Physics 143a – Workshop 3

On General Angular Momentum Operators

Week Summary

- **Angular Momentum Operators and Basis states:** For the orthonormal basis states $\{|j,m\rangle\}$ where j represents the total-angular momentum[1] and m represents the z angular momentum of the state, we have the eigenvalue-eigenket relations

$$\hat{\boldsymbol{J}}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle \tag{1}$$
$$\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle, \tag{2}$$

  where $\hat{\boldsymbol{J}}^2$ is the total-angular momentum squared operator and $\hat{J}_z$ is the angular momentum operator in the z direction. We note that the values of m run from $m=-j$ to $m=j$ in integer steps, so that the state $|j,m\rangle$ has a $2j+1$ degeneracy with respect to $\hat{\mathbf{J}}^2$. The raising and lowering operators $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ act on these kets according to

$$\hat{J}_\pm|j,m\rangle = N^\pm_{j,m}|j,m\pm 1\rangle, \tag{3}$$

  where

$$N^+_{j,m} \equiv \hbar\sqrt{j(j+1)-m(m+1)} \tag{4}$$
$$N^-_{j,m} \equiv \hbar\sqrt{j(j+1)-m(m-1)} \tag{5}$$

  We note that Eq.(3) gives us $\hat{J}_+|j,m=j\rangle = 0$ and $\hat{J}_-|j,m=-j\rangle = 0$ as we expect.

- **Spin- j Angular Momentum:** The angular momentum operator $\hat{J}_z$ in the z basis states is given by the $(2j+1)\times(2j+1)$ matrix

$$\hat{J}_z = \hbar\begin{pmatrix} j & & & \\ & j-1 & & \\ & & \ddots & \\ & & & -j \end{pmatrix} \tag{6}$$

  In this basis, the raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ are

$$\hat{J}_+ = \begin{pmatrix} 0 & N^+_{j,\,j-1} & & \\ & 0 & \ddots & \\ & & \ddots & N^+_{j,\,-j} \\ & & & 0 \end{pmatrix}, \quad \hat{J}_- = \begin{pmatrix} 0 & & & \\ N^-_{j,\,j} & 0 & & \\ & \ddots & \ddots & \\ & & N^-_{j,\,-j+1} & 0 \end{pmatrix}. \tag{7}$$

  The $\hat{J}_x$ and $\hat{J}_y$ operators can then be determined from the equations

$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-), \quad \hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-). \tag{8}$$

[1] This value is always positive and currently in the course is equivalent to the spin of the particle we're studying. We note also that although j *represents* total angular momentum, it is not *equivalent to* total angular momentum. Total angular momentum is $\sqrt{j(j+1)}$ in units of $\hbar$.

1 Problems

1. **Spin-1 Angular Momentum**
Consider a system of angular momentum $j = 1$, whose state space is spanned by the basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ of three eigenvectors common to $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$. The state of the system is

$$|\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle, \tag{9}$$

where α, β, and γ are three given complex parameters.

   (a) What is the probability of finding $2\hbar^2$ and $\hbar$ if, respectively, total-angular momentum squared and z-angular momentum are measured one after the other?

   (b) Calculate the mean value of $\langle \mathbb{J} \rangle = (\langle \hat{J}_x \rangle, \langle \hat{J}_y \rangle, \langle \hat{J}_z \rangle)$in terms of α, β, and γ.

   (c) Give expressions for the mean values of $\langle \hat{J}_x^2 \rangle$, $\langle \hat{J}_y^2 \rangle$, and $\langle \hat{J}_z^2 \rangle$ in terms of the same quantities.

2 Solutions to Problems

(a) For simplicity we will assume that $|\psi\rangle$ is already normalized.

If total angular momentum squared is measured, the probability of getting $2\hbar^2$ is 1, since all of the eigenstates of $|\psi\rangle$ have $j = 1$ (corresponding to $j(j+1)\hbar^2 = 2\hbar^2$). Such a measurement does not disturb the state because $|\psi\rangle$ is an eigenstate of $\hat{\boldsymbol{J}}^2$. Thus after this first measurement, the system is still in the state $|\psi\rangle$.

If we then measure $\hat{J}_z$, the probability of getting $+\hbar$ is equal to the probability amplitude squared between the state $|1,1\rangle$ (i.e., the eigenstate corresponding to $+\hbar$) and $|\psi\rangle$. We thus have

$$\text{Prob}(J_z = +\hbar) = |\langle 1,1|\psi\rangle|^2 = |\alpha|^2. \tag{10}$$

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(b) To calculate the mean value of $\boldsymbol{J}$ we need to first compute the mean values of the x, y, and z angular momentum operators. This in turn requires us to compute the operators themselves. From Eq.(6), we know that the $\hat{J}_z$ operator (in the $|j,m\rangle$ basis) is

$$\hat{J}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{11}$$

Thus the average of J_z is

$$\langle\psi|\hat{J}_z|\psi\rangle = (\alpha^*, \beta^*, \gamma^*)\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \hbar\left(|\alpha|^2 - |\gamma|^2\right) \tag{12}$$

From Eq.(7), we know that the raising and lowering operators are

$$\hat{J}_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{J}_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{13}$$

And with Eq.(8), we obtain

$$\hat{J}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad \hat{J}_y = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}. \tag{14}$$

So the average value of J_x is

$$\begin{aligned} \langle\hat{J}_x\rangle &= (\alpha^*, \beta^*, \gamma^*)\frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \\ &= \frac{\hbar}{\sqrt{2}}(\alpha^*, \beta^*, \gamma^*) \begin{pmatrix} \beta \\ \alpha + \gamma \\ \beta \end{pmatrix} \\ &= \hbar\sqrt{2}\,\text{Re}\left[\beta(\alpha^* + \gamma^*)\right], \end{aligned} \tag{15}$$

and the average value of J_y is

$$\langle \hat{J}_y \rangle = (\alpha^*, \beta^*, \gamma^*) \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{16}$$

$$= \frac{\hbar}{i\sqrt{2}} (\alpha^*, \beta^*, \gamma^*) \begin{pmatrix} \beta \\ -\alpha + \gamma \\ -\beta \end{pmatrix}$$

$$= \hbar\sqrt{2}\, \mathrm{Im}\left[\beta(\alpha^* - \gamma^*)\right]. \tag{17}$$

Therefore

$$\langle \boldsymbol{J} \rangle = \hbar \left(\sqrt{2}\, \mathrm{Re}\left[\beta(\alpha^* + \gamma^*)\right], \sqrt{2}\, \mathrm{Im}\left[\beta(\alpha^* - \gamma^*)\right], |\alpha|^2 - |\gamma|^2 \right). \tag{18}$$

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(c) We follow a similar procedure as in the previous part, except now we compute the average of the square of each angular momentum operator . We find

$$\langle \psi | \hat{J}_z^2 | \psi \rangle = (\alpha^*, \beta^*, \gamma^*) \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \hbar \left(|\alpha|^2 + |\gamma|^2 \right), \tag{19}$$

$$\langle \hat{J}_x^2 \rangle = (\alpha^*, \beta^*, \gamma^*) \frac{\hbar^2}{4} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{\hbar^2}{2} \left[|\alpha|^2 + |\gamma|^2 + 2|\beta|^2 + 2\, \mathrm{Re}(\alpha^* \gamma) \right], \tag{20}$$

and

$$\langle \hat{J}_y^2 \rangle = (\alpha^*, \beta^*, \gamma^*) \frac{\hbar^2}{4} \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{\hbar^2}{2} \left[|\alpha|^2 + |\gamma|^2 + 2|\beta|^2 - 2\, \mathrm{Re}(\alpha^* \gamma) \right]. \tag{21}$$

We thus note that we find

$$\langle \hat{J}_x^2 \rangle + \langle \hat{J}_y^2 \rangle + \langle \hat{J}_z^2 \rangle = \hbar^2 \left[2|\alpha|^2 + 2|\gamma|^2 + |\beta|^2 \right] = 2\hbar^2. \tag{22}$$

as expected from $\boldsymbol{J}^2 |j, m\rangle = 2\hbar^2 |j, m\rangle$.

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30 min - Practice Exam

1. **(Eigenvalues and Eigenvectors)**
   Find the normalized eigenvalues and eigenvectors of the matrix

$$\hat{A} = \frac{1}{5}\begin{pmatrix} 4 & 3i \\ -3i & 4 \end{pmatrix}. \tag{23}$$

   Is $\hat{A}$ unitary? How do we know?

2. **(States and Measurements)**

$$|\phi\rangle = \frac{1+i}{\sqrt{6}}|+,\mathbf{z}\rangle + \frac{2}{\sqrt{6}}|-,\mathbf{z}\rangle. \tag{24}$$

What is $\Delta_\phi \hat{S}_x$ for this state?

3. **(Change of Bases)**
In the $|\pm, \mathbf{z}\rangle$ basis, the spin operator $\hat{\mathbf{S}} \cdot \mathbf{n}$ is written as

$$\left[\hat{\mathbf{S}} \cdot \mathbf{n}\right]_{\text{in } z \text{ basis}} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}. \tag{25}$$

Write explicitly in "bra and ket" notation the change of basis matrix U which would take $\hat{\mathbf{S}} \cdot \mathbf{n}$ to the $|\pm, \mathbf{y}\rangle$ basis with the formula

$$\left[\hat{\mathbf{S}} \cdot \mathbf{n}\right]_{\text{in } y \text{ basis}} = U \left[\hat{\mathbf{S}} \cdot \mathbf{n}\right]_{\text{in } z \text{ basis}} U^{\dagger}. \tag{26}$$

Compute U and find $\left[\hat{\mathbf{S}} \cdot \mathbf{n}\right]_{\text{in } y \text{ basis}}$. What would we expect for this result when $\theta = \pi/2$ and $\phi = \pi/2$?

4. **(Commutation Identities)**
If the quantity
$$AB[C,D] + A[B,D]C + [A,D]BC \tag{27}$$
(where A, B, C, and D are operators) can be reduced to the single commutator $[X_1, X_2]$, what are X_1 and X_2?

3 Answers to Exam

1. The eigenvalues are $\lambda_1 = 7/5$ and $\lambda_2 = 1/5$ and the corresponding eigenstates are

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{28}$$

   $\hat{A}$ is not unitary because $\hat{A}\hat{A}^\dagger \neq \mathbb{I}$.

2. $\Delta_\phi \hat{S}_x = \dfrac{\hbar\sqrt{5}}{6}$.

3.

$$\begin{aligned} U_{z\to y \text{ basis}} &= \begin{pmatrix} \langle +, \mathbf{y}|+, \mathbf{z}\rangle & \langle +, \mathbf{y}|-, \mathbf{z}\rangle \\ \langle -, \mathbf{y}|+, \mathbf{z}\rangle & \langle -, \mathbf{y}|-, \mathbf{z}\rangle \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \end{aligned} \tag{29}$$

$$\left[\hat{\mathbf{S}} \cdot \mathbf{n}\right]_{\text{in } y \text{ basis}} = \frac{\hbar}{2} \begin{pmatrix} \sin\theta\sin\phi & \cos\theta - i\sin\theta\cos\phi \\ \cos\theta + i\sin\theta\cos\phi & -\sin\theta\sin\phi \end{pmatrix}. \tag{30}$$

$$. \tag{31}$$

   For $\theta = \pi/2$ and $\phi = \pi/2$ we have $\mathbf{n} = \mathbf{y}$, so $\hat{\mathbf{S}} \cdot \mathbf{n} = \hat{S}_y$. Thus $\left[\hat{\mathbf{S}} \cdot \mathbf{n}\right]_{\text{in } y \text{ basis}}$ at $\theta = \pi/2$ and $\phi = \pi/2$ should have the same form as $\hat{S}_z$ in the z basis, namely

$$\left[\hat{\mathbf{S}} \cdot \mathbf{n}\right]_{\text{in } y \text{ basis}} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{32}$$

   This result is confirmed by Eq.(31).

4. $X_1 = ABC$ and $X_2 = D$.