Understanding Symmetry Factors

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(Dated: April 1, 2012)

(Version 1: 4/1/2012) In these notes we provide a more developed approach to symmetry factors than is found in current textbooks. The intent is leave with a clear understanding of why symmetry factors are necessary whenever we represent perturbative expansions in terms of simple diagrams.

I. INTRODUCTION

In these notes we will provide analytical justification for the symmetry factors which are routinely associated with the diagrammatic representation of $n-$point function and interaction processes in statistical field theory. In such diagrams the symmetry factors arise from rearrangements of propagators which can be equivalently obtained by rearrangements of vertices, or vice versa. Since these two types of rearrangements are equivalent there is often an overcounting factor when we transcribe the diagram of the process into an analytical expression. This overcounting factor of course would not be present if we began with the analytic result and derived the expression for a process without the use of diagrams. This analytical derivation is what we provide below.

Before we begin a detailed analysis of certain example diagrams we must first discuss some necessary preliminaries. We will restrict our attention to a $'\phi^{3}'$ theory and we will replace the continuous integrals with discrete summations, noting that translating between the two is simple. Then, every term which is represented by a diagram comes from the perturbative expansion of the interacting vacuum-to-vacuum transition amplitude $Z[L]$:

$$\frac{Z[L]}{Z[L=0]} = \exp\left[\frac{h}{3!}\sum_r \left(\frac{\partial}{\partial L_r}\right)^3\right] \times \exp\left[\frac{1}{2}\sum_{l,m} L_l L_m M^{-1}_{lm}\right] \tag{1}$$

$$= \sum_{V=0}^{\infty}\frac{1}{V!}\left[\frac{h}{3!}\sum_r \left(\frac{\partial}{\partial L_r}\right)^3\right]^V \times \sum_{P=0}^{\infty}\frac{1}{P!}\left[\frac{1}{2}\sum_{l,m} L_l L_m M^{-1}_{lm}\right]^P \tag{2}$$

where although we have used a multiplication sign we mean to say that the source derivatives *act* on the sources. Each diagram that we can draw is composed of V vertices, P propagators, and $E = 2P - 3V$ external sources. The convention for the subsequent analysis is that we form diagrams and select the particular terms from the above expansion which corresponds to our diagram.

In our derivation of symmetry factors we must decide between two computationally equivalent conventions: we can either keep the order of the propagators fixed and then combinatorially rearrange the vertex derivatives to act on the corresponding sources, or we can keep the vertices fixed and rearrange the sources with their conjoining propagators. A set choice must be made in order to ensure we are not overcounting contributions, a mistake which would inevitably always leave us with a symmetry factor of 1. In these notes we will choose the former convention in which the order of the sources and propagators are fixed and the vertices are rearranged.

In the subsequent discussion we will introduce combinatorial factors which partially eliminate factors from the exponential expansion. How exactly are these additional combinatorial factors generated? Whenever we choose a particular ordering of derivatives applied to the sources we multiply our result by a combinatorial factor to account for all of the other equivalent orderings we did not choose. The most basic application of this idea arises when we apply a single source derivative to a two-source-one-propagator (2S-1P) term:

$$\frac{\partial}{\partial L_r}\frac{1}{2}\sum_{k,l} L_k L_l M^{-1}_{kl} = \frac{1}{2}\times 2\sum_l L_l M^{-1}_{rl} \tag{3}$$

$$= \sum_l L_l M^{-1}_{rl}. \tag{4}$$

There are two possible sources upon which the derivative can be applied and we must consequently multiply our final result by two after we choose one of the sources to differentiate. A more rigorous interpretation of the generation of such factors is that they arise from a simple application of the product rule from calculus. The above result always applies: the first derivative applied to any such two source-one propagator term always generates a factor of 2 which cancels the factor of $\frac{1}{2}$ in front of the sum. Consequently, in the subsequent analysis, we will not explain the cancellations of these $\frac{1}{2}$ factors, only removing them when a derivative is applied to the corresponding source.

Another application of the above idea arises when we have three derivatives each of which must act on one source in a collection of three 2S-1P terms. In this case the combinatorial factor is easily written down.

$$\left(\frac{\partial}{\partial L_r}\frac{\partial}{\partial L_r}\frac{\partial}{\partial L_r}\right)\frac{1}{2}\sum_{i,j}L_iL_jM_{ij}^{-1}\times\frac{1}{2}\sum_{k,l}L_kL_lM_{kl}^{-1}$$
$$\times\frac{1}{2}\sum_{m,n}L_mL_nM_{mn}^{-1}$$
$$=3!\sum_j L_jM_{rj}^{-1}\times\sum_l L_lM_{rl}^{-1} \quad (5)$$
$$\times\sum_n L_nM_{rn}^{-1} \quad (6)$$

With these results established we can now move on to a computation of the values of certain diagrams including their associated symmetry factors.

II. DIAGRAMS AND THEIR ANALYTIC EXPRESSIONS

A. Example 1

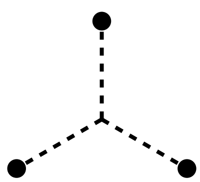

FIG. 1: $V = 1$, $P = 3$, $E = 3$

This diagram has $V = 1$ and $P = 3$ so the term in the expansion which corresponds to this diagram is

$$\frac{h}{3!}\sum_r\left(\frac{\partial}{\partial L_r}\right)^3\times\frac{1}{3!}\times\frac{1}{2}\sum_{i,j}L_iL_jM_{ij}^{-1}$$
$$\times\frac{1}{2}\sum_{k,l}L_kL_lM_{kl}^{-1}\times\frac{1}{2}\sum_{m,n}L_mL_nM_{mn}^{-1} \quad (7)$$

For convention we keep the propagators fixed and move around the vector derivatives that are applied to them. Ignoring the factors of 2 which cancel the factors of 1/2 in each 2S-1P term we find that a factor of 3! is generated from the number of equivalent ways of arranging the 3 identical source derivatives such that the analytic expression corresponding to the diagram is produced. The final result is

$$\frac{h}{3!}\sum_r\sum_j\sum_l\sum_n L_jL_lL_nM_{rj}^{-1}M_{rl}^{-1}M_{rn}^{-1} \quad (8)$$

Thus indicating that the symmetry factor associated with this diagram is 3!. This value can be intuited from the diagram by arguing that a permutation of the external sources in the diagram leaves the associated analytical result unchanged and consequently generates a symmetry factor of $\boxed{3!}$.

B. Example 2

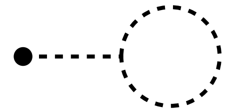

FIG. 2: $V = 1$, $P = 2$, $E = 1$

The above diagram is asssociated with the term

$$\frac{h}{3!}\sum_r\left(\frac{\partial}{\partial L_r}\right)^3\times\frac{1}{2}\times\frac{1}{2}\sum_{i,j}L_iL_jM_{ij}^{-1}$$
$$\times\frac{1}{2}\sum_{k,l}L_kL_lM_{kl}^{-1} \quad (9)$$

To find the number of equivalent ways of ordering the derivatives we will use a brute force method of listing the possibilities. In this method we will denote the particular derivatives by r_1, r_2, and r_3 to differentiate among them even though they are equivalent by definition. Indeed it is their equivalence which results in a combinatorial factor. The possible applications of derivatives to sources are as follows

L_i	L_j	L_k	L_l
r_1	r_2	r_3	
r_1	r_3	r_2	
r_1		r_2	r_3
	r_3	r_1	r_2
r_3	r_2	r_1	
	r_2	r_1	r_3

In the above table r_k underneath a source means that derivative number k is applied to the source. We read the table horizontally from left to right to determine a possible ordering of derivatives on sources. From the table we see that there are six equivalent ways to order the derivatives. We can understand this result more intuitively by realizing that the first derivative has 2 possible sets of sources to act upon and the remaining two derivatives can act on the remaining sources in 3 ways to generate the diagram. Hence there are $3 \times 2 = 6$ equivalent ways of ordering the derivatives. Incorporating this combinatorial factor gives us the final expression for the diagram.

$$\frac{h}{2}\sum_r\sum_j L_jM_{rj}^{-1}M_{rr}^{-1} \quad (10)$$

Indicating that the symmetry factor is $\boxed{2}$. This symmetry factor can be seen from the diagram by realizing that we can switch the endpoints of a propagator making up the loop without changing the value of the diagram.

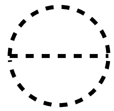

FIG. 3: $V = 2$, $P = 3$, $E = 0$

C. Example 3

The term which corresponds to the above diagram is

$$\frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r}\right)^3 \times \frac{h}{3!} \sum_s \left(\frac{\partial}{\partial L_s}\right)^3 \times \frac{1}{3!} \times \frac{1}{2} \times \sum_{i,j} L_i L_j M^{-1}_{ij} \times \frac{1}{2} \sum_{k,l} L_k L_l M^{-1}_{kl} \times \frac{1}{2} \sum_{m,n} L_m L_n M^{-1}_{mn}. \tag{11}$$

In this case the combinatorial factors are easily derived. There are 3!×3! equivalent ways of ordering the derivatives to generate the relevant expression. Each 3! corresponds to the number of ways of rearranging the derivatives associated with each vertex. The resulting expression is

$$\frac{1}{2} \times \frac{1}{3!} h^2 \sum_r \sum_s M^{-1}_{rs} M^{-1}_{rs} M^{-1}_{rs} \tag{12}$$

which tells us that the symmetry factor is $\boxed{2 \times 3!}$. We can understand the source of this value from the diagram. there are 3! equivalent ways to rearrange the propagators and we can switch the vertices (or equivalently flip the endpoints of the propagators) and end up with the same diagram.

D. Example 4

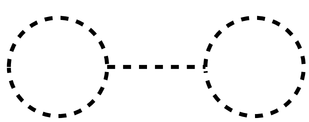

FIG. 4: $V = 2$, $P = 3$, $E = 0$

The term which corresponds to the above diagram is the same as the term in Example 3:

$$\frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r}\right)^3 \times \frac{h}{3!} \sum_s \left(\frac{\partial}{\partial L_s}\right)^3 \times \frac{1}{3!} \times \frac{1}{2} \times \sum_{i,j} L_i L_j M^{-1}_{ij} \times \frac{1}{2} \sum_{k,l} L_k L_l M^{-1}_{kl} \times \frac{1}{2} \sum_{m,n} L_m L_n M^{-1}_{mn}. \tag{13}$$

To generate one of the symmetric vertices in the figure we must differentiate two sources attached to the same propagator and another source attached to another propagator. This differentiation can be done in a number of equivalent orders. In this sense the derivatives associated with a single vertex act on one pair of two source one propagator (2S-1P) terms. This pair is distinguished by order so that there are $3 \times 2 = 6$ ordered pairs which can be chosen from the above three 2S-1P terms. Choosing a particular ordered pair at random we find that there are 3 equivalent derivative orderings:

L_i	L_j	L_k	L_l
r_1	r_2	r_3	
r_1	r_3	r_2	
r_1		r_2	r_3

Alternatively, we could claim that the pairs of 2S-1P terms need not be ordered in which case we would have 3 such terms and 6 possible derivative ordering for each term. In either case we find that the net combinatorial factor is 6×3. We then end up with the term

$$\frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r}\right)^3 \times \frac{h}{2} \sum_s M^{-1}_{ss} \sum_l L_l L_j M^{-1}_{sl} \times \frac{1}{2} \sum_{m,n} L_m L_n M^{-1}_{mn}. \tag{14}$$

To generate the last vertex we find tha there are 3 ways to order the corresponding derivatives

L_l	L_m	L_n
r_1	r_2	r_3
r_2	r_3	r_1
r_3	r_1	r_2

we say there are 3 ways instead of 3×2 because the ordering of the last two derivatives is irrelevant. With the application of the above derivatives we end up with the final result

$$\frac{h^2}{2^3} \times \sum_r \sum_s M^{-1}_{ss} M^{-1}_{sr} M^{-1}_{rr}, \tag{15}$$

indicating that we have a symmetry factor of 2^3. We can understand this result from the diagram: switching the end points of both propagators which make up the two loops (2×2) and switching the two vertices (2) results in a symmetry factor $2 \times 2 \times 2 = 2^3$.

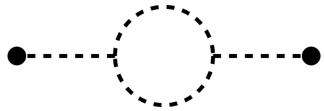

FIG. 5: $V = 2$, $P = 4$, $E = 2$

E. Example 5

The expression which corresponds to the above diagram is

$$\frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r}\right)^3 \times \frac{h}{3!} \sum_s \left(\frac{\partial}{\partial L_s}\right)^3 \times \frac{1}{4!} \times \frac{1}{2} \times \sum_{i,j} L_i L_j M_{ij}^{-1} \times \frac{1}{2} \sum_{k,l} L_k L_l M_{kl}^{-1} \times \frac{1}{2} \sum_{m,n} L_m L_n M_{mn}^{-1} \times \frac{1}{2} \sum_{g,h} L_g L_h M_{gh}^{-1}. \tag{16}$$

To generate one of the two equivalent vertices we must order three 2S-1P terms out of four. There are $4 \times 3 \times 2$ ways of doing this, thus yielding a combinatorial factor of 4! the resulting term is

$$\frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r}\right)^3 \frac{h}{3!} \times \frac{1}{2} \sum_s \sum_j L_j M_{sj}^{-1} \sum_l L_l M_{sl}^{-1} \sum_n L_n M_n^{-1} \times \sum_{g,h} L_g L_h M_{gh}^{-1}. \tag{17}$$

From the diagram we know that the last term above must be differentiated by one of the three source derivatives, so if we apply one derivative we must multiply our result by 3 to account for the other possible derivative choices. With a single derivative set to apply on the last term there are two derivatives left which can be applied to any of the three one source-one propagator (1S-1P) terms. There are 3×2 ways to order these two derivatives on the three 1S-1P terms. So the net combinatorial factor for this step is $3 \times 3 \times 2$. We therefore end up with the term

$$\frac{h}{2^2} \sum_r \sum_s \sum_h \sum_j L_j L_h M_{sj}^{-1} M_{sr}^{-1} M_{sr}^{-1} M_{rh}^{-1} \tag{18}$$

indicating that the symmetry factor is $\boxed{2^2}$. We can understand this result intuitively from the diagram by noting that we can switch internal propagators which make up the loop (2) and the the endpoints of the diagram (2) without changing the result.

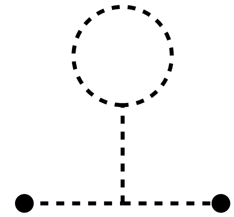

FIG. 6: $V = 2$, $P = 4$, $E = 2$

F. Example 6

The expression corresponding to the above diagram is the same as the one in Example 5

$$\frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r}\right)^3 \times \frac{h}{3!} \sum_s \left(\frac{\partial}{\partial L_s}\right)^3 \times \frac{1}{3!} \times \frac{1}{2} \times \sum_{i,j} L_i L_j M_{ij}^{-1} \times \frac{1}{2} \sum_{k,l} L_k L_l M_{kl}^{-1} \times \frac{1}{2} \sum_{m,n} L_m L_n M_{mn}^{-1} \times \frac{1}{2} \sum_{g,h} L_g L_h M_{gh}^{-1}. \tag{19}$$

The two vertices in the diagram are not equivalent, so before we begin constructing the analytical expression corresponding to a particular vertex choice we must realize that our result must be multiplied by a factor of 2 to account for the other ordering of vertices. We will produce the final expression for the diagram through two routes each of which begins with the construction of one of the vertices.

Beginning with the top vertex we see that we must choose two out of four of the 2S-1P terms to create the vertex. Claiming that the order of these pairs is not important we find that there are $\frac{4!}{2!2!} = 6$ ways of pairing these terms. In each pair there are $3 \times 2 = 6$ ways of ordering the derivatives to apply on the source terms. We can understand this result by listing the possible orderings for a particular pair of 2S-1P terms:

L_i	L_j	L_k	L_l
s_1	s_2	s_3	
s_1	s_3	s_2	
s_1		s_2	s_3
	s_3	s_1	s_2
s_3	s_2	s_1	
	s_2	s_1	s_3

After incorporating these combinatorial factors and multiplying the result by 2 to account for the opposite ordering of the vertices our expression is

$$2 \times \frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r}\right)^3 \times \frac{h}{4} \sum_s \sum_j L_j M_{sj}^{-1} M_{ss}^{-1} \times \frac{1}{2} \sum_{m,n} L_m L_n M_{mn}^{-1} \times \frac{1}{2} \sum_{g,h} L_g L_h M_{gh}^{-1}. \tag{20}$$

From the figure, we realize that one of the remaining derivatives must act on the 1S-1P term. There are three derivatives and therefore 3 ways to act on this term. The remaining two derivatives can act on the two 2S-1P terms in 2 orderings. The net combinatorial fact for this step is, then, $3 \times 2 = 6$ which when incorporated into the expression yields

$$\frac{h^2}{2^2} \sum_m \sum_h L_m L_h \sum_r \sum_s M_{sr}^{-1} M_{ss}^{-1} M_{mr}^{-1} M_{hr}^{-1} \tag{21}$$

indicating that the symmetry factor is $\boxed{2^2}$.

We can derive the above symmetry factor another way by beginning with the other vertex. To construct the bottom vertex from our original expression, we must order three derivatives applied to four possible sets of 2S-1P terms. There are $4 \times 3 \times 2$ ways to do this, thus yielding the combinatorial factor 4!. Including a factor of 2 to account for the other possible ordering of vertices, the resulting term is

$$2 \times \frac{1}{2} \times \frac{h}{3!} \sum_r \left(\frac{\partial}{\partial L_r} \right)^3 \times \frac{h}{3!} \sum_s \sum_j L_j M_{sj}^{-1} \sum_s \sum_l L_l M_{sl}^{-1} \sum_s \sum_n L_n M_{sn}^{-1} \times \frac{1}{2} \sum_{g,h} L_g L_h M_{gh}^{-1} \tag{22}$$

By inspection of the above diagram, we know that the last term above must be differentiate twice. There are 3 possible pairings of derivatives in this regard. The remaining derivative has three possible choices for sources thus giving us a combinatorial factor of 3 upon selection. Multiplying the above expression by the net factor of 9, for this step, and appropriately reducing the expression we find

$$\frac{h^2}{2^2} \sum_j \sum_n L_j L_n \sum_r \sum_s M_{sj}^{-1} M_{sr}^{-1} M_{sn}^{-1} M_{rr}^{-1} \tag{23}$$

which is the same expression we found in (23). The symmetry factor for this process can be intuited from the diagram by noting that we can switch the two sources (2) and the endpoints of the propagator which makes up the loop (2) without changing the diagram.

III. SYMMETRY FACTORS AS DERIVED FROM DIAGRAMS: A DISCUSSION

In the above discussion, after we showed how the numerical coefficients of the diagrammatic expression arose analytically we also claimed that these coefficients could be argued from the symmetry of the diagram. How is this so? How are the diagrammatic argument and the final form of the analytic expression connected? The answer exists in the explicit form of the argument of the exponential in the Gaussian integral. The coefficient of 1/2 in front of the quadratic term and the coefficient of 1/3! in front of the cubic term of the argument allow us to derive symmetry factors by a mere inspection of the relevant diagram because these factors force the symmetry factors associated with the simplest two-point and three-point diagrams to be 2 and 3! respectively. Consequently we can interpret these symmetry factors as arising from rearrangements of sources which do not change the diagram. This interpretation is extended to more complicated diagrams in which the rearrangement of propagator ends and/or sources (or sometimes equivalently, vertices and vertex derivatives) do not change the diagram and result in a numerical coefficient of 1/(symmetry factor) in the analytical expression of the diagram. We therefore claim that when certain rearrangements in a diagram do not result in a change in the diagram we divide by the symmetry of the rearrangement to obtain the correct analytical expression.

IV. CONCLUSION

[1] See, for example, R.L. Liboff, *Introductory Quantum Mechanics, 3rd Ed.* (Addison-Wesley, Reading, MA, 1998) problem 10.58, page 481.

[2] P. Morse and H. Feshbach, *Methods of Mathematical Physics* (McGraw-Hill, New York, 1953), page 1650.

[3] I learned these methods in conversation with Jeffrey Goldstone, who claims they are well known.