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Fundamental Physics: Quantum Field Theory

What is the topic? Quantum field theory refers to the quantum theory of fields in an analogous way to how quantum mechanics refers to the quantum theory of the mechanics. More precisely, it is the result of combining relativity and quantum mechanics in such a way to maintain the Lorentz invariance or covariance suggested by the former while keeping the consistent probabilistic interpretations necessary in the latter[1]. It forms the mathematical foundation for our modern theory of how elementary particles interact, and comprises the first arena in which the general methods of statistical field theory were developed.

1 Key Terms

- **Quantum Field:** A quantum field is an operator that takes on a different value at each point in spacetime. In QFT, quantum fields can have different spins and qualitatively, excitations in a quantum field are what we typically consider as particles. More precisely, particles are energy eigenstates of hamiltonian operators written in terms of the fields.

$$\phi(x),\ \Psi(x),\ \psi(x),\ A^a_\mu(x) \tag{1}$$

- **Spin:** Each quantum field has a spin angular momentum quantum number, and this quantum number defines the Lorentz transformation properties of the field which in tern determines the associated Lorentz invariant Lagrangian. By the spin-statistics theorem, particles with half-integral spin are fermions and particles with integer spin are bosons.

$$\text{Spin Statistics Theorem } = \begin{cases} \text{integral spin} & \to \text{boson} \\ \text{half-integral spin} & \to \text{fermions} \end{cases} \tag{2}$$

- **Lagrangian Density and Hamiltonian:** The Hamiltonian operator for a free field theory defines the energy eigenstates of the field and these energy eigenstates define represent single/multiparticle states. The Lagrangian density is a function of the various fields and field derivatives in the system, and constrains how the fields interact.

  Hamiltonian of Quantum Electrodynamics

$$H_{\text{QED}} = \int d^3x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \overline{\Psi}(i\gamma^\mu \partial_\mu + ie\gamma_\mu A^\mu + m)\Psi \right) = H_{\text{ED}} + H_{\text{Dirac}} + H_{\text{int}}, \tag{3}$$

  where

$$H_{\text{ED}} = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\, \hbar\omega_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \right) \tag{4}$$

  Lagrangian (density) of Scalar Quantum Chromodynamics

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4g^2} F^{a\mu\nu} F_{a\mu\nu} - (D_\mu \varphi)^\dagger (D^\mu \varphi) - m^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 \tag{5}$$

[1] The fact that combining relativity and quantum mechanics necessitates quantum field theory and not just relativistic quantum mechanics is an important point and was historically what impeded the full development of the subject. For example, this way Feynman's roadblock [**?**]

- **Correlation Function:** The expectation value of various products of fields between two vacuum states. As many important physical predictions (cross section/decay rates) in QFT are written in terms of correlation functions, it is an assumed calculational goal.

  Example Correlation Function
$$\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \tag{6}$$

- **Scattering Amplitude (Cross Sections/Decay Rates)**: Defines a sort of probability of interaction between two particle states. With a knowledge of the scattering amplitude (often denoted $\mathcal{M}$) we can compute the lifetime or decay rate of a single particle, or calculate the cross section for a two particle interaction.

  Scattering amplitude for $\phi\phi \to \phi\phi$ scattering
$$i\mathcal{M}(\phi\phi\to\phi\phi) = ig^2\left[\frac{1}{(k_1+k_2)^2-m^2}+\frac{1}{(k_1'-k_1)^2-m^2}+\frac{1}{(k_2'-k_1)^2-m^2}\right] \tag{7}$$

  Use of scattering amplitude in differential cross section and decay rate
$$\mathcal{M} \to \begin{cases} \dfrac{d\sigma}{d\Omega} & \text{Differential Cross Section} \\ \Gamma & \text{Decay Rate} \end{cases} \tag{8}$$

- **Dyson Series:** Dyson Series is a perturbation series in the Interaction picture which allows us to compute the correlation function for an interacting field theory. It is employed when we are studying QFT *without* Lagrangians and functional integrals.
$$i\frac{\partial}{\partial t}U(t,t') = V(t)U(t,t') \tag{9}$$
$$\begin{aligned} U(t,t') &= 1 - i\int_{t'}^{t} dt_1\, V(t_1)U(t_1,t') \\ &= \sum_{n=0}^{\infty}(-i)^n\int_{t'}^{t}dt_1\int_{t'}^{t_1}dt_2\ldots\int_{t'}^{t_{n-1}}dt_n\,\mathrm{T}\,V(t_1)V(t_2)\cdots V(t_n) \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t'}^{t}dt_1\int_{t'}^{t}dt_2\ldots\int_{t'}^{t}dt_n\,V(t_1)V(t_2)\cdots V(t_n) \end{aligned} \tag{10}$$

  This unitary operator can be used to derive
$$\langle\Omega|\mathrm{T}\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle = \frac{\langle 0|\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\mathrm{T}\exp\left[i\int d^4x\,\mathcal{H}_I(x)\right]|0\rangle}{\langle 0|\mathrm{T}\,\exp\left[i\int d^4x\,\mathcal{H}_I(x)\right]|0\rangle} \tag{11}$$

  where $|\Omega\rangle$ is the vacuum state of the interacting theory and $|0\rangle$ is the vacuum state of the non-interacting theory, and the subscript I means we're considering the operators in the interaction picture.

- **Functional Integral:** The functional integral is an extension of the path integral in quantum mechanics made relevant to the study of quantum field interactions. The functional integral is the generating functional for correlation functions ($Z[J]$ for disconnected correlation functions; $W[J]$ for connected correlation functions). For interacting field theories, the functional can be written as a perturbation series with which we can associate Feynman Diagrams.

$$\mathcal{Z}_{\text{Yang-Mills}} = \int \mathcal{D}A \exp\left[-\frac{i}{4g^2}\int d^dx\, F^{\mu\nu}F_{\mu\nu}\right] \tag{12}$$

- **Feynman Diagrams:** A diagrammatic method for organizing the terms in a scattering amplitude for a particle interaction process. Each part of the diagram represents a mathematical term that when multiplied/integrated (depends on context) gives us the mathematical expression for the scattering amplitude of the process.

  Feynman diagrams exist for scattering amplitudes, terms in the functional integral and correlation functions. In general, for quantum and statistical field theories any perturbative result can be written as a series of Feynman diagrams.

- **LSZ Reduction Formula:** A method for converting correlation functions into scattering amplitudes. From [1] we have

$$\begin{aligned}\langle f|i\rangle &= \langle k_{1'}\ \ldots, k_{n'}|k_1 \ldots k_n\rangle \\ &= i^{n+n'}\int d^4x_1\, e^{ik_1x_1}\left(-\partial_1^2 + m^2\right)\ldots \\ &\quad \int d^4x_{1'}\, e^{ik_{1'}x_{1'}}\left(-\partial_{1'}^2 + m^2\right)\ldots \\ &\quad \times \langle 0|\mathrm{T}\varphi(x_1)\cdots\varphi(x_{1'})\cdots|0\rangle\end{aligned} \tag{13}$$

- **Gauge Theory:** A theory of spin-1 quantum fields where the fields are written in the basis of generators of a symmetry group. For such theories, we take the unitary transformation to be locally dependent on spacetime.

  Definition of Unitary transformation:

$$U(x) = \exp\left[i\alpha^a(x)T^a\right] \qquad \text{Unitary Transformation for group with generators } T^a \tag{14}$$

  The field transformation

$$A^\mu(x) \to U(x)A(x)U^{-1}(x) \tag{15}$$

  leaves the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{16}$$

  invariant.

- **Non-Abelian Gauge Theory:** A gauge theory, where the generators which define the symmetry group do not commute. Is it inaccurate to say most gauge theories are non-abelian gauge theories?

$$U(x) = \exp\left[i\alpha^a(x)T^a\right] \quad \text{where} \quad \left[T^a, T^b\right] \neq 0 \tag{17}$$

$$\mathcal{L}_{\text{non-abelian}} = -\frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu} - \overline{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi \tag{18}$$

- **Operator Product Expansion:** (I'm not entirely sure how best to define this because I've never used it) A way to write a single operator as the sum of products of many other operators.

- **Symmetry Breaking:** Symmetry breaking refers to the destruction (in some way) of a previous symmetry in a theory. *Spontaneous* symmetry breaking (the most discussed form in textbooks) occurs when the physical properties/manifestation of a theory does not bear out the symmetries which define the laws of the theory. The common example is a ferromagnet which is not rotationally invariant even though the laws of electromagnetism are.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \varphi^\dagger \partial^\mu \varphi + \mu^2 \varphi^\dagger \varphi - \frac{\lambda}{4}(\phi^\dagger \varphi)^2 \quad \longrightarrow \quad \langle \varphi^\dagger \varphi \rangle \neq 0 \tag{19}$$

- **Higgs Mechanism:** The mechanism by which gauge particles in a spontaneously broken theory obtain a mass and a longitudinal degree of freedom. This is the mechanism by which the $W^\pm$ and Z^0 bosons in the Standard model obtain their mass.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(D_\mu \varphi)^\dagger \partial^\mu \varphi + \mu^2 \varphi^\dagger \varphi - \frac{\lambda}{4}(\phi^\dagger \varphi)^2 \quad \longrightarrow \quad \langle \varphi^\dagger \varphi \rangle \neq 0 \tag{20}$$

$$\longrightarrow \quad \mathcal{L}_{\text{mass}} = -\frac{1}{2}m^2 A^\mu A_\mu + \cdots \tag{21}$$

- **Standard Model:** The modern theory of particle interactions. It is often represented as a Lagrangian, and it defines how the higgs, leptons (electrons, muons, taus), their associated neutrinos, the quarks (up/down, charm/strange, top/bottom), and gauge bosons (photons, W^+, W^-, and Z^0 bosons, gluons), and all of their many anti-particles interact. It does not model the massive nature of neutrinos, does not describe gravity, nor does it seem to account for the interactions/particles which make up dark matter,

2 What questions does it answer/Why these questions are important.

- **Perturbative Scattering Amplitude:** allows us to perturbatively compute the scattering amplitude of elementary particles, which in turn allows us to determine decay rates and differential cross sections (Famous examples: Klein-Nishina Formula ($e^- + \gamma \to e^- + \gamma$), Pion Decay Rate ($\pi \to 2\gamma$), Beta Decay Rate ($n \to p + e + \bar{\nu}$))
- **Quantum Corrections (QFT):** allows us to compute quantum corrections to physical quantities such as the energy levels of the hydrogen atom and the magnetic moment of electron
- **Critical Exponents (SFT):** allows us to determine the critical exponents in a theory beyond the mean field theory approximation.
- **Renormalization Group:** allows us to see how the of a theory evolve/change according to the energy or length scale at which we investigate the theory.
- **Language of Standard Model:** serves as the fundamental language of the standard model of particle physics (the current best description of elementary particle interactions)

Why are these questions(and their answers) important?

3 Features of QFT systems

The fact that the mathematics of quantum field theory is applicable in contexts having nothing to do with either relativity and quantum mechanics suggests that what we're dealing with is really a mathematical framework which happened to emerge naturally from those two topics, but is not at all dependent on them (Much in the same way that modern calculus emerged from the study of mechanics). Considering that the methods of QFT are relevant in condensed matter systems and looking at the similarity between the two subjects there appear to be three requirements for the methods of QFT/SFT to be applicable:

- **A Dynamic Equilibrium Configuration/Exactly Solvable (and Non trivial) Unperturbed System:** Qualitatively we want a system which has an unperturbed description which is completely solvable and dynamically interesting. Mathematically, this often boils down to admitting a solution of the system in terms of wave or oscillatory functions. This is necessary because (in QFT and SFT) physical observables are expanded in terms of the free propagators.
- **Uncertainty/Probabilistic Description:** In order for the basic methods of QFT to be applicable the system must be probabilistic in some way. The functional integral appears in both SFT and QFT and we could see that what unites the two is a sum over configurations picture made necessary by an inherently probabilistic system.
- [2] **Continuous Degrees of Freedom:** The system must have continuous degrees of freedom, in order for the continuous scale relationships (i.e. the renormalization group relationships) to exist. It is of course possible, to have a renormalization group for discrete systems (like the Ising Model), but fields are by definition continuous. For example, this is how statistical mechanics differs from statistical field theory.

In much the same way we use derivatives whenever we're trying to study how quantities change with respect to one another, and we use integration whenever we're trying to determine the sum of a variable result, we use the methods of quantum field theory whenever our system contains all of the above features.

4 Illustrative Examples

We give examples of a few systems which can be represented as a path integral and can therefore be studied using the methods of quantum field theory.

- **Brownian Motion of a Particle:** The brownian motion of a particle moving in an external potential can be formulated as a result of quantum field theory. First the propagator for the particle to move from a position x and time t to a position x' and time t' is

$$K(x_f, t_f; x_0, t_0) = \int_{x=x_0(t_0)}^{x=x_f(t_t)} \mathcal{D}x(t)\, \exp\left[-\frac{1}{2D}\int_0^T dt\, \left(\frac{1}{2}\dot{x}(t)^2 - V(x)\right)\right]. \tag{22}$$

This propagator determines the evolution of the probability density for the particle to be at position x at time t:

$$\rho(x,t) = \int dx_0\, K(x,t;x_0,t_0)\rho(x_0,t_0) \tag{23}$$

This density must in turn obey the evolution equation

$$\frac{\partial}{\partial t}\rho(x,t) = -D\frac{\partial^2}{\partial x^2}\rho(x,t) + V(x)\rho(x,t) \tag{24}$$

[2]This is something I added later. I?m not sure how necessary it is for the methods of QFT to be applicable.

- **Path Integral for Relativistic Particle in Space time:** The path integral for a relativistic particle in curved space time is dependent on the action in curved space time. Going for the most transparent approach, we could write the path integral as

$$K = \int \mathcal{D}X(\tau)\, \exp\left[-\frac{im}{\hbar}\int_{\tau_0}^{\tau_f} d\tau \left(-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu\right)^{1/2}\right]. \tag{25}$$

But (by [2]) using an integration factor $\eta(x)$ we can write this propagator as

$$K = \int \mathcal{D}X(\tau)\mathcal{D}\eta(\tau)\, \exp\left[\frac{i}{2\hbar}\int_{\tau_0}^{\tau_f} d\tau \left(\eta(\tau)^{-1} g_{\mu\nu}(X)\dot{X}^\mu(\tau)\dot{X}^\nu(\tau) - \eta(\tau)m^2\right)\right]. \tag{26}$$

Integrating over the $\eta(\tau)$ variable amounts to solving its equation of motion. Doing so gives us

$$-\eta^{-2} g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu - m^2 = 0 \qquad \eta = m^{-1}\left(-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu\right)^{1/2}, \tag{27}$$

which when substituted into the action yields the original path integral. This is an example of a $0+1$ dimensional quantum field theory for a d dimensional vector X^μ in Minkowski space.

- **Equilibrium Thermodynamics of Nonlinear String:** The thermodynamics of a nonlinear string is defined by a partition function which has the same form as the euclidean functional integral. First, the Hamiltonian for the nonlinear string is

$$H = \int_0^L dx\, \left(\frac{1}{2\lambda}\pi^2 + \frac{T}{2}(\partial_x y)^2 + gy^3\right), \tag{28}$$

where π is the conjugate momentum of the field y. We note that in equilibrium thermodynamics we need to write all terms in the hamiltonian in time-independent forms. The corresponding partition function is then

$$\begin{aligned} Z &= \int \mathcal{D}y(x)\mathcal{D}\pi(x)\, \exp\left[-\frac{1}{k_B T}\int_0^L dx\, \left(\frac{1}{2\lambda}\pi^2 + \frac{T}{2}(\partial_x y)^2 + gy^3\right)\beta\right] \\ &= c_0 \int \mathcal{D}y(x)\, \exp\left[-\frac{1}{k_B T}\int_0^L dx\, \left(\frac{T}{2}(\partial_x y)^2 + gy^3\right)\beta\right]. \end{aligned} \tag{29}$$

We note that integrating over the conjugate momentum results in a thermodynamically irrelevant constant factor in front on the partition function. This is why in most discussions of statistical field theory (which is what we're considering here because the fluctuation parameter is temperature), no reference is made to the conjugate momentum of any field variables.

5 Subjects which Build Upon it

I once heard a joke referring to the fact that high energy physicists learn quantum field theory in kindergarten. Of course, this is not precisely true, but it does get at the deeper truth that quantum field theory is foundational to all modern areas of research in quantum field theory. Indeed many of the extensions in quantum field theory come from this realm.

- **Supersymmetry:** The fundamental mathematical objects in supersymmetry are superfields which contain super-multiplets. Arguably the generators (which are responsible for supersymmetry transformations) are fundamental as well. More qualitatively, supersymmetry requires for each boson in nature there exists a fermion which is related to the boson through a supersymmetry transformation. Two good books for this subject (that I have used) are [3] and [4].

- **String Theory:** I know virtually nothing about this subject. It concerns quantum theory of strings apparently. A possible good book (which I haven't read) on the subject is [5] (This book was recommended by Delilah Gates according to a recommendation by Strominger). A book with lengthier descriptions is [2].

- **AdS/CFT:** AdS/CFT connects a gravitational theory in anti-de Sitter space to a conformal field theory. The most common field theory studied in this context is $\mathcal{N} = 4$ Super Yang Mills theory. Two good reviews for this subject are [6] and [7].

References

[1] M. Srednicki, *Quantum field theory*. Cambridge University Press, 2007.

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[6] J. Kaplan, "Lectures on ads/cft from the bottom up."

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