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2. **Simple Harmonic Motion**
   A particle undergoing simple harmonic motion has the position

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

This position can also be written as $x(t) = E\cos(\omega_0 t - \phi)$ where $E > 0$. What are E and ϕ in terms of A and B?

Solution: Using the sum of angles formula we find

$$\begin{aligned} x(t) &= E\cos(\omega_0 t - \phi) \\ &= E\cos\phi\cos(\omega_0 t) + E\sin\phi\sin(\omega_0 t). \end{aligned} \tag{1}$$

Equating this result to the given position, we find

$$E\cos\phi\cos(\omega_0 t) + E\sin\phi\sin(\omega_0 t) = A\cos(\omega_0 t) + B\sin(\omega_0 t), \tag{2}$$

which implies that $E\cos\phi = A$ and $E\sin\phi = B$. Using trigonometric identities and the condition $E > 0$, we then find

$$E = \sqrt{A^2 + B^2}, \qquad \phi = \tan^{-1}\frac{B}{A}. \tag{3}$$

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3. **Effective spring constant**
   Two springs with spring constants k_1 and k_2 are connected in parallel as shown in Fig. 1. What is the

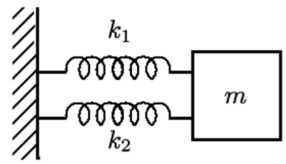

Figure 1

effective spring constant k_{eff}? In other words, if the mass is displaced by x, find the k_{eff} for which the force equals $F = -k_{\text{eff}}x$.

Solution: If the mass is displaced a distance x from its equilibrium position, the spring with spring constant k_1 exerts a restoring force $-k_1 x$. Similarly, the spring with spring constant k_2 exerts a restoring force $-k_2 x$ on the mass. Thus the net force on the mass is

$$F_{\text{net}} = -k_1 x - k_2 x = -(k_1 + k_2)x. \tag{4}$$

Therefore the effective spring constant is $k_{\text{eff}} = k_1 + k_2$.

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4. **Superposition**
   Let $x_1(t)$ and $x_2(t)$ be solutions to the differential equation

$$\ddot{x}^2(t) = b\,x(t).$$

where b is a constant. Is the linear combination $x(t) = c_1 x_1(t) + c_2 x_2(t)$, where c_1 and c_2 are constants, a solution to the differential equation? (Provide a calculation showing why or why not).

Solution: If x_1 and x_2 solve the differential equation $\ddot{x}^2(t) = b\,x(t)$, then we have

$$\ddot{x}_1^2(t) = b\,x_1(t) \qquad \ddot{x}_2^2(t) = b\,x_2(t). \tag{5}$$

For $x(t) = c_1 x_1(t) + c_2 x_2(t)$ to be a solution to the differential equation, it too must satisfy the equations above. Checking whether it does, we have

$$\begin{aligned}
\left[\frac{d^2}{dt^2}\left(c_1 x_1(t) + c_2 x_2(t)\right)\right]^2 &\stackrel{?}{=} b\left[c_1 x_1(t) + c_2 x_2(t)\right] \\
\left(c_1 \ddot{x}_1 + c_2 \ddot{x}_2\right)^2 &\stackrel{?}{=} bc_1 x_1 + bc_2 x_2 \\
c_1^2 \ddot{x}_1^2 + 2c_1 c_2 \ddot{x}_1 \ddot{x}_2 + c_2^2 \ddot{x}_2^2 &\stackrel{?}{=} bc_1 x_1 + bc_2 x_2 \\
c_1^2 b x_1 + 2c_1 c_2 \ddot{x}_1 \ddot{x}_2 + c_2 x_2 &\neq bc_1 x_1 + bc_2 x_2
\end{aligned} \tag{6}$$

where in the last line we used Eq.(5). Thus the linear combination $c_1 x_1(t) + c_2 x_2(t)$ does not satisfy the differential equation and is not a solution to it.

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5. **An Anti-Hookean oscillator**
   A particle of mass m and at a position x is subject to a force

$$F(x) = +kx,$$

   with $k > 0$.

   (a) What is the most general form of $x(t)$ in terms of the parameters of the system?
   (b) The particle begins at x_0 and after long times (i.e., $t \to \infty$), the particle is at $x = 0$. What is the particle's initial velocity?

Solution: The equation of motion of this system is $m\ddot{x} - kx = 0$, or

$$\ddot{x} - \frac{k}{m}x = 0. \tag{7}$$

Finding the general solution by guessing a general exponential and fixing its parameters, we find

$$x(t) = Ae^{t\sqrt{k/m}} + Be^{-t\sqrt{k/m}}. \tag{8}$$

If the particle begins at x_0, we have $A + B = x_0$. If the particle is at $x = 0$ when $t \to \infty$, then we have $A = 0$. Thus, the specific solution is

$$x(t) = x_0 e^{-t\sqrt{k/m}}, \tag{9}$$

and the particle's initial velocity is $v_0 = -x_0\sqrt{k/m}$.

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6. **Exponential force**
   A particle of mass m is attached to a spring of spring constant k and is subject to another external force $F(t) = F_0 e^{-bt}$. The equation of motion of the system is

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} e^{-bt}, \tag{10}$$

   where $\omega_0^2 = k/m$. Guess a solution of the form $x(t) = Ae^{\alpha t}$ and determine what A and α should be in order for this solution to satisfy the above differential equation.

   Solution: Guessing a solution of the form $x(t) = Ae^{\alpha t}$ and plugging it into the equation of motion we find

$$(\alpha^2 + \omega_0^2) A e^{\alpha t} = \frac{F_0}{m} e^{-bt}. \tag{11}$$

   In order for this equation to be true for all time, we need to take $\alpha = -b$ and $A = F_0/[m(\alpha^2 + \omega_0^2)] = F_0/[m(b^2 + \omega_0^2)]$. We therefore find the particular solution to the given differential equation is

$$x(t) = \frac{F_0/m}{\omega_0^2 + b^2} e^{-bt}. \tag{12}$$

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7. **Matrix algebra**
   We have the following system of two equations of motion

$$\begin{aligned} m\ddot{x}_1 &= -k_L x_1 + k_M(x_2 - x_1) \\ m\ddot{x}_2 &= -k_R x_2 - k_M(x_2 - x_1), \end{aligned}$$

where x_1 and x_2 are position variables and k_L, k_M, and k_R are spring constants. Say we want to write the two equations as a matrix equation where a 2×2 matrix multiplies a 2×1 matrix to yield a 2×1 matrix:

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

What should the values of the question marks be?

Solution: The equations of motion for x_1 and x_2 are

$$\begin{aligned} m\ddot{x}_1 &= -(k_L + k_M)x_1 + k_M x_2 \\ m\ddot{x}_2 &= -(k_R + k_M)x_2 + k_M x_1, \end{aligned}$$

Writing the above equation of motion as a matrix, we have

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_K - k_M & k_M \\ k_M & -k_R - k_M \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

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8. **Second derivatives**

   Say we have the function
$$f(x) = x^2 + \sin(2x).$$
   Compute the value of
$$\lim_{a \to 0} \frac{f(x+a) - 2f(x) + f(x-a)}{a^2}.$$

   Solution: The given limit is the definition of a second derivative. Thus for the provided function we have
$$\lim_{a \to 0} \frac{f(x+a) - 2f(x) + f(x-a)}{a^2} = \frac{d^2 f}{dx^2} = 2 - 4\sin(2x). \tag{13}$$