha$.

Finally, due to the exponential factor in the solutions to the dynamical equations and the implicit nature of Eq (22), we require a numerical analysis to compute the necessary arc length:

$$L_R = \int_0^{T_f} \sqrt{[\dot{x}_d(t)]^2 + [\dot{z}_d(t)]^2}\, dt \tag{23}$$

## IV. NUMERICAL ANALYSIS

Before we begin our numerical analysis we require additional information to complete our simulation; we will use $g = 9.81\text{m/s}^2$ but our $v_0$ and *alpha* and are arbitrary. Imagining an actual experimental setup with a ball launcher we can find an upper bound for $v_0$. If we were to launch the ball straight up in a modern day college classroom we can claim that the maximum height it could reach is $h_{max} = 3$ m and therefore by conservation of energy (which is less valid if we consider air resistance) we find

$$\begin{aligned} \frac{1}{2}mv_0^2 &= mgh_{max} \\ v_0 &= \sqrt{2h_{max}g} \\ &\approx 7.7\text{m/s} \end{aligned}$$

Lastly, we shall take $\alpha = 0.15\text{s}^{-1}$. Now our expansion parameter is

$$\left(\frac{\alpha v_0}{g}\right) = 0.1177 \tag{24}$$

With our imposed value of $\alpha$, we can use *Mathematica* to solve Eq (22) and find $T_F$ for various $\theta$. When we find $T_f$ we can then substitute our result into Eq (23) to determine $L_R$. The computation of $L_0$ is much simpler and only requires an evaluation of Eq (13).

Our calculated values of $L_0$ and $L_R$ for various values of $\theta$ are listed in Table I.

From the table we see that, matching our prediction, most errors are approximately of the order $\left(\frac{\alpha v_0}{g}\right) = 0.1177$. More precisely our errors have a mean $\bar{x}$ and

| $\theta$ | $L_R$ | $L_0$ | $\Delta L/L_R$ |
|---|---|---|---|
| 15° | 2.939 | 3.0577 | 0.0404 |
| 30° | 5.127 | 5.512 | 0.0751 |
| 45° | 6.316 | 6.937 | 0.0981 |
| 60° | 6.539 | 7.224 | 0.104 |
| 75° | 5.607 | 6.658 | 0.187 |
| 90° | 5.263 | 6.044 | 0.148 |

TABLE I: Computation of Percent Error for various angles

standard deviation $s_x$

$$\bar{x} = 0.109$$
$$s_x = 0.0521$$

Most interestingly we find that the error is best when $\theta$ is in the range $40^\circ < \theta < 60^\circ$ aka close to the value $\theta_0 \approx 56.46$ where $L_0$ is maximum.

## V. DISCUSSION

From the analysis of the previous section we see that dimensional analysis can provide a clear estimate of errors in perturbation theory prior to any difficult calculation. However, the $\theta$ dependence of the correction suggests that our analysis was incomplete and there is most likely a supplementary angular distribution in $\Delta L$. This conjecture is further confirmed by the fact that $\Delta L/L_R$ is closest to our approximate (angle independent) error $(\alpha v_0/g)$ when $L_0$ has no first order dependence on $\theta$ aka when $L_0$ is maximum. This suggests that the perturbation expansion initially provided in Eq (4) should be modified to express our new found $\theta$ dependence

$$L_R = \sum_{k=0}^{\infty} B_k(\theta) \left(\frac{\alpha v_0}{g}\right)^k L_0 \quad (25)$$

The exact form of $B_k(\theta)$ and whether this $\theta$ dependence could be absorbed into the perturbation parameter would require supplementary analysis.

[1] D. Morin *Introduction to Classical Mechanics: With Problems and Solutions, 3rd Ed.* (Cambridge University Press, Cambridge, MA, 2008) problem 3.19, pages 75-84.

Our method is similar to but not an exact replication of Morin's calculation