Blackbody Radiation of the Eyeball

We consider Purcell's back of the envelope problem [1] concerning the blackbody radiation emitted by the human eye, and use the problem as an opportunity to develop the formalism of blackbody radiation and answer some basic conceptual questions about the formalism.

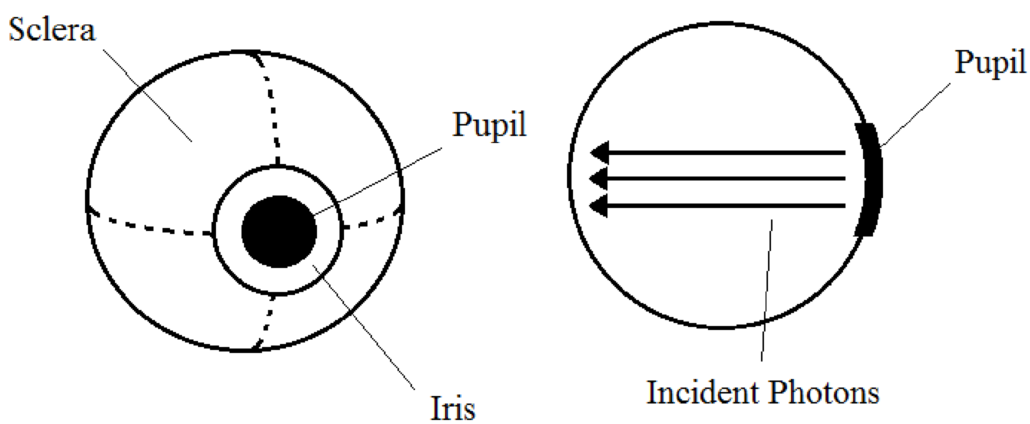
1 Electromagnetic Energy in Eyeball

In this section we present the relevant problem from [1] followed by a discussion of its solution.The problem statement is as follows.

> Electromagnetic radiation inside your eyeball consists of ordinarily two components: (a) 310° K blackbody radiation and (b) visible photons that have entered through the pupil. In order of magnitude, what is the ratio of the total energy in the second form to that in the first when you have your eyes open in a well lighted room?

We want to calculate the quantity $E_{\text{vis.}}/E_{\text{bb}}$ where $E_{\text{vis.}}$ is the energy of visible light entering into you eye through your pupil, and E_{bb} is the energy of blackbody radiation due to your interior body temperature of 310° K. We begin by idealizing the eyeball as a sphere of volume V_{eye} and where the pupil comprises a surface of area A_{pupil}. Such a depiction is presented in Fig. 1a.

We compute the energy due to black body radiation first. From standard thermodynamics we know



(a) Basic Anatomy of Human Anatomy

(b) Photons passing through pupil

Figure 1: Light enters the eye through the pupil and then travels through the eye to reach the rods and cones in the back. For this problem we model the eye as a perfect sphere.

that if we model the interior of eye as the interior of a black body at temperature T_{eye} then the associated energy is

$$E_{\text{bb}} = \frac{4\sigma}{c} T_{\text{eye}}^4 V_{\text{eye}} \tag{1}$$

where σ is the Stefan-Boltzmann constant defined as

$$\sigma = \frac{\pi^2 k_B^4}{60\pi^4 c^2} = 5.67 \times 10^{-8}\,\mathrm{Jm^{-2}s^{-1}K^{-4}}. \tag{2}$$

Next, we calculate the energy coming from photons streaming into the pupil from a light bulb in a well lit room. The physical situation is depicted in Fig. 1b: photons enter the eye through the pupil and remain within the eye for the amount of time it takes them to travel to the rods and cones in the back. By this setup, in order to compute $E_{\text{vis.}}$ we need to calculate the rate at which visible light energy enters the eye and also the time during which it is present in the eye.

The light comes from a light bulb situated, hypothetically, in the center of the room and we assume our eye looks directly at it. The light bulb is presumed to be incandescent meaning that it produces its light through thermal radiation, and we can therefore model the frequency spectrum of the emitted light with the black body radiation formula:

$$\rho(\nu, T)\, d\nu = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}\, d\nu \tag{3}$$

where $\rho(\nu)\, d\nu$ is volume density of states found between frequencies ν and $\nu + d\nu$. Later, because the tungstent filament in the light bulb is the source of the emitted radiation, we will set $T = T_{\text{tung.}}$ where $T_{\text{tung.}}$ is the temperature of the tungsten filament.

In order to calculate the rate at which energy enters the eye we need to find the flux emitted, that is the energy emitted per unit area per unit time, by the light bulb. First we write down the spectral energy density implied by Eq.(3).

$$\mathcal{E}(\nu, T) = h\nu\, \rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}. \tag{4}$$

We don't typically think of a light bulb as active as a photon source with a well defined volume. So the interpretation of $\mathcal{E}(\nu, T)$ is not exactly that it is the energy density of the photon source. Rather, we should simply use this result as an intermediary in the calculation of the more physical quantity energy flux.

Because the light travels at a speed c, the energy flux of photons is naively $\mathcal{E}(\nu, T)\, c$, but in this case and typically we are interested in calculating the flux in a particular direction. To calculate such a flux we imagine a surface set up right next to the radiation source. All photons traveling, even if only slightly, in the direction of the surface, pass through it.

To calculate the average energy flux through this surface we need to compute the energy flux of all the photons which can travel through the surface. This calculation requires us to only account for the angular restriction of the traveling photons. For example, we are interested only in the flux which lies perpendicular to the surface and the only photons which can pass through surface are the ones traveling in the forward direction. Incorporating these elements we find the spectral flux density

$$\begin{aligned}\mathcal{I}(\nu, T) = \langle \mathcal{E}(\nu, T) c_\perp \rangle &= \mathcal{E}(\nu, T) \times \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, \sin\theta c \cos\theta \\ &= \frac{1}{4}\mathcal{E}(\nu, T) c.\end{aligned} \tag{5}$$

Now, to calculate the energy flux associated with this *spectral* energy flux density, we need to integrate this result over the relevant frequency range. For this problem, the only frequency ranges the pupil absorbs exist within the visible spectrum. So the the energy flux within this bandwidth is

$$\begin{aligned}I(\nu_0, \nu_f, T) = \int_{\nu_0}^{\nu_f} d\nu\, \mathcal{I}(\nu, T) &= \int_{\nu_0}^{\nu_f} d\nu \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} \\ &= \frac{2\pi h}{c^2} \left(\frac{k_B T}{h}\right)^4 \int_{x_0}^{x_f} dx\, \frac{x^3}{e^x - 1}\end{aligned} \tag{6}$$

where $x \equiv h\nu/k_B T$, $\nu_0 = 430 \times 10^{12}$ Hz and $\nu_f = 790 \times 10^{12}$ Hz.

Lastly, to compute the energy contained within the eye due to the light bulb's energy flux we multiply the above result by the area of the pupil, A_{pupil} and the time it takes the photons to go from the entrance of the eye to the back. The latter quantity, dubbed the travel time, is d/c, where d is the diameter of the eyeball. The energy from visible light contained within the eyeball is therefore

$$E_{\text{vis.}} = \frac{1}{c}\, I(\nu_0, \nu_f, T_{\text{tung.}}) A_{\text{pupil}}\, d. \tag{7}$$

With the above results we can now compute the numerical values of E_{bb} and $E_{\text{vis.}}$ in turn. First, using Eq.(1) with $V_{\text{eye}} = 4\pi/3\,(12\,\text{mm})^3 = 7.3 \times 10^{-6}\,\text{m}^3$ and $T_{\text{eye}} = 310^\circ$ K, we find

$$\begin{aligned} E_{\text{bb}} &= \left(\frac{4}{3.086 \times 10^8\,\text{m/s}}\right)\left(\frac{5.67 \times 10^{-8}\,\text{J}}{\text{m}^2\,\text{s}\,\text{K}^4}\right)(310^\circ\,\text{K})^4\,(7.3 \times 10^{-6}\,\text{m}^3) \\ &= 5.06 \times 10^{-11}\,\text{J}. \end{aligned} \tag{8}$$

For the visible energy, we first compute the flux over the relevant frequency domain. With $T_{\text{tung.}}$ we find that the limits of the x integration in Eq.(6) are $x_0 = 8.25$ and $x_f = 15.16$. Then, the energy flux can be computed to be

$$I(x_0, x_f, T_{\text{tung.}}) = (80.64\ \text{W/m}^2) \times (0.2134) = 17.21\ \text{W/m}^2. \tag{9}$$

Using the eyeball properties $A_{\text{pupil}} = \pi(2.5\ \text{mm})^2 = 1.96 \times 10^{-5}\,\text{m}^2$, and $d = 24$ mm, the energy contained within the human eye from visible light is

$$\begin{aligned} E_{\text{vis.}} &= \left(\frac{1}{3.086 \times 10^8\,\text{m/s}}\right)\left(\frac{17.21\,\text{J}}{\text{m}^2\,\text{s}}\right)(1.96 \times 10^{-6}\,\text{m}^2)\,(24 \times 10^{-3}\text{m}) \\ &= 2.71 \times 10^{-14}\,\text{J}. \end{aligned} \tag{10}$$

The ratio between the two energies is thus

$$E_{\text{vis.}}/E_{\text{bb}} = 5.3 \times 10^{-4}. \tag{11}$$

2 Background

2.1 Conceptual Questions

Q: What is the so called "black body spectrum"?

A: The black body spectrum is the spectrum of radiation emitted by an object whose production of light is defined by its temperature (See Definition of Black Body Radiation➡). A black body is typically defined as an object which absorbs and emits photons but does not reflect them. The frequency spectrum of a black body is thermally defined, with higher frequencies dominating the spectrum at higher temperatures and lower frequencies dominating the spectrum at lower temperatures.

In general, an object can be considered a black body when the frequency spectrum of its radiation is thermally defined. Examples of (imperfect) black body radiators include incandescent light bulbs, the sun and other stars, and heated metals that comprise stove tops. (See Examples of Black Body Radiators➡). A fluorescent light bulb cannot be reasonably modeled by a black body because its radiation is not produced via a thermal process.

We should also note that an object does not need to emit photons to be defined by the black body spectrum. Even if a black body did not emit any photons, its interior photons would still exhibit the black body spectrum if they were in thermal equilibrium.

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Q: Could we compute the number of photons contained in a black body?

A: No. Photons are the wrong way to think about a black body because they connote a particulate picture of electromagnetic radiation, akin to say an ideal gas, when in fact the more appropriate picture is one akin to a collection of oscillators. Indeed, the black body spectrum of an object is computed by calculating the macrostate properties of electromagnetic quantum oscillators in thermal equilibrium. Thus when talking about black body radiation,it is more appropriate to say that higher modes of electromagnetic field are excited by increasing the temperature of the black body, rather than saying that increasing the temperature increases the energies of the individual photons.

From another frame, it is not correct to talk about an exact number of photons that a black body contains because photon number is constantly in flux due to creation from and annihilation into the vacuum.

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Q: Why can we talk about an energy density for black body radiation but not a photon number density?

A: The photons which are contained in or emitted by a black body are presumed to have different energies so at most we could speak of the relavent proportions of photons emitted at different frequencies and not simply the total number of photons at a particular frequency per unit volume. Consequently, a naive calculation of photon number density would yield a physically uninterpretable result. Computing "Total Photon Number" analogously to the computation of total energy we find

$$N = \sum_{\omega} N_{\omega} = \sum_{\omega} \frac{g_{\omega}}{e^{\hbar\omega/k_B T} - 1} \tag{12}$$

$$= \int_0^{\infty} \frac{1}{e^{\hbar\omega/k_B T} - 1} \frac{V\,\omega^2}{\pi^2 c^3} d\omega. \tag{13}$$

This quantity is the sum of the occupation number for photons over all possible frequencies. In this way N is the total occupation number for photons contained in a volume V. But this number cannot be connected to any notion of a number density or a number of photons because N is not constant with temperature[1]. Since N changes with temperature, if we were to interpret N as a discrete number of photons we would inevitably find that some temperatures yield fractional photon numbers. Consequently, we must abandon any interpretation of N as definitive of photon number and we have to conclude that photon number density is not a physical quantity.

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Q: What is the difference between a thermal equilibrium system of massive identical bosons and a thermal equilibrium system with photons?

A: A thermal equilibrium system of massive identical bosons can be considered as a system of particles with a definite number of particles, while a thermal equilibrium system of photons is a system of oscillator modes where a particle-number interpretation does not apply. Also the massive system of identical bosons can be either relativistic or non-relativistic, while the system of photons, as a system of massless particles, can only be relativistic.

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Q: What happens if we take the mass of an identical boson system to zero?

[1] Unlike a particulate calculation, this calculation of N does not make use of the chemical potential which ensures constancy of N.

A: A system of identical particles (either fermions or bosons) is inherently quantum mechanical, so if we have a system of massless identical bosons we would have a relativistic quantum system. Such a system would need to be modeled using the tools of quantum field theory and consequently we need to represent the system as a collection of quantum oscillators similar to the oscillators which exist at the foundation of the BB radiation calculation.

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Q: In what way is the "black body spectrum" of radiation a result of quantum field theory?

A: The underlying assumption of the black body radiation calculation is that the electromagnetic field comprises a collection of quantum oscillators whose relative mode densities are thermally defined. The assumption that the quantum electromagnetic field (and other fundamental fields) is essentially a system of coupled quantum oscillators is the fundamental premise of quantum field theory. In essence then we can see the standard result for the energy spectrum of a black body as a basic result of the thermodynamics of quantum fields.

[QUESTION: Since the black body radiation spectrum is essentially a result from the thermodynamics applied to quantum fields can we compute other thermal quantities related to quantum fields? For example
- Entropy of Black Body Radiation?
- Analogous quantities for scalar quantum fields?
- Gravitational Fields?
And last but not least, how do these results relate to Hawking's formula for Black Hole Entropy?]

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Q: What is the Grand Canonical Formalism and how is it used?

A: The grand canonical ensemble is a formalism used to compute the thermal statistical properties based on a system which can exchange particles with a larger environment.

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Q: Is the chemical potential of an electromagnetic radiation system zero because we are dealing with a quantum system or because photons are massless?

A: I think both in a way.

We take the chemical potential to be zero when we can no longer talk about particle number conservation, i.e. when the energy cost for adding or removing particles is zero. This is the case for electromagnetic radiation assuming we were to interpret the momentum modes as defining a quantum system of massless particles. In such a system, particles can be freely created or destroyed because they are coupled to a quantum vacuum. Since such a quantum vacuum only exists when we attempt to model relativistic quantum particles, we can see the existence of a quantum vacuum and the corrollary of a zero chemical potential as a consequence of the fact that we are attempting to model a thermal distribution of photons which are inherently both quantum and relativistic.

By this explanation if we were considering any relativistic quantum system we would have to take the chemical potential to be zero. This is something I don't know for sure. But we should be able to check with a Klein Gordon Field Theory.

[QUESTION: An interesting follow up question is how self interactions would change the thermal spectrum of radiation for such particles. For example, what would be the radiation spectrum of a mass less ϕ^3 theory?]

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Q: Why can we use the grand canonical ensemble, a technique used to analyze systems with a variable number of particles, to derive results for fermions and bosons in systems in which particle number is fixed.

A: All schemes in thermodynamics and theoretical and approximate.

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Q: Does your eyeball, as a black body, have electromagnetic energy in a dark room?

A: All hot material objects composed of molecules and atoms emit electromagnetic radiation. The basic physical process is that in a heated environment the molecules and atoms of an object vibrate and then from this vibration the charged constituents of the molecules emit electromagnetic radiation. However, most objects we come into contact with on a daily basis reflect more light than they emit through blackbody radiation process. So although your basketball, for example, can be seen as a "hot black body" which emits light through a thermal process, this thermally emitted light is largely washed out by the light reflected off the basketball.

But if there is no light to reflect, as is the case in a dark room, and the object is maintained at a constant temperature then the black body radiation spectrum well describes the frequency spectrum of radiation. So in a dark room, your eyeball would emit electromagnetic radiation with a frequency spectrum defined by the black body spectrum and we could say that it emits electromagnetic energy. However, your eyeball is not special in this regard. The same frequency spectrum would be emitted by your arm, your head, etc.

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Q: Does a light bulb emit energy outside of the visible spectrum?

A: Yes it does. Statistical mechanics tells us that an idealized black body emits radiation at all frequencies although many frequencyies comprise such small portions of the emission spectrum that they can be considered altogether absent. The light bulb, however, emits most of it radiation outside the visible spectrum.

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Q: How is a light bulb related to a black body?

A: In an incandescent light bulb a tungsten filament is heated to temperatures of 2,000-3,300 K. The heated filament then emits radiation, and because this radiation is thermally defined, we can approximate its spectrum by the black body spectrum.

Fluorescent light bulbs, however, do not emit radiation through a thermal process and therefore do not have a radiation spectrum which can be reasonably approximated by the black body spectrum.

2.2 Blackbody Radiation Formula

The Hamiltonian for the electromagnetic field in vacuum (that is far from the presence of sources) can be written as

$$H = \sum_{\vec{k},\lambda} \frac{1}{2}\left(|\vec{\pi}_\lambda(\vec{k},t)|^2 + \omega(\vec{k})|\vec{A}_\lambda(\vec{k},t)|^2\right) \tag{14}$$

where λ is a polarization label, and $\vec{k}$ is the wave number vector. This form of the Hamiltonian shows that the electromagnetic field in vacuum can be seen as a collection of independent oscillators, where there is one oscillator for each mode and polarization.

Because we are primarily concerned with a quantum system of light we promote the momentum and coordinate variables to quantum operators. Using the standard result for the average energy of a single mode quantum harmonic oscillator, we find that the average energy for each mode $\vec{k}$ and polarization λ is

$$\langle E_{\vec{k},\,\lambda}\rangle = \frac{\hbar\omega(\vec{k})}{2} + \frac{\hbar\omega(\vec{k})}{e^{\beta\hbar\omega(\vec{k})}-1}. \tag{15}$$

The average energy when considering all the modes is therefore

$$\begin{aligned} E = \sum_{\vec{k},\lambda}\langle E_{\vec{k},\,\lambda}\rangle &= \sum_{\lambda=\pm}\frac{V}{(2\pi)^3}\int d^3\vec{k}\,\langle E_{\vec{k},\,\lambda}\rangle \\ &= \frac{2V}{(2\pi)^3}\int d^3\vec{k}\,\langle E_{\vec{k},\,\lambda}\rangle = VE_0 + \frac{2V}{(2\pi)^3}\int d^3\vec{k}\,\frac{\hbar\omega(\vec{k})}{e^{\beta\hbar\omega(\vec{k})}-1} \end{aligned} \tag{16}$$

Ignoring the zero point energy term, using $\omega(\vec{k}) = kc$, and $d^3\vec{k} = 4\pi k^2 dk$, we find

$$E = \frac{Vc}{\pi^2}\int_0^\infty dk\,\frac{\hbar k^3}{e^{\beta\hbar kc}-1} \tag{17}$$

or with $k = 2\pi\nu/c$

$$E = \frac{8\pi V}{c^3}\int_0^\infty d\nu\,\frac{h\nu^3}{e^{\beta h\nu}-1} \tag{18}$$

Remarks

- Counting Photons: When we compute the average energy for each mode and polarization, $\langle E_{\vec{k},\lambda}\rangle$, we take the number of internal modes, $n_{\vec{k},\lambda}$, from zero to infinity Is this allowed given the photon interpretation of a single internal mode?

  Yes.

  If $n_{\vec{k},\lambda} = 1$ represents a single photon with wave number $\vec{k}$ and polarization λ, then taking $n_{\vec{k},\lambda}$ to infinity connotes an infinite number of such photons. The essence of the above procedure then is to consider the contributions from an infinite number of photons in each possible momentum mode.

  Such a consideration is quite different from the analogous one in situations with a finite number of bosons. In those situations we take the energies of the particles to infinity but let the number of particles remain constant. The case of photons and the quantized electromagnetic field, is different however, because with photons we are no longer dealing with discrete particles which can be separated from their vacuum. Photons can be created are destroyed because as quantum constituents of the electromagnetic field and they are inextricably tied to the quantum vacuum.

  Moreover a more accurate characterization of the electromagnetic field is in terms of modes. These modes can, however, be represented as photon particle states which allows us to connect a summation over all field modes to a summation over all possible particle states. This is why it is OK to take $n_{\vec{k},\lambda}$ to infinity.

- Vacuum and the Quantum: Why do we consider the dynamics of the electromagnetic field to be quantum mechanical rather than classical? Is it because the process by which photons are emitted (i.e. electromagnetic radiation is produced) is inherently quantum mechanical? Or is it because in some temperature regimes, low temperature for example, we cannot treat the field as classical but must acknowledge its quantum mechanical properties.

2.3 Derivation of Hamiltonian

We can derive the electromagnetic hamiltonian Eq.(14) by starting from the Lagrangian of electromagnetic fields

$$L = \int d^3\vec{x} \left(\frac{\epsilon_0}{2} \vec{E}^2 - \frac{1}{2\mu_0} \vec{B}^2 \right) \tag{19}$$

Writing this Lagrangian in terms of the vector potential alone we then find

$$L = \int d^3\vec{x} \left(\frac{\epsilon_0}{2} \left(\frac{\partial \vec{A}}{\partial t} \right)^2 - \frac{1}{2\mu_0} (\nabla \times \vec{A})^2 \right) \tag{20}$$

To find the associated Hamiltonian, we employ the standard procedure to write the conjugate field momentum as $\vec{\pi} = \epsilon_0 \partial \vec{A}/\partial t$. The Hamiltonian is then

$$H = \int d^3\vec{x} \left(\frac{1}{2\epsilon_0} \vec{\pi}^2 + \frac{1}{2\mu_0} (\nabla \times \vec{A})^2 \right). \tag{21}$$

We express this Hamiltonian in momentum space using the Fourier transforms

$$\vec{\pi}(\vec{x},t) = \sum_{\lambda=\pm} \int \frac{d^3\vec{k}}{(2\pi)^3} \, \vec{\pi}_\lambda(\vec{k},t) \, e^{i\vec{k}\cdot\vec{x}}, \qquad \vec{A}(\vec{x},t) = \sum_{\lambda=\pm} \int \frac{d^3\vec{k}}{(2\pi)^3} \, \vec{A}_\lambda(\vec{k},t) \, e^{i\vec{k}\cdot\vec{x}}. \tag{22}$$

The parameter λ defines the two transverse polarizations of the electromagnetic field, and $\vec{k}$ is the wave number. Substituting these forms into Eq.(21) and using the Coulomb gauge condition $\nabla \cdot \vec{A} = 0$, we find

$$H = \sum_{\lambda=\pm} \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{1}{2\epsilon_0} |\vec{\pi}(\vec{p},t)|^2 + \frac{1}{2\mu_0} \vec{k}^2 |\vec{A}(\vec{p},t)|^2 \right). \tag{23}$$

We now rescale these fields according to

$$\vec{\pi}(\vec{p},t) \to \sqrt{V\epsilon_0} \, \vec{\pi}(\vec{p},t), \qquad \vec{A}(\vec{p},t) \to \sqrt{\frac{V}{\epsilon_0}} \, \vec{A}(\vec{p},t) \tag{24}$$

where V is the volume of our system. We thus find

$$H = \sum_{\lambda=\pm} \frac{V}{(2\pi)^3} \int d^3\vec{p} \left(\frac{1}{2} |\vec{\pi}(\vec{p},t)|^2 + \frac{1}{2} \vec{k}^2 c^2 |\vec{A}(\vec{p},t)|^2 \right). \tag{25}$$

As a side point: Why must we sum over the polarizations of the vector potential when we write out the Fourier transform? The answer is best seen through example. For circularly polarized light propagating in the z direction, the electric field has the form

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t)\hat{x} + E_0 \sin(kz - \omega t)\hat{y}. \tag{26}$$

The corresponding vector potential is then

$$\vec{A}(z,t) = -\frac{E_0}{\omega} \hat{x} \, \sin(kz - \omega t) - \frac{E_0}{\omega} \hat{y} \, \cos(kz - \omega t). \tag{27}$$

Light represented as a propagation of electromagnetic waves always has two polarizations due to the presence of the orthogonal electric and magnetic fields. Hence the vector potential written as a sum of oscillating modes will always need to take into account both polarizations.

References

[1] E. M. Purcell, “The back of the envelope,” *American Journal of Physics*, vol. 51, no. 8, pp. 685–685, 1983.