MITES 2010: Physics III
Survey of Modern Physics
Final Exam Solutions

Exercises

1. **Problem 1. Consider a particle with mass m that moves in one-dimension. Its position at time t is $x(t)$. As a function of its position, Its potential energy is $U(x) = C$, where C is a constant.**
   (a) Derive the equation of motion for the particle using Newtons second law.
   (b) Derive the equation of motion for the particle, but this time, using the Lagrangian.
   (c) Solve the equation of motion (i.e., what are $x(t)$ and $\frac{dx}{dt}$?). Define all the free parameters in your solution (i.e. what are their physical meanings?)

   Solution: (a)From Newton's Second Law we find

$$
\begin{aligned}
m\ddot{x} &= F(x) \\
&= -\frac{d}{dx}U(x) \\
&= -\frac{d}{dx}C \\
&= 0 \qquad \Longrightarrow \qquad \boxed{\ddot{x} = 0}
\end{aligned}
$$

(b) From the definition of the Lagrangian we find that the Lagrangian of this system has the form

$$
\begin{aligned}
\mathcal{L} &= \mathrm{KE} - \mathrm{PE} \\
&= \frac{1}{2}m\dot{x}^2 - C
\end{aligned}
$$

From this Lagrangian we may derive equation of motion using the Euler-Lagrange (E-L) equation.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}(-C) \\
&= 0 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) \\
&= \frac{d}{dt}(m\dot{x}) \\
&= m\ddot{x}
\end{aligned}
$$

and so

$$
0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 - m\ddot{x} \qquad \Longrightarrow \qquad \boxed{\ddot{x} = 0}
$$

(c) Our equation of motion $\ddot{x} = 0$ is a simple second order differential equation so we may solve it by separation of variables.

$$\begin{aligned}
\ddot{x} &= 0 \\
\frac{d\dot{x}}{dt} &= 0 \\
d\dot{x} &= 0\,dt \\
\int_{\dot{x}_0}^{\dot{x}(t)} d\dot{x} &= 0 \\
\dot{x}(t) - \dot{x}_0(t) &= 0 \qquad \Longrightarrow \qquad \boxed{\dot{x}(t) = \dot{x}_0}
\end{aligned}$$

This equation for velocity provides us with another differential equation for the positon $x(t)$.

$$\begin{aligned}
\dot{x} &= \dot{x}_0 \\
\frac{dx}{dt} &= \dot{x}_0 \\
d\dot{x} &= \dot{x}_0\,dt \\
\int_{x_0}^{x(t)} dx &= \int_0^t \dot{x}_0\,dt' \\
x(t) - x_0 &= \dot{x}_0 t \qquad \Longrightarrow \qquad \boxed{x(t) = x_0 + \dot{x}_0 t}
\end{aligned}$$

In our solutions, the free parameter $\dot{x}_0$ represents the initial velocity of particle and x_0 represents the initial position.

2. **Problem 2. Hanging mass The potential energy for a mass hanging from a spring is $V(y) = ky^2/2 + mgy$, where $y = 0$ corresponds to the position of the spring when nothing is hanging from it.**
 (a) What is the net force acting on the particle?
 (b) Write down the Lagrangian for this system.
 (c) Use the Euler-Lagrange equation to derive the equation of motion for this system. DO NOT use Newtons second law to derive it. What kind of motion does this correspond to?
 (d) Solve the equation of motion, thereby obtaining y(t). What do all the free parameters in your solution physically mean?

 Solution: (a)

$$\begin{aligned}
F(y) &= -V'(y) \\
&= -\frac{d}{dy}(ky^2/2 + mgy) \\
&= \boxed{-ky - mg}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{L} &= \text{KE} - \text{PE} \\
&= \boxed{\frac{1}{2}m\dot{x}^2 - \frac{1}{2}ky^2 - mgy}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial}{\partial y}(-\frac{1}{2}ky^2 - mgy) \\
&= -ky - mg \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) \\
&= m\ddot{x}
\end{aligned}$$

$$0 = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = -ky - mg - m\ddot{y} \qquad \Longrightarrow \qquad \boxed{\ddot{y} + \frac{k}{m}y + g = 0}$$

If we let $\delta = y + mg/k$ then we have $\ddot{\delta} + \omega^2\delta = 0$ where $\omega^2 = k/m$. This is the simple harmonic oscillator equation which models all free undamped oscilatory motion.

d) The solution to the SHO EOM, $\ddot{\delta} + \omega^2\delta = 0$, is

$$\delta(t) = A\sin\omega t + B\cos\omega t$$

And from our definition $\delta = y + mg/k$, we find that $y(t)$ is

$$\boxed{y(t) = A\sin\omega t + B\cos\omega t - \frac{mg}{k}} \qquad \omega = \sqrt{\frac{k}{m}}$$

. In order to find what A and B represent, we solve for the initial conditions of our mass. Setting $t = 0$ in our position equation, we find

$$y_0 = y(t=0) = A\sin 0 + B\cos 0 - mg/k = B - mg/k \qquad \Longrightarrow \boxed{B = y_0 + \frac{mg}{k}}$$

B represents the initial displacement from the equilibrium position $y_{eq} = -mg/k$. Similarly, differentiating $y(t)$ and then plugging in $t = 0$ gives us

$$v_0 = \dot{y}(t=0) = A\omega\cos 0 - B\omega\sin 0 = A\omega \qquad \Longrightarrow \boxed{A = \frac{v_0}{\omega}}$$

A represents the inital velocity of the bead divided by the angular frequency. All of this information gives us a general solution $x(t)$ of the form

$$y(t) = \frac{v_0}{\omega}\sin\omega t + \left(y_0 + \frac{mg}{k}\right)\cos\omega t$$

3. **Problem 3. Zipper problem (Model for unzipping of DNA). A zipper has N links; each link has a state in which it is closed with energy 0 and a state in which it is open with energy ϵ. We require, however, that the zipper can only unzip from the left end, and that the link number s can only open if all links to the left $(1, 2, \ldots, s-1)$ are already open. (a.) Show that the partition function can be summed in the form**

$$\mathcal{Z} = \frac{1 - \mathbf{exp}\left[-\frac{(N+1)\epsilon}{k_B T}\right]}{1 - \mathbf{exp}\left[-\frac{\epsilon}{k_B T}\right]}$$

(b.) In the limit $\epsilon >> k_B T$, find the average number of open links. This model is a very simplified model of the unwinding of two-stranded DNA molecules.

Solution: (a)To obtain a feel for the problem we will begin by considering the case of $N = 2$. If we have two links connected successively, then there are three possible configurations: one with both links closed; one with the left link open and the right link closed; one with both links open. We do not have a state with the right link open and the left link closed because we require that the links open from the left. Considering the fact that each open link has an energy ϵ associated with it we know that the energy of the first configuration is 0, the energy of the second configuration is ϵ, and the energy of the last configuration is 2ϵ. This collection of configurations and associated energies generates a partition function of the form

$$\mathcal{Z} = e^{-\beta\cdot 0} + e^{-\beta\epsilon} + e^{-2\beta\epsilon}$$

We can easily generalize this example to our case of arbitrary N. We realize the partition function must then take on the form

$$\begin{aligned} \mathcal{Z} &= e^{-\beta\cdot 0} + e^{-\beta\epsilon} + e^{-2\beta\epsilon} + \cdots + e^{-(N-1)\beta\epsilon} + e^{-N\beta\epsilon} \\ &= 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} + \cdots + e^{-(N-1)\beta\epsilon} + e^{-N\beta\epsilon} \end{aligned}$$

which is a finite geometric series. We can sum the series using the formula

$$1 + r + r^2 + \cdots + r^{N-1} + r^N = \frac{1 - r^{N+1}}{1 - r^N}$$

with $r = e^{-\beta\epsilon}$. Therefore, we find that

$$\boxed{\mathcal{Z} = \frac{1 - e^{-\beta(N+1)\epsilon}}{1 - e^{-\beta\epsilon}}}$$

$$\mathcal{Z} = \sum_{i=0}^{N} e^{-\beta\epsilon_i}.$$

(b) To calculate the average number of open links, we will calculate the average energy of our various configurations and divide this result by ϵ, the energy of one open link. This procedure works because the only nonzero energy in our various configurations comes from the number of open links.

$$\langle n \rangle = \frac{\langle E \rangle}{\epsilon}$$

Computing the average energy, we find

$$
\begin{aligned}
\langle E \rangle &= -\frac{\partial \log \mathcal{Z}}{\partial \beta} \\
&= -\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \beta} \\
&= -\frac{1}{\mathcal{Z}} \frac{\partial}{\partial \beta} \frac{1-e^{-\beta(N+1)\epsilon}}{1-e^{-\beta\epsilon}} \\
&= -\frac{1}{\mathcal{Z}} \left[\frac{\epsilon(N+1)e^{-(N+1)\beta\epsilon}(1-e^{-\beta\epsilon}) - \epsilon e^{-\beta\epsilon}(1-e^{-(N+1)\beta\epsilon})}{(1-e^{-\beta\epsilon})^2} \right] \\
&= -\frac{1-e^{-\beta\epsilon}}{1-e^{-\beta(N+1)\epsilon}} \left[\frac{\epsilon(N+1)e^{-(N+1)\beta\epsilon}(1-e^{-\beta\epsilon}) - \epsilon e^{-\beta\epsilon}(1-e^{-(N+1)\beta\epsilon})}{(1-e^{-\beta\epsilon})^2} \right] \\
&= -\frac{\epsilon(N+1)e^{-(N+1)\beta\epsilon}}{1-e^{-(N+1)\beta\epsilon}} + \frac{\epsilon e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}}
\end{aligned}
$$

Now, if we take $\beta\epsilon >> 1$, then we can take the following approximations

$$
\begin{aligned}
1 - e^{-\beta\epsilon} &\approx 1 \\
e^{-\beta\epsilon} &>> e^{-(N+1)\beta\epsilon}
\end{aligned}
$$

So that our average energy becomes

$$\langle E \rangle \approx \epsilon e^{-\beta\epsilon}$$

and the average number of open links is $\boxed{\langle n \rangle = e^{-\beta\epsilon}}$.

(c.) Give a physical meaning of what $\epsilon >> k_B T$ means. In particular, give a physical justification for what the approximation $\epsilon >> k_B T$ means and why you can ignore some of the terms in your partition function in (b) under this approximation. Recall that $k_B T$ is the typical thermal energy in a system. [Hint: How accessible are the states with energy much larger than the typical thermal energy?]

Solution: ϵ is defined as the energy necessary to open one link in our zipper, so $\epsilon >> k_B T$ means that it requires a lot of energy to open one link. Consequently, it will take a lot more energy to open multiple links. Therefore, if we consider this problem as a describing a state with a limited amount of energy, then the approximation $\epsilon >> k_B T$ restricts the number of states the system can access. Since it requires such a large amount of energy to open mutliple states, the most probable states will be the ones with the lowest energy. This allows us to shorten our partition function so that it only contains the configurations with the lowest energy

$$
\begin{aligned}
\mathcal{Z} &= 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} + \cdots + e^{-(N-1)\beta\epsilon} + e^{-N\beta\epsilon} \\
&\approx 1 + e^{-\beta\epsilon}
\end{aligned}
$$

4. **Problem 4: Applying the rules of Quantum Mechanics.**
Consider a particle whose energy can only take on the following values:

$E_1 = 3\hbar\omega,\ \ E_2 = 6\hbar\omega,\ \ E_3 = 9\hbar\omega,\ \ E_4 = 0$

These are the only allowed energies that the particle can have. They have corresponding eigenstates:
$|E_1\rangle, |E_2\rangle, |E_3\rangle, |E_4\rangle$. $\hat{H}$ **is the Hamiltonian operator - operator representing measurement of the particle's energy.**

Consider a particle in the following indeterminate state.
$|\psi\rangle = \frac{-1}{\sqrt{6}}|E_1\rangle + \frac{e^{i\theta}}{\sqrt{6}}|E_2\rangle + \sqrt{\frac{2}{3}}e^{i\varphi}|E_4\rangle$

(i.) **If you measure the energy of 10,000 identical particles, all in the above 'indeterminate' state just before your measurement, how many of them do you expect to yield energy value of E_2?**

Solution: The probability of obtaining E_2 is $\left|\frac{e^{i\theta}}{\sqrt{6}}\right|^2 = \frac{1}{6}$. So, out of 10,000 measurements, we expect

$$\frac{10,000}{6} \approx \boxed{1667 \text{ particles}}$$

to yield energy E_2.

(ii.) **On average, what value of energy would your measurement yield?**

Solution:

$$\begin{aligned} E_{avg} &= |c_1|^2 E_1 + |c_2|^2 E_2 + |c_3|^2 E_3 + |c_4|^2 E_4 \\ &= \frac{1}{6}3\hbar\omega + \frac{1}{6}6\hbar\omega + 0 \cdot 9\hbar\omega + \frac{2}{3} \cdot 0 \\ &= \boxed{\frac{3}{2}\hbar\omega} \end{aligned}$$

(iii.) **For which value of θ is the probability of a particle having energy E_2 zero?**

Solution: $\boxed{\text{There is no value of } \theta \text{ for which the probability of obtaining } E_2 \text{ is zero}}$. First of all, this is because the coefficient of the state $|E_2\rangle$ is proportional to $e^{i\theta}$ and $e^{i\theta}$ never equals zero. Second of all, the magnitude of $e^{i\theta}$ is always 1 regardless of θ, so the probability of being in E_2 is independent of θ Prob(E_2)=$|e^{i\theta}/\sqrt{6}|$.

(iv.) **Suppose you measure the energy of a particle. You find that it has energy E_1. Does this mean that the particle was in state $|E_1\rangle$ just before your measurement? Suppose you measure the energy of the same particle for the second time. What is the probability that you get E_4 as the energy this second time?**

Solution: $\boxed{\text{No, the particle was not necessarily in the state } |E_1\rangle \text{ before the measurement}}$. A postulate of quantum mechanics states that a quantum state is in a superposition of many states before a measurement and only collapses to one state after the measurement. Also, since the state is now in $|E_1\rangle$ then there is $\boxed{\text{zero probability that it is in any other state}}$.

(v.) **Suppose you measure the energy of the particle and find that it has energy E_2. Write down the ket-vector representing the particle's state immediately after this measurement.**

Solution: Since we measured energy and obtained the value E_2 we know that the particle's state has now collapsed to the state corresponding to E_2. That is, the particle is in state $\boxed{|\psi\rangle = |E_2\rangle}$.

5. **Problem 5. Sequential measurements**
An operator $\hat{A}$ representing observable A, has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2 respectively. Operator $\hat{A}$ representing observable B, has two normalized eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. The eigenstates are related by

$$\psi_1 = \frac{3\phi_1 + 4\phi_2}{5} \qquad\qquad \psi_2 = \frac{4\phi_1 - 3\phi_2}{5} \tag{1}$$

(a.) How do we know that observables $\hat{A}$ and $\hat{B}$ do not commute (i.e. $\hat{A}$ and $\hat{B}$ are incompatible observables)?

Solution: We know that the observables $\hat{A}$ and $\hat{B}$ do not commute[1] because they do not have the same eigenstates (i.e. $\phi_1 \neq \psi_1$ and $\phi_2 \neq \psi_2$). Compatible observables (observables with commuting operators) must have the same eigenstate because the order of their operation does not matter. For example, let $|B_1\rangle$ be an eigenstate of $\hat{B}$ with eigenvalue b_1. Then we can perform the following operation for commuting operators $\hat{B}$ and $\hat{C}$

$$\begin{aligned} 0 &= [\hat{C}, \hat{B}]|B_1\rangle \\ &= \hat{C}\hat{B}|B_1\rangle - \hat{B}\hat{C}|B_1\rangle \\ \hat{B}\hat{C}|B_1\rangle &= \hat{C}\hat{B}|B_1\rangle \\ \hat{B}(\hat{C}|B_1\rangle) &= b_1(\hat{C}|B_1\rangle) \end{aligned}$$

So we see that $\hat{C}|B_1\rangle$ must be an eigenvector of $\hat{B}$ with eigenvalue b_1. This means that we can write $\hat{C}|B_1\rangle$ as a multiple of $|B_1\rangle$ such as

$$\hat{C}|B_1\rangle = k|B_1\rangle$$

But, this is just the definition of an eigenvector $|B_1\rangle$ of $\hat{C}$ with eigenvalue k. So we see that commuting operators share the same eigenvectors. Conversely, noncommuting operators will have different eigenvectors.

(b.) **Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?**

Solution: By one of the postulates of quantum mechanics, the state has collapsed into the state corresponding to eigenvalue a_1. So, $\boxed{\text{the state is in } \psi_1}$.

(c.) **If B is now measured, what are the possible results, and what are their probabilities?**

Solution: The state is currently in ψ_1.

$$\psi_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2$$

[1]Commuting observables satisfy $\hat{A}\hat{B} = \hat{B}\hat{A}$ or $[\hat{A}, \hat{B}] = 0$

From this state, there are two possible results upon measurement of B. These results and their associated probabilities are

Results	Probability
b_1	$\left(\frac{3}{5}\right)^2 = \frac{9}{25}$
b_2	$\left(\frac{4}{5}\right)^2 = \frac{16}{25}$

which fits our normalization requirement because $9/25 + 16/25 = 25/25 = 1$

(d.) **Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note that the answer would be quite different if I had told you the outcome of the B measurement.)**

Solution: There are two possible ways to get to a_1 from our initial a_1 measurement. We can obtain b_1 from the B measurement and then obtain a_1 from the A measurement. Or, we can obtain b_2 from the B measurement and then obtain a_1 from the A measurement. In order to calculate the total probability of obtaining a_1, we must calculate the probability of these independent paths and then add them. But first, we must write ϕ_1 and ϕ_2 in terms of ψ_1 and ψ_2 in order to obtain the probabilities for the second measurement of A after our measurement of B. Solving Eq (1)by elimination, we find for ϕ_1

$$\begin{aligned}
\psi_1 &= \frac{3\phi_1 + 4\phi_2}{5} \\
\frac{5\psi_1 - 4\phi_2}{3} &= \phi_1
\end{aligned}$$

Substituting this result into our other equation, we find

$$\begin{aligned}
\psi_2 &= \frac{4\phi_1 - 3\phi_2}{5} \\
5\psi_2 &= \frac{4}{3}(5\psi_1 - 4\phi_2) - 3\phi_2 \\
&= \frac{20}{3}\psi_1 - \frac{16}{3}\phi_2 - 3\phi_2 \\
&= \frac{20}{3}\psi_1 - \frac{25}{3}\phi_1 \\
25\phi_2 &= \frac{20}{3}\psi_1 - 5\psi_2 \\
\phi_2 &= \frac{4\psi_1 - 3\psi_2}{5}
\end{aligned}$$

Now, solving for ϕ_1

$$\begin{aligned}
5\psi_1 &= 3\phi_1 + 4(\frac{4}{5}\psi_1 - \frac{3}{5}\psi_2) \\
&= 3\phi_1 + \frac{16}{5}\psi_1 - \frac{12}{5}\psi_2 \\
\frac{9}{5}\psi_1 &= 3\phi_1 - \frac{12}{5}\psi_2 \\
\phi_1 &= \frac{3\psi_1 + 4\psi_2}{5}
\end{aligned}$$

So, in summary, we have

$$\phi_1 = \frac{3\psi_1 + 4\psi_2}{5} \qquad\qquad \phi_2 = \frac{4\psi_1 - 3\psi_2}{5}$$

From this result, we may now construct our probabilities. There are two paths to get to a_1 from a measurement of B. These paths and their associated probabilities are

Path	Probability
1) From b_1 to a_1	$\frac{9}{25} \times \frac{9}{25} = \frac{81}{625}$
2) From b_2 to a_1	$\frac{16}{25} \times \frac{16}{25} = \frac{256}{625}$

For each path, we multiplied the probability of obtaining our first measured value of B by the probability of obtaining our second measured value of A. The total probability of obtaining a_1 is the sum of these probabilities and is therefore $\frac{81}{625} + \frac{256}{625} = \boxed{\frac{337}{625}}$ We can check that this procedure for calculating probability is consistent with normalization by calculating the probability other two possible paths. Instead of obtaining a_1, we could have obtained a_2 in our second measurement and the paths and probabilities associated with this value are

Path	Probability
3) From b_1 to a_2	$\frac{9}{25} \times \frac{16}{25} = \frac{144}{625}$
4) From b_2 to a_2	$\frac{16}{25} \times \frac{9}{25} = \frac{144}{625}$

We have listed all possible outcomes of our measurements. The sum of the probabilities for theses outcomes must be one.

$$\text{Prob(Path 1)} + \text{Prob(Path 2)} + \text{Prob(Path 3)} + \text{Prob(Path 4)} = \frac{81}{625} + \frac{256}{625} + \frac{144}{625} + \frac{144}{625} = \frac{625}{625} = 1$$

Our outcomes are normalized, so we know that our procedure is correct.

6. **Problem 6. Eigenstates and Eigenvalues of a two level system.**
The Hamiltonian $\hat{H}$ of a particle with two possible energy states is represented by the 2×2 matrix

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

(a.) What are the possible values of energy for this particle (i.e. eigenvalues of $\hat{H}$)?

Solution: We employ the standard procedure for the calculation of eigenvalues.

$$\begin{aligned} 0 &= \det(\hat{H} - E\hat{I}) \\ &= \det \begin{pmatrix} \hbar\omega - E & 0 \\ 0 & 2\hbar\omega - E \end{pmatrix} \\ &= (\hbar\omega - E)(2\hbar\omega - E) \qquad \Longrightarrow \qquad \boxed{E_1 = \hbar\omega,\ E_2 = 2\hbar\omega} \end{aligned}$$

(b.) For each eigenvalue of $\hat{H}$ you found in (a), find the corresponding eigenstate of $\hat{H}$.

Solution: $\underline{E_1 = \hbar\omega}$

$$\begin{aligned} 0 &= (\hat{H} - E_1\hat{I})|E_1\rangle \\ &= \begin{pmatrix} \hbar\omega - \hbar\omega & 0 \\ 0 & 2\hbar\omega - \hbar\omega \end{pmatrix} |E_1\rangle \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \hbar\omega \end{pmatrix} |E_1\rangle \end{aligned}$$

The last line suggests that $|E_1\rangle$ cannot have a second component. So we have

$$|E_1\rangle = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \boxed{|E_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Where we set $\alpha_1 = 1$ in order to normalize our eigenstate.

For the other eigenvalue, $E_2 = 2\hbar\omega$, we have

$$\begin{aligned} 0 &= (\hat{H} - E_2\hat{I})|E_2\rangle \\ &= \begin{pmatrix} \hbar\omega - 2\hbar\omega & 0 \\ 0 & 2\hbar\omega - 2\hbar\omega \end{pmatrix} |E_2\rangle \\ &= \begin{pmatrix} -\hbar\omega & 0 \\ 0 & 0 \end{pmatrix} |E_2\rangle \end{aligned}$$

The last line suggests that $|E_2\rangle$ cannot have a first component. So we have

$$|E_1\rangle = \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix} \quad \Longrightarrow \quad \boxed{|E_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Where we set $\alpha_2 = 1$ in order to normalize our eigenstate.

(c.) The two eigenstates you found in (b,) represent two orthogonal vectors. What is the physical significance of this fact? Explain your answer in terms of the measurement of energy of a particle that is in an arbitrary state before your measurement.

Solution: The fact that the two eigenstates are orthogonal (i.e. $\langle E_1|E_2\rangle = \langle E_2|E_1\rangle = 0$) signifies the fact that when we are in one eigenstate of an observable then we can only be in that eigenstate. There is no overlap between the states so that if we measure the energy and obtain eigenvalue E_1 then when we perform another measurement we will obtain E_1 again because the probability of the particle being in state E_2 is zero.

$$\text{Probability of measuring } E_2 \text{ after obtaining } E_1 = |\langle E_2|E_1\rangle|^2 = 0$$