Dysonian Redux

Revisiting EM shift of Spin Zero Particles

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Abstract

We revisit an early paper of Freeman Dyson in a modern context.

I. ENTER THE DYSON

In 1948 Dyson left England and came to Cornell as a graduate student and he could not have arrived at a better time. Quantum Electrodynamics, developed nearly two decades prior, was beginning to obtain experimental confirmation of its supposedly spurious virtual processes and the US was at the forefront of theoretical research in this reemerging field. It seems that the interplay of calculational work and the need to "get the numbers out" during the WWII war research developed a community of theoretical physicists in America whose work was tantamount to phenomenology. As such, the experimental investigations, which themselves were bolstered by war research in radar, were closely followed and studied by those who had the intellectual abilities to best explain them.

Willis Lamb, a rare theorist turned experimentalist, initiated one of these investigations when he sought to precisely measure an anomaly of the grand edifice of Dirac's electron theory. The Dirac equation, incorporating both relativity and spin, theoretically determined the fine structure of energy levels in hydrogen, a problem which was previously only solvable by old perturbative methods applied to non-relativistic quantum mechanics. Still, it took only a decade after the Dirac equation was postulated to find experimental deviations from its predictions; the degeneracy of two energy levels established by the Dirac equation was found to only be approximate. However, the results of the experiments were not so conclusive as to initiate a complete overhaul of theory and theorists, so enamored by Dirac's theory, assumed such discrepancies arose from experimental error. It took the experimental genius of Willis Lamb in 1947 to make this discrepancy explicit and decisively measure the shift which now bears his name. Consequently, theoretical physicists encouraged by this unambiguous result quickly sought to explain the phenomena and in what can be considered the quickest time between problem formulation and solution, Hans Bethe, on a train ride leaving a conference presenting Lamb's results, found an answer.

Using the quantum radiation theory of the previous decade, Hans Bethe calculated the separation in hydrogen energy levels and concretely established the necessity of the renormalization procedure in quantum theory. However, his formulation ignored relativity and the spin of the electron, and it was realized a calculation incorporating these effects would produce a more accurate result. Immediately recognizing the calculational difficulty of this spin - relativistic solution, Bethe handed the problem to one of his graduate students. Still the formalism was too difficult and as a toy theory Bethe decided to consider an analysis of relativity without spin. This was the problem Bethe gave to Dyson as his first foray into theoretical physics. Dyson did well with the problem and was able to apply previous methods, now considered "old perturbation theory", and obtain a finite answer. His solution demonstrated a fluency with quantum field theory which was a portent of things to come and gave Bethe confidence in his new student. Dyson's analysis may be found in the PROLA under the somewhat ubiquitous title "Electromagnetic Shift of Energy Levels.[1]"

II. FORMULATION

We intend to present the solution using modern perturbation theory. This analysis essential boils down to considering the various ways a spinless particle can interact with an external electromagnetic potential to one loop order. There are essentially five processes, besides the external scalar mass renormalizations, to consider. One of these processes requires a ϕ^4 term in the original lagrangian, an aspect which did not appear in Dyson's analysis because the renormalizability criterion which mandates such a term was not developed at the time his paper was written. As a result, Dyson does not consider this particular process and, to be consistent, neither will we. Another process involves all the corrections to the external potential (vacuum polarization). Dyson did compute the affects of this phenomenon in his Appendix but determined them to be sufficiently small to leave out of the final answer and therefore so will we. However, we did compute this affect using modern methods and found it to be in perfect agreement with his characteristic "1/120" result. The remaining three processes are calculated below.

III. CONTRIBUTING TERMS

To obtain the Lamb Shift due to radiative corrections we must effectively compute the loop corrected vertex for the scalar-electromagnetic interaction. There are three processes.

A. Left-handed Interaction

Here the extended photon connects to the left side of the diagram with a four point scalar-scalar-photon-photon vertex. The incoming electron and the extended photon annihilate into an electron and a photon. The created photon is then absorbed by the created electron. Using the standard scalar QED interaction rules we have

$$
\begin{aligned}
i\mathbf{V}_I^{\mu}(k',k) &= (ie)(-2ie^2)\left(\frac{1}{i}\right)^2 g^{\mu\nu}\int \frac{d^4l}{(2\pi)^4}\frac{g_{\nu\rho}(l+k+k')^{\rho}}{(l^2+\lambda^2)((l+k')^2+m^2)} \\
&= -2e^3 g^{\mu\nu}\int \frac{d^4l}{(2\pi)^4}\frac{g_{\nu\rho}(l+k+k')^{\rho}}{(l^2+\lambda^2)((l+k')^2+m^2)}
\end{aligned}
$$

where we have included a finite photon mass,λ, to define an infrared cutoff for our theory. This cutoff will later be used to obtain an experimental result. Using the standard loop techniques, we derive the following

$$
\begin{aligned}
\tilde{\Delta}_{\lambda}(l^2)\tilde{\Delta}_m((l+k')^2) &= \int_0^1 dx\,\left(x((l+k')^2+m^2)+(1-x)(l^2+\lambda^2)\right)^{-2} \\
&= \int_0^1 dx\,(l^2+2xl\cdot k'+xk'^2+xm^2+(1-x)\lambda^2)^{-2} \\
&= \int_0^1 dx\,(q^2+D_1)^{-2}
\end{aligned}
$$

Where

$$
q = l + xk' \qquad (1)
$$

$$
D_1 = x(1-x)k'^2 + xm^2 + (1-x)\lambda^2 \qquad (2)
$$

So that by making the change of variables from l to q we have

$$
i\mathbf{V}_I^{\mu}(k',k) = -2e^3\int_0^1 dx\int \frac{d^4q}{(2\pi)^4}\frac{(q+(1-x)k'+k)^{\mu}}{(q^2+D_1)^2}
$$

Now we take our result from $4 \to d$ dimensions and Wick Rotate the temporal component so that we have a Euclidean integration. We also take $e^2 \to e^2\tilde{\mu}^{\epsilon}$, where $\epsilon = 4-d$, to ensure that the coupling remains dimensionless. Our result is

$$
\mathbf{V}_I^{\mu}(k',k) = -2e^3\int_0^1 dx\int \frac{d^d\bar{q}}{(2\pi)^d}\frac{(\bar{q}+(1-x)k'+k)^{\mu}}{(\bar{q}^2+D_1)^2}
$$

The linear term integrates to zero and using a formula from the Appendix, we have

$$
\mathbf{V}_I^{\mu}(k',k) = -\frac{e^3}{8\pi^2}\int_0^1 dx\,((1-x)k'+k)\left(\frac{2}{\epsilon}-\ln(D_1/\mu^2)\right)
$$

B. Right-handed Interaction

This process is essentially the mirror image of the Left-handed Interaction. Instead of connecting to the extended photon on the left side of the diagram, the scalar creates a virtual scalar and photon which then annihilate with the extended photon to create the final scalar state. The essential relation is that they may be transformed into each other with $k \leftrightarrow k'$. This transformation yields the following vertex

$$\begin{aligned} \mathbf{V}^{\mu}_{II}(k',k) &= -\frac{e^3}{8\pi^2}\int_0^1 dx\,((1-x)k+k')\left(\frac{2}{\epsilon}-\ln(D_2/\mu^2)\right) \\ D_2 &= x(1-x)k^2+xm^2+(1-x)\lambda^2 \end{aligned}$$

C. Symmetric Interaction

This process is pictorially equivalent to the one-loop electron vertex in spinor QED. We have a standard self energy graph, a virtual scalar-photon loop, with the internal scalar line cut in half by its connection to the extended photon. The standard rules yield

$$i\mathbf{V}^{\mu}_{III}(k',k) = (ie)^3\left(\frac{1}{i}\right)^3\int\frac{d^4l}{(2\pi)^4}\frac{g^{\sigma\nu}(2k+l)_\nu(2k'+l)_\sigma(k'+k+l)^\mu}{(l^2+\lambda^2)((l+k')^2+m^2)((l+k')^2+m^2)} \tag{3}$$

$$\tag{4}$$

$$\begin{aligned} \sum_i^3\tilde{\Delta}_{m_i}(l_i^2) &= \int dF_3\left(x_1((l+k)^2+m^2)+x_2((l+k')^2+m^2)\right) \\ &\quad+\int dF_3\left((1-x_1-x_2)(l^2+\lambda^2)\right)^{-3} \\ &= \int dF_3\ \left(l^2+2l\cdot(kx_1+k'x_2)+x_1k^2+x_2k'^2\right) \\ &\quad+\int dF_3\left((x_1+x_2)m^2+x^3\lambda^2\right)^{-3} \\ &= \int dF_3\ \left(l^2+2l\cdot(kx_1+k'x_2)+(kx_1+k'x_2)^2\right) \\ &\quad+\int dF_3\left(x_1k^2+x_2k'^2-(kx_1+k'x_2)^2+(1-x_3)m^2+x_3\lambda^2\right)^{-3} \\ &= \int dF_3\ \left(q^2+D_3\right)^{-3} \end{aligned}$$

where

$$q = l + x_1 k + x_2 k' \quad (5)$$

$$D_3 = x_1 k'^2 + x_2 k^2 - (k' x_1 + k x_2)^2 + (1 - x_3) m^2 + x_3 \lambda^2 \quad (6)$$

So we may write

$$i\mathbf{V}^{\mu}_{III}(k', k) = e^3 \int dF_3 \int \frac{d^4 q}{(2\pi)^4} \frac{N^{\mu}}{(q^2 + D_3)}$$

Where N^{μ} is the numerator of Eq (4).

$$\begin{aligned}
N^{\mu} &= (l + 2k) \cdot (l + 2k')(2l + k' + k)^{\mu} \\
&= (q + (2 - x_1)k - x_2 k') \cdot (q + (2 - x_2)k' - x_1 k) \\
&\quad \times (2q + (1 - 2x_1)k + (1 - 2x_2)k')^{\mu} \\
&= (q \cdot q + (2 - x_2)k' \cdot q - x_1 k \cdot q + (2 - x_1)k \cdot q + (2 - x_1)(2 - x_2)k \cdot k' \\
&\quad - x_1(2 - x_1)k^2 - x_2 k' \cdot q - x_2(2 - x_2)k'2^2 + x_1 x_2 k' \cdot k) \\
&\quad \times (2q + (1 - 2x_1)k + (1 - 2x_2)k')^{\mu} \\
&= (q \cdot q + [2(1 - x_2)k' + 2(1 - x_1)k] \cdot q + [(2 - x_1)(2 - x_2) + x_1 x_2]k' \cdot k \\
&\quad - x_1(2 - x_1)k^2 - x_2(2 - x_2)k'2^2 + x_1 x_2 k' \cdot k) \\
&\quad \times (2q + (1 - 2x_1)k + (1 - 2x_2)k')^{\mu} \\
&= 4[(1 - x_1)k + (1 - x_2)k'] \cdot q\, q^{\mu} + q^2((1 - 2x_1)k + (1 - 2x_2)k')^{\mu} \\
&\quad + [((2 - x_1)(2 - x_2) + x_1 x_2)k' \cdot k \\
&\quad - x_1(2 - x_1)k^2 - x_2(2 - x_2)k'2^2 + x_1 x_2 k' \cdot k] \\
&\quad \times [(1 - 2x_1)k + (1 - 2x_2)k']^{\mu}
\end{aligned}$$

In the last line we dropped cubic and linear in q terms because they integrate to zero. We will subsequently label the q dependent term of N^{μ} as $N^{\mu}_{q^2}$ and the q-independent term as

$\tilde{N}^\mu$. $N^\mu_{q^2}$ can be further simplified so that it is written as a coefficient multiple of q^2.

$$\begin{aligned}
N^\mu_{q^2} &= 4[(1-x_1)k+(1-x_2)k']\cdot q\, q^\mu + ((1-2x_1)k+(1-2x_2)k')^\mu q^2 \\
&= 4[(1-x_1)k+(1-x_2)k']_\sigma q^\sigma\, q^\mu + \text{'' ''} \\
&= \frac{4}{d}[(1-x_1)k+(1-x_2)k']_\sigma\, g^{\sigma\mu} q^2 + \text{'' ''} \\
&= \left(\frac{4}{d}[(1-x_1)k+(1-x_2)k'] + ((1-2x_1)k+(1-2x_2)k')^\mu\right) q^2
\end{aligned} \tag{7}$$

In the third line, we used the fact that symmetric integration of $q^\sigma q^\mu$ is identical to integration of $g^{\sigma\mu}q^2/d$ in d dimensions.

Now, we perform the standard transition to d dimension, Wick Rotation, and the dimensional fixing of the coupling constant to obtain

$$\mathbf{V}^\mu_{III}(k'k) = e^3\tilde{\mu}^\epsilon \int dF_3 \int \frac{d^d\bar{q}}{(2\pi)^d}\frac{N^\mu_{q^2}+\tilde{N}^\mu}{(\bar{q}^2+D_3)^3} \tag{8}$$

Since $N^\mu_{q^2}$ is of order q^2 when we integrate it, using the formula in the appendix, we have

$$\begin{aligned}
\int \frac{d^d\bar{q}}{(2\pi)^d}\frac{N^\mu_{q^2}}{(\bar{q}^2+D_3)^3} &= \frac{1}{(4\pi)^2}[2/\epsilon - 1/2 - \ln(D_3/\mu^2)][(1+\epsilon/4)[(1-x_1)k+(1-x_2)k'] \\
&\quad +(1-2x_1)k+(1-2x_2)k']^\mu \\
&= \frac{1}{(4\pi)^2}[2/\epsilon - 1/2 - \ln(D_3/\mu^2)][(2-3x_1)k+(2-3x_2)k']^\mu \\
&\quad + \frac{1}{(4\pi)^2}[(1-x_1)k+(1-x_2)k']^\mu/2
\end{aligned}$$

and for $\tilde{N}^\mu$ of order q^0

$$\int \frac{d^d\bar{q}}{(2\pi)^d}\frac{\tilde{N}^\mu}{(\bar{q}^2+D_3)^3} = \frac{1}{(4\pi)^2}\frac{\tilde{N}^\mu}{2D_3}$$

So, in all, we have

$$\begin{aligned}
\mathbf{V}^\mu_{III}(k',k) &= \frac{e^3}{16\pi^2}\int dF_3 \left[\left(\frac{2}{\epsilon} - \frac{1}{2} - \ln(D_3/\mu^2)\right)((2-3x_1)k+(2-3x_2)k')^\mu\right] \\
&\quad + \frac{e^3}{16\pi^2}\int dF_3 \left[[(1-x_1)k+(1-x_2)k']^\mu/2 + \tilde{N}^\mu/(2D_3)\right]
\end{aligned} \tag{9}$$

IV. RENORMALIZATION CONDITIONS

The scalar-scalar-photon vertex to first loop order consists of the correction terms evaluated above and tree level results with a renormalization constant.

$$\mathbf{V}^\mu(k',k) = Z_1 e(k+k')^\mu + \mathbf{V}^\mu_I(k',k) + \mathbf{V}^\mu_{II}(k',k) + \mathbf{V}^\mu_{III}(k',k)$$

Where $Z_1 = 1 + eO(\alpha)$ and we define it such that as $k \to k'$, $\mathbf{V}^\mu(k', k)$ goes to its tree level result. Specifically,

$$\mathbf{V}^\mu(k', k)|_{k'=k} = 2ek^\mu \tag{10}$$

Thus Z_1 acts to cancel all divergent terms and additional numerical factors. The exact form of Z_1 is irrelevant to bound-state problems so in the subsequent analysis we will just use general arguments to obtain a vertex function which obeys Eq (10). We begin by noting that for on-shell external scalars $D_1 = D_2$ because

$$\begin{aligned} D_1 &= x(1-x)k'^2 + xm^2 + (1-x)\lambda^2 \\ &= -x(1-x)m^2 + xm^2 + (1-x)\lambda^2 \\ &= -x(1-x)k^2 + xm^2 + (1-x)\lambda^2 \\ &= D_2 \end{aligned}$$

Therefore we may consolidate $\mathbf{V}^\mu_I(k', k)$ and $\mathbf{V}^\mu_{II}(k', k)$ into one term.

$$\mathbf{V}^\mu_I(k', k) + \mathbf{V}^\mu_{II}(k', k) = -\frac{e^3}{8\pi^2}(k+k')^\mu \int_0^1 dx\,(2-x)\left(\frac{2}{\epsilon} - \ln(D_1/\mu^2)\right)$$

But, this term is just a constant multiple of $(k+k')$ so it may be absorbed into Z_1. In fact, any term which is a constant multiple of the tree level result is really just a renormalization of $(k+k')^\mu$ and cannot affect the result of scattering experiments. Therefore, we are left with $\mathbf{V}^\mu_{III}(k', k)$ as our corrected vertex. We could have anticipated this result from the fact that the ends of the left and right handed interactions look like self energy diagrams and thus cannot affect the vertex in an on-shell renormalization scheme. For $\mathbf{V}^\mu_{III}(k', k)$ we integrate over the Feynman Parameters in the linear in k and k' terms to obtain.

$$\begin{aligned} \mathbf{V}^\mu_{III}(k', k) &= \frac{e^3}{16\pi^2}\left[\left(\frac{2}{\epsilon} - \frac{1}{2}\right) + \frac{1}{3}\right](k+k')^\mu \\ &\quad - \frac{e^3}{16\pi^2}\int dF_3 \left[\ln(D/\mu^2)((2-3x_1)k + (2-3x_2)k')^\mu - \tilde{N}^\mu/(2D)\right] \end{aligned}$$

Where we have relabeled D_3 as D because it is the only "D" remaining in the analysis. Once again we may absorb the coefficient of the $(k+k')$ term into Z_1 so that our vertex may be written as

$$\mathbf{V}^\mu(k',k) = \mathcal{Z}e(k+k')^\mu - \frac{e^3}{16\pi^2}\int dF_3 \left[\ln(D/\mu^2)((2-3x_1)k + (2-3x_2)k')^\mu - \tilde{N}^\mu/(2D)\right] \tag{11}$$

where we have used $\mathcal{Z}$ to signify the incorporation of the previous factors into a new renormalization constant. As we previously stated, the exact form of the constant is irrelevant to observables as long as the vertex fits the renormalization condition. So we may choose our vertex in such a way that the condition is immediately satisfied and take $\mathcal{Z}$ to be implicitly defined by the form of this new vertex. Doing so, we have

$$\begin{aligned}\mathbf{V}^\mu(k',k) \;=\; & e(k+k')^\mu \\ & -\frac{e^3}{16\pi^2}\int dF_3 \left[\ln(D/D_0)((2-3x_1)k + (2-3x_2)k')^\mu + \tilde{N_0}^\mu/2D_0 - \tilde{N}^\mu/(2D)\right]\end{aligned}$$

Where D now written in terms of $q = k' - k$ or, equivalently, $k \cdot k = -\frac{1}{2}q^2 - m^2$ is

$$\begin{aligned} D \;&=\; x_1k^2 + x_2k'^2 - (x_1k + x_2k')^2 + x_3\lambda^2 + (x_1+x_2)m^2 \\ &=\; (x_1^2 + x_2^2)m^2 - 2x_1x_2k\cdot k' + x_3\lambda^2 \\ &=\; (1-x_3)^2m^2 + x_1x_2q^2 + x^3\lambda^2 \end{aligned}$$

and D_0 and N_0 are D and N, respectively, evaluated at $q^2 = 0$. In this form it is evident that if we take $k' = k$ the integral becomes zero and we obtain the tree level result as desired.

V. VERTEX FUNCTION

We want an explicit form for our vertex. D is simple enough that we may just plug in our previous result but $\tilde{N}^\mu$ will take some work.

$$
\begin{aligned}
\tilde{N}^\mu &= [((2-x_1)(2-x_2)+x_1x_2)k\cdot k' - x_1(2-x_1)k^2 - x_2(2-x_2)k'^2] \\
\times &\quad [(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= [-(1/2)((2-x_1)(2-x_2)+x_1x_2)q^2 \\
&\quad -m^2((2-x_1)(2-x_2)+x_1x_2 - x_1(2-x_1) - x_2(2-x_2))] \\
&\quad \times[(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= \text{''''} - m^2(4-2x_1-2x_2+x_1x_2+x_1x_2-2x_1+x_1^2-2x_2+x_2^2] \\
&\quad \times[(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= \text{''''} - m^2[4x_3+2x_1x_2+x_1^2+x_2^2][(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= \text{''''} - m^2[4x_3+x_3^2-1+2(1-x_3)][(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= \text{''''} - m^2[x_3^2+2x_3+1][(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= -\big([(2-x_1)(2-x_2)+x_1x_2]q^2/2 + m^2(x_3+1)^2\big) \\
&\quad \times[(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= -\big([4-2x_1-2x_2+2x_1x_2]q^2/2 + m^2(x_3+1)^2\big) \\
&\quad \times[(1-2x_1)k^\mu + (1-2x_2)k'^\mu] \\
&= -\big([1+x_3+x_1x_2]q^2 + m^2(x_3+1)^2\big) \\
&\quad \times[(1-2x_1)k^\mu + (1-2x_2)k'^\mu]
\end{aligned}
$$

Now, we separate the antisymmertic parameter integration from the symmetric parameter integration by defining

$$k' + k = \chi \tag{12}$$

$$k' - k = \phi \tag{13}$$

So that

$$k' = \frac{\chi+\phi}{2} \qquad\qquad k = \frac{\chi-\phi}{2}$$

and

$$\begin{aligned}(1-2x_1)(\chi-\phi)/2+(1-2x_2)(\chi+\phi)/2 &= \frac{\chi}{2}(2-2x_1-2x_2)+\frac{\phi}{2}(2x_1-2x_2)\\ &= \chi x_3+\phi(x_1-x_2)\end{aligned}$$

The last term above must integrate to zero because the integration measure dF_3 is symmetric upon $x_1 \leftrightarrow x_2$ but an integrand containing $(x_1 - x_2)$ is not. Therefore, we may neglect this term and from this analysis $\tilde{N}$ is fully determined.

The logarithmic term in the vertex function has a similar ”symmetric+antisymmetric” prefactor and we may simplify it accordingly

$$\begin{aligned}(2-3x_1)k+(2-3x_2)k' &= (2-3x_1)(\chi-\phi)/2+(2-3x_2)(\chi+\phi)/2\\ &= (4-3x_1-3x_2)\chi/2+(3x_1-3x_2)\phi/2\\ &= (1+3x_3)\chi/2+(3x_1-3x_2)\phi/2\end{aligned}$$

Once again we drop the antisymmetric term. Consolidating our results we finally have the one loop vertex function for on shell scalars

$$\begin{aligned}\mathbf{V}^{\mu}(k',k) &= e(k'+k)^{\mu}F(q^2)\\ F(q^2) &= 1-\frac{e^2}{32\pi^2}\int dF_3\left[(1+3x_3)\ln\left(1+\frac{x_1x_2q^2}{m^2(1-x_3)^2+x_3\lambda^2}\right)\right]\\ &\quad -\frac{e^2}{32\pi^2}\int dF_3\left[\frac{q^2(1+x_3+x_2x_1)x_3+m^2(x_3+1)^2x_3}{m^2(1-x_3)^2+x_1x_2q^2+x_3\lambda^2}-\frac{m^2(x_3+1)^2x_3}{m^2(1-x_3)^2+x_3\lambda^2}\right]\end{aligned}$$

VI. LAMB SHIFT ESTIMATION

To compute the Lamb Shift we will focus on the low energy expansion of our above form factor. This is the low energy region of the relativistic result and will subsequently act as the high energy region of a non relativistic result. The photon mass λ acts as the regulator which defines a lower cutoff for this domain of energy. To obtain the full Lamb Shift we will also have to include the purely non relativistic result which does not require the full framework of quantum field theory. Proceeding with the calculation, we Taylor expand $F(q^2)$ in the $|q^2| << m^2$ limit. The necessary equations are

$$
\begin{aligned}
\ln\left(1+\frac{x_1x_2}{m^2(1-x_3)^2+x_3\lambda^2}\right) &\approx \frac{x_1x_2q^2}{m^2(1-x_3)^2+x_3\lambda^2} \\
\frac{q^2(1+x_3+x_2x_1)x_3+m^2(x_3+1)^2x_3}{m^2(1-x_3)^2+x_1x_2q^2+x_3\lambda^2} &\approx \frac{q^2(1+x_3+x_2x_1)x_3}{m^2(1-x_3)^2+x_3\lambda^2}+\frac{m^2(x_3+1)^2x_3}{m^2(1-x_3)^2+x_3\lambda^2} \\
&\quad -\frac{m^2q^2(x_3+1)^2x_1x_2x_3}{(m^2(1-x_3)^2+x_3\lambda^2)^2}
\end{aligned}
$$

So now

$$
\begin{aligned}
F(q^2) &\approx 1-\frac{e^2}{32\pi^2}\int dF_3[\frac{(1+3x_3)x_1x_2q^2}{m^2(1-x_3)^2+x_3\lambda^2}+\frac{q^2(1+x_3+x_2x_1)x_3}{m^2(1-x_3)^2+x_3\lambda^2} \\
&\quad +\frac{m^2(x_3+1)^2x_3}{m^2(1-x_3)^2+x_3\lambda^2}-\frac{m^2q^2(x_3+1)^2x_1x_2x_3}{(m^2(1-x_3)^2+x_3\lambda^2)^2}-\frac{m^2(x_3+1)^2x_3}{m^2(1-x_3)^2+x_3\lambda^2}] \\
&= 1-\frac{e^2q^2}{32\pi^2}\int dF_3\left[\frac{(1+4x_3)x_1x_2+(1+x_3)x_3}{m^2(1-x_3)^2+x_3\lambda^2}-\frac{m^2q^2(x_3+1)^2x_1x_2x_3}{(m^2(1-x_3)^2+x_3\lambda^2)^2}\right]
\end{aligned}
$$

Recalling the definition of our integration measure

$$\int dF_3 = 2\int_0^1 dx_3\int_0^{1-x_3} dx_1\int_0^{1-x_3-x_2} dx_2\delta(x_1+x_2+x_3-1)$$

we have the following two results

$$
\begin{aligned}
\int dF_3\cdot 1 &= 2\int_0^1 dx_3\int_0^{1-x_3} dx_1\int_0^{1-x_3-x_2} dx_2\delta(x_1+x_2+x_3-1) \\
&= 2\int_0^1 dx_3\,(1-x_3) \\
\int dF_3\cdot x_1x_2 &= 2\int_0^1 dx_3\int_0^{1-x_3} dx_1\int_0^{1-x_3-x_2} dx_2\delta(x_1+x_2+x_3-1)x_1x_2 \\
&= 2\int_0^1 dx_3\int_0^{1-x_3} dx_1\,x_1(1-x_3-x_1) \\
&= 2\int_0^1 dx_3\frac{(1-x_3)^3}{6}
\end{aligned}
$$

Incorporating these results, we're left with only one integration

$$F(q^2) = 1-\frac{e^2q^2}{16\pi^2}\int_0^1 dx_3\left[\frac{(1+4x_3)(1-x_3)^3/6+(1+x_3)(1-x_3)}{m^2(1-x_3)^2+x_3\lambda^2}+\frac{m^2(x_3+1)^2x_3(1-x_3)^3/6}{(m^2(1-x_3)^2+x_3\lambda^2)^2}\right]$$

To simplify the subsequent analysis we make a change of variables

$$u = 1-x_3$$

So that the integral becomes

$$\begin{aligned} F(q^2) &= 1 - \frac{e^2 q^2}{16\pi^2}\int_0^1 du \left[\frac{(5-4u)u^3/6 + (2-u)u(1-u)}{m^2u^2 + (1-u)\lambda^2} - \frac{m^2(2-u)^2(1-u)u^3/6}{(m^2u^2+(1-u)\lambda^2)^2}\right] \\ &= 1 - \frac{e^2 q^2}{16\pi^2}\int_0^1 du \frac{U_1}{m^2u^2 + (1-u)\lambda^2} - \frac{m^2 U_2}{(m^2u^2+(1-u)\lambda^2)^2} \end{aligned} \tag{14}$$

Where

$$\begin{aligned} U_1 &= (5-4u)u^3/6 + (2-u)u(1-u) \\ &= 5u^3/6 - 2u^3/3 + 2u - 3u^2 + u^3/6 \\ &= 2u - 3u^2 + 11u^3/6 - 2u^4/3 \\ U_2 &= (2-u)^2(1-u)u^3/6 \\ &= (4-4u+u^2)(1-u)u^3/6 \\ &= (4-4u+u^2-4u+4u^2-u^3)u^3/3 \\ &= 2u^3/3 - 4u^4/3 + 5u^5/6 - u^6/6 \end{aligned}$$

Now, we invoke one last approximation

$$\begin{aligned} \frac{1}{(m^2u^2+(1-u)\lambda^2} &\approx \frac{1}{m^2u^2+\lambda^2} + O(\lambda^2) \\ \frac{1}{(m^2u^2+(1-u)\lambda^2)^2} &\approx \frac{1}{(m^2u^2+\lambda^2)^2} + O(\lambda^2) \end{aligned}$$

and with this we may easily compute Eq (14). The relevant integrals were computed with *Mathematica* and are tabulated in Appendix II. We only retained terms of $o(1/m^2)$. The result is

$$\begin{aligned} F(q^2) &= 1 - \frac{e^2q^2}{16\pi^2}[(2\,I_1 - 3\,I_2 + \frac{11}{6}\,I_3 - \frac{2}{3}\,I_4) \\ &\quad -m^2(\frac{2}{3}I_5 - \frac{4}{3}I_6 + \frac{5}{6}I_7 - \frac{1}{6}I_8)] \\ &= 1 - \frac{e^2q^2}{16\pi^2}[(\frac{1}{m^2}\ln(m^2/\lambda^2) - \frac{3}{m^2} + \frac{11}{12m^2} - \frac{2}{9m^2}) \\ &\quad -(-\frac{1}{3m^2} + \frac{1}{3m^2}\ln(m^2/\lambda^2) - \frac{4}{3m^2} + \frac{5}{12m^2} - \frac{1}{18m^2})] \\ &= 1 - \frac{e^2q^2}{16\pi^2}\left(\frac{2}{3m^2}\ln\left(\frac{m^2}{\lambda^2}\right) - \frac{1}{m^2}\right) \end{aligned}$$

So, finally, we have our small momentum form factor

$$F(q^2) = 1 + \frac{e^2}{24\pi^2}\left(\frac{q^2}{m^2}\right)\left[\ln\left(\frac{\lambda^2}{m^2}\right) + \frac{3}{2}\right]$$

Making the transition to non relativistic quantum mechanics, we may model this form factor as a perturbation to the coulomb potential in the hydrogen atom. As such it has the form

$$\begin{aligned}\delta\hat{H} &= \frac{e^2}{24\pi^2}\left[\ln\left(\frac{\lambda^2}{m^2}\right)+\frac{3}{2}\right]e\nabla^2 A^0(\mathbf{x}) \\ &= -\frac{e^2}{24\pi^2}\left[\ln\left(\frac{\lambda^2}{m^2}\right)+\frac{3}{2}\right]Ze^2\delta^3(\mathbf{x}) \\ &= -\frac{e^2}{24\pi^2}\left[\ln\left(\frac{\lambda^2}{m^2}\right)+\frac{3}{2}\right]Ze^2|\psi(0)|^2\end{aligned}$$

And with

$$\psi(0) = \frac{1}{\sqrt{\pi}}\left(\frac{m\alpha}{n}\right)^{3/2}\delta_{l,0}$$

we have

$$\left\langle n,l,m|\delta\hat{H}|n,l,m\right\rangle_{\text{High}} = -\frac{2\alpha^5 m}{3\pi n^3}[\ln(\lambda^2/m^2)+3/2]\delta_{l,0}$$

Adding this result to the non relativistic result with ultraviolet cutoff λ [2]

$$\left\langle n,l,m|\delta\hat{H}|n,l,m\right\rangle_{\text{Low}} = \frac{4\alpha^5 m}{3\pi n^3}[\ln\frac{\lambda}{2\left\langle E-E_n\right\rangle}+5/6]\delta_{l,0}$$

we have for the shift, without vaccuum polarization,

$$\delta E_{n,l} = \frac{4\alpha^5 m}{3\pi n^3}\left[\ln\frac{m}{2\left\langle E-E_n\right\rangle}+1/12\right]\delta_{l,0}$$

This result yields no shift for $l \neq 0$ states so we may write the total shift as coming from the s states. For the standard $2s$ evaluation we find

$$\delta E_{2,0} = h \times 9653.98\,\text{MHz}$$

This value is close to that achieved by Dyson ($\approx 1003\,\text{MHz}$).This was our goal; to compute Dyson's result in a modern context. But we find our result is farther from the experimentally verified ($\approx 1057\,\text{MHz}$) spin 1/2 result than Dyson's. Still, a paper written by Newcomb and Rohlrlich in 1951[3] suggests that our result is closer than Dyson's to the "true" spin zero result of $\approx 956\,\text{MHz}$.

VII. APPENDIX I

The following integrals are relevant to the dimensional regularization performed in our analysis

$$I(a,b) = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2+D)^b} = \frac{\Gamma(b-a-d/2)\,\Gamma(a+d/2)}{(4\pi)^{d/2}\,\Gamma(b)\,\Gamma(d/2)}\, D^{-(b-a-d/2)}$$

A. 1

$$\begin{aligned}
\tilde{\mu}^\epsilon \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^3} = \tilde{\mu}^\epsilon I(0,3) \;&=\; \tilde{\mu}^\epsilon \frac{\Gamma(3-d/2)\,\Gamma(d/2)}{(4\pi)^{d/2}\,\Gamma(3)\,\Gamma(d/2)}\, D^{-(3-d/2)} \\
&=\; \frac{\Gamma(1+\epsilon/2)}{2(4\pi)^2}\,\frac{1}{D}\,\left(\frac{4\pi\tilde{\mu}}{D}\right)^{\epsilon/2} \\
&=\; \frac{\epsilon}{2}\,\frac{\Gamma(\epsilon/2)}{2(4\pi)^2}\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2} \\
&=\; \frac{\epsilon}{2}\,\frac{1}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+O(\epsilon)\right]\frac{1}{D} \\
&\quad\;\times\left[1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)+O(\epsilon^2)\right] \\
&=\; \frac{1}{2(4\pi)^2}\,\frac{1}{D}
\end{aligned}$$

B. 2

$$\begin{aligned}
\tilde{\mu}^\epsilon \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{q}^2}{(\bar{q}^2+D)^3} = \tilde{\mu}^\epsilon I(1,3) \;&=\; \tilde{\mu}^\epsilon \frac{\Gamma(2-d/2)\,\Gamma(1+d/2)}{(4\pi)^{d/2}\,\Gamma(3)\,\Gamma(d/2)}\, D^{-(2-d/2)} \\
&=\; \frac{d}{2}\,\frac{\Gamma(\epsilon/2)}{2(4\pi)^2}\,\frac{1}{D}\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2} \\
&=\; (2-\epsilon/2)\,\frac{1}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+O(\epsilon)\right]\frac{1}{D} \\
&\quad\;\times\left[1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)+O(\epsilon^2)\right] \\
&=\; \frac{1}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\frac{1}{2}-\ln(D/\mu^2)\right]
\end{aligned}$$

where

$$\mu^2 = 4\pi\tilde{\mu}^2/e^\gamma$$

VIII. APPENDIX II

Integrals in Lamb Shift Calculation

$$
\begin{aligned}
I_1 &= \int_0^1 \frac{u^2}{m^2u^2+\lambda^2} \approx \frac{1}{2m^2}\ln(m^2/\lambda^2) \\
I_2 &= \int_0^1 \frac{u^3}{m^2u^2+\lambda^2} \approx 1/m^2 + O(\lambda) \\
I_3 &= \int_0^1 \frac{u^4}{m^2u^2+\lambda^2} \approx 1/2m^2 + O(\lambda^2) \\
I_4 &= \int_0^1 \frac{u^5}{m^2u^2+\lambda^2} \approx 1/3m^2 + O(\lambda^2) \\
I_5 &= \int_0^1 \frac{u^3}{(m^2u^2+\lambda^2)^2} \approx -1/2m^4 + \frac{1}{2m^4}\ln(m^2/\lambda^2) \\
I_6 &= \int_0^1 \frac{u^4}{(m^2u^2+\lambda^2)^2} \approx 1/m^4 + O(\lambda^2) \\
I_7 &= \int_0^1 \frac{u^5}{(m^2u^2+\lambda^2)^2} \approx 1/2m^4 + O(\lambda^2) \\
I_8 &= \int_0^1 \frac{u^6}{(m^2u^2+\lambda^2)^2} \approx 1/3m^4 + O(\lambda^2)
\end{aligned}
$$

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[1] F.J. Dyson, Phys. Rev. 73, 617 (1948

[2] S. Weinberg, *Quantum Theory of Fields: Foundations* (Cambridge University Press, 1995), page 590

[3] W. A. Newcomb, and F. Rohrlich, Phys. Rev. 81, 282-283 (1951)