Algebra Before Numbers

When you encounter a physics problems which defines physical quantities in terms of their numerical values there are two approaches to obtaining a solution: you can substitute the numbers into the problem immediately and manipulate the numerical quantities to find the final result; or you can solve the problem analytically *first* (that, is solve it using the symbolic representation of the physical quantities) and substitute in the numerical quantities only after you have obtained the final result.

It is almost always preferable to do the latter, namely it's better to perform the algebra before you substitute in numbers. Even if there are no variables given in the problem, it is generally better to define numerical quantities in terms of variables and then work with the variables in lieu of the numbers. There are three main reasons for adopting this "algebra before numbers" strategy.

- **Cleaner Solution:** Without numbers cluttering your work, you typically end up with a cleaner written solution which is easier to subsequently analyze. You also avoid the potential numerical mistakes arising from the repeated substitution and manipulation of numerical quantities.
- **Generalized Result:** Your solution is easier to generalize and can thus be easily applied to more cases than those defined by the numerical quantities of the given problem.
- **Can Check Your Result:** You can apply intuitive checks (like dimensional considerations and limiting cases) to check your solution.

1 A Loan Wolf

We can more precisely illustrate these advantages with an example problem.[1]

> Let's say you get a loan from a shady character. The amount of money you owe back for the loan at time t is represented by $\$(t)$. In general, if the rate of interest (continuously compounded monthly) on the loan is fixed at r, and if you start off owing an amount $\$(t = 0) = \$(0)$, then at the time t you owe
> $$\$(t) = \$(0)e^{rt}. \tag{1}$$
> In your particular case, you received a loan for 100 dollars at a monthly interest rate of $r = 0.5$. How much time would it take for the amount you owe to be 200 dollars?

We will construct the solution to this by first going through the "numbers before algebra" strategy, and then going through the "algebra before numbers" strategy. We will then compare both strategies to determine which informs us more about the nature of our solution.

1.1 Numbers Before Algebra

We get a loan of $\$(0) = 100$ dollars at a rate of $r = 0.25$ per month. We want to know how much time t (in months) it takes the amount we owe to grow to 200 dollars. Using Eq.(1) (and denoting dollar amounts as pure numerical quantities) we have the equation.

$$200 = 100e^{(0.5t}. \tag{2}$$

[1] Given the students in this course I'm probably preaching to the choir, but this example of interest rates provides a good rebuttal to the common "When will I ever have to use this stuff?" question.

Dividing both sides by 100, taking the natural logarithm and finally dividing by 0.25, we find

$$t = \frac{\ln 2}{0.5} \text{ months} \approx 1.4 \text{ months}. \quad (3)$$

So it takes less than a month and a half to owe twice as much as you borrowed.

1.2 Algebra Before Numbers

Now, let's solve the problem more abstractly by representing all quantities symbolically until the end. Let's say we want to find the time t_f it takes an initial loan of $\$(0)$ to grow to an amount $\$(t_f)$ given that it has an interest rate of r. By Eq.(1), the fundamental equation is

$$\$(t_f) = \$(0)e^{rt_f}. \quad (4)$$

Dividing both sides by $\$(0)$, taking the natural logarithm, and finally dividing by r we find

$$t_f = \frac{1}{r} \ln \frac{\$(t_f)}{\$(0)} \quad (5)$$

computed in months. Substituting the numerical quantities $r = 0.5$, $\$(t_f) = 200$ dollars, and $\$(0) = 100$ dollars we reproduce Eq.(3), but Eq.(5) affords us theoretical possibilities not contained in Eq.(3).

For one, the algebraic result Eq.(5) is more general than the bare numerical result Eq.(3); it applies to any case where we want to find how much time it would take for us to owe a multiple of what we initially owed. Second, with the algebraic result we can check our answer to ensure it is consistent with what we expect intuitively. Here are some possible checks we could perform.

- The lower the interest rate the more time it takes for the amount you owe to reach a new value:

$$r \to 0 \text{ gives } t_f \to \infty \quad (6)$$

- The time it takes to reach a new value depends only on the ratio between the new value and the initial value:

$$t_f \text{ is a function of } \$(t_f)/\$(0) \quad (7)$$

- It is not possible for the amount of money you owe at a later time to be less than the amount of money you started off owing:

$$\$(t_f) < \$(0) \text{ leads to } t_f < 0 \text{ (an impossible result)} \quad (8)$$

These checks are not possible with only our numerical expression Eq.(3). Thus solving the problem algebraically first not only gives us a general formula, but it also allows us to theoretically check our result against expectations.

The Takeaway: This problem was primarily mathematical, not physical, but the essence of the lesson applies when answering questions in physics. When solving a problem which includes numerical quantities, it is better to solve the problem algebraically (or more generally, analytically) before you substitute in numbers.