Central Force Motion - With and Without Relativity

In these notes we consider the central force problem in non-relativistic and relativistic contexts. The central force in this case is a fixed Newtonian potential and using the lagrangian formalism we obtain the equations of motion, the trajectory solution and Kepler's laws. We apply a similar procedure to the relativistic problem (without the entire formalism of general relativity) and show that the orbit precesses.

1 Non-Relativistic Central Force Problem

We begin with the action for a particle of mass m in the presence of the gravitational field of a much larger (and stationary) mass M:

$$S = \int_{t_0}^{t_1} dt\, \left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}\right], \tag{1}$$

where G is the gravitational constant, θ is the polar angle, and r is the distance between m and M in the $x-y$ plane. The assumption that M is stationary is not necessary to the problem; we chose it because it yields the same solution as the general case with less algebra. Now, applying the Euler-Lagrange equations to the integrand of the action above, we find the dynamical equations

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0\,; \qquad m\ddot{r} = mr\dot{\theta}^2 - \frac{GMm}{r^2}. \tag{2}$$

The first equation is a constraint which states that the angular momentum of our system is conserved. Defining L as the angular momentum and $J \equiv L/m$ as the angular momentum per unit mass we have the condition

$$J = r^2\dot{\theta} \qquad \text{(Conservation of Angular Momentum)}. \tag{3}$$

Using this result to eliminate $\dot{\theta}$ in the second equation in Eq.(2) and moving all the terms to one side we find

$$m\ddot{r} - \frac{mJ^2}{r^3} + \frac{GMm}{r^2} = 0. \tag{4}$$

Multiplying this equation by $\dot{r}$ and extracting a time derivative then gives us

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\dot{r}^2 + \frac{J^2}{r^2}\right) - \frac{GMm}{r}\right] = 0. \tag{5}$$

This condition is another conservation equation. From prior experience we know that the quantity in the parentheses is the total energy of our system. Defining E as this total energy and $\epsilon \equiv E/m$ as the energy per unit mass m, we have the condition

$$\epsilon = \frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \frac{\alpha}{r} \qquad \text{(Conservation of Energy)} \tag{6}$$

where we defined $\alpha \equiv GM$. With Eq.(6) and Eq.(3) we can now orbtain a differential equation for our orbital motion in terms of coordinate variables. Solving for $\dot{r}^2$ in Eq.(6) we have

$$\left(\frac{dr}{dt}\right)^2 = 2\epsilon + \frac{2\alpha}{r} - \frac{J^2}{r^2}.$$

Multiplying this equation by $(dt/d\theta)^2$ as defined by Eq.(3) we have

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{J^2}\left(2\epsilon + \frac{2\alpha}{r} - \frac{J^2}{r^2}\right), \tag{7}$$

which is the orbital equation we were looking for. This equation is easily solved by a sequence of redefinitions. First we define $u \equiv 1/r$. This changes Eq.(7) to

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2\epsilon}{J^2} + \frac{2\alpha}{J}u - u^2.$$

Completing the square on the right hand side above gives us

$$\left(\frac{du}{d\theta}\right)^2 = -\left(u - \frac{\alpha}{J^2}\right)^2 + \frac{\alpha^2}{J^4} + \frac{2\epsilon}{J^2}.$$

Defining the term in the parentheses as y and the sum of the second and third terms on the RHS as A we have the differential equation

$$\left(\frac{dy}{d\theta}\right)^2 + y^2 = A^2.$$

This equation is easily solved by recognizing it is similar to the definition of energy for a simpler harmonic oscillator. With this analogy we then find

$$y(\theta) = A\cos(\theta - \theta_0) \tag{8}$$

where

$$A = \frac{\alpha}{J^2}\sqrt{1 + \frac{2\epsilon J^2}{\alpha^2}}. \tag{9}$$

Tracing the sequence of definitions back to $r(\theta)$ we find the solution

$$r(\theta) = \frac{J^2/\alpha}{1 + e\cos(\theta)} \tag{10}$$

where we set $\theta_0 = 0$ because the starting angle of the trajectory is not important and

$$e = \sqrt{1 + \frac{2\epsilon J^2}{\alpha^2}} \tag{11}$$

is the eccentricity of the orbit. We note that for elliptical (i.e. bound) orbits the total energy of the system is negative. This is the main result for the non relativistic central force problem. In the next subsection we will use the collection of results derived in this section to derive Kepler's laws.

1.1 Kepler's Laws

Kepler's three laws are as follows:

(1) **Elliptical Orbits:** The orbit of a particle forms an elliptical conic section over one period of motion.

(2) **Equal Areas in Equal Times:** The orbit of a particle with its focus at the origin sweeps out equal areas of the face of the ellipse over equal times.

(3) **Period Squared $\propto$ Major axis cubed:** The period of the orbit of a particle is proportional to the cube of the length of the major axis with a proportionality constant independent of the properties of the particle.

We will show how the results we obtained in the previous section produce each of the three laws above in turn. The first law is manifest in Eq.(10). Plotting this result parametrically in the $x-y$ plane we have the image depicted in Fig. 1. A specific requirement for the Eq.(10) to define an ellipse is that the eccentricity Eq.(11) must be greater than zero but less than 1. That is the energy per unit mass of the system is negative and must satify

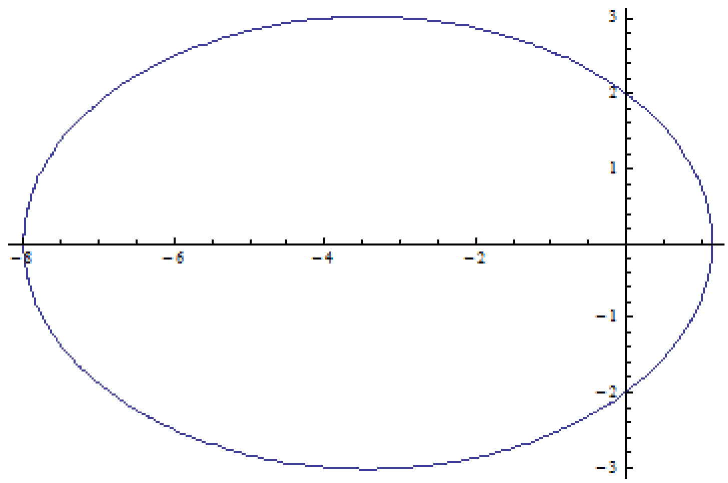

Figure 1: Plot of the orbit defined by Eq.(10) in the $x-y$ plane: The figure depicts a parametric plot of $\vec{r}(t) = r(t)(\cos t, \sin t)$ where $r(t) = 2/(1 + 3/4 \cos t)$ a form analogous to Eq.(10)

$$-\frac{\alpha^2}{2J^2} < \epsilon < 0. \tag{12}$$

If the energy hits the lower limit ($\epsilon = -G^2M^2/2J^2$), the orbit is a circle and if the orbit hits the upper limit ($\epsilon = 0$) the orbit is an unbound parabola. Below the lower limit, the particle's orbit is not closed and it eventually hits the source of the gravitational field. Above the upper limit the the particle's orbit is an unbound hyperbola.

We derive Kepler's second law by integrating Eq.(3). From the geometry of the polar coordinate system, we know that the infinitesimal area of a polar plot which spans an angle $d\theta$ is

$$dA = \frac{1}{2} r(\theta)^2 d\theta. \tag{13}$$

Using this result Eq.(3) becomes

$$\frac{dA}{dt} = \frac{J}{2} \tag{14}$$

which means that the rate at which the spanned area changes during an orbit is constant in time. Integrating this equation gives

$$A(t) = \frac{J}{2} t. \tag{15}$$

We can interpret this result as the statement "If a particle travels for a time Δt then the polar area it covers is $\Delta A = J\Delta t/2$ no matter where the particle is in its orbital trajectory.

There are at least two routes we can go through to derive the third law. Using Eq.(3) and Eq.(10) we can try to compute the period explicitly as a function of the orbital parameters and then try to find a way to

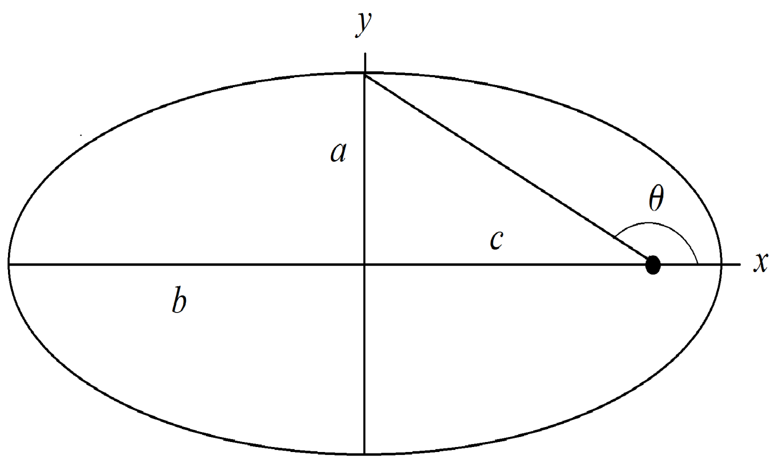

Figure 2: Relationship between major and minor axes: The semi-major axes is b and the semi-minor axes is a. c is the distance between the focal point and the center of the ellipse.

relate these parameters to the geometry of the orbit. By this route the period of the orbit is

$$T = \frac{1}{J}\int_0^{2\pi} d\theta \left(\frac{J^2/\alpha}{1+e\cos\theta}\right)^2 . \tag{16}$$

But this integral is tedious to compute even with algebraic manipulation software.

The other route uses a previous result to skip the computation of this integral. Integrating Eq.(14) over the entire period of the orbit we find

$$T = \frac{2}{J}A \tag{17}$$

where A is the total area covered by the particle over a full orbit. That is A is the total area of the ellipse. From planar geometry we know that the total area of an ellipse πab where a is the length of the semi-minor axis and b is the length of the semi major axis. Therefore we have

$$T = \frac{2}{J}\pi ab. \tag{18}$$

This is equivalent to the result Eq.(16). Now what we need to do is reduce Eq.(18) to something akin to Kepler's third law. We do this by relating the elliptical parameters a and b to the physical parameters J and ϵ which define Eq.(10).

An example elliptical orbit is shown in Fig. 2. From the figure we see that the length of the semi-major axis b is

$$b = \frac{r(\theta=0)+r(\theta=\pi)}{2} = \frac{J^2}{\alpha}\frac{1}{1-e^2} \tag{19}$$

where we obtained the last line by using Eq.(10). We can find the length of the semi-minor axis by defining a vector $\vec{R}$ which has its origin at the center of the ellipse. This vector is then defined in terms of θ as

$$\vec{R} = c\hat{x} + r(\theta)\cos\theta\hat{x} + r(\theta)\sin\theta\hat{y} \tag{20}$$

where c is the length between the focus and the center of the ellipse. The length of the semi-minor axis is then the magnitude of $\vec{R}$ when $\vec{R}$ has no x component. To find this magnitude we must first compute the value of c. Using the geometry of the figure we find

$$c = b - r(\theta = 0) = \frac{r(\theta = \pi) - r(\theta = 0)}{2} = \frac{J^2}{\alpha}\frac{e}{1-e^2}. \tag{21}$$

Now, the value of θ which makes $\vec{R}$ an exclusively y component vector is defined by the condition

$$\frac{c}{r(\theta_1)} = -\cos\theta_1 \tag{22}$$

Substituting Eq.(21) and Eq.(10) into this condition then yields

$$\frac{e}{1-e^2}(1 + e\cos\theta_1) = -\cos\theta_1 \quad \Longrightarrow \quad \cos\theta_1 = -e. \tag{23}$$

We can now determine the value of a. From Eq.(20) we have

$$a \equiv |\vec{R}(\theta_1)| = \frac{J^2}{\alpha}\frac{\sin\theta_1}{1+e\cos\theta_1} = \frac{J^2}{\alpha}\frac{1}{\sqrt{1-e^2}}. \tag{24}$$

The values of b and a can be further reduced by making use of Eq.(11). Doing so we find

$$b = \frac{\alpha}{2|\epsilon|}, \qquad a = \frac{J}{\sqrt{2|\epsilon|}}. \tag{25}$$

Writing a in terms of b gives us $a = J\sqrt{b/\alpha}$, so that Eq.(18) becomes $T = 2\pi b\sqrt{b/\alpha}$ and squaring it gives us

$$T^2/b^3 = \frac{4\pi^2}{\alpha}, \tag{26}$$

which is Kepler's third law.

2 Relativistic Central Force Problem

Now that we have obtained a solution for the non-relativistic central force problem we now turn to the relativistic case. For this case our primary concern will be to see how the orbit of a relativistic particle differs from that of a non-relativistic particle in the same potential. We begin as we did in the previous section with the appropriate lagrangian. A particle of mass m in the presence of the Newtonian scalar field Φ has the action

$$\mathcal{A} = -mc^2 \int d\tau \left(1 + \frac{\Phi}{c^2}\right) = -mc^2 \int \sqrt{-\frac{\eta_{\mu\nu}}{c^2}dx^\mu dx^\nu}\left(1 + \frac{\Phi}{c^2}\right), \tag{27}$$

[NEED TO JUSTIFY THIS LAGRANGIAN] where τ is the proper time, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$, and c is the speed of light. Restricting our particle motion to exist in a plane and choosing polar coordinate variables we have the action

$$\mathcal{A} = -mc^2 \int \sqrt{dt^2 - dr^2 - r^2 d\theta^2}\left(1 + \frac{\Phi(r)}{c^2}\right), \tag{28}$$

where $\dot{q} = dq/dt$ and we made the potential's dependence on r explicit. We know from standard newtonian gravity that this potential is $\Phi(r) = -GM/r$ but for generality we do not substitute this value yet. Now, working some voodoo calculus. We can factor out $c\,dt$ from the square root to obtain a more useful form of the action. We have

$$\mathcal{A} = -mc^2 \int_0^t dt\, B(r)\sqrt{1 - (\dot{r}^2 + r^2\dot{\theta}^2)/c^2}\,, \tag{29}$$

where

$$B(r) = 1 + \frac{\Phi(r)}{c^2}. \tag{30}$$

As a check, we note that upon expanding Eq.(29) to zeroth order in $1/c^2$ we obtain Eq.(1). To obtain the orbital equation for this relativistic case we need to obtain the dynamical equations defined by the lagrangian. Implementing the standard Euler-Lagrange algorithm we find the differential equations

$$\frac{d}{dt}\left[mr^2\dot{\theta}\gamma B(r)\right] = 0 \tag{31}$$

$$\frac{d}{dt}\left[m\dot{r}\gamma B(r)\right] = mc^2\left(\frac{r\dot{\theta}^2}{c^2}\gamma B(r) - \frac{B'(r)}{\gamma}\right) \tag{32}$$

where $\gamma^{-1} \equiv \sqrt{1-(\dot{r}^2 + r^2\dot{\theta}^2)/c^2}$. Eq.(31) is a conservation equation. Setting the quantity within the parentheses to be $J\,m$ where J is the angular momentum per unit pass and noting that $\Delta t/\gamma = \Delta\tau$ the proper time of the particle we have

$$\gamma\dot{\theta} = \frac{d\theta}{d\tau} = \frac{J}{r^2B(r)}. \tag{33}$$

Multiplying Eq.(32) by γ and inserting the above result we then find

$$\frac{d}{d\tau}\left[B(r)\frac{dr}{d\tau}\right] = \frac{J^2}{B(r)r^3} - c^2B'(r). \tag{34}$$

This differential equation can be presented in conservation (i.e. integral) form by multiplying through with an integration factor. The Inspection suggests that the appropriate factor is $B(r)dr/d\tau$. Multiplying Eq.(34) by this factor we find

$$B^2(r)\frac{dr}{d\tau}\frac{d^2r}{d\tau^2} + B(r)B'(r)\left(\frac{dr}{d\tau}\right)^3 = \frac{J^2}{r^3}\frac{dr}{d\tau} - c^2B(r)B'(r)\frac{dr}{d\tau} \tag{35}$$

or

$$\frac{d}{d\tau}\left[\frac{1}{2}B^2(r)\left(\frac{dr}{d\tau}\right)^2\right] = \frac{d}{d\tau}\left[-\frac{J^2}{2r^2} - \frac{1}{2}c^2B^2(r)\right]. \tag{36}$$

Moving all the terms to one side, integrating the equation, and setting the integration constant to be ϵ, the energy per unit mass of the orbit we find the conservation condition

$$\epsilon = \frac{1}{2}B^2(r)\left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{2r^2} + \frac{1}{2}c^2B^2(r) \tag{37}$$

Now Eq.(33) and Eq.(37) represent fairly simplified forms of the dynamical equations, but what we really want is an equation analogous to Eq.(7) which we can solve to give the geometrical trajectory of the particle. We get this by using Eq.(33) to eliminate the τ variable in Eq.(7). Taking $d\tau = d\theta\, r^2B(r)/J$, and simplifying we find

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{J^2}\left(2\epsilon - \frac{J^2}{r^2} - c^2B^2(r)\right). \tag{38}$$

This result can be further simplified by defining $u \equiv 1/r$ to yield

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2\epsilon}{J^2} - u^2 - \frac{c^2}{J^2}B^2(1/u). \tag{39}$$

Eq.(38) (or its equivalent, Eq.(39)) is the general orbital equation for a relativistic particle moving in a plane in the presence of a gravitational potential $\Phi(r) = c^2(B(r) - 1)$. Later we will use this general equation to compute the properties of orbital motion when the potential is not purely the Newtonian gravity result. But for the case we are considering now we can substitute in the definition $\Phi(r) = -\alpha/r$. This replacement gives us

$$\begin{aligned}\left(\frac{du}{d\theta}\right)^2 &= \frac{2\epsilon}{J^2} - u^2 - \frac{c^2}{J^2}(1 - \alpha u/c^2)^2. \\ &= -\left(1 + \frac{\alpha^2}{c^2 J^2}\right) u^2 + \frac{2\alpha}{J^2} u + \frac{2\epsilon}{J^2} - \frac{c^2}{J^2}. \end{aligned} \tag{40}$$

Completing the square on the RHS, we get

$$\left(\frac{dy}{d\theta}\right)^2 = -(1+\delta)^2 y^2 + A_\delta^2 \tag{41}$$

where

$$(1+\delta)^2 \equiv 1 + \frac{\alpha^2}{c^2 J^2}, \qquad y \equiv u - \frac{\alpha/J^2}{(1+\delta)^2}, \qquad A_\delta \equiv \frac{\alpha/J^2}{1+\delta}\sqrt{1 + \frac{(2\epsilon - c^2)J^2}{\alpha^2}(1+\delta)^2}. \tag{42}$$

Solving Eq.(41) yields, simply,

$$y(\theta) = A_\delta \cos[(1+\delta)(\theta - \theta_0)],$$

where θ_0 is an unimportant initial condition. Tracking back through our sequence of redefinitions we ultimately find for $r(\theta)$

$$r(\theta) = \frac{(1+\delta)^2 J^2/\alpha}{1 + (1+\delta)e_\delta \cos[(1+\delta)\theta]}, \tag{43}$$

where

$$e_\delta = \sqrt{1 + \frac{(2\epsilon - c^2)J^2}{\alpha^2}(1+\delta)^2}. \tag{44}$$

The main physical property we should take away from this solution is that the angular period for our motion is no longer 2π as it was in the non relativistic case. That is because the coefficient of θ in cosine's argument is not 1 the period is not $2\pi/1 = 2\pi$. Physically this means that when θ completes a full revolution in polar space and goes from $\theta = 0$ to $\theta = 2\pi$, the particle at $\theta = 2\pi$ does not return to its position at $\theta = 0$. Indeed, after one revolution the particle is offset from its original position by a small angle. This angle is called the rate of precession because it defines the angle by which our orbit precesses per revolution. We calculate it by computing the angular extent by which $r(\theta = 0)$ differs from $r(\theta = 2\pi)$. Computing the two quantities we find that the comparison comes down to a comparison of $\cos(2\pi)$ and $\cos(2\pi\delta)$. Since $\cos(2\pi) = 1$, we find that the offset angle incurred after each period of revolution is

$$\phi_{\text{preces.}} = 2\pi\delta = 2\pi\left(\sqrt{1 + \frac{\alpha^2}{c^2 J^2}} - 1\right) \simeq \pi \frac{\alpha^2}{c^2 J^2}. \tag{45}$$

We note that this quantity is purely relativistic and vanishes in the non-relativistic ($c \to \infty$) limit as we should expect. Fig. 3 shows what a precessional orbit looks like over a single revolution of θ. As we can see, the coordinates of the particle do not return to their positions at $\theta = 0$ and hence it would not be correct to define such an orbit as an ellipse even though the energy of the system is in the expected elliptical regime. In general Eq.(43) does not correspond to any conic section. It is also interesting to ask if the particle does not return to its original position after a single revolution of θ how many revolutions would it take for it to do so? This number is calculated simply by taking $r(\theta = 0) = r(\theta = 2\pi\, N)$ and solving for N. Doing so

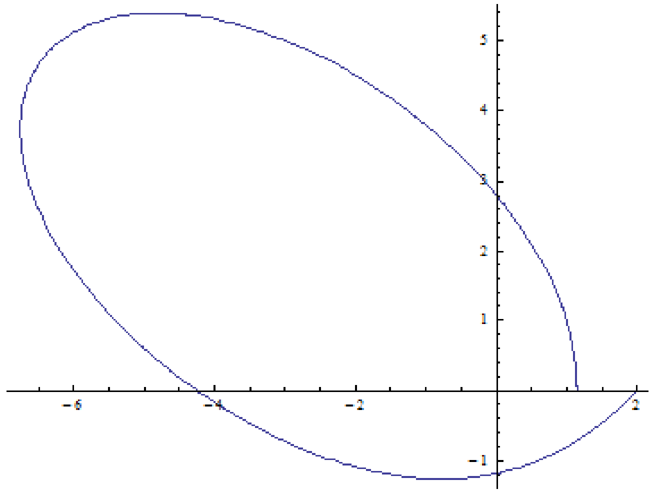

Figure 3: Precessional orbit: Orbit of a particle in $x - y$ space defined by $r(t) = 2/(1 + 3/4\ \cos[(1 + 1/4)t])$ plotted from $t = 0$ to $t = 2\pi$. The x intercept point closest to the origin defines $r(t = 0)$. We see it would not be correct to describe this orbit as an ellipse.

we find $N = 1/\delta$. For our plot in Fig. 3 this means after $N = 1/\delta = 4$ full angular revolutions we expect to return to our starting point. We indeed find this as we see by plotting multiple periods as in Fig. 4 Eq.(45) is the main physical result for this section. In later sections we will use the formalism developed here to compute the properties of orbit of a particle in a self-coupled scalar theory of gravity.

References

[1] T. Padmanabhan, *Gravitation: foundations and frontiers*. Cambridge University Press, 2010.

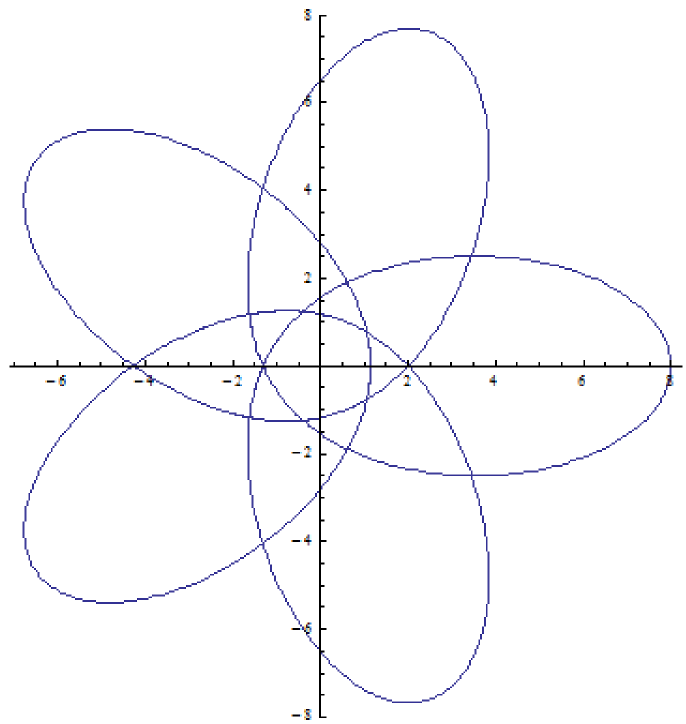

Figure 4: Precessional orbit many revolutions: Orbit of a particle in $x - y$ space defined by $r(t) = 2/(1 + 3/4\ \cos[(1 + 1/4)t])$ plotted from $t = 0$ to $t = 2\pi/(1/4) = 8\pi$.