Dynamic Analysis and Simulation of a 3 Rigid Body Spacecraft with 5 DOF's

ME 5010 Advanced Mechanics of Particles

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Introduction

A dynamic analysis can be performed on a mechanical system using Newton's Laws. When a system becomes too complex, however, using only Newton's Laws to perform a dynamic analysis can be tedious and difficult. Luckily there is an alternative, Lagrange's equations. Lagrange's equations are derived from Newton's Laws and allow for a systematic way of solving the equations of motion.

The system that is to be analyzed is the 3 rigid body spacecraft with 5 degrees of freedom, see Figure 1. The spacecraft is far away from any gravity fields thus gravity can be ignored. The spacecraft consists of a rigid central body called the "bus" and two panels that are connected to the bus via a single degree of freedom joint. The solar panels are modeled as rigid thin rods and their elasticity is modeled via a torsional spring placed at rotational joints between the panels and the bus. The properties of the spacecraft are summarized in Table 1.

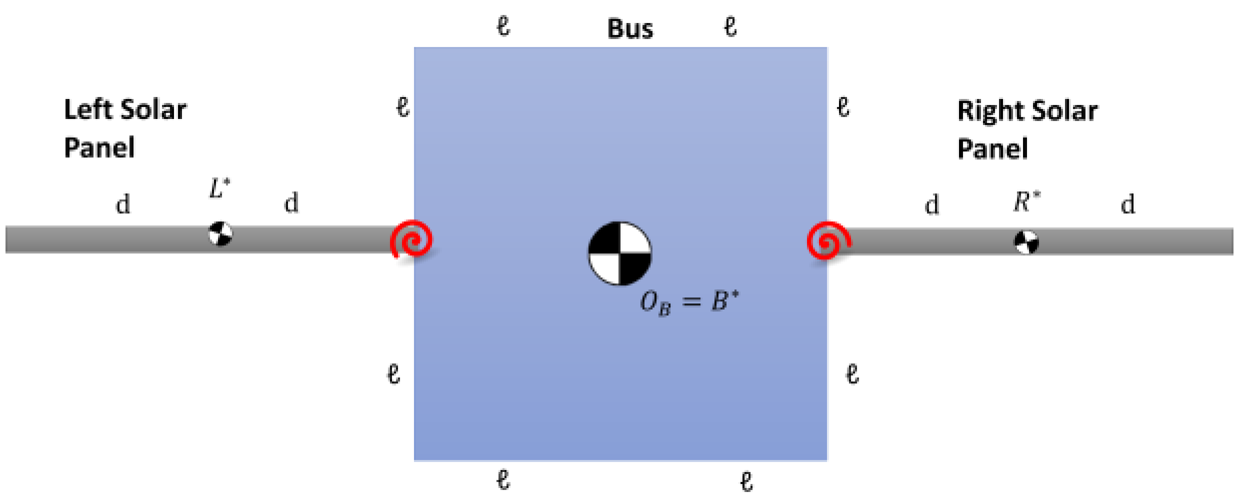


Figure 1 Three rigid body spacecraft with 5 degrees of freedom

Table 1 System Properties

Property	Symbol	Value
Mass of Bus	m_B	$122\ kg$
Mass of each panel	m_p	$2.7\ kg$
Inertia of Bus	I_B	$10\ kgm^2$
Half-length of sides of Bus	l	$0.3\ m$
Half-length of panel	d	$0.45\ m$
Natural frequency of panels	f_n	$6.3\ Hz$

Theory

The equations of motion can be obtained for the system by using Lagrange's Equations.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k \tag{1}$$

Where $\mathcal{L}$ is the Lagrangian of the system, q_k is selected degrees of freedom and Q_k is the generalized forces.

$$\mathcal{L} = \sum_{i=1}^{N} T - \sum_{i=1}^{N} U \tag{2}$$

The Lagrangian, $\mathcal{L}$, is defined by Equation 2 where T is the kinetic energy and U is the potential energy of the system, and N is the number of masses. The kinetic energy of this system for mass can be obtained by the following,

$$T = \frac{1}{2}\mathrm{m}\boldsymbol{v}^2 \tag{3}$$

$$T = \frac{1}{2} I \, \dot{\theta}^2 \tag{4}$$

where m is the mass, $\boldsymbol{v}$ is the velocity of the of the Nth mass with respect to the inertial frame, I is the moment of inertia and $\dot{\theta}$ is the angular velocity. The potential energy of this system can be defined by the following,

$$U = \frac{1}{2} k \, \theta^2 \tag{5}$$

where k is the stiffness of the spring and θ is the angular displacement of the spring.

Problem Formulation

Figure 2 shows a detailed schematic of the 3 rigid body system that is to be analyzed. The system has five degrees of freedom x , y , $\theta_B, \theta_L and\ \theta_R$The system was separated into 4 different frames, where N is the inertial frame, P is the frame fixed on the bus, E is the frame fixed on the left panel and R is the frame fixed on the right panel.

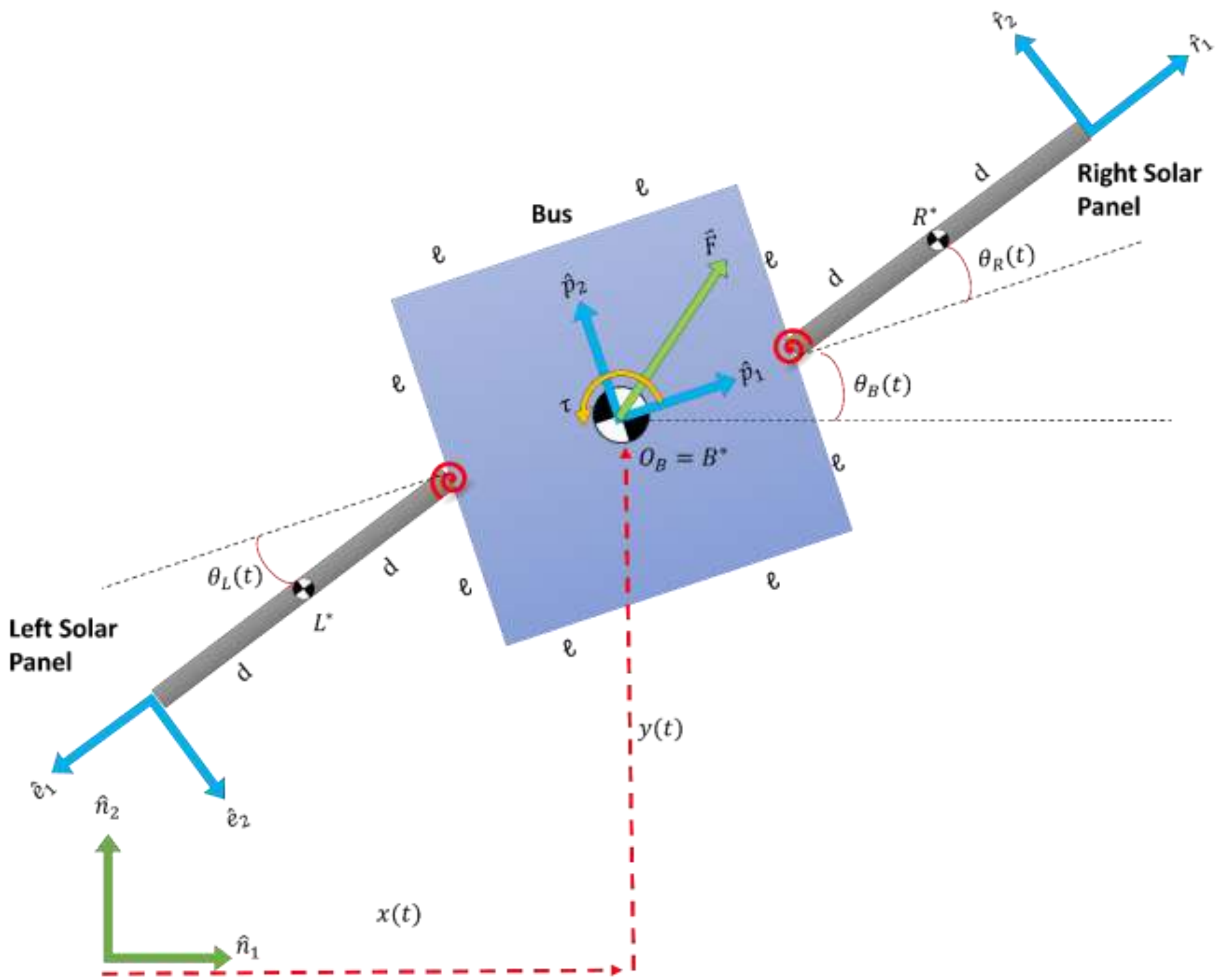


Figure 2 Labeled Diagram of the System

To find the equations of motion using Lagrange's Equations the position vectors for every center of mass must be identified. The following are the position vectors for every mass,

$$\vec{R}_B = x\,\hat{n}_1 + y\,\hat{n}_2$$

$$\vec{R}_L = x\,\hat{n}_1 + y\,\hat{n}_2 - l\,\hat{p}_1 + d\hat{e}_1$$

$$\vec{R}_R = x\,\hat{n}_1 + y\,\hat{n}_2 + l\,\hat{p}_1 + d\hat{r}_1$$

were $\vec{R}_{\mathrm{B}}, \vec{R}_L, \vec{R}_R$ are the position vectors for the bus, left panel, and right panel respectively. By taking the time derivative of the position vector with respect to the inertial frame N, the following velocities are obtained.

$$\vec{V}_B = \dot{x}\,\hat{n}_1 + \dot{y}\,\hat{n}_2$$

$$\vec{V}_L = \dot{x}\,\hat{n}_1 + \dot{y}\,\hat{n}_2 - l\dot{\theta}_B\hat{p}_2 + d\big(\dot{\theta}_B + \dot{\theta}_L\big)\hat{e}_2$$

$$\vec{V}_R = \dot{x}\,\hat{n}_1 + \dot{y}\,\hat{n}_2 + l\dot{\theta}_B\hat{p}_2 + d\big(\dot{\theta}_B + \dot{\theta}_R\big)\hat{r}_2$$

With the velocities in hand the kinetic energy can be computed using Equation 3 and Equation 4 and results in the following,

$$\begin{aligned} T = \frac{1}{2}\, m_B(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\, I_B\dot{\theta}_B^2 \\ + \frac{1}{2} m_p \Big[\, 2\dot{x}^2 + 2\dot{y}^2 + 2l^2\dot{\theta}_B^2 + d^2\big(\dot{\theta}_B + \dot{\theta}_L\big)^2 \\ + 2\dot{x}d\big(\dot{\theta}_B + \dot{\theta}_L\big)\sin(\theta_B + \theta_L) + \ 2l\,\dot{\theta}_B d\big(\,\dot{\theta}_B + \ \dot{\theta}_L\big)\cos(\theta_L) \\ - 2\dot{y}d\big(\dot{\theta}_B + \ \dot{\theta}_L\big)\cos(\theta_B + \theta_L) + d^2\big(\dot{\theta}_B + \dot{\theta}_R\big)^2 \\ - 2\dot{x}d\big(\dot{\theta}_B + \dot{\theta}_R\big)\sin(\theta_B + \theta_R) + 2l\dot{\theta}_B d\big(\dot{\theta}_B + \dot{\theta}_R\big)\cos(\theta_R) \\ + 2\dot{y}d\big(\dot{\theta}_B + \dot{\theta}_R\big)\cos(\theta_B + \theta_R)\Big] + \frac{1}{2} I_p\big(\dot{\theta}_B + \dot{\theta}_R\big)^2 + \frac{1}{2} I_p\big(\dot{\theta}_B + \dot{\theta}_L\big)^2 \end{aligned} \tag{6}$$

Using Equation 5 the potential energy can be computed and results in the following,

$$U = \frac{1}{2} k\,\theta_r^2 + \frac{1}{2} k\,\theta_L^2 \tag{7}$$

since the spacecraft is away from any gravity fields, there is no potential energy due to gravity. Once the kinetic and potential energy are found, the Lagrangian can be computed by plugging in Equation 6 and Equation 7 into Equation 2 which yield the following

$$\begin{aligned} \mathcal{L} = \frac{1}{2}\, m_B(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\, I_B\dot{\theta}_B^2 \\ + \frac{1}{2} m_p \Big[\, 2\dot{x}^2 + 2\dot{y}^2 + 2l^2\dot{\theta}_B^2 + d^2\big(\dot{\theta}_B + \dot{\theta}_L\big)^2 \\ + 2\dot{x}d\big(\dot{\theta}_B + \dot{\theta}_L\big)\sin(\theta_B + \theta_L) + \ 2l\,\dot{\theta}_B d\big(\,\dot{\theta}_B + \ \dot{\theta}_L\big)\cos(\theta_L) \\ - 2\dot{y}d\big(\dot{\theta}_B + \ \dot{\theta}_L\big)\cos(\theta_B + \theta_L) + d^2\big(\dot{\theta}_B + \dot{\theta}_R\big)^2 \\ - 2\dot{x}d\big(\dot{\theta}_B + \dot{\theta}_R\big)\sin(\theta_B + \theta_R) + 2l\dot{\theta}_B d\big(\dot{\theta}_B + \dot{\theta}_R\big)\cos(\theta_R) \\ + 2\dot{y}d\big(\dot{\theta}_B + \dot{\theta}_R\big)\cos(\theta_B + \theta_R)\Big] + \frac{1}{2} I_p\big(\dot{\theta}_B + \dot{\theta}_R\big)^2 + \frac{1}{2} I_p\big(\dot{\theta}_B + \dot{\theta}_L\big)^2 \\ - \frac{1}{2} k\,\theta_r^2 - \frac{1}{2} k\,\theta_L^2 \end{aligned} \tag{8}$$

Once the Lagrangian is obtained, the equations of motion can be found by using Equation 1, Lagrange's Equation, and solving for all five the degrees of freedom with the following terms:

$$\dot{q} = \dot{x}, \dot{y}, \dot{\theta}_B, \dot{\theta}_L, \dot{\theta}_R$$

$$q = x\,, y\,, \theta_B, \theta_L, \theta_R$$

$$Q_k = \ F_x\,, F_y, \tau\,, 0\,, 0\,, 0$$

Solving for all five Lagrange's Equations yields the following equations of motion:

$\boldsymbol{X}$ EOM

$$m_p\left[2\ddot{x} - d\left(\ddot{\theta}_B + \ddot{\theta}_R\right)\sin(\theta_B + \theta_R) - d\left(\dot{\theta}_B + \dot{\theta}_R\right)^2\cos(\theta_B + \theta_R) + d\left(\ddot{\theta}_B + \ddot{\theta}_L\right)\sin(\theta_B + \theta_L) + d\left(\dot{\theta}_B + \dot{\theta}_L\right)^2\cos(\theta_B + \theta_L)\right] + m_b\ddot{x} = \vec{F}_x\hat{n}_1$$

$\boldsymbol{Y}$ EOM

$$m_p\left[2\ddot{y} - d\left(\ddot{\theta}_B + \ddot{\theta}_L\right)\cos(\theta_B + \theta_L) + d\left(\dot{\theta}_B + \dot{\theta}_L\right)^2\sin(\theta_B + \theta_L) + d\left(\ddot{\theta}_B + \ddot{\theta}_R\right)\cos(\theta_B + \theta_R) - d\left(\dot{\theta}_B + \dot{\theta}_R\right)^2\sin(\theta_B + \theta_R)\right] + m_B\ddot{y} = \vec{F}_y\hat{n}_2$$

$\boldsymbol{\theta_B}$ EOM

$$m_p\left[2l^2\ddot{\theta}_B + d^2\left(\ddot{\theta}_B + \ddot{\theta}_L\right) + \ddot{x}dsin(\theta_B + \theta_L) + ld\left(2\ddot{\theta}_B + \ddot{\theta}_L\right)\cos(\theta_L) - ld\left(2\dot{\theta}_B + \dot{\theta}_L\right)\dot{\theta}_l\sin(\theta_L) - \ddot{y}dcos(\theta_B + \theta_L) + d^2\left(\ddot{\theta}_B + \ddot{\theta}_R\right) - \ddot{x}dsin(\theta_B + \theta_R) + ld\left(2\ddot{\theta}_B + \ddot{\theta}_R\right)\cos(\theta_R) - ld\left(2\dot{\theta}_B + \dot{\theta}_R\right)\dot{\theta}_R\sin(\theta_R) + \ddot{y}dcos(\theta_B + \theta_R)\right] + I_B\ddot{\theta}_b + I_p\left(\ddot{\theta}_B + \ddot{\theta}_L\right) + I_p\left(\ddot{\theta}_B + \ddot{\theta}_R\right) = \tau$$

$\boldsymbol{\theta_L}$ EOM

$$m_p\left[d^2\left(\ddot{\theta}_B + \ddot{\theta}_L\right) + \ddot{x}dsin(\theta_B + \theta_L) - \ddot{y}dcos(\theta_B + \theta_L) + l\ddot{\theta}_Bdcos(\theta_L) - l\dot{\theta}_B\dot{\theta}_Ldsin(\theta_L) + l\dot{\theta}_Bd\left(\dot{\theta}_B + \dot{\theta}_L\right)\sin(\theta_L)\right] + I_p\left(\ddot{\theta}_B + \ddot{\theta}_L\right) + k\theta_L = 0$$

$\boldsymbol{\theta_R}$EOM

$$m_p\left[d^2\left(\ddot{\theta}_B + \ddot{\theta}_R\right) - \ddot{x}dsin(\theta_B + \theta_R) + l\ddot{\theta}_Bdcos(\theta_r) + \ddot{y}dcos(\theta_B + \theta_R) - l\dot{\theta}_B\dot{\theta}_Rdsin(\theta_r) + l\dot{\theta}_Bd\left(\dot{\theta}_B + \dot{\theta}_R\right)\sin(\theta_R)\right] + I_p\left(\ddot{\theta}_B + \ddot{\theta}_R\right) + k\theta_R = 0$$

Simulation Setup

To simulate the response of the system the equations of motion must be first in the matrix form of:

$$\underline{\underline{M}}(q)\ddot{q} + \underline{C}(q,\dot{q}) + \underline{K}q = \underline{Q} \tag{9}$$

Where $\underline{\underline{M}}$ is the mass matrix, $\ddot{q}$ is acceleration of the degrees of freedom matrix, $\underline{C}$ the Coriolis matrix, $\underline{K}$ is the stiffness matrix, q is the degrees of freedom matrix and $\underline{Q}$ is the generalized applied force matrix and are defined as the following:

$$\underline{\underline{M}} = \begin{bmatrix} m_B + 2m_p & 0 & m_pd(S_{BL} - S_{BR}) & m_pdS_{BL} & -m_pdS_{BR} \\ 0 & m_B + 2m_p & m_pd(C_{BR} - C_{BL}) & -m_pdC_{BL} & m_pdC_{BR} \\ m_pd(S_{BL} - S_{BR}) & m_pd(C_{BR} - C_{BL}) & \begin{matrix} 2m_pd^2 + 2m_pldC_L \\ +2m_pldC_R + 2m_pl^2 \\ +2I_p + I_B \end{matrix} & m_pd^2 + m_pldC_L + I_p & m_Pd^2 + m_pldC_R + I_p \\ m_pdS_{BL} & -m_pdC_{BL} & m_pd^2 + m_pldC_L + I_p & m_pd^2 + I_p & 0 \\ -m_pdS_{BR} & m_pdC_{BR} & m_pd^2 + m_pldC_R + I_p & 0 & m_pd^2 + I_p \end{bmatrix}$$

$$\ddot{q} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta}_B \\ \ddot{\theta}_L \\ \ddot{\theta}_R \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} m_p d\left(\dot{\theta}_B + \dot{\theta}_L\right)^2 C_{BL} - m_p d\left(\dot{\theta}_B + \dot{\theta}_R\right)^2 C_{BR} \\ m_p d\left(\dot{\theta}_B + \dot{\theta}_L\right)^2 S_{BL} - m_p d\left(\dot{\theta}_B + \dot{\theta}_R\right)^2 S_{BR} \\ -m_p ld\left(2\dot{\theta}_B + \dot{\theta}_L\right)\dot{\theta}_L S_L - m_p ld\left(2\dot{\theta}_B + \dot{\theta}_R\right)\dot{\theta}_R S_R \\ m_p ld\ \dot{\theta}^2 S_L \\ m_p ld\dot{\theta}^2 S_R \end{bmatrix}$$

were

$$S_{BL} = \sin(\theta_B + \theta_L)$$

$$S_{BR} = \sin(\theta_B + \theta_R)$$

$$C_{BL} = \cos(\theta_B + \theta_L)$$

$$C_{BR} = \cos(\theta_B + \theta_R)$$

$$S_L = \sin(\theta_L)$$

$$S_R = \sin(\theta_R)$$

$$C_L = \cos(\theta_L)$$

$$C_R = \cos(\theta_R)$$

$$\underline{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix}$$

$$q = \begin{bmatrix} x \\ y \\ \theta_B \\ \theta_L \\ \theta_R \end{bmatrix}$$

$$\underline{Q} = \begin{bmatrix} F_x \\ F_y \\ \tau \\ 0 \\ 0 \end{bmatrix}$$

To use MATLAB to simulate the response, Equation 9 must be solved for $\ddot{q}$ which yields the following:

$$\ddot{q} = \underline{\underline{M}}(q)^{-1}\left[\underline{Q} + \underline{C}(q,\dot{q}) - \underline{K}q\right] \tag{10}$$

The state equations for the system are the following:

$$\begin{aligned}
X_1 &= x & \dot{X}_1 &= \dot{x} \\
X_2 &= y & \dot{X}_2 &= \dot{y} \\
X_3 &= \theta_B & \dot{X}_3 &= \dot{\theta}_B \\
X_4 &= \theta_L & \dot{X}_4 &= \dot{\theta}_L \\
X_4 &= \theta_L & \dot{X}_4 &= \dot{\theta}_L \\
X_5 &= \theta_R & \dot{X}_5 &= \dot{\theta}_R \\
X_6 &= \dot{x} & \dot{X}_6 &= \ddot{x} = \ddot{q}_1 \\
X_7 &= \dot{y} & \dot{X}_7 &= \ddot{y} = \ddot{q}_2 \\
X_8 &= \dot{\theta}_B & \dot{X}_8 &= \ddot{\theta}_B = \ddot{q}_3 \\
X_9 &= \dot{\theta}_L & \dot{X}_9 &= \ddot{\theta}_L = \ddot{q}_4 \\
X_{10} &= \dot{\theta}_L & \dot{X}_{10} &= \ddot{\theta}_L = \ddot{q}_5
\end{aligned}$$

The reason the states were chosen in this order was for state $\dot{X}_6$ through $\dot{X}_{10}$ and the values of Equation 10 to be aligned.

Simulation

Each simulation was run with various initial conditions for θ_L and θ_R. The initial values of these two degrees of freedom determine the symmetry of the system. In the symmetric case, θ_R was set to 6° and θ_L to -6° to set both panels in the same direction, see Figure 3. Similarly, in the antisymmetric case, the angles of the panels were chosen to set both panels in the opposite direction as shown in Figure 4.

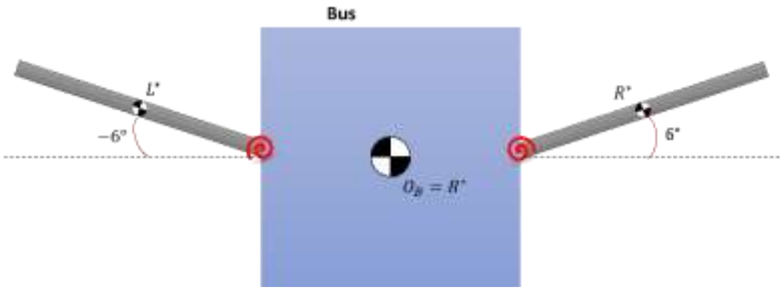


Figure 3 Symmetric case

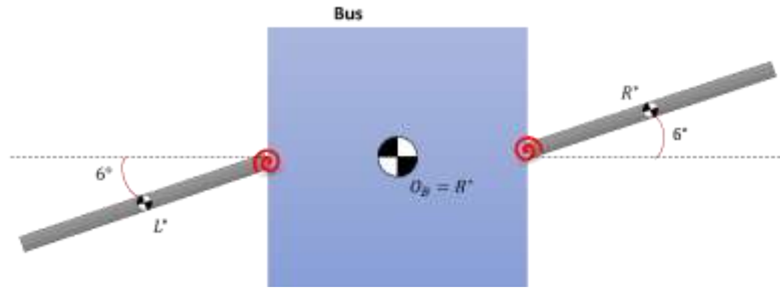


Figure 4 Antisymmetric Case

Half of the simulations were also run with an initial velocity in the x-direction. The initial velocity $\dot{x}_0$ was set to 10m/s, and this provided a translational motion for the bus. These simulations are shown in figures 7, 8, 11, and 12.

The simulations also included cases with external force applied and no force applied. The cases with an external force applied had a force F_x, F_y and τ of 10N. These simulations are shown in figures 9-12.

The simulation was run in MATLAB using the ODE5 differential equation solver with a step size of 0.01 seconds. This step size was chosen because it sufficiently captures the small angle motion of the panels without accumulating too much integration error.

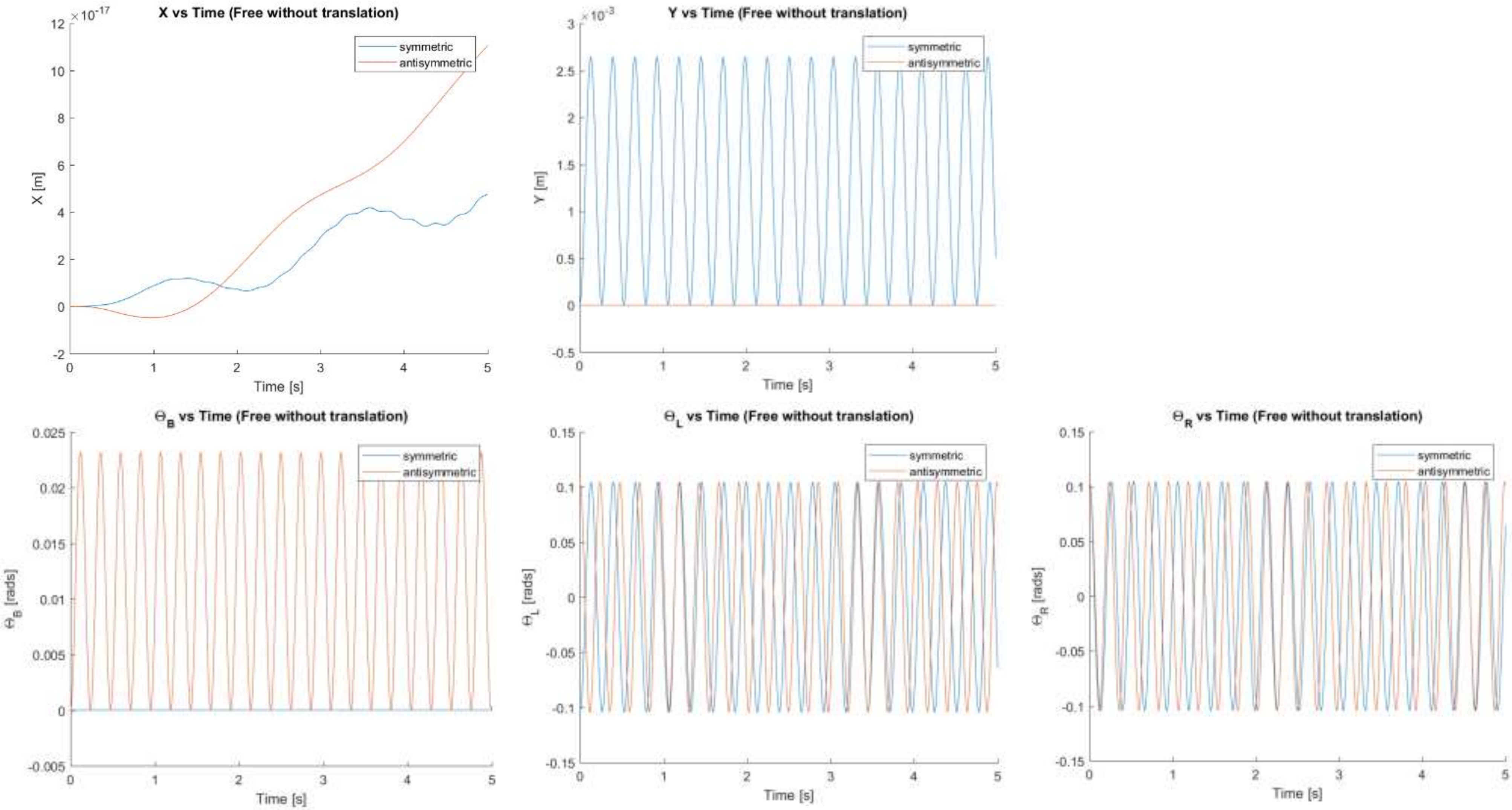


Figure 5 Displacements of MATLAB simulation with free motion and no translation

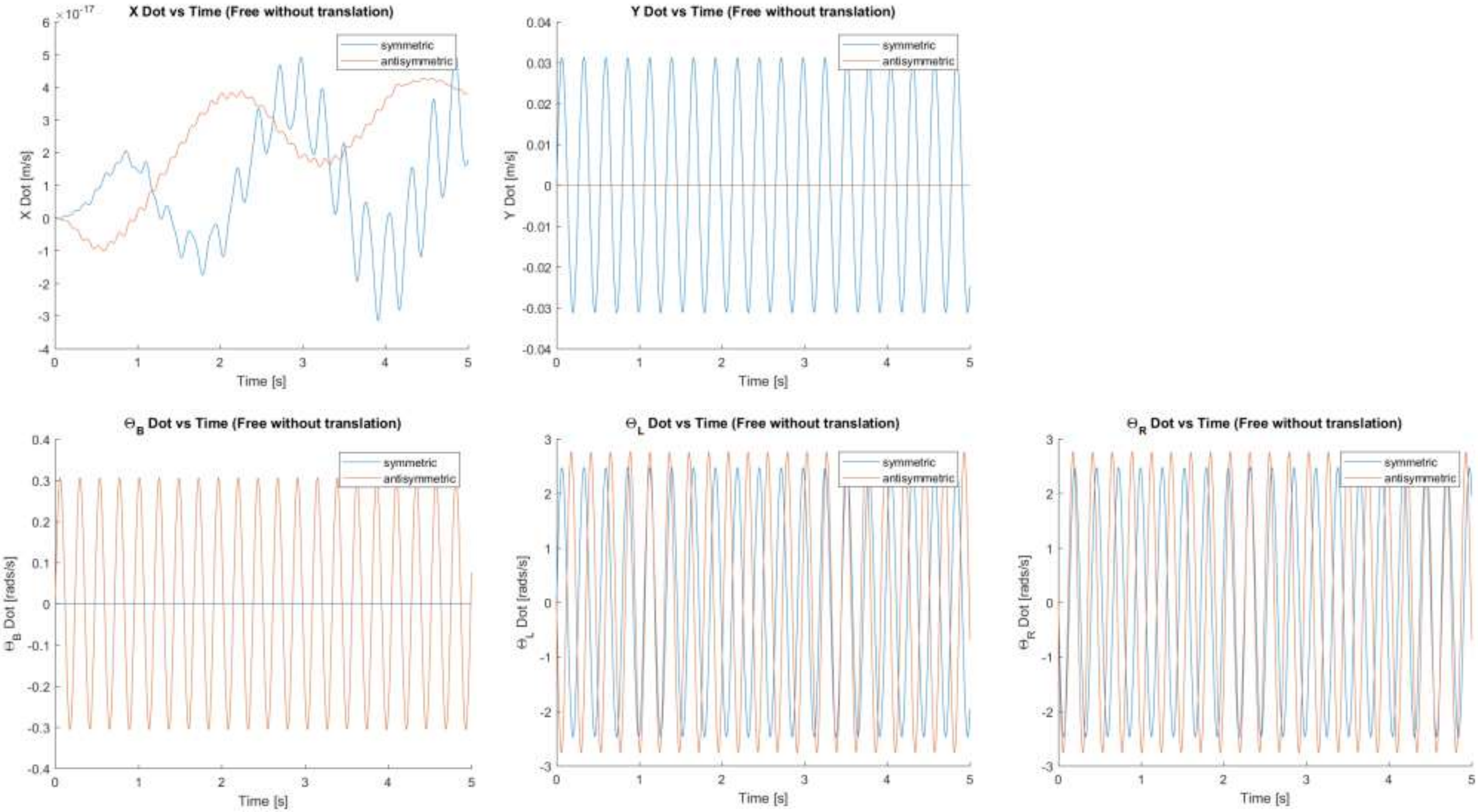


Figure 6 Velocities of MATLAB simulation with free motion and no translation

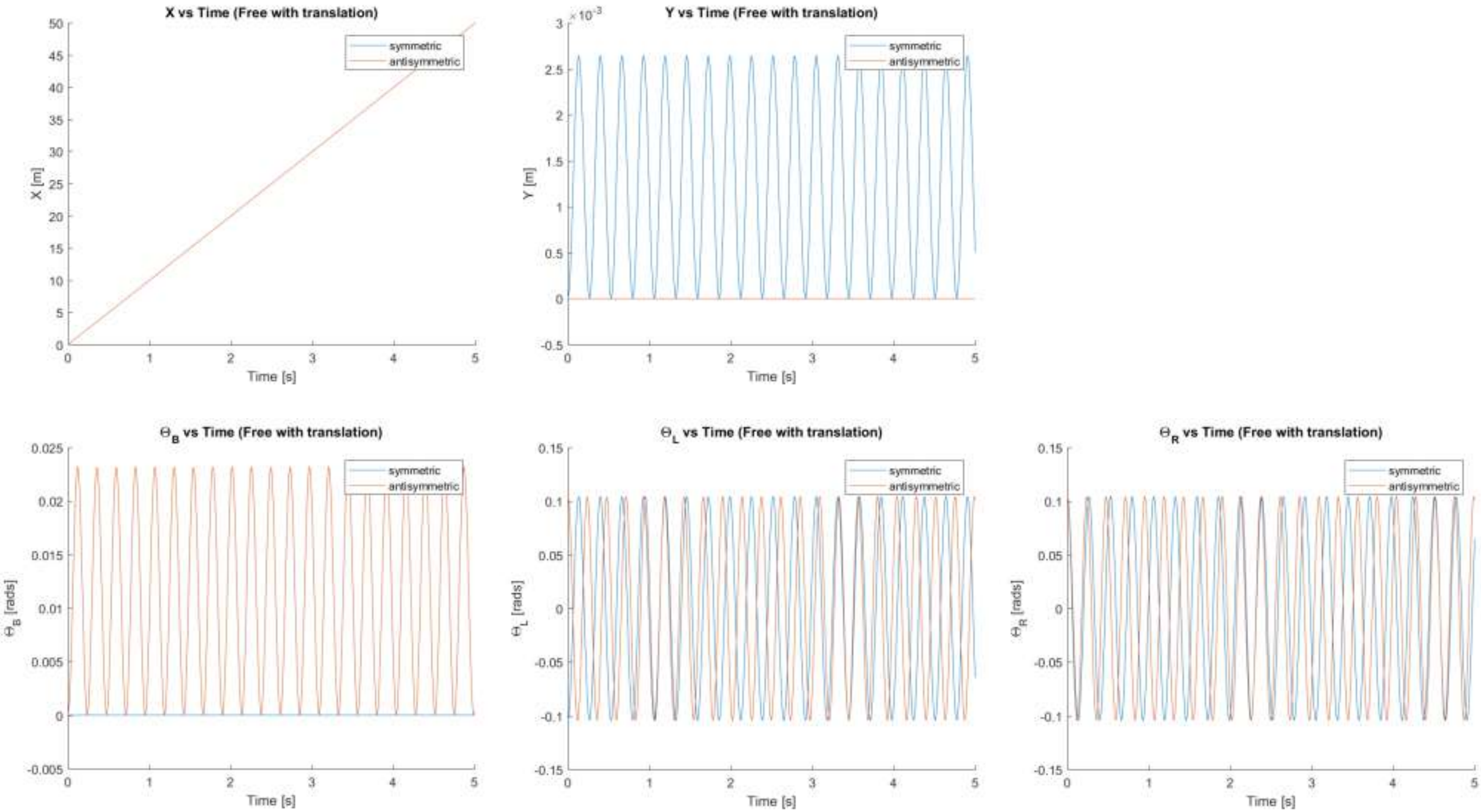


Figure 7 Displacements of MATLAB simulation with free motion and translation

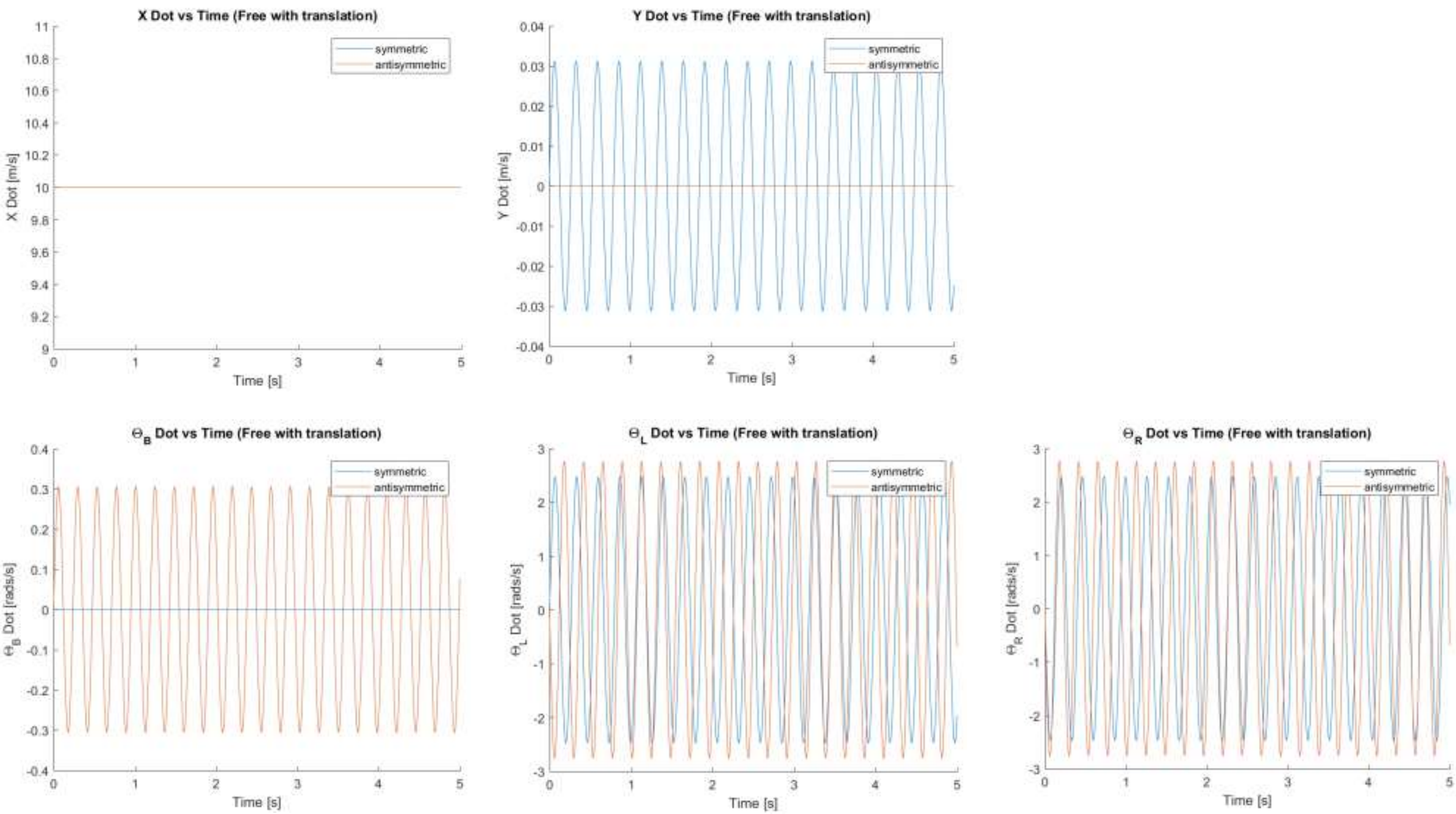


Figure 8 Velocities of MATLAB simulation with free motion and translation

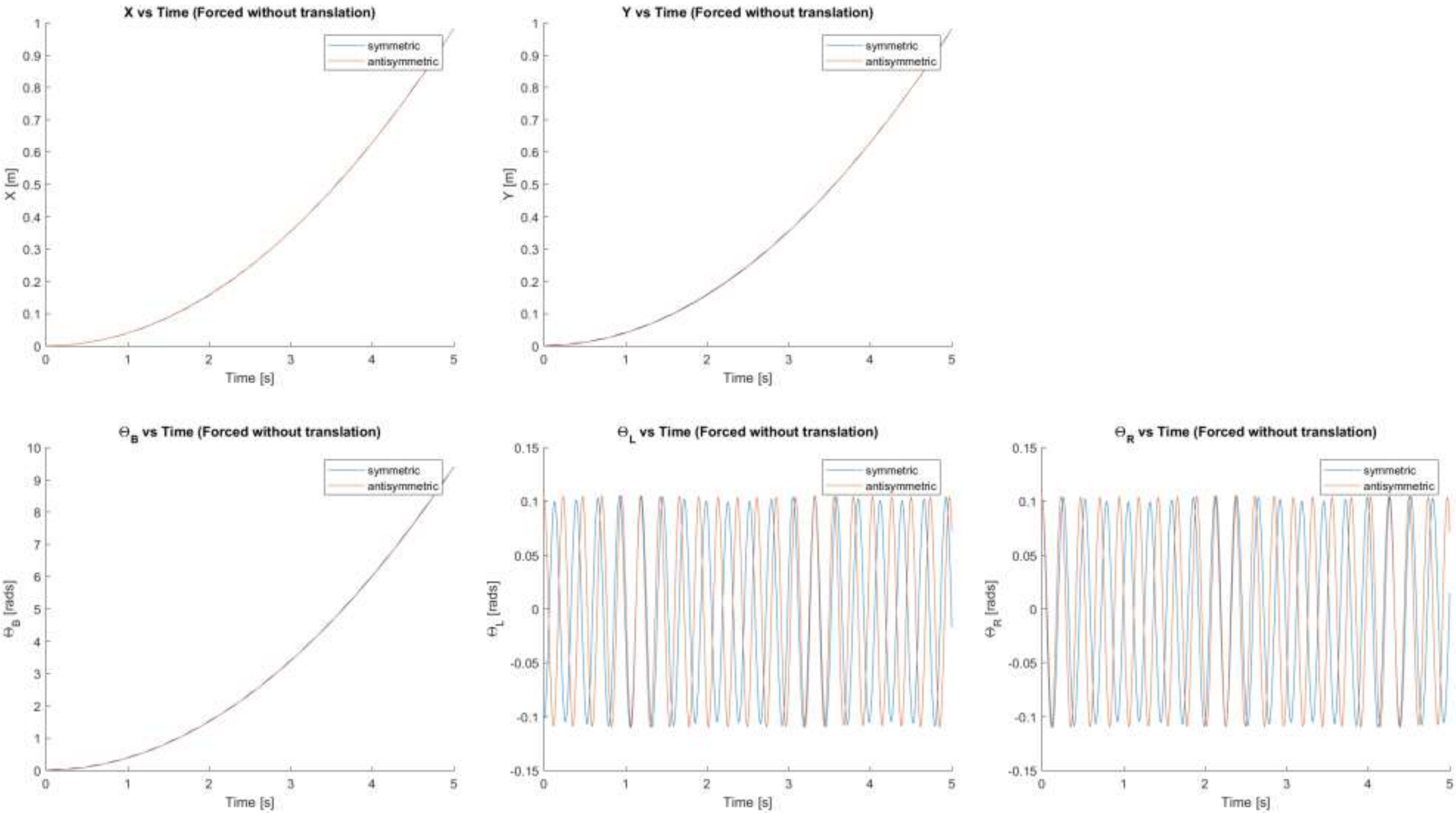


Figure 9 Displacements of MATLAB simulation with forced motion and no translation

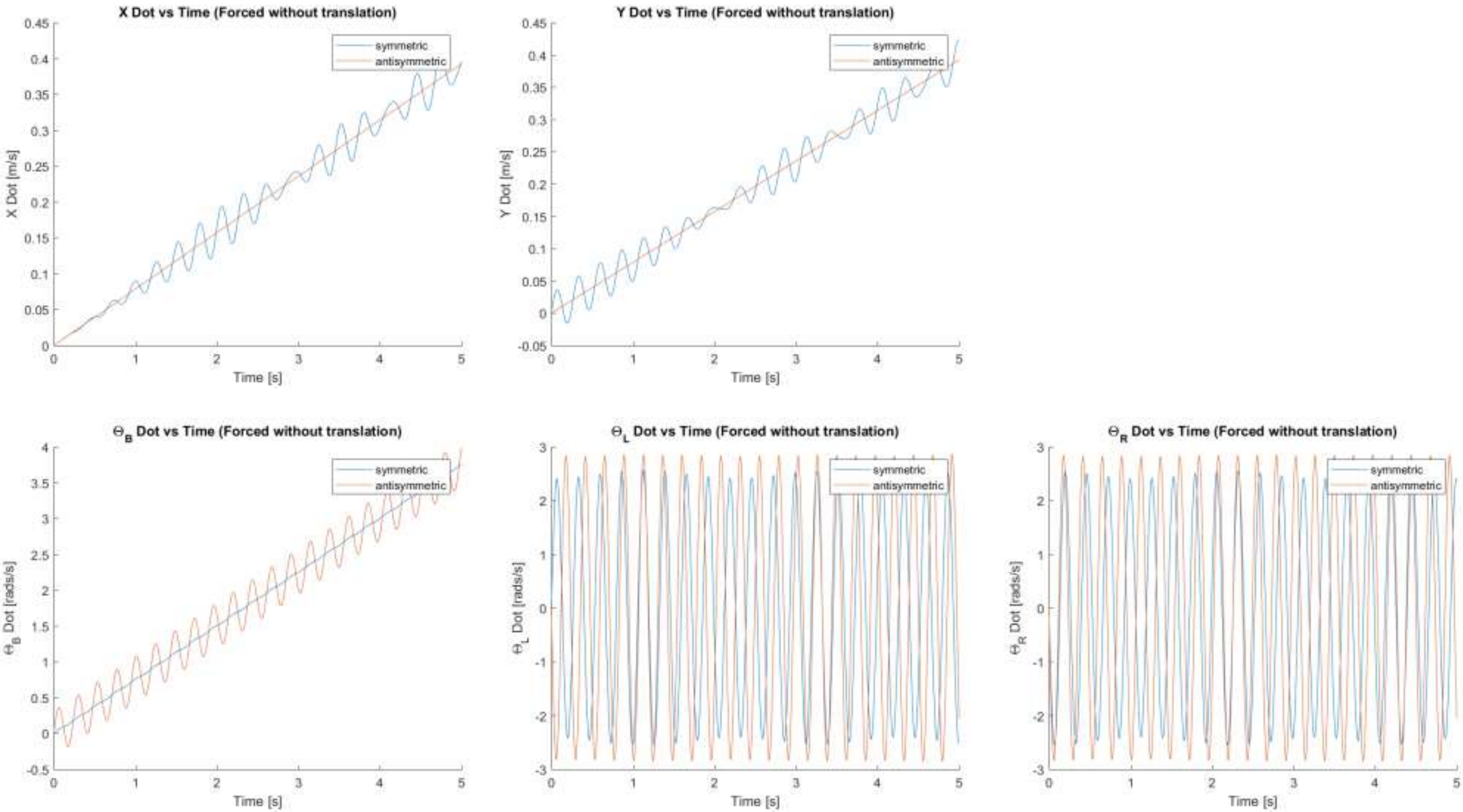


Figure 10 Velocities of MATLAB simulation with forced motion and no translation

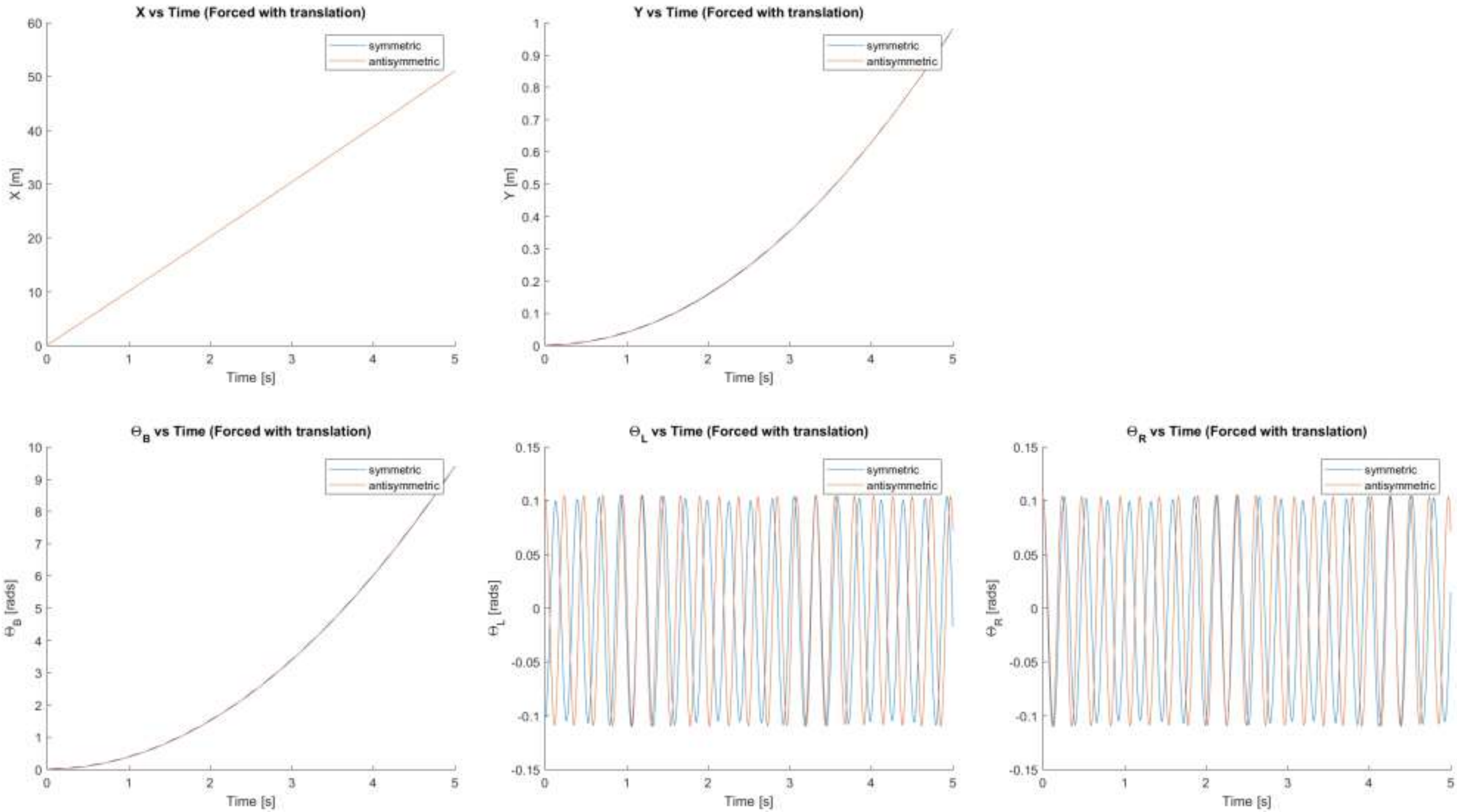


Figure 11 Displacements of MATLAB simulation with forced motion and translation

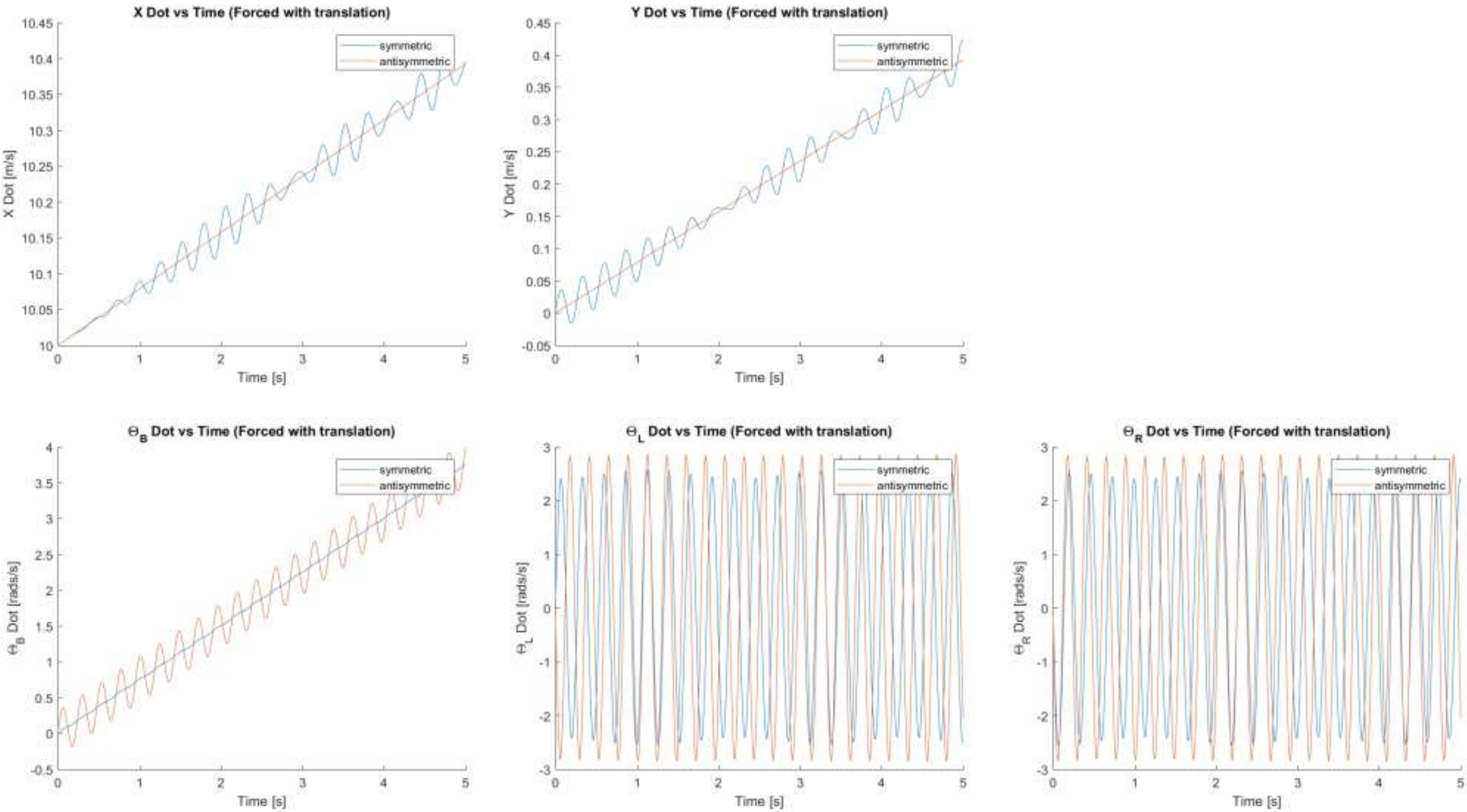


Figure 12 Velocities of MATLAB simulation with forced motion and translation

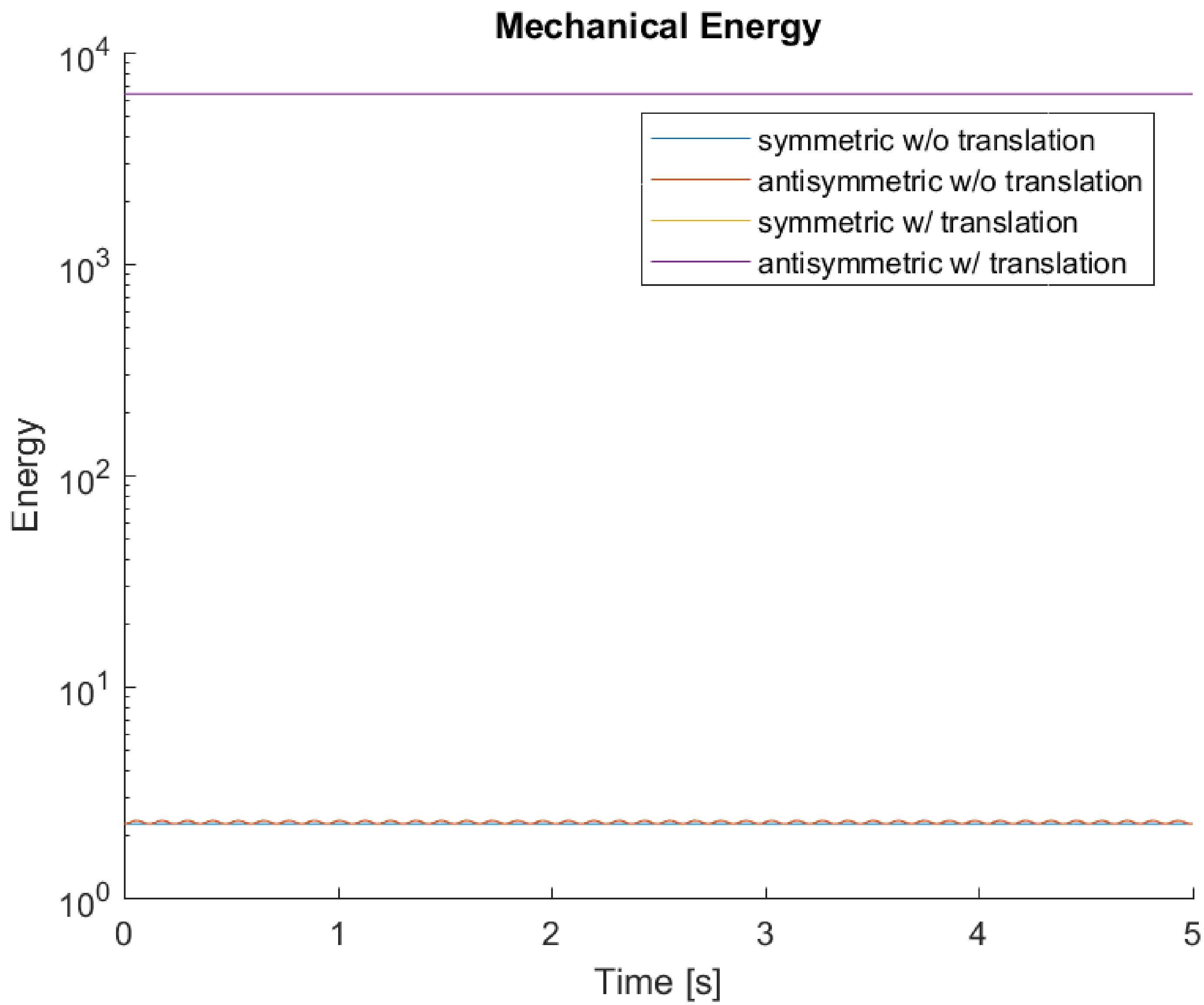


Figure 13 Mechanical Energy for Simulations

Discussion

To verify that the equations of motions are valid the total mechanical energy of the system must be conservative throughout the simulation. The total mechanical energy was computed for each simulation using the following relationship,

$$E = T + U \tag{11}$$

where E is the total mechanical energy and T and U are kinetic energy and potential energy respectively. Figure 13 shows a plot of the total mechanical energy of the system for the simulations, from the plot it is evident that the mechanical energy of the system remains constant thus implying mechanical energy is conserved. Since the total mechanical energy is conserved throughout the simulation then the equations of motion can be validated.

The simulations also show that when the bus has an initial velocity and a force applied, as shown in Figures 11-12, the mass of the panels does not affect the dynamics of the system. Conversely, the other simulations show that the mass of the panels and the initial conditions of the panels affect the overall dynamics of the system by introducing an oscillation in various degrees of freedom. These oscillations depend on the mass of the panels relative to the mass of the base, and the motion of the panels relative to the motion of the base. When most of the mass and force is on the base, the panels have a negligible effect on the overall dynamics of the system.

Conclusion

The equations of motion for a satellite with 5 degrees of freedom were accurately obtained using Lagrange's equations. These equations of motion were verified by simulating their motion in MATLAB and observing the response of the system for various cases. The simulations of the symmetric and antisymmetric cases supported the validity of the equations since they showed conservation of momentum. The equations were also verified by finding the equations for kinetic and potential energy and plotting the conservation of energy. The MATLAB simulations showed that energy was constant and conserved for all cases with no external force applied, and this further supports that the equations of motion derived from Lagrange's equations are valid.

Appendix 1 MATLAB code

```
close all; clear all; clc;
%% This script simulates the EOM of  Planar 3 rigid body
spacecraft
% With 5 DOF
% Created By Francisco Moxo and David Garcia: 5-15-2018
% Modified By 5/16/2018 by: Francisco Moxo


%% Initial positions and velocities
% X0 = [X , Y , ThetaB, ThetaL, ThetaR, Xdot , Ydot , ThetaBdot,
ThetaLdot, ThetaRdot]
X0 =    [0 0 0 -6*pi/180 6*pi/180  0 0 0 0 0;
    0 0 0 6*pi/180 6*pi/180  0 0 0 0 0;
    0 0 0 -6*pi/180 6*pi/180  10 0 0 0 0;
    0 0 0 6*pi/180 6*pi/180  10 0 0 0 0];
[qstring, q2string, lgdstring, Tstring] = plotlgd();
[numset ,j] = size(X0);
stepsize = .01; %stepsize for ODE 5


%% Free Motion with various Initial conditions
y1=[]; %Contains All the Simulation data For Free Motion
for k = 1:numset
    [t , yout] = ODESolver(X0(k,:),stepsize,0,0,0);

    y1 = [y1,yout]; % Saves all data in matriix y

    for i  = 1:10
        if k < 3
            figure(i);
        else
            figure(i+10);
        end
        hold on
        plot(t, yout(:,i));
        %Saves all images in PNG format to subfolde /simImg
        if ~mod(k,2)
            title(strcat(qstring(i),' vs Time
(Free',Tstring(k/2),')'));
            ylabel(lgdstring{i});
            xlabel('Time [s]');
            legend('symmetric','antisymmetric');
            filename =
strcat(pwd,'/simImg/free',Tstring(k/2),num2str(i-
1),q2string(i),'vsTime.png');
            saveas(gcf,filename{1});
```

```
        end
    end
end

%% Constant Forces with various Initial conditions
y2=[]; %Contains All the Simulation data for applied forces
% Continous Forces To be applied to System
Fx =[10 10 10 10]; % [N]
Fy =[10 10 10 10]; % [N]
Tau =[10 10 10 10]; % [Nm]
for k = 1:numset
    [t , yout] = ODESolver(X0(k,:),stepsize,
Fx(k),Fy(k),Tau(k));

    y2 = [y2,yout]; % Saves all data in matriix y

    for i  = 1:10
        if k < 3
            figure(i+20);
        else
            figure(i+30);
        end
        hold on
        plot(t, yout(:,i));
        %Saves all images in PNG format to subfolde /simImg
        if ~mod(k,2)
            title(strcat(qstring(i),' vs Time
(Forced',Tstring(k/2),')'));
            ylabel(lgdstring{i});
            xlabel('Time [s]');
            legend('symmetric','antisymmetric');
            filename =
strcat(pwd,'/simImg/forced',Tstring(k/2),num2str(i-
1),q2string(i),'vsTime.png');
            saveas(gcf,filename{1});
        end
    end
end

%% Conservation of Energy Check
figure(41)
hold on
for k = 1:numset
    E = TotalMechEng(y1(:,10*(k-1)+1:10*k));
    loglog(t,E);
end
set(gca, 'YScale', 'log');
```

```
title('Mechanical Energy');
ylabel('Energy');
xlabel('Time [s]');
legend('symmetric w/o translation','antisymmetric w/o
translation',...
    'symmetric w/ translation','antisymmetric w/ translation');
filename = strcat(pwd,'/simImg/','Energy.png');
saveas(gcf,filename);

%% Functions for Ploting
function [E] = TotalMechEng(yout)
% Variables
x  =yout(:,1);
y  =yout(:,2);
tb =yout(:,3);
tl =yout(:,4);
tr =yout(:,5);
xd =yout(:,6);
yd =yout(:,7);
bd =yout(:,8);
ld =yout(:,9);
rd =yout(:,10);
% Constants
mb = 122;  % [kg]
mp = 2.7;  % [kg]
ib = 10;   % [kg m^2]
ip = 0.13; % [kg m^2]
d  = 0.3;  % [m]
l  = 0.45; % [m]
fn = 6.3;  % [Hz]
k = ip*(2*pi*fn)^2;
% Variables to simplify matrix
sbl = sin(tb+tl);
cbl = cos(tb+tl);
sbr = sin(tb+tr);
cbr = cos(tb+tr);
sb = sin(tb);
cb = cos(tb);
sr = sin(tr);
cr = cos(tr);
sl = sin(tl);
cl = cos(tl);

% Kinetic Energy
T= .5*mb.*(xd.^2+yd.^2)+...
```

```
.5*mp.*(xd.^2+yd.^2+l.^2*bd.^2+d.^2.*(bd+ld).^2+2*d.*xd.*(bd+ld)
.*sbl+...
    2*l.*d.*bd.*(bd+ld).*cl-2*d.*yd.*(bd+ld).*cbl+...
    xd.^2+yd.^2+l^.2.*bd.^2+d.^2.*(bd+rd).^2-
2*d.*xd.*(bd+rd).*sbr+...
    2*l.*d.*bd.*(bd+rd).*cr+2*d.*yd.*(bd+rd).*cbr)...
    +.5*ib.*bd.^2+.5*ip.*((bd+rd).^2+(bd+ld).^2);


%potential Energy
U= .5*k.*tr.^2+.5*k.*tl.^2;

%Total Kinetic Energy
E = T + U;
end
function [qstring, q2string, lgdstring, Tstring] = plotlgd()
qstring={'X','Y','\Theta_B','\Theta_L','\Theta_R','X Dot','Y
Dot','\Theta_B Dot','\Theta_L Dot','\Theta_R Dot'};
q2string={'X','Y','Theta_B','Theta_L','Theta_R','X Dot','Y
Dot','Theta_B Dot','Theta_L Dot','Theta_R Dot'};
lgdstring ={'X [m]','Y [m]','\Theta_B [rads]','\Theta_L
[rads]','\Theta_R [rads]','X Dot [m/s]','Y Dot [m/s]','\Theta_B
Dot [rads/s]','\Theta_L Dot [rads/s]','\Theta_R Dot [rads/s]'};
Tstring={' without translation',' with translation'};

end
function [t , y] = ODESolver(X0,stepsize,fx,fy,tau)

%% Constants and Initial Conditions
param.mb = 122;  % [kg]
param.mp = 2.7;  % [kg]
param.ib = 10;   % [kg m^2]
param.ip = 0.13; % [kg m^2]
param.d  = 0.3;  % [m]
param.l  = 0.45; % [m]
param.fn = 6.3;  % [Hz]
param.fx = fx;    % [N]
param.fy = fy;    % [N]
param.tau = tau;   % [N-m]
%% State Variables
t= (0:stepsize:5);
% [t,y] = ode45(@diffyq,[0 20],X0,[],param);

% qstring={'X','Y','\Theta_B','\Theta_L','\Theta_R','X Dot','Y
Dot','\Theta_B Dot','\Theta_L Dot','\Theta_R Dot'};
y = ode5(@diffyq,t ,X0',param);
```

```
% for i  = 1:5
%     figure;
%     plot(t, y(:,i));
%     title([qstring(i),'vs Time'])
% end

end
function xdot = diffyq(t,x,param)
%% Function Sets up the State Space Representaion of 5 Dof
System
mb = param.mb;
mp = param.mp;
ib = param.ib;
ip = param.ip;
d  = param.d;
l  = param.l;
fn = param.fn;
fx = param.fx;
fy = param.fy;
tau = param.tau;

k = ip*(2*pi*fn)^2;
% Variables to simplify matrix
sbl = sin(x(3)+x(5));
cbl = cos(x(3)+x(5));
sbr = sin(x(3)+x(4));
cbr = cos(x(3)+x(4));

sb = sin(x(3));
cb = cos(x(3));
sr = sin(x(4));
cr = cos(x(4));
sl = sin(x(5));
cl = cos(x(5));

% Mass Matrix [5x5]
M = [mb+2*mp,0,mp*d*(sbl-sbr),mp*d*sbl,-mp*d*sbr;
    0,mb+2*mp,-mp*d*(cbl-cbr),-mp*d*cbl,mp*d*cbr;
    mp*d*(sbl-sbr),-mp*d*(cbl-
cbr),2*mp*d^2+2*mp*l*d*cl+2*mp*l*d*cr+ib+2*mp*l^2+2*ip,mp*d^2+mp
*l*d*cl+ip,mp*d^2+mp*l*d*cr+ip;
    mp*d*sbl,-mp*d*cbl,mp*d^2+ip+mp*l*d*cl,ip+mp*d^2,0;
    -mp*d*sbr,mp*d*cbr,mp*l*d*cr+mp*d^2+ip,0,mp*d^2+ip];

% Generalized Forces Matrix [5x1]
Q = [fx,fy,tau,0,0]';
```

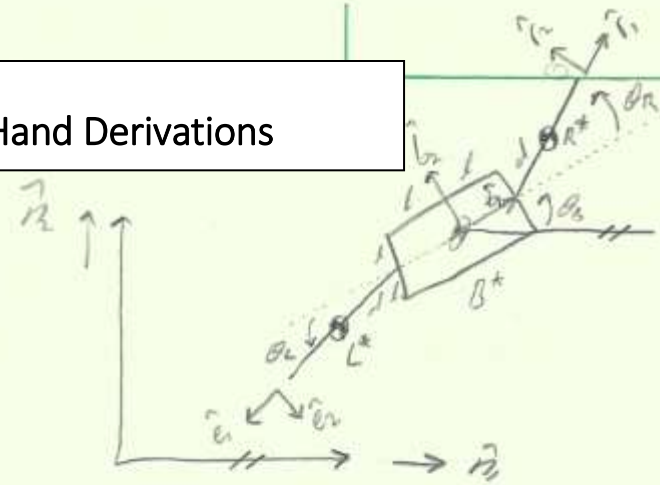
```
% Stiffness Matrix [5x5]
K = [zeros(3),zeros(3,2);
    zeros(2,3),[k,0;0,k]];
% q =[ x , y ,z ,Tb, Tl, Tr] [5x1]
q = x(1:5);

%Coriolis Matrix [5x1]
C = [mp*d*(x(8)+x(10))^2*cbl-mp*d*(x(8)+x(9))^2*cbr;
    mp*d*(x(8)+x(10))^2*sbl-mp*d*(x(8)+x(9))^2*sbr;
    -mp*l*d*(2*x(8)+x(10))*x(10)*sl-
mp*l*d*(2*x(8)+x(9))*x(9)*sr;
    mp*l*d*x(8)^2*sl;
    mp*l*d*x(8)^2*sr];
% Solves for q double dots i.e the accelerations
qddot = M\(Q-K*q-C);

%Concatenates q dot and q double dot
xdot = [x(6:10);qddot];

end
```

Appendix 2: Hand Derivations



$$\vec{R}_B = x\hat{n}_1 + y\hat{n}_2$$
$$^{N}\vec{v}^{R} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2$$

$$\vec{R}_R = x\hat{n}_1 + y\hat{n}_2 + l\hat{b}_1 + d\hat{r}_1$$
$$^{N}\vec{v}^{R} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 + l\,\frac{^{N}d\hat{b}_1}{dt} + d\,\frac{^{N}d\hat{r}_1}{dt}$$

$$^{N}\vec{\omega}^{B} = \dot{\theta}_B\hat{b}_3$$
$$\frac{^{N}d\hat{b}_1}{dt} = {}^{N}\vec{\omega}^{B}\times\hat{b}_1 = \dot{\theta}_B\hat{b}_2$$
$$^{N}\vec{\omega}^{R} = (\dot{\theta}_R + \dot{\theta}_B)\hat{r}_3$$
$$\frac{^{N}d\hat{r}_1}{dt} = (\dot{\theta}_R + \dot{\theta}_B)\hat{r}_3\times\hat{r}_1 = (\dot{\theta}_R + \dot{\theta}_B)\hat{r}_2$$

$$^{N}\vec{v}^{R} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 + l\dot{\theta}_B\hat{b}_2 + d(\dot{\theta}_R + \dot{\theta}_B)\hat{r}_2$$

$$\vec{R}_L = x\hat{n}_1 + y\hat{n}_2 - l\hat{b}_1 + d\hat{e}_1$$
$$^{N}\vec{v}^{L} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 - l\dot{\theta}_B\hat{b}_2 + d\,\frac{^{N}d\hat{e}_1}{dt}$$

$$^{N}\vec{\omega}^{E} = (\dot{\theta}_B + \dot{\theta}_L)\hat{e}_3$$
$$\frac{^{N}d\hat{e}_1}{dt} = (\dot{\theta}_B + \dot{\theta}_L)\hat{e}_3\times\hat{e}_1 = (\dot{\theta}_B + \dot{\theta}_L)\hat{e}_2$$

$$^{N}\vec{v}^{L} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 - l\dot{\theta}_B\hat{b}_2 + d(\dot{\theta}_B + \dot{\theta}_L)\hat{e}_2$$

$$I_B = \frac{1}{12}M_b\left((2l)^2 + (2l)^2\right)$$
$$= \frac{1}{12}M_b(8l^2)$$
$$= \frac{2}{3}M_b l^2$$

$$I_p = \frac{1}{3} M_p (2d)^2$$
$$= \frac{4}{3} M_p d^2$$

$$T_B = \frac{1}{2} M_b \tilde{v}^B \cdot \tilde{v}^B + \frac{1}{2} I_B \dot\theta_B^2$$
$$= \frac{1}{2} M_b (\dot x \hat n_1 + \dot y \hat n_2) \cdot (\dot x \hat n_1 + \dot y \hat n_2) + \frac{1}{2} I_B \dot\theta_B^2$$
$$= \frac{1}{2} M_b (\dot x^2 + \dot y^2) + \frac{1}{2} \left(\frac{2}{3} M_b l^2\right) \dot\theta_B^2$$
$$= \frac{1}{2} M_b (\dot x^2 + \dot y^2) + \frac{1}{3} M_b l^2 \dot\theta_B^2$$

$$T_R = \frac{1}{2} M_p \tilde{v}^R \cdot \tilde{v}^R + \frac{1}{2} I_p (\dot\theta_B + \dot\theta_R)^2$$
$$= \frac{1}{2} M_p (\dot x \hat n_1 + \dot y \hat n_2 + l \dot\theta_B \hat b_2 + d(\dot\theta_R + \dot\theta_B) \hat r_2) \cdot (\dot x \hat n_1 + \dot y \hat n_2 + l \dot\theta_B \hat b_2 + d(\dot\theta_R + \dot\theta_B) \hat r_2)$$
$$+ \frac{1}{2} I_p (\dot\theta_B + \dot\theta_R)^2$$
$$= \frac{1}{2} M_p \Big(\dot x^2 + \dot x l \dot\theta_B \cos(\theta_B + \tfrac{\pi}{2}) + \dot x d(\dot\theta_R + \dot\theta_B) \cos(\theta_B + \theta_R + \tfrac{\pi}{2})$$
$$+ \dot y^2 + \dot y l \dot\theta_B \cos(\theta_B) + \dot y d(\dot\theta_R + \dot\theta_B) \cos(\theta_B + \theta_R)$$
$$+ \dot x l \dot\theta_B \cos(-\tfrac{\pi}{2} - \theta_B) + \dot y l \dot\theta_B \cos(-\theta_B) + l^2 \dot\theta_B^2 + l \dot\theta_B d(\dot\theta_R + \dot\theta_B) \cos(\theta_R)$$
$$+ \dot x d(\dot\theta_R + \dot\theta_B) \cos(-\tfrac{\pi}{2} - \theta_B - \theta_R) + \dot y d(\dot\theta_R + \dot\theta_B) \cos(-\theta_B - \theta_R)$$
$$+ l \dot\theta_B d(\dot\theta_R + \dot\theta_B) \cos(-\theta_R) + d^2 (\dot\theta_R + \dot\theta_B)^2 \Big) + //$$
$$= \frac{1}{2} M_p \Big(\dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + d^2 (\dot\theta_R + \dot\theta_B)^2$$
$$+ 2 \dot x l \dot\theta_B \cos(\theta_B + \tfrac{\pi}{2}) + 2 \dot x d (\dot\theta_R + \dot\theta_B) \cos(\theta_B + \theta_R + \tfrac{\pi}{2})$$
$$+ 2 \dot y l \dot\theta_B \cos(\theta_B) + 2 \dot y d (\dot\theta_R + \dot\theta_B) \cos(\theta_B + \theta_R)$$
$$+ 2 l \dot\theta_B d (\dot\theta_R + \dot\theta_B) \cos(\theta_R) \Big) + \frac{1}{2} \left(\frac{4}{3} M_p d^2\right) (\dot\theta_B + \dot\theta_R)^2$$
$$= \frac{1}{2} M_p \Big(\dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + \frac{7}{3} d^2 (\dot\theta_B + \dot\theta_R)^2$$
$$+ 2 \dot x l \dot\theta_B \cos(\theta_B + \tfrac{\pi}{2}) + 2 \dot x d (\dot\theta_R + \dot\theta_B) \cos(\theta_B + \theta_R + \tfrac{\pi}{2})$$
$$+ 2 \dot y l \dot\theta_B \cos(\theta_B) + 2 \dot y d (\dot\theta_R + \dot\theta_B) \cos(\theta_B + \theta_R)$$
$$+ 2 l \dot\theta_B d (\dot\theta_R + \dot\theta_B) \cos(\theta_R) \Big)$$

$$T_L = \tfrac{1}{2} M_7 \tilde{v}^L \cdot \tilde{v}^L + \tfrac{1}{2} I_7 (\dot\theta_B + \dot\theta_L)^2$$

$$= \tfrac{1}{2} M_7 (\dot x \hat n_1 + \dot y \hat n_2 - l\dot\theta_B \hat b_2 + d(\dot\theta_B + \dot\theta_L)\hat e_2) \cdot (\dot x \hat n_1 + \dot y \hat n_2 - l\dot\theta_B \hat b_2 + d(\dot\theta_B + \dot\theta_L)\hat e_2)$$

$$+ \tfrac{1}{2} I_7 (\dot\theta_B + \dot\theta_L)^2$$

$$= \tfrac{1}{2} M_7 (\dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + d^2 (\dot\theta_B + \dot\theta_L)^2$$

$$- 2\dot x l \dot\theta_B \cos(\theta_B + \tfrac{\pi}{2}) + 2\dot x d (\dot\theta_B + \dot\theta_L) \sin(\theta_B + \theta_L)$$

$$- 2\dot y l \dot\theta_B \cos(\theta_B) - \dot y d (\dot\theta_B + \dot\theta_L) \cos(\theta_B + \theta_L)$$

$$+ 2 l \dot\theta_B d (\dot\theta_B + \dot\theta_L) \cos(\theta_L)) + \tfrac{1}{2} (\tfrac{3}{4} M_7 d^2)(\dot\theta_B + \dot\theta_L)^2$$

$$= \tfrac{1}{2} M_7 (\dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + \tfrac{7}{3} d^2 (\dot\theta_B + \dot\theta_L)^2$$

$$+ 2\dot x l \dot\theta_B \sin(\theta_B) + 2\dot x d (\dot\theta_B + \dot\theta_L) \sin(\theta_B + \theta_L)$$

$$- 2\dot y l \dot\theta_B \cos(\theta_B) - 2\dot y d (\dot\theta_B + \dot\theta_L) \cos(\theta_B + \theta_L)$$

$$+ 2 l \dot\theta_B d (\dot\theta_B + \dot\theta_L) \cos(\theta_L))$$

$$T_R = \tfrac{1}{2} M_7 (\dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + d^2 (\dot\theta_R + \dot\theta_R)^2$$

$$- 2\dot x l \dot\theta_B \sin(\theta_B) - 2\dot x d (\dot\theta_R + \dot\theta_B) \sin(\theta_B + \theta_R)$$

$$+ 2\dot y l \dot\theta_B \cos(\theta_B) + 2\dot y d (\dot\theta_R + \dot\theta_B) \cos(\theta_B + \theta_R)$$

$$+ 2 l \dot\theta_B d (\dot\theta_R + \dot\theta_B) \cos(\theta_R)) + \tfrac{1}{2} I_7 (\dot\theta_B + \dot\theta_R)^2$$

$$T_L = \tfrac{1}{2} M_7 (\dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + d^2 (\dot\theta_B + \dot\theta_L)^2$$

$$+ 2\dot x l \dot\theta_B \sin(\theta_B) + 2\dot x d (\dot\theta_B + \dot\theta_L) \sin(\theta_B + \theta_L)$$

$$- 2\dot y l \dot\theta_B \cos(\theta_B) - 2\dot y d (\dot\theta_B + \dot\theta_L) \cos(\theta_B + \theta_L)$$

$$+ 2 l \dot\theta_B d (\dot\theta_B + \dot\theta_L) \cos(\theta_L)) + \tfrac{1}{2} I_7 (\dot\theta_B + \dot\theta_L)^2$$

$$T_B = \tfrac{1}{2} M_B (\dot x^2 + \dot y^2) + \tfrac{1}{2} I_B \dot\theta_B^2$$

$U_L = \frac{1}{2} k \theta_L^2$

$U_R = \frac{1}{2} k \theta_R^2$

$$
\begin{aligned}
\mathcal{L} &= T - U \\
&= \tfrac{1}{2} M_b (\dot{x}^2 + \dot{y}^2) \\
&\quad + \tfrac{1}{2} M_7 \big(2\dot{x}^2 + 2\dot{y}^2 + 2l^2 \dot{\theta}_B^2 + d^2(\dot{\theta}_B + \dot{\theta}_R)^2 + d^2(\dot{\theta}_B + \dot{\theta}_L)^2 \\
&\qquad - 2\dot{x} d (\dot{\theta}_R + \dot{\theta}_B) \sin(\theta_B + \theta_R) + 2\dot{x} d (\dot{\theta}_B + \dot{\theta}_L) \sin(\theta_B + \theta_L) \\
&\qquad + 2\dot{y} d (\dot{\theta}_R + \dot{\theta}_B) \cos(\theta_B + \theta_R) - 2\dot{y} d (\dot{\theta}_B + \dot{\theta}_L) \cos(\theta_B + \theta_L) \\
&\qquad + 2 l \dot{\theta}_B d (\dot{\theta}_R + \dot{\theta}_B) \cos(\theta_R) + 2 l \dot{\theta}_B d (\dot{\theta}_B + \dot{\theta}_L) \cos(\theta_L) \big) \\
&\quad + \tfrac{1}{2} I_7 (\dot{\theta}_B + \dot{\theta}_R)^2 + \tfrac{1}{2} I_7 (\dot{\theta}_B + \dot{\theta}_L)^2 + \tfrac{1}{2} I_B \dot{\theta}_B^2 \\
&\quad - \tfrac{1}{2} k \theta_L^2 - \tfrac{1}{2} k \theta_R^2
\end{aligned}
$$

DOF's = 5

$\theta_B, \theta_L, \theta_R, x, y$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_L}\right) - \frac{\partial \mathcal{L}}{\partial \theta_L} = Q_{\theta_L}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_L} = \frac{1}{2} M_7 \Big(2d^2(\dot{\theta}_B + \dot{\theta}_L) + 2\dot{x} d \sin(\theta_B + \theta_L) - 2\dot{y} d \cos(\theta_B + \theta_L) + 2 l \dot{\theta}_B d \cos(\theta_L)\Big) + I_7(\dot{\theta}_B + \dot{\theta}_L)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_L}\right) = \frac{1}{2} M_7 \Big(2d^2(\ddot{\theta}_B + \ddot{\theta}_L) + 2\ddot{x} d \sin(\theta_B + \theta_L) + 2\dot{x} d (\dot{\theta}_B + \dot{\theta}_L)\cos(\theta_B + \theta_L) - 2\ddot{y} d \cos(\theta_B + \theta_L) + 2\dot{y} d(\dot{\theta}_B + \dot{\theta}_L)\sin(\theta_B + \theta_L) + 2 l \ddot{\theta}_B d \cos(\theta_L) - 2 l \dot{\theta}_B \dot{\theta}_L d \sin(\theta_L)\Big) + I_7(\ddot{\theta}_B + \ddot{\theta}_L)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_L} = \frac{1}{2} M_7 \Big(2\dot{x} d(\dot{\theta}_B + \dot{\theta}_L)\cos(\theta_B + \theta_L) + 2\dot{y} d(\dot{\theta}_B + \dot{\theta}_L)\sin(\theta_B + \theta_L) - 2 l \dot{\theta}_B d(\dot{\theta}_B + \dot{\theta}_L)\sin(\theta_L)\Big) - k\theta_L$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_L}\right) - \frac{\partial \mathcal{L}}{\partial \theta_L} = \frac{1}{2} M_7 \Big(2d^2(\ddot{\theta}_B + \ddot{\theta}_L) + 2\ddot{x} d \sin(\theta_B + \theta_L) - 2\ddot{y} d \cos(\theta_B + \theta_L) + 2 l \ddot{\theta}_B d \cos(\theta_L) - 2 l \dot{\theta}_B \dot{\theta}_L d \sin(\theta_L) + 2 l \dot{\theta}_B d(\dot{\theta}_B + \dot{\theta}_L)\sin(\theta_L)\Big) + k\theta_L$$

$$Q_{\theta_L} = \frac{\partial {}^N\vec{v}^{\,B}}{\partial \dot{\theta}_L}\cdot \vec{F}_1 + \frac{\partial {}^N\vec{\omega}^{\,B}}{\partial \dot{\theta}_L}\cdot \vec{M}_1 = 0$$

EOM:

$$\boxed{\frac{1}{2} M_7 \Big(2d^2(\ddot{\theta}_B + \ddot{\theta}_L) + 2\ddot{x} d \sin(\theta_B + \theta_L) - 2\ddot{y} d \cos(\theta_B + \theta_L) + 2 l \ddot{\theta}_B d \cos(\theta_L) - 2 l \dot{\theta}_B \dot{\theta}_L d \sin(\theta_L) + 2 l \dot{\theta}_B d(\dot{\theta}_B + \dot{\theta}_L)\sin(\theta_L)\Big) + k\theta_L = 0}$$

$$\frac{d}{dt}\left(\frac{\partial S}{\partial \dot{x}}\right) - \frac{\partial S}{\partial x} = Q_x$$

$$\frac{\partial S}{\partial \dot{x}} = M_b \dot{x}$$
$$+\tfrac{1}{2} M_7 \left(4\dot{x} - 2d(\dot{\theta}_R + \dot{\theta}_B)\sin(\theta_B + \theta_R) + 2d(\dot{\theta}_B + \dot{\theta}_L)\sin(\theta_B + \theta_L)\right)$$

$$\frac{d}{dt}\left(\frac{\partial S}{\partial \dot{x}}\right) = M_b \ddot{x}$$
$$+\tfrac{1}{2} M_7 \left(4\ddot{x} - 2d(\ddot{\theta}_R + \ddot{\theta}_B)\sin(\theta_B + \theta_R) - 2d(\dot{\theta}_R + \dot{\theta}_B)^2 \cos(\theta_B + \theta_R)\right.$$
$$\left. + 2d(\ddot{\theta}_B + \ddot{\theta}_L)\sin(\theta_B + \theta_L) + 2d(\dot{\theta}_B + \dot{\theta}_L)^2 \cos(\theta_B + \theta_L)\right)$$

$$\frac{\partial S}{\partial x} = 0$$

$$Q_x = \frac{\partial \tilde{v}^A}{\partial \dot{x}} \cdot \vec{F}_1 + \frac{\partial \tilde{\omega}^B}{\partial \dot{x}} \cdot \vec{M}_1$$
$$= \hat{n}_1 \cdot \vec{F} \qquad + 0$$
$$= \hat{n}_1 \cdot \vec{F}$$

EOM:

$$\boxed{\begin{aligned}&\tfrac{1}{2} M_7 \left(4\ddot{x} - 2d(\ddot{\theta}_R + \ddot{\theta}_B)\sin(\theta_B + \theta_R) - 2d(\dot{\theta}_R + \dot{\theta}_B)^2 \cos(\theta_B + \theta_R)\right. \\ &\quad \left. + 2d(\ddot{\theta}_B + \ddot{\theta}_L)\sin(\theta_B + \theta_L) + 2d(\dot{\theta}_B + \dot{\theta}_L)^2 \cos(\theta_B + \theta_L)\right) + M_b \ddot{x} = \hat{n}_1 \cdot \vec{F}\end{aligned}}$$

$$+2ld(\ddot{\theta}_B+\ddot{\theta}_L)\cos(\theta_L)-2ld\dot{\theta}_L(\dot{\theta}_B+\dot{\theta}_L)\sin(\theta_L)$$

$$+2l\ddot{\theta}_B d\cos(\theta_L)-2l\dot{\theta}_B\dot{\theta}_L d\sin(\theta_L))$$

$$+I_7(\ddot{\theta}_B+\ddot{\theta}_R)+I_7(\ddot{\theta}_B+\ddot{\theta}_L)+I_B\ddot{\theta}_B$$

$$Q_{\theta_B}=\frac{\partial \vec{v}}{\partial \dot{\theta}_B}\cdot\vec{F}_1+\frac{\partial \vec{\omega}}{\partial \dot{\theta}_B}\cdot\vec{M}_1$$

$$=\hat{b}_3\cdot\tau\,\hat{b}_3$$

$$=\tau$$

EOM:

$$\boxed{\begin{aligned}&\tfrac{1}{2}M_7\big(4l^2\ddot{\theta}_R+4d^2(\ddot{\theta}_B+\ddot{\theta}_L)-2\ddot{x}d\sin(\theta_B+\theta_R)+2\ddot{x}d\sin(\theta_B+\theta_L)+2\ddot{y}d\cos(\theta_B+\theta_R)\\&\quad-2\ddot{y}d\cos(\theta_B+\theta_L)+2ld(\ddot{\theta}_R+\ddot{\theta}_B)\cos(\theta_R)-2ld\dot{\theta}_R(\dot{\theta}_R+\dot{\theta}_B)\sin(\theta_R)\\&\quad+2l\ddot{\theta}_B d\cos(\theta_R)-2l\dot{\theta}_B\dot{\theta}_R d\sin(\theta_R)+2ld(\ddot{\theta}_B+\ddot{\theta}_L)\cos(\theta_L)\\&\quad-2ld\dot{\theta}_L(\dot{\theta}_B+\dot{\theta}_L)\sin(\theta_L)+2l\ddot{\theta}_B d\cos(\theta_L)-2l\dot{\theta}_B\dot{\theta}_L d\sin(\theta_L)\big)\\&+I_7(\ddot{\theta}_B+\ddot{\theta}_R)+I_7(\ddot{\theta}_R+\ddot{\theta}_L)+I_B\ddot{\theta}_B=\tau\end{aligned}}$$

1 $ld(\ddot{\theta}_R+\ddot{\theta}_B)\,c\,\theta_R$

2 $-ld\dot{\theta}_R(\dot{\theta}_R+\dot{\theta}_B)\,s\,\theta_R$

1 $+l\ddot{\theta}_B d\,c\,\theta_R$

2 $-l\dot{\theta}_B\dot{\theta}_R d\,s\,\theta_R$

3 $+ld(\ddot{\theta}_B+\ddot{\theta}_L)\,c\,\theta_L$

4 $-ld\dot{\theta}_L(\dot{\theta}_B+\dot{\theta}_L)\,s\,\theta_L$

3 $+l\ddot{\theta}_B d\,c\,\theta_L$

4 $-l\dot{\theta}_B\dot{\theta}_L d\,s\,\theta_L$

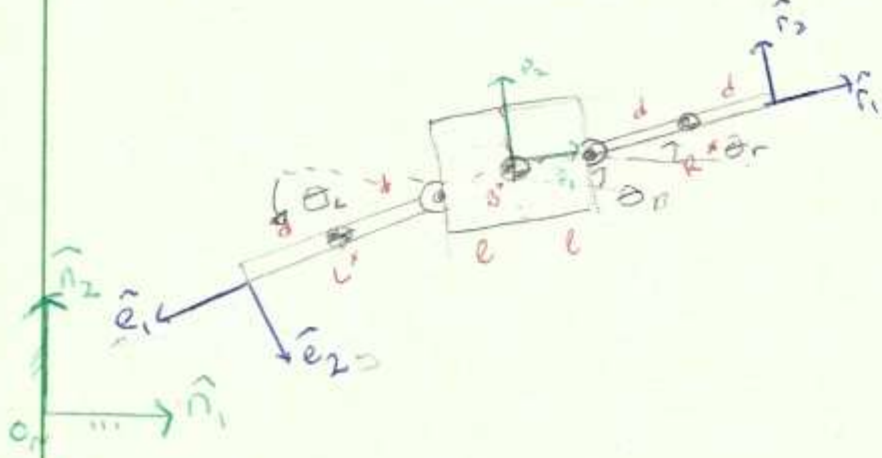
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_B}\right) - \frac{\partial \mathcal{L}}{\partial \theta_B} = Q_{\theta_B}$$

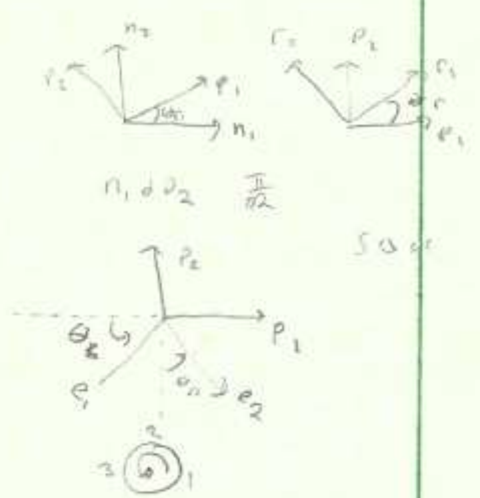
$$\frac{\partial \mathcal{L}}{\partial \dot\theta_B} = \tfrac{1}{2} M_7 \Big(4l^2 \dot\theta_B + 4d^2(\dot\theta_B + \dot\theta_L)$$
$$- 2\dot x d \sin(\theta_B+\theta_R) + 2\dot x d \sin(\theta_B + \theta_L)$$
$$+ 2\dot y d \cos(\theta_B+\theta_R) - 2\dot y d \cos(\theta_B+\theta_L)$$
$$+ 2ld(\dot\theta_R + \dot\theta_B)\cos(\theta_R) + 2l\dot\theta_B d\cos(\theta_R)$$
$$+ 2ld(\dot\theta_B+\dot\theta_L)\cos(\theta_L) + 2l\dot\theta_B d\cos(\theta_L)\Big)$$
$$+ I_7(\dot\theta_B + \dot\theta_R) + I_7(\dot\theta_B + \dot\theta_L) + I_B\dot\theta_B$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_B}\right) = \tfrac{1}{2} M_7 \Big(4l^2 \ddot\theta_B + 4d^2(\ddot\theta_B + \ddot\theta_L)$$
$$- 2\ddot x d \sin(\theta_B+\theta_R) - 2\dot x d(\dot\theta_B + \dot\theta_R)\cos(\theta_B+\theta_R)$$
$$+ 2\ddot x d \sin(\theta_B+\theta_L) + 2\dot x d(\dot\theta_B+\dot\theta_L)\cos(\theta_B+\theta_L)$$
$$+ 2\ddot y d\cos(\theta_B+\theta_R) - 2\dot y d(\dot\theta_B+\dot\theta_R)\sin(\theta_B+\theta_R)$$
$$- 2\ddot y d\cos(\theta_B+\theta_L) + 2\dot y d(\dot\theta_B+\dot\theta_L)\sin(\theta_B+\theta_L)$$
$$+ 2ld(\ddot\theta_R + \ddot\theta_B)\cos(\theta_R) - 2ld\dot\theta_R(\dot\theta_R+\dot\theta_B)\sin(\theta_R)$$
$$+ 2l\ddot\theta_B d\cos(\theta_R) - 2l\dot\theta_B\dot\theta_R d\sin(\theta_R)$$
$$+ 2ld(\ddot\theta_B + \ddot\theta_L)\cos(\theta_L) - 2ld\dot\theta_L(\dot\theta_B+\dot\theta_L)\sin(\theta_L)$$
$$+ 2l\ddot\theta_B d\cos(\theta_L) - 2l\dot\theta_B\dot\theta_L d\sin(\theta_L)\Big)$$
$$+ I_7(\ddot\theta_B + \ddot\theta_R) + I_7(\ddot\theta_B + \ddot\theta_L) + I_B\ddot\theta_B$$

$$\frac{\partial \mathcal{L}}{\partial \theta_B} = \tfrac{1}{2} M_7 \Big(-2\dot x d(\dot\theta_R + \dot\theta_B)\cos(\theta_B+\theta_R) + 2\dot x d(\dot\theta_B+\dot\theta_L)\cos(\theta_B+\theta_L)$$
$$- 2\dot y d(\dot\theta_R+\dot\theta_B)\sin(\theta_B+\theta_R) + 2\dot y d(\dot\theta_B+\dot\theta_L)\sin(\theta_B+\theta_L)\Big)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_B}\right) - \frac{\partial \mathcal{L}}{\partial \theta_B} = \tfrac{1}{2} M_7 \Big(4l^2\ddot\theta_B + 4d^2(\ddot\theta_B + \ddot\theta_L) - 2\ddot x d\sin(\theta_B+\theta_R)$$
$$+ 2\ddot x d\sin(\theta_B+\theta_L) + 2\ddot y d\cos(\theta_B+\theta_R) - 2\ddot y d\cos(\theta_B+\theta_L)$$
$$+ 2ld(\ddot\theta_R + \ddot\theta_B)\cos(\theta_R) - 2ld\dot\theta_R(\dot\theta_R+\dot\theta_B)\sin(\theta_R)$$
$$+ 2l\ddot\theta_B d\cos(\theta_R) - 2l\dot\theta_B\dot\theta_R d\sin(\theta_R)$$





$$\vec{R}_B = x\hat{n}_1 + y\hat{n}_2$$

$$\boxed{{}^{N}\vec{V}^{B} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2}$$

$$\vec{R}_R = x\hat{n}_1 + y\hat{n}_2 + l\hat{p}_1 + d\hat{r}_1$$

$${}^{N}\vec{V}^{R} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 + l\dot{\theta}_B\hat{p}_2 + \frac{{}^{N}d}{dt}r_1$$

$$\boxed{{}^{N}\vec{V}^{R} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 + l\dot{\theta}_B\hat{p}_2 + d(\dot{\theta}_B + \dot{\theta}_r)\hat{r}_2}$$

$$\frac{{}^{N}d}{dt}\hat{r}_1 = (\dot{\theta}_B + \dot{\theta}_r)\hat{r}_3 \times r_1$$

$$\frac{{}^{N}d}{dt}\hat{r}_1 = (\dot{\theta}_B + \dot{\theta}_r)\hat{r}_3 \times \hat{r}_1$$

$$\hat{r}_2 = (\dot{\theta}_B + \dot{\theta}_r)\hat{r}_2$$

$$\vec{R}_L = x\hat{n}_1 + y\hat{n}_2 - l\hat{p}_1 + d\hat{e}_1$$

$${}^{N}\vec{V}^{L} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 - l\dot{\theta}_B\hat{p}_2 + d\frac{{}^{N}d}{dt}\hat{e}_1$$

$$\boxed{{}^{N}\vec{V}^{L} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 - l\dot{\theta}_B\hat{p}_2 + d(\dot{\theta}_B + \dot{\theta}_L)\hat{e}_2}$$

$$\frac{{}^{N}d}{dt}\hat{e}_1 = {}^{N}\vec{\omega}^{E} \times \hat{e}_1 = (\dot{\theta}_B + \dot{\theta}_L)\hat{e}_3 \times \hat{e}_1$$

$$\frac{{}^{N}d}{dt}\hat{e}_1 = (\dot{\theta}_B + \dot{\theta}_L)e_3 \times e_1$$

$$\frac{{}^{N}d}{dt}\hat{e}_1 = (\dot{\theta}_B + \dot{\theta}_L)\hat{e}_2$$

n = # DoF = 5 $(x, y, \theta_B, \theta_R, \theta_L)$

N = # Particles = 3

Since $g=0 \Rightarrow$ P.E U only comes from springs

$$\boxed{U = \frac{1}{2}k\theta_r^2 + \frac{1}{2}k\theta_L^2}$$

$$T = \sum_{i=1}^{N=3} T_i = T_B + T_R + T_L + T_{I^*}$$

$$T_B = \tfrac{1}{2} M_B \, {}^{N}\vec{V}^{B} \cdot {}^{N}\vec{V}^{B} = \tfrac{1}{2} M_B \left(\dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 \right)\left(\dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 \right)$$

$$= \tfrac{1}{2} M_B \left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos(\tfrac{\pi}{2}) \right)$$

$$\boxed{T_B = \tfrac{1}{2} M_B \left(\dot{x}^2 + \dot{y}^2 \right)} \qquad T_{B^*} = \tfrac{1}{12} M_B (8l^2) \Rightarrow$$

$$T_R = \tfrac{1}{2} M_P \, {}^{N}\vec{V}^{R} \cdot {}^{N}\vec{V}^{R} = \tfrac{1}{2} M_P \left(\dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 + l\dot{\theta}_B \hat{b}_2 + d(\dot{\theta}_B + \dot{\theta}_r)\hat{r}_2 \right)^2$$

$$\begin{aligned} T_R = \tfrac{1}{2} M_P \Big[& \dot{x}^2 + \dot{y}^2 + l^2\dot{\theta}_B^2 + d^2(\dot{\theta}_B + \dot{\theta}_r)^2 + 2\dot{x} l \dot{\theta}_B \, c(\theta_B + \tfrac{\pi}{2}) \\ \rightarrow & + 2\dot{x} d(\dot{\theta}_B + \dot{\theta}_r) \, c(\theta_B + \theta_r - \tfrac{\pi}{2}) + 2\dot{y} l \dot{\theta}_B \, c(\theta_B) \\ & + 2\dot{y} d(\dot{\theta}_B + \dot{\theta}_r) \, c(\theta_B + \theta_r) + 2 l \dot{\theta}_B d(\dot{\theta}_B + \dot{\theta}_r) \, c(\theta_r) \Big] \end{aligned}$$

$$T_L = \tfrac{1}{2} M_P \, {}^{N}\vec{V}^{L} \cdot {}^{N}\vec{V}^{L} = \tfrac{1}{2} M_P \left(\dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 - l\dot{\theta}_B \hat{b}_2 + d(\dot{\theta}_B + \dot{\theta}_L)\hat{e}_2 \right)^2$$

$$\begin{aligned} T_L = \tfrac{1}{2} M_P \Big[& \dot{x}^2 + \dot{y}^2 + l^2\dot{\theta}_B^2 + d^2(\dot{\theta}_B + \dot{\theta}_L)^2 - 2\dot{x} l \dot{\theta}_B \, c(\theta_B + \tfrac{\pi}{2}) \\ & + 2\dot{x} d(\dot{\theta}_B + \dot{\theta}_L) \, c(\theta_B + \theta_L + \tfrac{\pi}{2}) - 2\dot{y} l \dot{\theta}_B \, c(\theta_B) \\ & + 2\dot{y} d(\dot{\theta}_B + \dot{\theta}_L) \, c(\theta_B + \theta_L + \pi) - 2 l \dot{\theta}_B d(\dot{\theta}_B + \dot{\theta}_L) \, c(\theta_L + \pi) \end{aligned}$$

$$\begin{aligned} T_L = \tfrac{1}{2} M_P \Big[& \dot{x}^2 + \dot{y}^2 + l^2\dot{\theta}_B^2 + d^2(\dot{\theta}_B + \dot{\theta}_L)^2 - 2\dot{x} l \dot{\theta}_B \, s(\theta_B) \\ & + 2\dot{x} d(\dot{\theta}_B + \dot{\theta}_L) \, s(\theta_B + \theta_L) - 2\dot{y} l \dot{\theta}_B \, c(\theta_B) \\ & - 2\dot{y} d(\dot{\theta}_B + \dot{\theta}_L) \, c(\theta_B + \theta_L) + 2 l \dot{\theta}_B d(\dot{\theta}_B + \dot{\theta}_L) \, c(\theta_L) \end{aligned}$$

$$\begin{aligned} T_r = \tfrac{1}{2} M_P \Big[& \dot{x}^2 + \dot{y}^2 + l\dot{\theta}_B^2 + d^2(\dot{\theta}_B + \dot{\theta}_r)^2 + 2\dot{x} l \dot{\theta}_B \, s(\theta_B) \\ & - 2\dot{x} d(\dot{\theta}_B + \dot{\theta}_r) \, s(\theta_B + \theta_r) + 2\dot{y} l \dot{\theta}_B \, c(\theta_B) \\ & + 2\dot{y} d(\dot{\theta}_B + \dot{\theta}_r) \, c(\theta_B + \theta_r) + 2 l \dot{\theta}_B d(\dot{\theta}_B + \dot{\theta}_r) \, c(\theta_r) \end{aligned}$$

From Rotation

$$\boxed{T_I^* = \tfrac{1}{2} I_B \dot\theta_B^2 + \tfrac{1}{2} I_p (\dot\theta_B + \dot\theta_r)^2 + \tfrac{1}{2} I_p (\dot\theta_B + \dot\theta_L)^2}$$

1 $T_L + T_R = \frac{1}{2} M_p [\, \dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + d^2(\dot\theta_B + \dot\theta_L)^2 - 2\dot x l \cancel{\dot\theta_B}\, s(\theta_B)$

2 $+ 2\dot x d(\dot\theta_B + \dot\theta_L) s(\theta_B + \theta_L) - 2\dot y \cancel{l} \dot\theta_B\, c(\theta_B)$

3 $- 2\dot y d(\dot\theta_B + \dot\theta_L) c(\theta_B + \theta_L) + 2 l \dot\theta_B d(\dot\theta_B + \dot\theta_L) c(\theta_L)$

1 $+ \dot x^2 + \dot y^2 + l \dot\theta_B^2 + d^2(\dot\theta_B + \dot\theta_r)^2 - 2\dot x \cancel{l} \dot\theta_B\, s(\theta_B)$

2 $- 2\dot x d(\dot\theta_B + \dot\theta_r) s(\theta_B + \theta_r) + 2\dot y \cancel{l} \dot\theta_B\, c(\theta_B)$

3 $+ 2\dot y d(\dot\theta_B + \dot\theta_r) c(\theta_B + \theta_r) + 2 l \dot\theta_B d(\dot\theta_B + \dot\theta_r) c(\theta_r) \,]$

$\Rightarrow \Sigma T = \frac{1}{2} M_B(\dot x^2 + \dot y^2) + \frac{1}{2} I_B \dot\theta_B^2$

$+ \frac{1}{2} M_p [\, \dot x^2 + \dot y^2 + l^2 \dot\theta_B^2 + d^2(\dot\theta_B + \dot\theta_L)^2 + 2\dot x d(\dot\theta_B + \dot\theta_L) s(\theta_B + \theta_L)$

$+ 2 l \dot\theta_B d(\dot\theta_B + \dot\theta_L) c(\theta_L) - 2\dot y d(\dot\theta_B + \dot\theta_L) c(\theta_B + \theta_L)$

$+ \dot x^2 + \dot y^2 + l \dot\theta_B^2 + d^2(\dot\theta_B + \dot\theta_r)^2 - 2\dot x d(\dot\theta_B + \dot\theta_r) s(\theta_B + \theta_r)$

$+ 2 l \dot\theta_B d(\dot\theta_B + \dot\theta_r) c(\theta_r) + 2\dot y d(\dot\theta_B + \dot\theta_r) c(\theta_B + \theta_r) \,]$

$+ \frac{1}{2} I_p (\dot\theta_B + \dot\theta_r)^2 + \frac{1}{2} I_p (\dot\theta_B + \dot\theta_L)^2$

$\mathcal{L} = T - U$

$\mathcal{L} = \frac{1}{2} M_B(\dot x^2 + \dot y^2) + \frac{1}{2} I_B \dot\theta_B^2$

$+ \frac{1}{2} M_p [\, 2\dot x^2 + 2\dot y^2 + 2 l^2 \dot\theta_B^2 + d^2(\dot\theta_B + \dot\theta_L)^2$

$+ 2\dot x d(\dot\theta_B + \dot\theta_L) s(\theta_B + \theta_L) + 2 l \dot\theta_B d(\dot\theta_B + \dot\theta_L) c(\theta_L)$

$- 2\dot y d(\dot\theta_B + \dot\theta_L) c(\theta_B + \theta_L) - 2\dot x d(\dot\theta_B + \dot\theta_r) s(\theta_B + \theta_r) + d^2(\dot\theta_B + \dot\theta_r)^2$

$+ 2 l \dot\theta_B d(\dot\theta_B + \dot\theta_r) c(\theta_r) + 2\dot y d(\dot\theta_B + \dot\theta_r) c(\theta_B + \theta_r) \,]$

$+ \frac{1}{2} I_p (\dot\theta_B + \dot\theta_r)^2 + \frac{1}{2} I_p (\dot\theta_B + \dot\theta_L)^2 - \frac{1}{2} k \theta_r^2 - \frac{1}{2} k \theta_L^2$

EOM: y

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = Q_y$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = M_B \dot{y} + \tfrac{1}{2} M_p [4\dot{y} - 2d(\dot{\theta}_B + \dot{\theta}_L)c(\theta_B + \theta_L)$$
$$+ 2d(\dot{\theta}_B + \dot{\theta}_r)c(\theta_B + \theta_r)]$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = M_B \ddot{y} + \tfrac{1}{2} M_p [4\ddot{y} - 2d(\ddot{\theta}_B + \ddot{\theta}_L)c(\theta_B + \theta_L)$$
$$+ 2d(\dot{\theta}_B + \dot{\theta}_L)^2 s(\theta_B + \theta_L)$$
$$+ 2d(\ddot{\theta}_B + \ddot{\theta}_r)c(\theta_B + \theta_r)$$
$$- 2d(\dot{\theta}_B + \dot{\theta}_r)^2 s(\theta_B + \theta_r)]$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0$$

EOM of Y

$$M_B \ddot{y} + M_p [2\ddot{y} - d(\ddot{\theta}_B + \ddot{\theta}_L)c(\theta_B + \theta_L) + d(\dot{\theta}_B + \dot{\theta}_L)^2 s(\theta_B + \theta_L)$$
$$+ d(\ddot{\theta}_B + \ddot{\theta}_r)c(\theta_B + \theta_r) - d(\dot{\theta}_B + \dot{\theta}_r)^2 s(\theta_B + \theta_r)]$$
$$= Q_y$$

EOM: θ_B PART 1

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_B}\right) - \frac{\partial \mathcal{L}}{\partial \theta_B} = Q_{\theta_B}$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta_B} = I_B \dot\theta_B + \tfrac{1}{2} M_P [4 l^2 \dot\theta_B + d^2(2\dot\theta_B + 2\dot\theta_L) + 2\dot x d s(\theta_B + \theta_L)$$
$$+ 2 l d (2\dot\theta_B + \dot\theta_L) c(\theta_L) - 2\dot y d c(\theta_B + \theta_L) + d^2(2\dot\theta_B + 2\dot\theta_R)$$
$$- 2\dot x d s(\theta_B + \theta_r) + 2 l d (2\dot\theta_B + \dot\theta_r) c(\theta_r)$$
$$+ 2\dot y d c(\theta_B + \theta_r)] + \tfrac{1}{2} I_P (2\dot\theta_B + 2\dot\theta_r) + \tfrac{1}{2} I_P (2\dot\theta_B + 2\dot\theta_L)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_B}\right) = I_B \ddot\theta_B + \tfrac{1}{2} M_P [4 l^2 \ddot\theta_B + d^2(2\ddot\theta_B + 2\ddot\theta_L)$$
$$+ 2\ddot x d s(\theta_B + \theta_L) + 2\dot x d(\dot\theta_B + \dot\theta_L) c(\theta_B + \theta_L)$$
$$+ 2 l d (2\ddot\theta_B + \ddot\theta_L) c(\theta_L) - 2 l d (2\dot\theta_B + \dot\theta_L)\dot\theta_L s(\theta_L)$$
$$- 2\ddot y d c(\theta_B + \theta_L) + 2\dot y d(\dot\theta_B + \dot\theta_L) s(\theta_B + \theta_L) + d^2(2\ddot\theta_B + 2\ddot\theta_R)$$
$$- 2\ddot x d s(\theta_B + \theta_r) - 2\dot x d(\dot\theta_B + \dot\theta_r) c(\theta_B + \theta_r)$$
$$+ 2 l d (2\ddot\theta_B + \ddot\theta_r) c(\theta_r) - 2 l d (2\dot\theta_B + \dot\theta_r)\dot\theta_r s(\theta_r)$$
$$+ 2\ddot y d c(\theta_B + \theta_r) - 2\dot y d(\dot\theta_B + \dot\theta_r) s(\theta_B + \theta_r)]$$
$$+ I_P(\ddot\theta_B + \ddot\theta_r) + I_P(\ddot\theta_B + \ddot\theta_L)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_B} = \tfrac{1}{2} M_P [2\dot x d(\dot\theta_B + \dot\theta_L) c(\theta_B + \theta_L) + 2\dot y d(\dot\theta_B + \dot\theta_L) s(\theta_B + \theta_L)$$
$$- 2\dot x d(\dot\theta_B + \dot\theta_r) c(\theta_B + \theta_r) - 2\dot y d(\dot\theta_B + \dot\theta_r) s(\theta_B + \theta_r)]$$

EOM: θ_B Part 2

$$\text{EOM}: \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_B}\right) - \frac{\partial \mathcal{L}}{\partial \theta_B} = Q_{\theta_B}$$

$$
\begin{aligned}
&M_p[2l^2\ddot{\theta}_B + d^2(\ddot{\theta}_B + \ddot{\theta}_L) + \ddot{x}ds(\theta_B+\theta_L) + \cancel{\dot{x}d(\dot{\theta}_B+\dot{\theta}_L)c(\theta_B+\theta_L)} \\
&+ ld(2\ddot{\theta}_B + \ddot{\theta}_L)c(\theta_L) - ld(2\dot{\theta}_B + \dot{\theta}_L)\dot{\theta}_L s(\theta_L) \\
&- \ddot{y}dc(\theta_B+\theta_L) + \cancel{\dot{y}d(\dot{\theta}_B+\dot{\theta}_L)s(\theta_B+\theta_L)} + d^2(\ddot{\theta}_B + \ddot{\theta}_R) \\
&- \ddot{x}ds(\theta_B+\theta_r) - \cancel{\dot{x}d(\dot{\theta}_B+\dot{\theta}_r)c(\theta_B+\theta_r)} \\
&+ ld(2\ddot{\theta}_B + \ddot{\theta}_r)c(\theta_r) - ld(2\dot{\theta}_B + \dot{\theta}_r)\dot{\theta}_r s(\theta_r) \\
&+ \ddot{y}dc(\theta_B+\theta_r) - \cancel{\dot{y}d(\dot{\theta}_B+\dot{\theta}_r)s(\theta_B+\theta_r)} \\
&- \cancel{\dot{x}d(\dot{\theta}_B+\dot{\theta}_L)c(\theta_B+\theta_L)} - \cancel{\dot{y}d(\dot{\theta}_B+\dot{\theta}_L)s(\theta_B+\theta_L)} \\
&+ \cancel{\dot{x}d(\dot{\theta}_B+\dot{\theta}_r)c(\theta_B+\theta_r)} + \cancel{\dot{y}d(\dot{\theta}_B+\dot{\theta}_r)s(\theta_B+\theta_r)}] \\
&+ I_B\ddot{\theta}_B + I_p(\ddot{\theta}_B + \ddot{\theta}_r) + I_p(\ddot{\theta}_B + \ddot{\theta}_L) = Q_{\theta_B}
\end{aligned}
$$

$$
\begin{aligned}
&M_p[2l^2\ddot{\theta}_B + d^2(\ddot{\theta}_B + \ddot{\theta}_L) + \ddot{x}ds(\theta_B+\theta_L) + ld(2\ddot{\theta}_B + \ddot{\theta}_L)c(\theta_L) \\
&- ld(2\dot{\theta}_B + \dot{\theta}_L)\dot{\theta}_L s(\theta_L) - \ddot{y}dc(\theta_B+\theta_L) + d^2(\ddot{\theta}_B + \ddot{\theta}) \\
&- \ddot{x}ds(\theta_B+\theta_r) + ld(2\ddot{\theta}_B + \ddot{\theta}_r)c(\theta_r) - ld(2\dot{\theta}_B + \dot{\theta}_r)\dot{\theta}_r s(\theta_r) \\
&+ \ddot{y}dc(\theta_B+\theta_r)] + I_B\ddot{\theta}_B + I_p(\ddot{\theta}_B + \ddot{\theta}_r) + I_p(\ddot{\theta}_B + \ddot{\theta}_L) = Q_{\theta_B}
\end{aligned}
$$

EOM : θ_r

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_r}\right) - \frac{\partial \mathcal{L}}{\partial \theta_r} = Q_{\theta_r}$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta_r} = \tfrac{1}{2}M_P[-2\dot x d\, s(\theta_B+\theta_r) + 2l\dot\theta_B d\, s(\theta_r) + 2\dot y d\, c(\theta_B+\theta_r) + 2d^2(\dot\theta_B+\dot\theta_r)] + \tfrac{1}{2}I_P(2\dot\theta_r + 2\dot\theta_B)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_r}\right) = \tfrac{1}{2}M_P[-2\ddot x d\, s(\theta_B+\theta_r) - 2\dot x d(\dot\theta_B+\dot\theta_r)c(\theta_B+\theta_r)$$
$$+ 2l\ddot\theta_B d\, c(\theta_r) - 2l\dot\theta_B d\dot\theta_r s(\theta_r) + 2d^2(\ddot\theta_B+\ddot\theta_r)$$
$$+ 2\ddot y d\, c(\theta_B+\theta_r) - 2\dot y d(\dot\theta_B+\dot\theta_r)s(\theta_B+\theta_r)]$$
$$+ \tfrac{1}{2}I_P(2\ddot\theta_r + 2\ddot\theta_B)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_r} = \tfrac{1}{2}M_P[-2\dot x d(\dot\theta_B+\dot\theta_r)c(\theta_B+\theta_r) - 2l\dot\theta_B d(\dot\theta_B+\dot\theta_r)s(\theta_r)$$
$$-2\dot y d(\dot\theta_B+\dot\theta_r)s(\theta_B+\theta_r)] - k\theta_r$$

$$EOM = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_r}\right) - \frac{\partial \mathcal{L}}{\partial \theta_r} = Q_{\theta_r}$$

$$EOM_{\theta_r} = \tfrac{1}{2}M_P[-2\ddot x d\, s(\theta_B+\theta_r) - \cancel{2\dot x d(\dot\theta_B+\dot\theta_r)c(\theta_B+\theta_r)}$$
$$+ 2l\ddot\theta_B d\, c(\theta_r) - 2l\dot\theta_B d\dot\theta_r s(\theta_r) + 2d^2(\ddot\theta_B+\ddot\theta_r)$$
$$+ 2\ddot y d\, c(\theta_B+\theta_r) - \cancel{2\dot y d(\dot\theta_B+\dot\theta_r)s(\theta_B+\theta_r)}$$
$$+ \cancel{2\dot x d(\dot\theta_B+\dot\theta_r)c(\theta_B+\theta_r)} + 2l\dot\theta_B d(\dot\theta_B+\dot\theta_r)s(\theta_r)$$
$$+ \cancel{2\dot y d(\dot\theta_B+\dot\theta_r)s(\theta_B+\theta_r)}] + I_P(\ddot\theta_r+\ddot\theta_B) + k\theta_r = Q_{\theta_r}$$

EOM_{θ_r}

$$M_P[-\ddot x d\, s(\theta_B+\theta_r) + l\ddot\theta_B d\, c(\theta_r) + \ddot y d\, c(\theta_B+\theta_r) + d^2(\ddot\theta_B+\ddot\theta_r)$$
$$- l\dot\theta_B d\dot\theta_r s(\theta_r) + l\dot\theta_B d(\dot\theta_B+\dot\theta_r)s(\theta_r)]$$
$$+ I_P(\ddot\theta_r+\ddot\theta_B) + k\theta_r = Q_{\theta_r}$$

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x EOM:

$$M_p\left(2\ddot{x}-d(\ddot{\theta}_R+\ddot{\theta}_B)\sin(\theta_B+\theta_R)-d(\dot{\theta}_R+\dot{\theta}_B)^2\cos(\theta_B+\theta_R)\right.$$
$$\left.+d(\ddot{\theta}_B+\ddot{\theta}_L)\sin(\theta_B+\theta_L)+d(\dot{\theta}_B+\dot{\theta}_L)^2\cos(\theta_B+\theta_L)\right)+M_b\ddot{x}=\hat{n}_1\cdot\vec{F}$$

y EOM:

$$M_p[2\ddot{y}-d(\ddot{\theta}_B+\ddot{\theta}_L)\cos(\theta_B+\theta_L)+d(\dot{\theta}_B+\dot{\theta}_L)^2\sin(\theta_B+\theta_L)$$
$$+d(\ddot{\theta}_B+\ddot{\theta}_r)\cos(\theta_B+\theta_r)-d(\dot{\theta}_B+\dot{\theta}_r)^2 s(\theta_B+\theta_r)]$$
$$+M_B\ddot{y}=\hat{n}_2\cdot\vec{F}$$

θ_B EOM

$$M_p[2\ell^2\ddot{\theta}_B+d^2(\ddot{\theta}_B+\ddot{\theta}_L)+\ddot{x}d\sin(\theta_B+\theta_L)$$
$$\ell d(2\ddot{\theta}_B+\ddot{\theta}_L)\cos(\theta_L)-\ell d(2\dot{\theta}_B+\dot{\theta}_L)\dot{\theta}_L\sin(\theta_L)+d^2(\ddot{\theta}_r+\ddot{\theta}_B)$$
$$-\ddot{y}d\cos(\theta_B+\theta_L)-\ddot{x}d\sin(\theta_B+\theta_r)+\ell d(2\ddot{\theta}_B+\ddot{\theta}_r)\cos(\theta_r)$$
$$-\ell d(2\dot{\theta}_B+\dot{\theta}_r)\dot{\theta}_r\sin(\theta_r)+\ddot{y}d\cos(\theta_B+\theta_r)]$$
$$+I_B\ddot{\theta}_B+I_p(\ddot{\theta}_B+\ddot{\theta}_r)+I_p(\ddot{\theta}_B+\ddot{\theta}_L)=\tau$$

θ_r EOM

$$M_p[-\ddot{x}d\sin(\theta_B+\theta_r)+\ell\ddot{\theta}_B d\cos(\theta_r)+\ddot{y}d\cos(\theta_B+\theta_r)+d^2(\ddot{\theta}_B+\theta_r)$$
$$-\ell\dot{\theta}_B d\dot{\theta}_r\sin(\theta_r)+\ell\dot{\theta}_B d(\dot{\theta}_B+\dot{\theta}_r)\sin(\theta_r)]$$
$$+I_p(\ddot{\theta}_r+\ddot{\theta}_B)+k\theta_r)=0$$

θ_L EOM

$$M_p[d^2(\ddot{\theta}_B+\ddot{\theta}_L)+\ddot{x}d\sin(\theta_B+\theta_L)-\ddot{y}d\cos(\theta_B+\theta_L)+\ell\ddot{\theta}_B d\cos(\theta_L)$$
$$-\ell\dot{\theta}_B\dot{\theta}_L d\sin(\theta_L)+\ell\dot{\theta}_B d(\dot{\theta}_B+\dot{\theta}_L)\sin(\theta_L)]+I_p(\ddot{\theta}_B+\ddot{\theta}_L)+k\theta_L=0$$

Matrix Form

Matrix EOM

$$\underline{\underline{M}}(q)\ddot{q} + \underline{C}(q,\dot{q}) + \underline{\underline{k}}q = Q$$

$S(\theta_B+\theta_L) = S_{BL}$ etc.

$C(\theta_B+\theta_L) = C_{BL}$

$q = [X,\ y\ \theta_B\ \theta_L\ \theta_R]$

$$\underline{\underline{M}} = \begin{array}{c} \begin{array}{ccccc} \ddot{x} & \ddot{y} & \ddot{\theta}_B & \ddot{\theta}_L & \ddot{\theta}_r \end{array} \\ \begin{bmatrix} M_B + 2M_P & 0 & M_pd(S_{BL}-S_{BR}) & M_pdS_{BL} & -M_pdS_{BR} \\ 0 & M_B+2M_P & M_pd(C_{BR}-C_{BL}) & -M_pdC_{BL} & M_pdC_{BR} \\ M_pd(S_{BL}-S_{BR}) & M_pd(C_{BR}-C_{BL}) & \begin{matrix} 2M_pd^2 \\ +2M_pld\,C\theta_L \\ +2M_pld\,C\theta_r \\ +2M_pl^2 \\ +2I_p+I_B \end{matrix} & \begin{matrix} M_pd^2 \\ +M_pld\,C\theta_L \\ +I_p \end{matrix} & \begin{matrix} M_pd^2 \\ +M_pld\,C\theta_r \\ +I_p \end{matrix} \\ M_pdS_{BL} & -M_pdC_{BL} & \begin{matrix} M_pd^2 \\ +M_pld\,C\theta_L \\ +I_p \end{matrix} & \begin{matrix} M_pd^2 \\ +I_p \end{matrix} & 0 \\ -M_pdS_{BR} & M_pdC_{BR} & \begin{matrix} M_pld\,C\theta_R \\ +M_pd^2 \\ +I_p \end{matrix} & 0 & \begin{matrix} M_pd^2 \\ +I_p \end{matrix} \end{bmatrix} \end{array}$$

(Mass Matrix)

$$\underline{C} = \begin{bmatrix} M_pd(\dot{\theta}_B+\dot{\theta}_L)^2C_{BL} - M_pd(\dot{\theta}_B+\dot{\theta}_R)^2C_{BR} \\ M_pd(\dot{\theta}_B+\dot{\theta}_L)^2S_{BL} - M_pd(\dot{\theta}_B+\dot{\theta}_R)^2S_{BR} \\ -M_pld(2\dot{\theta}_B+\dot{\theta}_L)\dot{\theta}_L S\theta_L - M_pld(2\dot{\theta}_B+\dot{\theta}_R)\dot{\theta}_R S\theta_R \\ M_pld\,\dot{\theta}_B^2 S\theta_L \\ M_pld\,\dot{\theta}_B S\theta_R \end{bmatrix}$$

(Coriolis Matrix)

$$k = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&k&0 \\ 0&0&0&0&k \end{bmatrix} \qquad Q = \begin{bmatrix} F_x \\ F_y \\ \tau \\ 0 \\ 0 \end{bmatrix}$$

(Stiffness Matrix)

State Variables

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= \theta_b \\
x_4 &= \theta_r \\
x_5 &= \theta_\ell \\
x_6 &= \dot{x} \\
x_7 &= \dot{y} \\
x_8 &= \dot{\theta}_b \\
x_9 &= \dot{\theta}_r \\
x_{10} &= \dot{\theta}_\ell
\end{aligned}
\qquad
\begin{aligned}
\dot{x}_1 &= x_6 \\
\dot{x}_2 &= x_7 \\
\dot{x}_3 &= x_8 \\
\dot{x}_4 &= x_9 \\
\dot{x}_5 &= x_{10}
\end{aligned}
$$

$$
\begin{bmatrix} \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \end{bmatrix} = [M] \backslash \left(Q - [h][q] - C \right)
$$