NLSM, Gross-Neveu, and CP^N Models In Large N Limit in d = 2

We work through the solutions to the non linear sigma model, Gross-Neveu Model, and the CP^N model in the limit of a large number of components. In typical QFT parlance, by "solution" we mean an exact evaluation of the functional integral. The functional integral can be evaluated exactly in these cases because in the large N limit, higher than one-loop corrections go to zero and we can thus represent the functional integral as a determinant. These field theories have the interesting feature that quantum corrections result in properties (additional mass terms, broken chiral symmetry, and new gauge degrees of freedom, respectively) not present in the classical theory. This discussion follows [1].

1 NLSM model

The non-linear sigma model can be defined by a lagrangian and a constraint. If we have real scalar fields $n^i(x)$, then the nonlinear sigma model defining these fields is

$$\mathcal{L} = -\frac{1}{2g_0^2}\partial_\mu n^i(x)\partial^\mu n^i(x) \qquad \text{with } n^i(x)n^i(x) = 1. \tag{1}$$

We find the QFT solution for this lagrangian by computing the functional integral for the theory. Incorporating the constraint with a delta function, this functional integral is

$$Z = \int Dn\, \exp\left[-\frac{i}{2g_0^2}\int d^2x\, \partial_\mu n^i(x)\partial^\mu n^i(x)\right]\prod_x \delta(n^i(x)n^i(x) - 1). \tag{2}$$

Now, this integral would be tractable if only we didn't have the pesky integration over the delta function. Using a functional integral Fourier transform [1] , we can write this delta function as

$$\prod_x \delta(n^2(x) - 1) = \int D\lambda \exp\left[i\int d^2x\, \lambda(x)(n^2(x) - 1)\right], \tag{3}$$

where the product is taken over all spacetime points and λ is a real field. We then find for Eq.(**??**)

$$Z = \int Dn\, D\lambda\, \exp\left[-\frac{i}{2g_0^2}\int d^2x\, (\partial n)^2 + i\int d^2x\, \lambda(n^2 - 1)\right]. \tag{4}$$

Since λ is an arbitrary auxiliary field, we can define it any way we please, so we rescale it as $\lambda \to \lambda/2g_0^2$ to give it the same prefactor as the kinetic term. The integration measure doesn't change because we can redefine it according to various undetermined constants. we then have the functional integral

$$\begin{aligned}
Z &= \int Dn\, D\lambda\, \exp\left[-\frac{i}{2g_0^2}\int d^2x \left((\partial n)^2 - \lambda\, n^2\right) - \frac{i}{2g_0^2}\int d^2x\, \lambda\right] \\
&= \int Dn\, D\lambda\, \exp\left[\frac{i}{2g_0^2}\sum_k \int d^2x\, n^k(\partial^2 + \lambda)n^k - \frac{i}{2g_0^2}\int d^2x\, \lambda\right] \\
&= \int Dn\, D\lambda\, \exp\left[-\frac{i}{2g_0^2}\int d^2x\, \lambda\right]\prod_k \exp\left[\frac{i}{2g_0^2}\int d^2x\, n_k(\partial^2 + \lambda)n_k\right]
\end{aligned}$$

[1] Because $\int d\lambda\, e^{i\lambda A} = (2\pi)\delta(A)$, we also have $\int \prod_k d\lambda_k\, e^{i\lambda_k A_k} = \prod_k (2\pi)\delta(A_k)$.

$$= \int D\lambda \, \exp\left[-\frac{i}{2g_0^2}\int d^2x\,\lambda\right]\left\{\det\left(\partial^2+\lambda\right)\right\}^{-N/2}$$

$$= \int D\lambda \, \exp\left[-\frac{i}{2g_0^2}\int d^2x\,\lambda - \frac{N}{2}\mathrm{Tr}\ln\left(\partial^2+\lambda\right)\right]. \tag{5}$$

Now with $g_0^2 N$ fixed, we take $N \to \infty$. Because $g_0^2 N$ is fixed both the first and second terms in the exponential are of order N, and it consequently makes sense to evaluate the integral by steepest descent. In steepest descent we approximate an exponential integral, by finding the maximum value of the integrand. For the above case where the exponential has an argument proportional to $-N$, we want to find the value of $\lambda(x)$ which minimizes this argument. In essence, we have

$$Z = \exp\left[-\frac{i}{2g_0^2}\int d^2x\,\overline{\lambda} - \frac{N}{2}\mathrm{Tr}\ln\left(\partial^2+\overline{\lambda}\right)\right], \tag{6}$$

where $\overline{\lambda}(x)$ is a solution to the functional differential equation

$$\frac{\delta}{\delta\lambda(x)}\left[-\frac{i}{2g_0^2}\int d^2x\,\lambda - \frac{N}{2}\mathrm{Tr}\ln\left(\partial^2+\lambda\right)\right] = 0 \tag{7}$$

Attempting to find a solution to this equation, gives us the condition

$$\frac{N}{2}\mathrm{Tr}\frac{1}{\partial^2+\lambda} = -\frac{i}{2g_0^2}. \tag{8}$$

Expressing the left hand side in momentum space gives us

$$\begin{aligned}
\frac{i}{2g_0^2} &= \frac{N}{2}\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2-\lambda} \\
&= i\frac{N}{2}\int\frac{d^2\overline{k}}{(2\pi)^2}\frac{1}{\overline{k}^2-\lambda} \\
&= i\int_0^{\Lambda}\frac{d\overline{k}}{(2\pi)^2}(2\pi\overline{k})\frac{1}{\overline{k}^2-\lambda} \\
&= i\frac{N}{8\pi}\int_0^{\Lambda^2}du\frac{1}{u-\lambda} \\
&= i\frac{N}{8\pi}\ln\frac{\Lambda^2}{-\lambda}.
\end{aligned} \tag{9}$$

From the form of the last line, (and also the appearance of λ in the differential operator), we can define λ as a mass term according to

$$\lambda \equiv -m^2, \tag{10}$$

and we then find that this mass term depends on the ultraviolet renormalization scale Λ as

$$\ln\frac{\Lambda}{m} = \frac{2\pi}{Ng_0^2}. \tag{11}$$

Λ as a renormalization scale is theoretically infinite. We can make this result finite by allowing introducing a renormalization term. If we modify our original lagrangian to be

$$\mathcal{L} = -\frac{1}{2g^2}\partial_\mu n^i(x)\partial^\mu n^i(x) - \frac{1}{2}\delta_g^2\partial_\mu n^i(x)\partial^\mu n^i(x) \tag{12}$$

where g is a renormalized coupling and δ_g^2 is meant to subtract the infinite term, then going through the same exact manipulations as above, we would end up with the condition

$$\frac{N}{2\pi} \ln \frac{\Lambda}{m} = \frac{1}{g^2} + \delta_g^2. \tag{13}$$

Thus, defining δ_g^2 so that it removes the infinite scale Λ we have

$$\delta_g^2 = \frac{N}{2\pi} \ln \frac{\Lambda}{M}, \tag{14}$$

where M is an arbitrary renormalization scale. Inserting Eq.(14) into Eq.(13), we thus find that m depends on the renormalization mass scale M according to [2]

$$m = M \exp\left(-\frac{2\pi}{Ng^2}\right). \tag{17}$$

This mass term still leaves the original non linear sigma model $O(N)$ invariant because it preserves the VEV $\langle \vec{n} \rangle = 0$.

Now that we know the value of λ, we can complete our evaluation of the functional integral. We have

$$Z = \exp\left[\frac{im^2}{2g_0^2}\text{Vol}_2 - \frac{N}{2}\text{Tr}\ln\left(\partial^2 - m^2\right)\right] \tag{18}$$

Understanding Questions

– $m(g^2, M)$ is not a correlation function. Why can we apply the Callan-Symanzik equation in this case?

– What happens to the $1/g_0^2$ in the differential operator for the fields n_k? In one derivation they can be factored out of the operator, but then they contribute to a divergent term in the integral.

– So we evaluated the functional integral. What does that mean? What does a clear evaluation of the integral give us?

– So the mass m is not constant but depends on our renormalization scale? Let's solve for g. How exactly does m depend on renormalization scale?

2 Gross-Neveu Model

The Gross-Neveu Model demonstrates how certain fermionic field theories can lose their chiral symmetry when made quantum mechanical. For this model, the field theory exists in two-dimensions and is defined by the lagrangian

$$\mathcal{L} = \overline{\psi}_j i\partial\!\!\!/ \psi_j + \frac{1}{2}g^2(\overline{\psi}_j\psi_j)^2, \tag{19}$$

[2]The Callan-Symanzik equation for this theory is

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial M}\right] m(g^2, M) = 0 \tag{15}$$

which we can use with Eq.(17), to show that

$$\beta(g) = -\frac{g^3 N}{4\pi}, \tag{16}$$

which is the same value as that computed in [1].

where $j = 1, \ldots, N$. The kinetic term of the fermions is built from matrices γ^μ that satisfy a two-dimensional Dirac Algebra:

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \tag{20}$$

where σ^i are the Pauli sigma matrices. Defining γ^5 as

$$\gamma^5 = \gamma^0\gamma^1 = \sigma^3, \tag{21}$$

gives us a γ^5 which anticommutes with γ^μ.

We then see that Eq.(19) is invariant under the chiral transformation $\psi_j \to \gamma^5\psi_j$. We show this by first noting that $\overline{\psi}_i \to \overline{\gamma^5\psi_i} = \overline{\psi}_i\overline{\gamma^5}$, and with

$$\begin{aligned}
\overline{\gamma^5} &= \beta(\gamma^5)^\dagger\beta \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (22) \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^5, \qquad (23)
\end{aligned}$$

where the first line comes from Eq. 38.3 of [2] we have

$$\overline{\psi}_i \to -\overline{\psi}_i\gamma^5, \tag{24}$$

Thus a mass term transforms as $\overline{\psi}_i\psi_i \to -\overline{\psi}_i\gamma^5\gamma^5\psi_i = -\overline{\psi}_i\psi_i$ and is not invariant, while a quartic term, $(\overline{\psi}_i\psi_i)^2$, is invariant.

A theory is renormalizable if it does not contain any coupling constants of negative mass dimensionality ([2]). The partial derivative has mass dimension $[\partial_\mu] = +1$, and for $d = 2$ our lagrangian has mass dimensionality $[\mathcal{L}] = +2$. Thus we find the mass dimensionality of the fermion field is

$$[i\overline{\psi}_i\partial\!\!\!/\psi_i] = +2 \quad \Longrightarrow \quad [\psi_i] = +1/2. \tag{25}$$

Similarly we find

$$+2 = [g^2(\overline{\psi}_i\psi_i)^2] = 2[g] + 2\,[\overline{\psi}_i\psi_i] = 2[g] + 2. \tag{26}$$

And thus $[g] = 0$. Since the coupling constant has zero mass dimensionality, the theory is renormalizable.

Our goal is to study the quantum mechanical properties of the theory by computing the effective potential. We can achieve this by introducing a non-propagating auxiliary field into the lagrangian. For the lagrangian

$$\mathcal{L}_\sigma = -i\overline{\psi}_i\partial\!\!\!/\psi_i - \sigma\overline{\psi}_k\psi_k - \frac{1}{2g^2}\sigma^2, \tag{27}$$

we find that solving the Euler-Lagrange equation for σ yields

$$0 = \frac{\partial\mathcal{L}}{\partial\sigma} = -\overline{\psi}_j\psi_j - \frac{1}{g^2}\sigma \quad \Longrightarrow \quad \sigma = -g^2\overline{\psi}_j\psi_j, \tag{28}$$

which when re-substituted into Eq.(27) becomes Eq.(19).

To study Eq.(27) quantum mechanically, we perform the functional integral over only the fermion fields. Doing so we have

$$\begin{aligned}
&\int D\overline{\psi}_m D\psi_m \,\exp\left[i\int d^4x\,\left(-\overline{\psi}_k\partial\!\!\!/\psi_k - \sigma\overline{\psi}_k\psi_k\right)\right] \\
&= \int D\overline{\psi}_m D\psi_m \,\exp\left[i\int d^2x\,d^2y\,(\overline{\psi}_k)_\alpha(\mathcal{S}^{km})^{\alpha\beta}(\psi_m)_\beta\right]
\end{aligned}$$

$$\propto (\det\mathcal{S}) = \exp\left[\mathrm{Tr}\ln\mathcal{S}\right] \tag{29}$$

Where in the second line Roman indices stand for fermion number indices, and greek indices stand for spinor indices. The trace "Tr" is a trace over spinor indices, fermion indices, and momentum/position space coordinates. In position space $\mathcal{S}$ can be written as

$$\mathcal{S} = (-i\partial\!\!\!/_x - \sigma)\delta^2(x-y), \tag{30}$$

and in momentum space it becomes

$$\mathcal{S} = (-p\!\!\!/ - \sigma)(2\pi)^2\delta^2(p-q). \tag{31}$$

Computing the determinant of $\mathcal{S}$ in momentum space we find

$$\begin{aligned}
\det\mathcal{S} &= \exp\left[\sum_{k=1}^{N}\mathrm{Tr}_{\text{mom.}}\mathrm{tr}_{\text{spin}}\ln(-p\!\!\!/ - \sigma)(2\pi)^2\delta^2(0)\right] \\
&= \exp\left[N\mathrm{Vol}_2\int\frac{d^2p}{(2\pi)^2}\mathrm{tr}\ln(-p\!\!\!/ - \sigma)\right] \\
&= \exp\left[N\mathrm{Vol}_2\int\frac{d^2p}{(2\pi)^2}\ln(p^2+\sigma^2)\mathrm{tr}\frac{\mathbb{I}}{2}\right] \\
&= \exp\left[N\mathrm{Vol}_2\int\frac{d^2p}{(2\pi)^2}\ln(p^2+\sigma^2)\right].
\end{aligned} \tag{32}$$

To compute the integral in the brackets we employ the standard dimensional reduction algorithm to obtain

$$\begin{aligned}
\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+A} &= i\int\frac{d^2\overline{p}}{(2\pi)^2}\frac{1}{\overline{p}^2+A} \\
&= \frac{i\Gamma(1-d/2)\Gamma(d/2)}{(4\pi)^{d/2}\Gamma(1)\Gamma(d/2)}A^{-(1-d/2)} \\
&= \frac{i\Gamma(1-d/2)}{(4\pi)^{d/2}}A^{-1+d/2}.
\end{aligned} \tag{33}$$

Integrating this result gives us

$$\begin{aligned}
\int\frac{d^2p}{(2\pi)^2}\ln(p^2+A) &= \frac{i\Gamma(1-d/2)}{(4\pi)^{d/2}}\frac{A^{d/2}}{d/2} \\
&= \frac{i\Gamma(-d/2)}{(4\pi)^{d/2}}A^{d/2} \\
&= i\Gamma(-1+\epsilon/2)\left(\frac{A}{4\pi}\right)^{1-\epsilon/2} \\
&= i\frac{(-1)}{1}\left[\frac{2}{\epsilon}-\gamma+1+\mathcal{O}(\epsilon)\right]\left[1-\frac{\epsilon}{2}\ln\left(\frac{A}{4\pi}\right)+\mathcal{O}(\epsilon)\right]\frac{A}{4\pi} \\
&= i\frac{A}{4\pi}\ln\frac{A}{4\pi} - i\frac{A}{4\pi}\frac{2}{\epsilon} - i\frac{A}{4\pi}(1-\gamma).
\end{aligned} \tag{34}$$

In the minimal subtraction renormalization scheme the $1/\epsilon$ term is eliminated. Thus, we find (after renormalization), the determinant

$$\det\mathcal{S} = \exp\left[N\mathrm{Vol}_2\int\frac{d^2p}{(2\pi)^2}\ln(p^2+\sigma^2)\right]$$

$$= \exp\left[iN\mathrm{Vol}_2\left(\frac{\sigma^2}{4\pi}\ln\frac{\sigma^2}{4\pi} - \frac{\sigma^2}{4\pi}(1-\gamma)\right)\right]. \tag{35}$$

Including the original $\sigma^2/2g^2$ potential and equating this result to the effective potential definition ,

$$\exp\left[-i\mathrm{Vol}_2 V^{\mathrm{eff}}(\sigma)\right], \tag{36}$$

we find

$$V^{\mathrm{eff}}(\sigma) = \frac{\sigma^2}{2g^2} - N\frac{\sigma^2}{4\pi}\ln\frac{\sigma^2}{4\pi} + N\frac{\sigma^2}{4\pi}(1-\gamma). \tag{37}$$

Computing the critical point of this condition, gives us

$$0 = \partial V^{\mathrm{eff}}(\sigma)/\partial\sigma = \sigma\left[\frac{1}{g^2} - N\left\{\frac{1}{2\pi}\ln\frac{\sigma^2}{4\pi} - \frac{1}{2\pi}\gamma\right\}\right]. \tag{38}$$

This critical condition is solved when $\sigma^0 = 0$ or for

$$\sigma_0^2 = 4\pi\exp\left[2\pi\left(\frac{1}{Ng^2} + \frac{\gamma}{2\pi}\right)\right]. \tag{39}$$

Checking the equilibrium condition by computing the second derivative of the potential,

$$\partial^2 V^{\mathrm{eff}}(\sigma)/\partial\sigma^2 = \frac{1}{g^2} - N\left\{\frac{1}{2\pi}\ln\frac{\sigma^2}{4\pi} - \frac{\gamma}{2\pi}\right\} - \frac{N}{\pi}, \tag{40}$$

we see that only the $\sigma_0^2 \neq 0$ solution is stable. Thus $V^{\mathrm{eff}}(\sigma)$ has a local minimum at Eq.(39). This means that the vacuum state of Eq.(27) has a non-zero mass term (from quantum corrections) and thus the lagrangian breaks chiral symmetry.

When we incorporate higher order corrections into the effective potential calculation, we find a potential of the form

$$V^{\mathrm{eff}}(\sigma) = Nf(\sigma) + Ng^2h(\sigma) + \cdots, \tag{41}$$

where $f(\sigma)$ is the one-loop term we already calculated, $h(\sigma)$ is a two-loop term, and the ellipses stand in for even higher-loop order terms. In the limit of $N \to \infty$ with g^2N constant we see the two-loop term becomes subdominant; this trend can be argued to continue for even higher-loop order terms.

Thus we have showed that a large number of fermions coupled through a 4-fermion interaction, lead to quantum corrections to the effective action which break the chiral symmetry which was present in the classical lagrangian.

- **Contradiction?** To show that σ obtained a VEV, we integrated out the Dirac fields and thus eliminated them from the theory. But then we claimed that σ obtaining a VEV lead to the Dirac fields obtaining a mass term and thus breaking the original chiral symmetry. But we just integrated out the Dirac fields! Why would it matter that quantum corrections lead to σ obtaining a VEV when the fermionic agrangian is no longer relevant given that fermions have been integrated out of the theory?

One resolution could come from changing the renormalization scheme. In a Wilsonian renormalization scheme we could imagine integrating out the fermion fields within a certain energy scale and thus we would still get a VEV for σ while still having fermions in the theory.

3 CP^N Model

A complex projective space CP^N can be defined as the space of $N+1$-dimensional complex vectors of the form $(z_1,\ldots,z_{N+1})$ subject to the condition

$$\sum_j |z_j|^2 = 1 \tag{42}$$

with points related by an overall phase rotation acting as a symmetry. That is

$$(e^{i\alpha}z_1,\ldots,e^{i\alpha}z_{N+1}) \quad \text{is identified with}(z_1,\ldots,z_{N+1}) \tag{43}$$

We study the two dimensional field theory with coordinates on this space.

One theory involving the fields $z_j(x)$ is defined by a lagrangian where the fields are subject to the above mentioned constraint and also have the local symmetry

$$z_j(x) \to e^{i\alpha(x)} z_j(x) \tag{44}$$

independently at each point x. We can demonstrate that a lagrangian which bears this symmetry is

$$\mathcal{L} = -\frac{1}{g^2}\left[|\partial_\mu z_j|^2 - |z_j^*\partial_\mu z_j|^2\right]. \tag{45}$$

According to Eq.(44), we see that the derivatives of the fields z_j transform as

$$\partial_\mu z_j \to (\partial_\mu z_j + i\partial_\mu\alpha\, z_j)e^{i\alpha}, \tag{46}$$

and the complex conjugate of the field multiplied by the derivative transform as Therefore the first term transforms as

$$\begin{aligned} z_j^*\partial_\mu z_j &\to z_j^* e^{-i\alpha}(\partial_\mu z_j + i\partial_\mu\alpha\, z_j)e^{i\alpha} \\ &= z_j^*\partial_\mu z_j + i\partial_\mu\alpha. \end{aligned} \tag{47}$$

Now, we can consider the transformation of each term in Eq.(45) in turn. For the first term we have

$$\begin{aligned} |\partial_\mu z_j|^2 &\to (\partial_\mu z_j^* + i\partial_\mu\alpha\, z_j^*)e^{-i\alpha}(\partial_\mu z_j + i\partial_\mu\alpha\, z_j)e^{i\alpha} \\ &= |\partial_\mu z_j|^2 + i\partial_\mu\alpha\left(\partial_\mu z_j^*\, z_j - z_j^*\partial_\mu z_j\right) + \partial^\mu\alpha\,\partial_\mu\alpha. \end{aligned} \tag{48}$$

And for the second term we have

$$\begin{aligned} |z_j^*\partial_\mu z_j|^2 &\to \left(z_j^*\partial_\mu z_j + i\partial_\mu\alpha\right)\left(\partial_\mu z_j^*\, z_j - i\partial_\mu\alpha\right) \\ &= |z_j^*\partial_\mu z_j|^2 + i\partial_\mu\alpha\left(\partial_\mu z_j^*\, z_j - z_j^*\partial_\mu z_j\right) + \partial^\mu\alpha\,\partial_\mu\alpha. \end{aligned} \tag{49}$$

Thus it is clear that for the entire lagrangian we have the transformation

$$|\partial_\mu z_j|^2 - |z_j^*\partial_\mu z_j|^2 \to |\partial_\mu z_j|^2 - |z_j^*\partial_\mu z_j|^2, \tag{50}$$

and hence Eq.(45) is invariant under Eq.(44).

As an aside, we can show that the CP^N model for $N=1$ is equivalent to the NLSM for $N=3$. We do so by defining the NLSM fields as

$$n^k = z_i^*\sigma_{ij}^k z_j. \tag{51}$$

Where σ^k are the Pauli matrices which satisfy the normalization condition $\text{tr}[\sigma^k \sigma^k] = 3$. The lagrangian for the NLSM is

$$\mathcal{L} = \frac{1}{2g^2} \partial_\mu n^k \, \partial_\mu n^k \qquad \text{subject to the constraint} \quad (n^k n_k = 1) \tag{52}$$

Computing this kinetic term in terms of the CP^N model fields, we find

$$\begin{aligned} \partial_\mu n^k \, \partial_\mu n^k &= \left((\partial_\mu z_i^*) \sigma_{ij}^k z_j + z_i^* \sigma_{ij}^k \partial_\mu z_j \right) \left((\partial_\mu z_m^*) \sigma_{mn}^k z_n + z_m^* \sigma_{mn}^k \partial_\mu z_n \right) \\ &= (\partial_\mu z_i^*) \sigma_{ij}^k z_j \, (\partial_\mu z_m^*) \sigma_{mn}^k z_n + (\partial_\mu z_i^*) \sigma_{ij}^k z_j \, z_m^* \sigma_{mn}^k \partial_\mu z_n \\ &\quad z_i^* \sigma_{ij}^k (\partial_\mu z_j) \, (\partial_\mu z_m^*) \sigma_{mn}^k z_n + z_i^* \sigma_{ij}^k (\partial_\mu z_j) \, z_m^* \sigma_{mn}^k \partial_\mu z_n. \end{aligned} \tag{53}$$

To reduce this result we need to make use of the completeness relation for pauli matrices:

$$\sigma_{ij}^k \sigma_{mn}^k = \delta_{in} \delta_{mj} - \frac{1}{2} \delta_{ij} \delta_{mn}. \tag{54}$$

Analyzing each term in the lagrangian we have

$$\begin{aligned} (\partial_\mu z_i^*) z_j \, (\partial_\mu z_m^*) z_n \sigma_{ij}^k \sigma_{mn}^k &= (\partial_\mu z_i^*) z_j \, (\partial_\mu z_m^*) z_n \left(\delta_{in} \delta_{mj} - \tfrac{1}{2} \delta_{ij} \delta_{mn} \right) \\ &= (\partial_\mu z_i^* \, z_i)^2 - \tfrac{1}{2} (\partial_\mu z_i^* z_i)^2 \\ &= \tfrac{1}{2} (\partial_\mu z_i^* \, z_i)^2 \end{aligned} \tag{55}$$

$$\begin{aligned} (\partial_\mu z_i^*) z_j \, z_m^* (\partial^\mu z_n) \sigma_{ij}^k \sigma_{mn}^k &= (\partial_\mu z_i^*) z_j \, z_m^* (\partial^\mu z_n) \left(\delta_{in} \delta_{mj} - \tfrac{1}{2} \delta_{ij} \delta_{mn} \right) \\ &= |\partial_\mu z_j|^2 - \tfrac{1}{2} |z_i^* \partial_\mu z_i|^2 \end{aligned} \tag{56}$$

$$\begin{aligned} z_i^* (\partial_\mu z_j) (\partial^\mu z_m^*) z_n \sigma_{ij}^k \sigma_{mn}^k &= z_i^* (\partial_\mu z_j) (\partial^\mu z_m^*) z_n \left(\delta_{in} \delta_{mj} - \tfrac{1}{2} \delta_{ij} \delta_{mn} \right) \\ &= |\partial_\mu z_j|^2 - \tfrac{1}{2} |z_i^* \partial_\mu z_i|^2 \end{aligned} \tag{57}$$

$$\begin{aligned} z_i^* (\partial_\mu z_j) z_m^* (\partial^\mu z_n) \sigma_{ij}^k \sigma_{mn}^k &= z_i^* (\partial_\mu z_j) z_m^* (\partial^\mu z_n) \left(\delta_{in} \delta_{mj} - \tfrac{1}{2} \delta_{ij} \delta_{mn} \right) \\ &= (z_i^* \partial^\mu z_i)^2 - \tfrac{1}{2} (z_i^* \partial_\mu z_i)^2 \\ &= \tfrac{1}{2} (z_i^* \partial^\mu z_i)^2 \end{aligned} \tag{58}$$

Summing these terms and using the identity $z_j^* \partial_\mu z_j = -(\partial_\mu z_j^*) z_j$ (derived from the normalization constraint), we find

$$\begin{aligned} \frac{1}{2g^2} \partial_\mu n^k \, \partial_\mu n^k &= \frac{1}{2g^2} \left[-\tfrac{1}{2} |\partial_\mu z_i^* \, z_i|^2 + 2 |\partial_\mu z_j|^2 - |z_i^* \partial_\mu z_i|^2 - \tfrac{1}{2} |z_i^* \partial^\mu z_i|^2 \right] \\ &= \frac{1}{g^2} \left[|\partial_\mu z_j|^2 - |z_i^* \partial_\mu z_i|^2 \right], \end{aligned} \tag{59}$$

which is Eq.(45).

Returning to Eq.(45), we can write the lagrangian in a simpler form by introducing two lagrange multiplier fields: a scalar field λ and a vector field A_μ. Our new lagrangian is

$$\mathcal{L} = -\frac{1}{g^2} \left[|D_\mu z_j|^2 + \lambda (|z_j|^2 - 1) \right] \tag{60}$$

where the covariant derivative D_μ is defined as $D_\mu = \partial_\mu + iA_\mu$. Writing Eq.(104) in terms of the vector field A_μ, we have

$$\mathcal{L} = -\frac{1}{g^2}\left[|\partial_\mu z_j|^2 + iA_\mu\left[(\partial^\mu z_j^*)z_j - z_j^*(\partial^\mu z_j)\right] + A^\mu A_\mu + \lambda(|z_j|^2 - 1)\right] \tag{61}$$

We can check that Eq.(104) reproduces Eq.(45) by solving the equations of motion for each lagrange multiplier field. For λ, we get

$$0 = \frac{\delta}{\delta\lambda(x)}S = |z_j(x)|^2 - 1, \tag{62}$$

which establishes the normalization constraint. For A_μ we have

$$0 = \frac{\delta}{\delta A_\mu(x)}S = -\frac{1}{g^2}i\left[(\partial^\mu z_j^*)z_j - z_j^*(\partial^\mu z_j)\right] - \frac{2}{g^2}A^\mu, \tag{63}$$

which given the normalization constraint yields

$$A^\mu = iz_j^*\partial^\mu z_j. \tag{64}$$

Incorporating these constraints into Eq.(61), we find

$$\begin{aligned}\mathcal{L} &= -\frac{1}{g^2}\left[|\partial_\mu z_j|^2 + iA_\mu\left[(\partial^\mu z_j^*)z_j - z_j^*(\partial^\mu z_j)\right] + A^\mu A_\mu + \lambda(|z_j|^2 - 1)\right]\\ &= -\frac{1}{g^2}\left[|\partial_\mu z_j|^2 - 2iA_\mu z_j^*(\partial^\mu z_j) + A^\mu A_\mu\right]\\ &= -\frac{1}{g^2}\left[|\partial_\mu z_j|^2 + (z_j^*\partial^\mu z_j)^2\right]\\ &= -\frac{1}{g^2}\left[|\partial_\mu z_j|^2 - |z_j^*\partial^\mu z_j|^2\right]\end{aligned} \tag{65}$$

as we should.

Now we want to solve this theory by computing the partition function associated with it. To this end it will be useful to integrate by parts the covariant kinetic term in Eq.(104). The calculation proceeds as follows

$$\begin{aligned}|D_\mu z_j|^2 &= (\partial_\mu - iA_\mu)z_j^*\,(\partial^\mu + iA^\mu)z_j\\ &= \partial_\mu z_j^*(\partial^\mu + iA^\mu)z_j - iA_\mu z_j^*(\partial^\mu + iA^\mu)z_j\\ &= \partial_\mu\left[z_j^*(\partial^\mu + iA^\mu)z_j\right] - z_j^*\partial_\mu(\partial^\mu + iA_\mu) - iA_\mu z_j^*(\partial^\mu + iA^\mu)z_j\\ &= -z_j^*(\partial_\mu + iA_\mu)(\partial^\mu + iA^\mu)z_j = -z_j^*D^2 z_j\end{aligned} \tag{66}$$

where for the second to last in equality we eliminated the total divergence term based on the assumption the fields were localized in space. Thus Eq.(104) becomes

$$\mathcal{L} = \frac{1}{g^2}z_j^*\left(D^2 - \lambda\right)z_j + \frac{1}{g^2}\lambda. \tag{67}$$

Integrating over the z_j fields in the $N \to \infty$ limit we are left with the partition function

$$\int DA\,D\lambda\exp\left[-N\,\mathrm{Tr}\ln(D^2 - \lambda) + \frac{i}{g^2}\int d^2x\,\lambda\right]. \tag{68}$$

The prefactor of the trace in the exponentials argument should be $N+1$, but in the limit of $N \to \infty$ the $+1$ is unimportant. Integrating over both λ and A_μ, by computing the equations of motion, we have the system

$$\begin{aligned}
0 = \frac{\delta}{\delta\lambda} S &= N\,\mathrm{Tr}\frac{1}{D^2 - \lambda} + \frac{i}{g^2} \\
&= N \left\langle x \left| \frac{1}{(\partial_\mu + iA_\mu)^2 - \lambda} \right| x \right\rangle + \frac{i}{g^2} \\
0 = \frac{\delta}{\delta A_\mu} S &= -N\,\mathrm{Tr}\frac{2D_\mu}{D^2 - \lambda} \\
&= -N \left\langle x \left| \frac{2(\partial_\mu + iA_\mu)}{(\partial_\mu + iA_\mu)^2 - \lambda} \right| x \right\rangle .
\end{aligned} \tag{69}$$

One possible solution contains the case for which $A_\mu = 0$. Here, we find that we can move these equations to momentum space and have the system

$$\begin{aligned}
N \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \lambda} &= \frac{i}{g^2} \\
N \int \frac{d^2k}{(2\pi)^2} \frac{2k_\mu}{k^2 + \lambda} &= 0
\end{aligned} \tag{70}$$

The second equation is automatically satisfied, and the first equation is same as the one we found in the NSLM. Thus we can take $A_\mu = 0$ and $\lambda = m^2 > 0$ as out solutions where

$$m = M \exp\left[-\frac{2\pi}{g^2 N}\right]. \tag{71}$$

We found the $A_\mu = 0$ solution of this quantum field theory, so let's investigate the first order corrections to this result. In particular, we will expand the argument of Eq.(68) to lowest nonzero order in A_μ and see what results we obtain. Key to this calculation is factoring our differential operator into A_μ independent and a A_μ dependent terms. We do this in the standard way. Defining the basis-less operator K as $K = D^2 - m^2$,we have

$$K(x,y) = \left[(\partial^\mu + iA^\mu)(\partial_\mu + iA_\mu) - m^2\right]_x \delta^2(x-y). \tag{72}$$

And we can show that $K(x,y)$ can be defined as

$$K(x,y) = \int d^2z\, K_0(x,z)\widetilde{K}(z,y), \tag{73}$$

where

$$K_0(x,z) = (\partial^2 - m^2)_x \delta^2(x-z) \tag{74}$$

$$\begin{aligned}
\widetilde{K}(z,y) &= \delta^2(z-y) - \left[iA^\mu\partial_\mu + i\partial_\mu A^\mu - A^\mu A_\mu\right]_y \Delta(z-y) \\
&= \delta^2(z-y) - \left[2iA^\mu\partial_\mu + i(\partial_\mu A^\mu) - A^\mu A_\mu\right]_y \Delta(z-y).
\end{aligned} \tag{75}$$

Aside: For completeness let's show why this is the case

$$\begin{aligned}
K(x,z) &= \left[(\partial^\mu + iA^\mu)(\partial_\mu + iA_\mu) - m^2\right]_x \delta^2(x-z) \\
&= \left[\partial^2 - m^2 + iA^\mu\partial_\mu + i\partial^\mu A_\mu - A^\mu A_\mu\right]_x \delta^2(x-z)
\end{aligned} \tag{76}$$

and so with $(\partial^2 - m^2)\Delta(x-y) = -\delta^2(x-y)$ we have

$$\begin{aligned}
K(x,z)\delta^2(z-y) &= (\partial^2 - m^2)_x \delta^2(x-z)\delta^2(z-y) \\
&\quad + \left[iA^\mu\partial_\mu + i\partial^\mu A_\mu - A^\mu A_\mu\right]_x \delta^2(x-z)\delta^2(z-y) \\
&= (\partial^2 - m^2)_z \delta^2(x-z)\delta^2(z-y) \\
&\quad - \left[iA^\mu\partial_\mu + i\partial^\mu A_\mu - A^\mu A_\mu\right]_y \delta^2(x-z)(\partial^2 - m^2)_z\Delta(z-y) \\
&= \delta^2(x-z)(\partial^2 - m^2)_z\Big[\delta^2(z-y) - \big(iA^\mu\partial_\mu + i\partial^\mu A_\mu - A^\mu A_\mu\big)_y\Delta(z-y)\Big] \qquad (77)
\end{aligned}$$

Now, with K defined as $K = K_0\widetilde{K}$ where K_0 is independent of A_μ, we can now compute the functional determinant with a focus on only the part relevant to A_μ:

$$-N\,\mathrm{Tr}\ln(D^2 - m^2) = -N\,\mathrm{Tr}\ln K = -N\,\mathrm{Tr}\ln K_0 - N\,\mathrm{Tr}\ln\widetilde{K}. \qquad (78)$$

Focusing on the second term and noting that $\widetilde{K}$ can be defined abstractly as

$$\widetilde{K} = I - H \qquad (79)$$

where

$$\langle x|H|y\rangle \equiv H(x,y) = \Big[2iA^\mu\partial_\mu + i(\partial_\mu A^\mu) - A^\mu A_\mu\Big]_y \Delta(x-y), \qquad (80)$$

we find

$$\begin{aligned}
-N\,\mathrm{Tr}\ln\widetilde{K} &= -N\,\mathrm{Tr}\ln(I-H) \\
&= N\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Tr}\,H^n. \qquad (81)
\end{aligned}$$

We will evaluate this action for the lowest non vanishing terms in A_μ. For a general n, the trace is defined as

$$\mathrm{Tr}\,H^n = \int d^2x_1\ldots d^2x_n\,\langle x_1|H|x_2\rangle\cdots\langle x_n|H|x_1\rangle. \qquad (82)$$

For the first term in Eq.(82), we have

$$\begin{aligned}
\mathrm{Tr}\,H &= \int d^2x\,\Big(2iA^\mu\partial_\mu + i(\partial_\mu A^\mu) - A^\mu A_\mu\Big)_x \Delta(x-x) \\
&= \int d^2x\,\Big(i(\partial_\mu A^\mu) - A^\mu A_\mu\Big)_x \Delta(0). \qquad (83)
\end{aligned}$$

Moving to momentum space with the definition

$$A^\mu(y) = \int \frac{d^2k}{(2\pi)^2} e^{iky}\tilde{A}^\mu(k), \qquad (84)$$

this result becomes

$$\begin{aligned}
\mathrm{Tr}\,H &= \int \frac{d^2k}{(2\pi)^2}\Big[-k_\mu\tilde{A}^\mu(k)(2\pi)^2\delta^2(k) - \tilde{A}^\mu(k)\tilde{A}_\mu(-k)\Big]\Delta(0) \\
&= -\int \frac{d^2k}{(2\pi)^2}\tilde{A}^\mu(k)\tilde{A}_\mu(-k)\Delta(0). \qquad (85)
\end{aligned}$$

For the second term in Eq.(82) (while keeping only the lowest order in A^μ terms) we have

$$
\begin{aligned}
\mathrm{Tr}\, H^2 &= \int d^2y\, d^2z\, \langle y|H|z\rangle\langle z|H|y\rangle \\
&= \int d^2y\, d^2z \Big(2iA^\mu\partial_\mu + i(\partial_\mu A^\mu) - A^\mu A_\mu\Big)_z \Delta(y-z) \\
&\quad \Big(2iA^\mu\partial_\mu + i(\partial_\mu A^\mu) - A^\mu A_\mu\Big)_y \Delta(z-y) \\
&= \int d^2y\, d^2z \Big(2iA^\mu\partial_\mu + i(\partial_\mu A^\mu)\Big)_z \Delta(y-z) \\
&\quad \Big(2iA^\mu\partial_\mu + i(\partial_\mu A^\mu)\Big)_y \Delta(z-y).
\end{aligned} \tag{86}
$$

We move to momentum space with the propagators

$$
\Delta(y-z) = \int \frac{d^2l}{(2\pi)^2} e^{il(y-z)}\tilde{\Delta}(l^2)\,, \qquad \Delta(z-y) = \int \frac{d^2p}{(2\pi)^2} e^{ip(y-z)}\tilde{\Delta}(p^2) \tag{87}
$$

and the vector fields

$$
A^\mu(z) = \int \frac{d^2k}{(2\pi)^2} e^{ikz}\tilde{A}^\mu(k)\,, \qquad A^\mu(y) = \int \frac{d^2q}{(2\pi)^2} e^{iqy}\tilde{A}^\mu(q). \tag{88}
$$

With these momentum space representations, we have

$$
\begin{aligned}
\Big(2iA^\mu\partial_\mu + i(\partial_\mu A^\mu)\Big)_z \Delta(y-z) &= \int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2} e^{il(y-z)}e^{ikz}\Big[+2l_\mu\tilde{A}^\mu(k) - \tilde{A}^\mu(k)k_\mu\Big]\tilde{\Delta}(l^2) \\
&= \int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2} e^{ily}e^{i(k-l)z}\,(2l-k)_\mu\tilde{A}^\mu(k)\tilde{\Delta}(l^2)
\end{aligned} \tag{89}
$$

and similarly

$$
\Big(2iA^\mu\partial_\mu + i(\partial_\mu A^\mu)\Big)_y \Delta(y-z) = \int \frac{d^2q}{(2\pi)^2}\frac{d^2p}{(2\pi)^2} e^{ipz}e^{i(q-p)y}\,(2p-q)_\nu\tilde{A}^\nu(q)\tilde{\Delta}(p^2). \tag{90}
$$

Assembling these results, we have

$$
\begin{aligned}
\frac{1}{2}\mathrm{Tr}\, H^2 &= \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\int d^2y\, d^2z\, e^{i(q+l-p)y}\, e^{i(k+p-l)z}\tilde{A}^\mu(k)\tilde{A}^\nu(q) \\
&\quad (2l-k)_\mu(2p-q)_\nu\tilde{\Delta}(l^2)\tilde{\Delta}(p^2) \\
&= \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2}\,\tilde{A}^\mu(k)\tilde{A}^\nu(-k)(2l-k)_\mu(2l-k)_\nu\,\tilde{\Delta}(l^2)\,\tilde{\Delta}\big((l-k)^2\big)
\end{aligned} \tag{91}
$$

Adding these first two terms thus yields

$$
\begin{aligned}
\mathrm{Tr}\, H + \frac{1}{2}\mathrm{Tr}\, H^2 &= \int \frac{d^2k}{(2\pi)^2}\tilde{A}^\mu(k)\tilde{A}^\nu(-k)\int \frac{d^2l}{(2\pi)^2}\left[-\frac{\eta_{\mu\nu}}{l^2+m^2} + \frac{1}{2}\frac{(2l-k)_\mu(2l-k)_\nu}{(l^2+m^2)((l-k)^2+m^2)}\right] \\
&\equiv \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\tilde{A}^\mu(k)\tilde{A}^\nu(-k)\,\Pi_{\mu\nu}(k^2),
\end{aligned} \tag{92}
$$

where we defined

$$\Pi_{\mu\nu}(k^2) = \int \frac{d^2 l}{(2\pi)^2} \left[-\frac{2\eta_{\mu\nu}}{l^2+m^2} + \frac{(2l+k)_\mu (2l+k)_\nu}{(l^2+m^2)((l+k)^2+m^2)} \right] \tag{93}$$

and performed a trivial change of variables. Now the only work that remains is to evaluate Eq.(93). To do so, we follow the standard algorithm. First employing Feynman parameters, we write

$$\frac{1}{(l^2+m^2)((l+k)^2+m^2)} = \int_0^1 dx\, \frac{1}{(q^2+D)^2}, \tag{94}$$

where

$$q \equiv l + xk\,, \qquad D \equiv k^2 x(1-x) + m^2. \tag{95}$$

Thus we have

$$\begin{aligned}
\Pi_{\mu\nu}(k^2) &= \int \frac{d^2 l}{(2\pi)^2} \left[\frac{-2\eta_{\mu\nu}((l+k)^2+m^2) + (2l+k)_\mu(2l+k)_\nu}{(l^2+m^2)((l+k)^2+m^2)} \right] \\
&= \int_0^1 dx \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2+D)^2} \left[-2\eta_{\mu\nu}((q+(x-1)k)^2+m^2) + (2q+(1-2x)k)_\mu (2q+(1-2x)k)_\nu \right] \\
&= \int_0^1 dx \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2+D)^2} \Big[-2(\eta_{\mu\nu} q^2 - 2q_\mu q_\nu) - 2\eta_{\mu\nu}[(x-1)^2 k^2 + m^2] + (1-2x)^2 k_\mu k_\nu \Big] \\
&= \int_0^1 dx \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2+D)^2} \Big[-2\eta_{\mu\nu} q^2 (1-2/d) + N^0_{\mu\nu} \Big]
\end{aligned} \tag{96}$$

With the integration identities for a euclidean distance $\bar{q}^2$ we have

$$\lim_{d\to 2^+} (1-2/d) \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{\bar{q}^2}{(\bar{q}^2+D)^2} = -\frac{1}{4\pi} \tag{97}$$

$$\int \frac{d^2 \bar{q}}{(2\pi)^2} \frac{1}{(\bar{q}^2+D)^2} = \frac{1}{4\pi D}, \tag{98}$$

and so $\Pi_{\mu\nu}$ becomes

$$\begin{aligned}
\Pi_{\mu\nu}(k^2) &= \frac{i}{4\pi} \int_0^1 dx \left[2\eta_{\mu\nu} + \frac{N^0_{\mu\nu}}{D} \right] \\
&= \frac{i\eta_{\mu\nu}}{2\pi} + \frac{i}{4\pi} \int_0^1 dx\, \frac{-2\eta_{\mu\nu}[(x-1)^2 k^2 + m^2] + (1-2x)^2 k_\mu k_\nu}{k^2(1/4 - y^2) + m^2}
\end{aligned} \tag{99}$$

Changing variables with $y = x - 1/2$, we have

$$\begin{aligned}
\Pi_{\mu\nu}(k^2) &= \frac{i\eta_{\mu\nu}}{2\pi} + \frac{i}{4\pi} \int_{-1/2}^{1/2} dy\, \frac{-2\eta_{\mu\nu}[(y-1/2)^2 k^2 + m^2] + 4y^2 k_\mu k_\nu}{k^2(1/4-y^2)+m^2} \\
&= \frac{i\eta_{\mu\nu}}{2\pi} + \int_{-1/2}^{1/2} dy\, \frac{4y^2 k_\mu k_\nu - 2\eta_{\mu\nu}[(y^2+1/4)k^2 + m^2]}{k^2(1/4-y^2)+m^2} \\
&= \frac{i\eta_{\mu\nu}}{2\pi} + \frac{i}{4\pi} \int_{-1/2}^{1/2} dy\, \left[\frac{4y^2 k_\mu k_\nu}{k^2(1/4-y^2)+m^2} - 2\eta_{\mu\nu} \frac{[2k^2 y^2 + (-y^2+1/4)k^2 + m^2]}{k^2(1/4-y^2)+m^2} \right] \\
&= \frac{i\eta_{\mu\nu}}{2\pi} - \frac{i\eta_{\mu\nu}}{2\pi} + \frac{i}{\pi}(k_\mu k_\nu - k^2 \eta_{\mu\nu}) \int_{-1/2}^{1/2} dy\, \left[\frac{y^2}{k^2(1/4-y^2)+m^2} \right]
\end{aligned}$$

$$= \frac{i}{\pi}(k_\mu k_\nu - k^2\eta_{\mu\nu})\frac{1}{k^2}\int_{-1/2}^{1/2} dy \left[\frac{y^2}{1/4 + m^2/k^2 - y^2}\right] \tag{100}$$

Evaluating the integral and then taking the $m \gg k$ limit we have

$$\begin{aligned}
\int_{-1/2}^{1/2} dy \left[\frac{y^2}{1/4 + m^2/k^2 - y^2}\right] &= \sqrt{1 + \frac{4m^2}{k^2}} \tanh^{-1}\left[\left(1 + \frac{4m^2}{k^2}\right)^{-1/2}\right] - 1 \\
&\simeq \tanh^{-1}\left(\frac{k}{2m}\right)\left(\frac{2m}{k}\right) - 1 \\
&= \left(\frac{k}{2m} + \frac{1}{3}\left(\frac{k}{2m}\right)^3 + \mathcal{O}(k^5/m^5)\right)\frac{2m}{k} - 1 \\
&= \frac{1}{3}\left(\frac{k}{2m}\right)^2 + \mathcal{O}(k^4/m^4).
\end{aligned} \tag{101}$$

So we have finally,

$$\begin{aligned}
\Pi_{\mu\nu}(k^2) &= \frac{i}{\pi}(k_\mu k_\nu - k^2\eta_{\mu\nu})\frac{1}{k^2}\left[\frac{1}{3}\left(\frac{k}{2m}\right)^2 + \mathcal{O}(k^4/m^4)\right] \\
&= \frac{i}{12\pi m^2}(k_\mu k_\nu - k^2\eta_{\mu\nu})\left[1 + \mathcal{O}(k^2/m^2)\right].
\end{aligned} \tag{102}$$

Now with Eq.(92), Eq.(82), and Eq.(78) we find that the contribution to the action from this expansion is

$$i\frac{N}{12\pi m^2}\frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\tilde{A}^\nu(-k)\Big(k_\nu k_\mu - \eta_{\mu\nu}k^2\Big)\tilde{A}^\mu(k) = -i\int d^2x \left(\frac{N}{3\pi m^2}\right)\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{103}$$

And thus we see that the kinetic term for the vector field was generated through quantum corrections to the action. The best way to prove this equality is actually to begin on the right hand side and use multiple integration by parts to obtain the left hand side.

Reverse constructing the analysis, we thus see that the low energy quantum corrected lagrangian for the CPN model can be written as

$$\mathcal{L} = -\frac{1}{g^2}\left[|D_\mu z_j|^2 + m^2|z_j|^2\right] - \left(\frac{N}{3\pi m^2}\right)\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{104}$$

We see then that although we began with a classical theory of many *massless* complex scalars interacting through the lagrangian

$$\mathcal{L} = \frac{1}{g^2}\Big[|\partial_\mu z_i|^2 - |z_j^*\partial_\mu z_j|^2\Big], \tag{105}$$

when we made the theory quantum mechanical we found that it was equivalent to a theory of many *massive* complex scalars interacting with a massless abelian gauge field (Eq.(104)).

3.1 Remarks

- **Hidden Local Symmetry:** This work seems to be a simple example of the more general ideas termed "Hidden Local Symmetry" (HLS) in which a theory without any gauge fields can be equated to a theory which includes gauge fields. However, the HLS protocol seems to be implemented at tree level and is fundamentally connected to spontaneous symmetry breaking, two features which are not manifestly present in the CP^N model.

- **Necessity of a Gauge Field:** The central message of the CP^N model seems to be that in gauge invariant theories, there must be an *effective* (i.e., quantum mechanical) gauge field even when there is no such field classically. The CP^N procedure is as follows

  1. We begin with a NLSM with complex scalar fields instead of real scalar fields.
  2. The complex scalar fields bear a global $U(1)$ symmetry, which we want to promote to a local symmetry.
  3. But we don't want to introduce a gauge field into the classical lagrangian. Instead, we include a constraint in the lagrangian which allows us to bypass the inclusion of classical gauge fields.
  4. However (!), we find that when we make the theory quantum mechanical we get a gauge field anyway.

  In other words,

  > **the quantum manifestation of a classical gauge symmetry seems to necessitate the existence of a propagating gauge field.**

  We showed this in the case of a $U(1)$ symmetry. There should be a manifestation of this result for non-abelian gauge theories.

- **Implications for Induced Gravity:** There is an analogous idea for general relativity termed "Induced Gravity" ([3]). The idea is we start with a field theory without any propagating (and hence, physical) gravitational field. We then mandate diffeomorphism invariance and through quantum corrections we induce gravity (or something like it) quantum mechanically. Thus, we find that **the quantum manifestation of diffeomorphism invariance necessitates the existence of a propagating gravitational field.**

  – This seems like an OK starting assumption but it is inadequate as a means of justifying the ideas behind induced gravity. For one, the gravitational theory which results from Induced Gravity does not appear to be the same theory as Einstein's Gravity (See [4] for criticisms).

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