MITES 2010: Physics III
Survey of Modern Physics
Midterm Solutions

Exercises

1. **Exercise 1. Consider a particle with mass m that moves in one-dimension. Its position at time t is $x(t)$. As a function of its position, Its potential energy is $U(x) = C$, where C is a constant.**
   (a) Derive the equation of motion for the particle using Newtons second law.
   (b) Derive the equation of motion for the particle, but this time, using the Lagrangian.
   (c) Solve the equation of motion (i.e., what are $x(t)$ and $\frac{dx}{dt}$?). Define all the free parameters in your solution (i.e. what are their physical meanings?)

   Solution: (a)From Newton's Second Law we find

$$\begin{aligned} m\ddot{x} &= F(x) \\ &= -\frac{d}{dx}U(x) \\ &= -\frac{d}{dx}C \\ &= 0 \qquad \Longrightarrow \qquad \boxed{\ddot{x} = 0} \end{aligned}$$

   (b) From the definition of the Lagrangian we find that the Lagrangian of this system has the form

$$\begin{aligned} \mathcal{L} &= \text{KE} - \text{PE} \\ &= \frac{1}{2}m\dot{x}^2 - C \end{aligned}$$

   From this Lagrangian we may derive equation of motion using the Euler-Lagrange (E-L) equation.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}(-C) \\ &= 0 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) \\ &= \frac{d}{dt}(m\dot{x}) \\ &= m\ddot{x} \end{aligned}$$

   and so

$$0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 - m\ddot{x} \qquad \Longrightarrow \qquad \boxed{\ddot{x} = 0}$$

(c) Our equation of motion $\ddot{x} = 0$ is a simple second order differential equation so we may solve it by separation of variables.

$$\begin{aligned}
\ddot{x} &= 0 \\
\frac{d\dot{x}}{dt} &= \\
d\dot{x} &= 0\,dt \\
\int_{\dot{x}_0}^{\dot{x}(t)} d\dot{x} &= 0 \\
\dot{x}(t) - \dot{x}_0(t) &= 0 \qquad \Longrightarrow \qquad \boxed{\dot{x}(t) = \dot{x}_0}
\end{aligned}$$

This equation for velocity provides us with another differential equation for the positon $x(t)$.

$$\begin{aligned}
\dot{x} &= \dot{x}_0 \\
\frac{dx}{dt} &= \dot{x}_0 \\
d\dot{x} &= \dot{x}_0\,dt \\
\int_{x_0}^{x(t)} dx &= \int_0^t \dot{x}_0\,dt' \\
x(t) - x_0 &= \dot{x}_0 t \qquad \Longrightarrow \qquad \boxed{x(t) = x_0 + \dot{x}_0 t}
\end{aligned}$$

In our solutions, the free parameter $\dot{x}_0$ represents the initial velocity of particle and x_0 represents the initial position.

2. **Exercise 2: A Lincoln Continental is twice as long as a VW Beetle, when they are at rest. As the Continental overtakes the VW, going through a speed trap, a (stationary) policeman observes that they both have the same length that the Lincoln Continental is twice as long as the . The VW is going at half the speed of light. How fast is the Lincoln going? Leave your answer as a multiple of c.**

   Solution: We know that the Lincoln Continental is twice as long as the VW Beetle when both are at rest ($L_{LC} = 2L_{VW}$). We also know that from the ground frame they have the same length ($\bar{L}_{LC} = \bar{L}_{VW}$) when they are moving with speeds v_{LC} and v_{VW} respectively. Lastly, we know that

$v_{VW} = c/2$. Using this information, we can solve for the velocity of the lincoln continenental v_{LC}.

$$
\begin{aligned}
\frac{L_{LC}}{\gamma_{LC}} &= \frac{L_{VW}}{\gamma_{VW}} \\
\frac{2L_{VW}}{\gamma_{LC}} &= \frac{L_{VW}}{\gamma_{VW}} \\
\frac{2}{\gamma_{LC}} &= \frac{1}{\gamma_{VW}} \\
2\sqrt{1-\frac{v_{LC}^2}{c^2}} &= \sqrt{1-\frac{v_{VW}^2}{c^2}} \\
4\left(1-\frac{v_{LC}^2}{c^2}\right) &= \left(1-\frac{1}{4}\right) \\
\left(1-\frac{v_{LC}^2}{c^2}\right) &= \frac{3}{16} \\
\frac{v_{LC}^2}{c^2} &= 1-\frac{3}{16} \\
&= \frac{13}{16} \implies \boxed{v_{LC} = \frac{\sqrt{13}}{4}c}
\end{aligned}
$$

3. **Exercise 3: A sailboat is manufactured so that the mast leans at an angle $\bar{\theta}$ with respect to the deck. An observer standing on a dock sees the boat go by at speed v. What angle does this observer say the mast makes?**

   Solution: A fundamental effect of special relativity is that lengths (of a moving object) parallel to the direction of motion are contracted. So, in this problem we realize that the horizontal extent of the bottom ray of the angle must be contracted due to the motion of the boat. Labeling the length of the mast as L we can write the height $\bar{H}$ and original horizontal extent $\bar{X}$ as

$$
\begin{aligned}
\bar{H} &= L\sin\bar{\theta} \\
\bar{X} &= L\cos\bar{\theta}
\end{aligned}
$$

   Due to length contraction, the value of $\bar{X}$ becomes the smaller value X in the moving reference frame and we find that the relation between X and $\bar{X}$ is

$$X = \frac{\bar{X}}{\gamma}$$

   So that $X = L\cos\bar{\theta}/\gamma$. Lengths perpendicular to the direction of motion are not changed by moving to different reference frames so we find that $H = \bar{H} = L\sin\bar{\theta}$. Using this information we can define the angle θ of the mast in the moving reference frame as

$$
\begin{aligned}
\tan\theta &= \frac{H}{X} \\
&= \frac{L\sin\bar{\theta}}{L\cos\bar{\theta}/\gamma} \\
&= \gamma\tan\bar{\theta} \implies \boxed{\theta = \tan^{-1}(\gamma\tan\bar{\theta})}
\end{aligned}
$$

   When we get an algebraic result such as this one, it is useful to check **limiting cases** to make sure that our formula correctly predicts what physical intuition tells us should be true. For example, we will consider the formula for $v = 0$ and $v \to c$ and make sure it provides a result which mirrors what we think should happen. For $v = 0$ we know that the sailboat is not moving and therefore length contraction is absent. Therefore θ should be equal to $\bar{\theta}$, which is what we find when we plug $v = 0 \Leftrightarrow \gamma = 1$ into our formula. Conversely, as $v \to c$ the horizontal length is further contracted until it only appears as though we have a stick of length L standing straight up. This reasoning matches the $\theta = \pi/2$ result we get if we plug $v = c \Leftrightarrow \gamma = \infty$ into our above formula.

4. **Exercise 4. Explain why oscillations occur so frequently (no pun intended!) when an object is just slightly perturbed from its stable equilibrium position. For this problem, consider only onedimensional motion. As a starting point, assume a particle of mass m, whose position is $x(t)$, and has potential energy V(x). In the process of answering this question, derive the equation for small oscillations about the stable equilibrium position. [Hint: Use Taylor series (see front page of this exam) to explain this.]**

   Solution: Physically, oscillations occur so frequently because particles which are perturbed from stable equilibrium always tend to move back towards that stable equilibrium. When we construct a mathematical model to represent this phenomenon, we find that the potential near an equilibrium point becomes a positive quadratic and the force therefore becomes linearly restoring. Specifically, from the theory of Taylor Series we can expand any potential function about an arbitrary point x_0.

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \frac{1}{3!}V'''(x_0)(x - x_0)^3 + \cdots$$

   If we then allow x_0 to define an equilibrium point where $F(x_0) = 0 = -V'(x_0)$ then the second term vanishes and we have

$$V(x) = V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \frac{1}{3!}V'''(x_0)(x - x_0)^3 + \cdots$$

   Next, if we only consider a domain of x which is very close to x_0(i.e. if we conside $|x - x_0| << 1$), then the quadratic term is much larger than the cubic term and the cubic term (and higher order terms) can be ignored. We then have

$$V(x) \approx V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2$$

   The first term on the right hand side is a constant. It sets the reference point for potential energy but the dynamics of the system are determined by $V'(x) = -F(x)$. So, any constants in the potential energy are irrelevant to any subsequent motion and may therefore be ignored. So we can write

$$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2$$

   From this formula for potential enegry we may derive the force and therfore the EOM

$$\begin{aligned} F(x) &= -V'(x) \\ &= -\frac{d}{dx}\left(\frac{1}{2}V''(x_0)(x - x_0)^2\right) \\ &= -V''(x_0)(x - x_0) \end{aligned}$$

So the EOM is

$$
\begin{aligned}
m\ddot{x} &= F(x) \\
&= -V''(x_0)(x-x_0) \\
0 &= m\ddot{x} + V''(x_0)(x-x_0) \\
&= \ddot{x} + \frac{V''(x_0)}{m}(x-x_0)
\end{aligned}
$$

If we define $\delta = x - x_0$ (the distance from the equilibrium position) then we have

$$
\begin{aligned}
\dot{\delta} &= \dot{x} - \frac{d}{dx}x_0 \\
&= \dot{x} \\
\ddot{\delta} &= \ddot{x}
\end{aligned}
$$

So our Equation of Motion can be written as $\ddot{\delta} + \dfrac{V''(x_0)}{m}\delta = 0$ which is just the basic equation for the harmonic oscillator with $\dfrac{V''(x_0)}{m} = \omega^2$.

5. **Exercise 5: In a given reference frame, event 1 happens at $x = 0$, $ct = 0$, and event 2 happens at $x = 2$, $ct = 1$. Find a frame in which the two events are simultaneous.**

   Solution: One of the fundamental effects of relativity is the loss of simultaneity. A corollary to this effect, potentially called the restoration of simultaneity, is depicted in this problem. Our job is to find another frame $\bar{O}$ in which the two events $(x_1, t_1) = (0, 0)$ and $(x_2, t_2) = (2, 1/c)$ occur simultaneously (i.e. $\bar{t}_1 = \bar{t}_2$). Assuming the original frame of the events is our rest frame we have the following Lorentz transformation for the time in a frame $\bar{O}$ moving with velocity v with respect to the rest frame.

$$
\bar{t} = \gamma\left(t - \frac{vx}{c^2}\right)
$$

   Requiring that our two events be simultaneous in this $\bar{O}$ frame, we find

$$
\begin{aligned}
\bar{t}_1 &= \bar{t}_2 \\
\gamma\left(t_1 - \frac{vx_1}{c^2}\right) &= \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \\
\gamma\left(0 - \frac{v \cdot 0}{c^2}\right) &= \gamma\left(\frac{1}{c} - \frac{2v}{c^2}\right) \\
0 &= \left(\frac{1}{c} - \frac{2v}{c^2}\right) \\
\frac{1}{c} &= \frac{2v}{c^2} \quad \Longrightarrow \quad \boxed{v = c/2}
\end{aligned}
$$

   So the $\bar{O}$ frame moves away from the rest frame with relative velocity $c/2$ in the x direction.

Problems

1. **Problem 1: A and B travel at $\frac{4}{5}c$ and $\frac{3}{5}c$ with respect to the ground. How fast should c travel so that she sees A and B approaching her at the same speed? What is this speed?**

   Solution: Let us label the velocity of A relative to C as $+u$. Then, according to the problem, the velocity of B relative to A must be $-u$. Using the relativistic transformations for velocity we then have

$$
\begin{aligned}
+u &= \frac{v_C - v_A}{1 - \frac{v_C v_A}{c^2}} \\
-u &= \frac{v_C - v_B}{1 - \frac{v_B v_A}{c^2}}
\end{aligned}
$$

   Where the letters label the velocities of the corresponding particle. From the problem statement we know $v_B = 3c/5$ and $v_A = 4c/5$ and we are trying to find v_C so, we have

$$
\begin{aligned}
u &= u \\
\frac{v_C - v_A}{1 - \frac{v_C v_A}{c^2}} &= \frac{v_B - v_C}{1 - \frac{v_B v_A}{c^2}} \\
\frac{v_C - \frac{4c}{5}}{1 - \frac{4v_C}{5c}} &= \frac{\frac{3c}{5} - v_C}{1 - \frac{3v_C}{5c}} \\
\left(v_C - \frac{4c}{5}\right)\left(1 - \frac{3v_C}{5c}\right) &= \left(1 - \frac{4v_C}{5c}\right)\left(\frac{3c}{5} - v_C\right) \\
v_C - \frac{4c}{5} + \frac{12v_C}{25} - \frac{3v_C^2}{5c} &= -v_C + \frac{4v_C^2}{5c} - \frac{12v_C}{25} + \frac{3c}{5} \\
0 &= -2v_C + \frac{7v_C^2}{5c} - \frac{24v_C}{25} + \frac{7c}{5} \\
&= \frac{7v_C^2}{5c} - \frac{74v_C}{25} + \frac{7c}{5} \\
&= 1/25c\left(35v_C^2 - 74v_C c + 35c^2\right) \\
&= (5v_C - 7c)(7v_C - 5c)
\end{aligned}
$$

   The last line produces two solutions: $v_C = 7c/5$ and $v_C = 5c/7$. But, since the speed of an object can never exceed the speed of light the first solution is extraneous and we have $\boxed{v_C = 5c/7}$

2. **Problem 2. Hanging mass The potential energy for a mass hanging from a spring is $V(y) = ky^2/2 + mgy$, where $y = 0$ corresponds to the position of the spring when nothing is hanging from it.**
   (a) What is the net force acting on the particle?
   (b) Write down the Lagrangian for this system.
   (c) Use the Euler-Lagrange equation to derive the equation of motion for this system. DO NOT use Newtons second law to derive it. What kind of motion does this correspond to?
   (d) Solve the equation of motion, thereby obtaining y(t). What do all the free parameters in your solution physically mean?

Solution: (a)

$$\begin{aligned} F(x) &= -V'(y) \\ &= -\frac{d}{dy}(ky^2/2 + mgy) \\ &= \boxed{-ky - mg} \end{aligned}$$

(b)

$$\begin{aligned} \mathcal{L} &= \text{KE} - \text{PE} \\ &= \boxed{\frac{1}{2}m\dot{x}^2 - \frac{1}{2}ky^2 - mgy} \end{aligned}$$

(c)

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial}{\partial y}(-\frac{1}{2}ky^2 - mgy) \\ &= -ky - mg \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} &= \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) \\ &= m\ddot{x} \end{aligned}$$

$$0 = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = -ky - mg - m\ddot{y} \qquad \Longrightarrow \qquad \boxed{\ddot{y} + \frac{k}{m}y + g = 0}$$

If we left $\delta = y + mg/k$ then we have $\ddot{\delta} + \omega^2\delta = 0$. This is the simple harmonic oscillator equation which models all free undamped oscilatory motion.

d) The solution to the SHO EOM $\ddot{\delta} + \omega^2\delta = 0$ is

$$\delta(t) = A\sin\omega t + B\cos\omega t$$

And from our definition $\delta = y + mg/k$, we find that $y(t)$ is

$$\boxed{y(t) = A\sin\omega t + B\cos\omega t - \frac{mg}{k}}$$

. In order to find what A and B represent, we solve for the initial conditions of our mass. Setting $t = 0$ in our position equation, we find

$$y_0 = y(t=0) = A\sin 0 + B\cos 0 - mg/k = B - mg/k \qquad \Longrightarrow \boxed{B = y_0 + \frac{mg}{k}}$$

B represents the initial displacement from the equilibrium position $y_{eq} = -mg/k$. Similarly, differentiating $y(t)$ and then plugging in $t = 0$ gives us

$$v_0 = \dot{y}(t=0) = A\omega\cos 0 - B\omega\sin 0 = A\omega \qquad \Longrightarrow \boxed{A = \frac{v_0}{\omega}}$$

A represents the inital velocity of the bead divided by the angular frequency. All of this information gives us a general solution $x(t)$ of the form

$$y(t) = \frac{v_0}{\omega}\sin\omega t + \left(y_0 + \frac{mg}{k}\right)\cos\omega t$$

3. **Problem 3. Bead on a rotating hoop: A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with constant angular speed ω around a vertical diameter (See Fig. 3). Find the equation of motion for the angle θ shown (angle is in radians). What are the equilibrium positions? What is the frequency of small oscillations about the stable equilibrium? There is one value of ω that is quite special. What is it and why is it special?**

   Solution: We will use the Lagrangian method to solve this problem. First, we must define our kinetic energy. We will do this by considering the simplest motions of the system. First, let us assume that $\omega = 0$ so that the bead can only move through the θ variable. From a consideration of polar coodinates it is clear that the kinetic energy contribution due to a change in θ is $\frac{1}{2}m(R\dot{\theta})$.

   Now, let θ be fixed at an arbitary angle $\theta = \theta_1$ and let us let $\omega \neq 0$. Then our former kinetic energy contribution disappears (because θ is constant) and we must find an alternate expression to account for the motion of the bead. If we let $\theta_1 = \pi/2$ then the bead points in a directions which is perpendicular to the spin axis and we may write the kinetic energy as $\frac{1}{2}mR^2\dot{\theta}^2$. Alternatively, if we let $\theta_1 = 0$ then the bead remains at the bottom of the hoop and has no kinetic energy[1]. Therefore we find that this part of the kinetic energy is defined by how far away (specifically the perpendicular distance) the bead is from the spin axis. In particular, the kinetic energy is defined by the perpendicular extent, $R\sin\theta$, of the bead from the spin axis. Allowing this perpendicular extent to be our radius of rotational motion we find that this part of the kinetic energy is $\frac{1}{2}m(R\sin\theta_1\omega)^2$. So, in general, we find that the kinetic energy of the bead is

$$\text{KE} = \frac{1}{2}mR^2(\sin^2\theta\omega^2 + \dot{\theta}^2)$$

   Now, we will find the potential energy. The bead is in a gravitational field so all of its potential energy must arise from the its height/vertical position. We can define this quantity with $R\cos\theta$. Also, the bead is constrained to move along the hoop so let us define zero potential energy to be at the bottom of the hoop $\theta = 0$ and maximum potential energy to be at the top of the hoop $\theta = \pi$. With these considerations we can write our potential energy function as[2]

$$\text{PE} = mgR(1 - \cos\theta)$$

   We may now write our Lagraingian.

$$\mathcal{L} = \frac{1}{2}mR^2(\sin^2\theta\omega^2 + \dot{\theta}^2) - mgR(1 - \cos\theta)$$

   Using the E-L equations to generate the EOM for θ we find

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{d}{d}\left(mR^2\dot{\theta}\right) \\
&= mR^2\ddot{\theta}
\end{aligned}$$

[1] We are considering the bead as so small that any rotational motion about its center of mass contributes a neglible amount to its kinet energy.

[2] The constant mgR is arbitary; $-mgR\cos\theta$ is the only essential part of the potential. We include the constant only to keep the potential energy exclusively positive.

So our EOM becomes

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} \implies \boxed{\ddot{\theta} = \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta}$$

where we divided both sides of the E-L equation by mR^2 to simplify the result. In order, to find the equilibrium positions in this system, we must find the values of θ where $\ddot{\theta} = 0$. Setting the right hand side of our EOM to zero we find.

$$\begin{aligned} 0 &= \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta \\ &= \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right) \end{aligned}$$

The right hand side is zero if $\sin\theta = 0$ (i.e. if $\theta = 0, \pi$) or if the quantity in the parentheses is zero. So we find that our equilibrium positions occur at the angles

$$\boxed{\theta_1 = 0} \qquad \boxed{\theta_2 = \cos^{-1}\left(\frac{g}{R\omega^2}\right)} \qquad \boxed{\theta_3 = \pi}$$