Tollman-Oppenheimer-Volkoff Equation

Stars as composite structures of charged particles and radiation are in general quite complicated entities. But by making a few simplifying assumptions we can develop toy models of stars which give us a better theoretical understanding of their properties. Such a model is the one defined by the hydrostatic equilibrium equation

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{1}$$

for an ideal fluid held in a sphere by its own gravitational force. This equation is essentially non-relativistic as it is grounded in flat-spacetime and the assumption that individual particle speed does not approach c. In this section we will extend this model to the general relativistic regime and derive what is known as the Tollman-Oppenheimer-Volkoff (TOV) equation. This equation relates the radial variations in the pressure of a star to its mass and pressure density properties.

0.1 Euler Equation for Static Ideal Fluid In Curved Spacetime

As a first step towards obtaining the general relativistic equation for stellar structure, we will use the relativistic Euler equation to obtain the first condtion which relates pressure variations to the metric of space time. First, we introduce the assumptions of the system. We assume our system is characterized by an ideal fluid. Thus, in curved spacetime, the stress-energy tensor is

$$T^{\alpha\beta} = \rho\, u^\alpha\, u^\beta + \left(g^{\alpha\beta} + u^\alpha\, u^\beta/c^2\right) P \tag{2}$$

and the corresponding Euler equation is

$$(\rho + P)u^\alpha \nabla_\alpha u^\beta/c^2 + (g^{\alpha\beta} + u^\alpha u^\beta/c^2)\partial_\alpha P = 0. \tag{3}$$

We also take our fluid to be static which amounts to the condition that u^α has no nonzero space components that is $u^\alpha = (u^0, \mathbf{0})$. Therefore, by the identity $u^\alpha u^\beta g_{\alpha\beta} = -c^2$ we have

$$u^0 = c\,(-g_{00})^{-1/2}. \tag{4}$$

Moreover, because the fluid, which is static and time independent, is the source of spacetime curvature, we expect the spacetime metric, not to mention the mass density and pressure density, to be static also:

$$\partial g_{\alpha\beta}/\partial x^0 = 0\,, \qquad \partial\rho/\partial x^0 = 0\,, \qquad \partial P/\partial x^0 = 0 \tag{5}$$

With these simplifications, the covariant derivative term in the relativistic Euler equation becomes

$$\begin{aligned} u^\alpha \nabla_\alpha u^\beta &= u^0 \nabla_0 u^\beta \\ &= u^\alpha \partial_\alpha u^\beta + \Gamma^\beta_{\lambda\alpha} u^\lambda u^\alpha \\ &= u^0\, \partial_0 u^\beta + \Gamma^\beta_{00}\, u^0 u^0 \\ &= \Gamma^\beta_{00}\, u^0 u^0 \\ &= -\frac{1}{2} g^{\beta\gamma} \frac{\partial g_{00}}{\partial x^\gamma} u^0 u^0. \end{aligned} \tag{6}$$

The second term in Eq.(3) can also be reduced:

$$
\begin{aligned}
(g^{\alpha\beta} + u^\alpha u^\beta)\partial_\alpha P &= g^{\alpha\beta}\partial_\alpha P + u^\alpha u^\beta \partial_\alpha P/c^2 \\
&= g^{\alpha\beta}\partial_\alpha P + u^0 u^\beta \partial_0 P/c^2 \\
&= g^{\alpha\beta}\partial_\alpha P.
\end{aligned} \tag{7}
$$

Collecting these reductions into the Euler equation we then have

$$
-\frac{1}{2}(\rho + P)\, g^{\beta\gamma}\frac{\partial g_{00}}{\partial x^\gamma}\, u^0 u^0 + g^{\gamma\beta}\partial_\gamma P = 0 \tag{8}
$$

which upon substituting the derived value of u^0 can be rearranged to yield

$$
\frac{\partial_\gamma P}{\rho + P} = -\frac{1}{2g_{00}}\frac{\partial g_{00}}{\partial x^\gamma}. \tag{9}
$$

0.2 Einstein's Equation for Ideal Fluid

To complete our analysis of the gravitational properties of a star modeled as an ideal fluid, we need to consider Einstein's equation. Specifically, in accord with our previous assumptions, we need to consider Einstein's equation is the presence of a static source. Because of the numerous factors of c we will have to contend with in this analysis, we will take $c = 1$.

The first step towards considering Einstein's equations under these conditions, is writing down the metric. What we are looking for is the most general metric for a static spherically symmetric source. Writing the metric as a proper time, we can write this general metric as [1]

$$
d\tau^2 = B(r)\, dt^2 - A(r)\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2. \tag{10}
$$

where t is the time coordinate and r, θ, and ϕ are the standard spherical coordinates. By definition $d\tau^2 = -g_{\alpha\beta}dx^\alpha\, dx^\beta$ so we can abstract from our proper time, the metric components

$$
g_{00} = -B(r), \qquad g_{rr} = A(r), \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2\sin^2\theta, \tag{11}
$$

where all other components vanish. With our general metric we can now compute the Riemann Tensor and use Einstein's Equation to solve for $A(r)$ and $B(r)$. But first we will put the equation in a form which makes our subsequent analysis simpler.

Einstein's Equation is typically written as

$$
R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\, R = -8\pi G\, T_{\alpha\beta}. \tag{12}
$$

Contracting both sides of this equation with $g^{\alpha\beta}$ we find

$$
R - 2R = -R = -8\pi G\, T \tag{13}
$$

where $T = T^\alpha_{\ \alpha}$ is the spacetime trace of $T_{\alpha\beta}$. We can then replace the Ricci scalar in Einstein's Equation with the trace of the stress energy tensor to obtain

$$
R_{\alpha\beta} = -8\pi G\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right). \tag{14}
$$

With this form of Einsteins equation we substitute a computation of the Ricci scalar, which is difficult to compute, for a computation of the stress-energy trace, which is comparatively simpler to compute.

Ordinarily it would be tedious work to compute the Ricci tensor from the metric, but fortunately we can use *Mathematica* to forego an algebraic analysis. Using *Mathematica* we find

$$R_{00} = \frac{1}{4A(r)}\left(\frac{A'(r)B'(r)}{A(r)} + \frac{B'(r)^2}{B(r)}\right) - \frac{B'(r)}{rA(r)} - \frac{B''(r)}{2A(r)} \tag{15}$$

$$R_{rr} = -\frac{A'(r)}{rA(r)} - \frac{A'(r)B'(r)}{4A(r)B(r)} - \frac{B'(r)^2}{4B(r)^2} + \frac{1}{2}\frac{B''(r)}{B(r)} \tag{16}$$

$$R_{\theta\theta} = -1 + \frac{1}{A(r)} + \frac{r}{2A(r)}\left(\frac{B'(r)}{B(r)} - \frac{A'(r)}{A(r)}\right) \tag{17}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}. \tag{18}$$

The stress-energy tensor components are easier to compute. We recall that the stress energy tensor is defined as

$$T_{\alpha\beta} = \rho u_\alpha\, u_\beta + (g_{\alpha\beta} + u_\alpha u_\beta)P$$

and the trace $T = 3P - \rho$. We also recall that u_α only has a time component, and given our general metric it is $u_0 = (-g^{00})^{-1/2} = (-g_{00})^{1/2} = \sqrt{B(r)}$. The individiual components of the right side of Eq.(14) are then

$$\begin{aligned} T_{00} - \frac{1}{2}g_{00}\,T &= \rho(u_0)^2 + \left(g_{00} + (u_0)^2\right)P - \frac{1}{2}g_{00}(3P-\rho) \\ &= \rho B + (-B+B)P + \frac{1}{2}B(3P-\rho) \\ &= \frac{1}{2}B\,(3P+\rho) \end{aligned} \tag{19}$$

$$\begin{aligned} T_{rr} - \frac{1}{2}g_{rr}\,T &= g_{rr}P - \frac{1}{2}g_{rr}(3P-\rho) \\ &= AP - \frac{1}{2}A(3P-\rho) \\ &= \frac{1}{2}A\,(\rho - P) \end{aligned} \tag{20}$$

$$\begin{aligned} T_{\theta\theta} - \frac{1}{2}g_{\theta\theta}\,T &= \frac{1}{2}g_{\theta\theta}(\rho - P) \\ &= \frac{1}{2}r^2\,(\rho - P) \end{aligned} \tag{21}$$

$$\begin{aligned} T_{\phi\phi} - \frac{1}{2}g_{\phi\phi}\,T &= \frac{1}{2}g_{\theta\theta}(\rho - P) \\ &= \frac{1}{2}r^2\sin^2\theta\,(\rho - P). \end{aligned} \tag{22}$$

Einsteins Equations for this system are then

$$\frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} - \frac{B''}{2A} = -4\pi G\,B(3P+\rho) \tag{23}$$

$$\frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} = -4piG\,A(\rho - P) \tag{24}$$

$$-1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} = -4\pi G(\rho - P)r^2 \tag{25}$$

There is an additional equation from the $\phi\,\phi$ component of Einstein's equations, but it is identical to the $\theta\,\theta$ equation. With this system of equations we can immediately determine how A relates to the properties of the star. We find

$$\frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2} + \frac{R_{00}}{2B} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{r^2A^2} = -8\pi G\rho \tag{26}$$

and so

$$\begin{aligned} -8\pi G\rho(r)\,r^2 &= -\frac{A'(r)}{A^2(r)}r - \left(1 - \frac{1}{A(r)}\right) \\ &= -\frac{d}{dr}\left[r\left(1 - \frac{1}{A(r)}\right)\right] \end{aligned} \tag{27}$$

Integrating both sides of this equation we find

$$r\left(1 - \frac{1}{A(r)}\right) = 8\pi G \int_0^r dr\, r^2\rho(r) = 2GM(r) \tag{28}$$

or

$$A(r) = \left(1 - \frac{2GM(r)}{r}\right)^{-1} \tag{29}$$

where $M(r) \equiv 4\pi \int_0^r r^2\rho(r)\,dr$. Now recalling Eq.(9) and that $g_{00} = B(r)$ we have the radial equation

$$\frac{1}{B(r)}\frac{dB(r)}{dr} = -\frac{2}{\rho + P}\frac{dP}{dr}. \tag{30}$$

From Eq.(25) we have

$$-1 - \frac{rA'}{2A^2} + \frac{1}{A}\left(1 + \frac{rB'}{2B}\right) = -4\pi G(\rho - P)r^2 \tag{31}$$

which, upon substitution of A and B becomes,

$$-1 - 4\pi G\rho\, r^2 + \frac{GM}{r} - r(1 - 2GM/r)\left(1 - \frac{rP'}{\rho + P}\right) = -4\pi G(\rho - P)r^2 \tag{32}$$

where we used the first line of Eq.(28) in the substitution for A'/A^2. Rearranging this equation to solve for P' we find

$$\frac{rP'(r)}{\rho(r) + P(r)} = -\frac{G(4\pi P(r)r^2 + M(r)/r}{1 - 2GM(r)/r} \tag{33}$$

which can be rewritten in the suggestive form

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}\left[1 + \frac{P(r)}{\rho(r)}\right]\left[1 + \frac{4\pi GP(r)r^3}{M(r)}\right]\left[1 - \frac{2GM(r)}{r}\right]^{-1}. \tag{34}$$

This is the TOV equation for stellar structure. Restoring factors of c^2 using dimensional analysis and noting that the energy density in non-relativistic regimes has a pre-dominate mass density contribution, we see that this result reduced to the non-relativistic hydrostatic equation in the limit of $c \to \infty$.

References

[1] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, vol. 1. Wiley New York, 1972.