Chapter 1

One-Dimensional Problems

2025/3/17

主題一

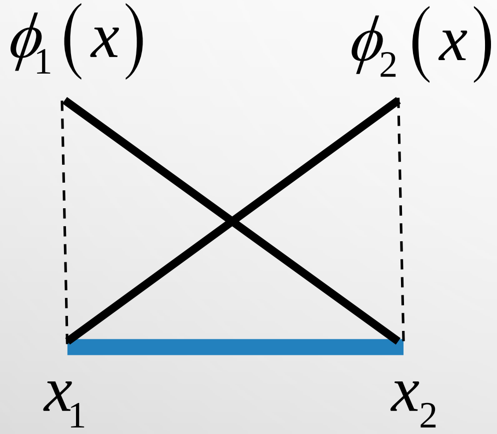
Lagrange Interpolation Functions

◆ Lagrange Interpolation Functions

- 線性近似函數

$$\phi_1^e(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\phi_2^e(x) = \frac{x - x_1}{x_2 - x_1}$$

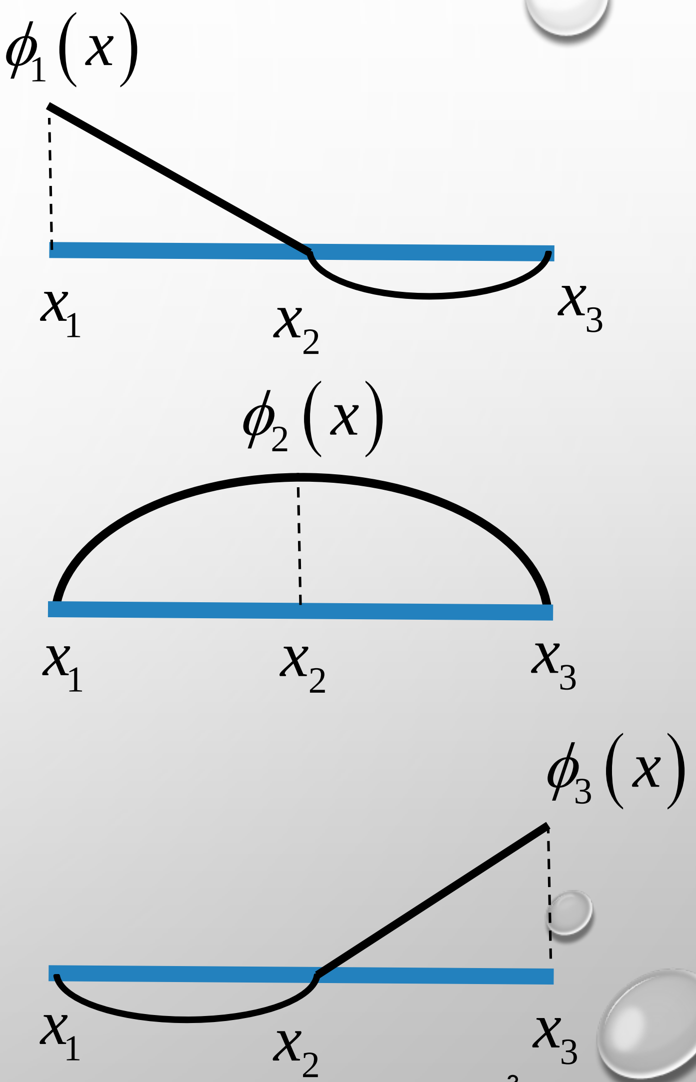


- 二次近似函數

$$\phi_1^e(x) = \frac{(x_2 - x)(x_3 - x)}{(x_3 - x_1)(x_2 - x_1)}$$

$$\phi_2^e(x) = \frac{(x_3 - x)(x_1 - x)}{(x_3 - x_2)(x_1 - x_2)}$$

$$\phi_3^e(x) = \frac{(x_2 - x)(x_1 - x)}{(x_2 - x_3)(x_1 - x_3)}$$



◆ Lagrange Interpolation Functions

➢ 對於一維 n個節點的 Lagrange 插值函數

$$\phi_i^e(\bar{x}) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{\bar{x}-\bar{x}_j}{\bar{x}_i-\bar{x}_j}, \qquad i = 1,2,3,\ldots,n$$

其中:

- $\phi_i(\bar{x})$ 是對應第i個節點的形狀函數
- $\bar{x}$ 為局部座標(通常範圍為[-1,1])
- 形狀函數滿足插值性質

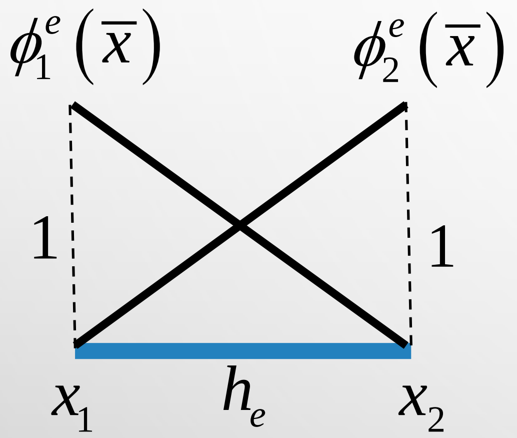
$$\phi_i(\bar{x}_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

✓ Partition unity

$$\sum_{i=1}^{n} \phi_i^e(x) = 1$$

Lagrange Interpolation Functions

- Linear Largrange Interpolation Function



假設:

$$\phi_1^e(\bar{x}) = a_1 + b_1\bar{x}$$
$$\phi_2^e(\bar{x}) = a_2 + b_2\bar{x}$$

1. 當 $\bar{x} = 0,\quad \phi_1^e(0) = 1$

$$a_1 = 1$$

當 $\bar{x} = h_e,\quad \phi_1^e(0) = 0$

$$b_1 = -\frac{1}{h_e}$$

2. 當 $\bar{x} = 0,\quad \phi_2^e(0) = 0$

$$a_2 = 0$$

當 $\bar{x} = h_e,\quad \phi_2^e(0) = 1$

$$b_2 = \frac{1}{h_e}$$

✓ 因此線性插值函數通式

$$\phi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e}$$

$$\phi_2^e(\bar{x}) = \frac{\bar{x}}{h_e}$$

Lagrange Interpolation Functions

- Quadratic Largrange Interpolation Function

假設二階插值函數(n=3)

$$\phi_i^e(\bar{x}) = \prod_{\substack{j=1 \\ j\neq i}}^{3} \frac{\bar{x}-\bar{x}_j}{\bar{x}_i-\bar{x}_j}$$

其中:

$$x_1 = 0$$

$$x_2 = \alpha \cdot h_e$$

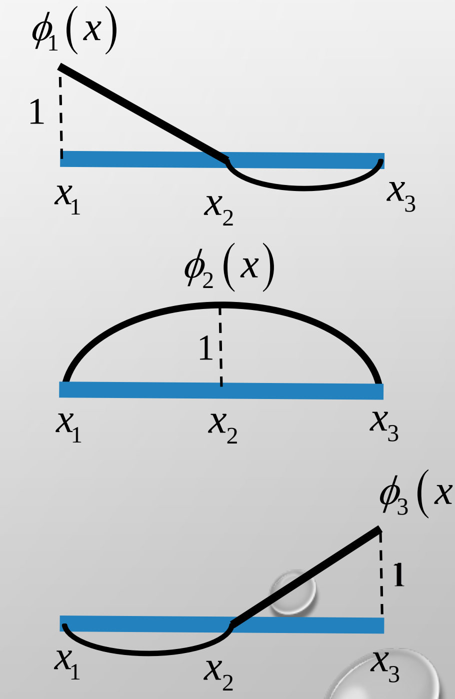
$$x_3 = h_e$$

✓ 因此二階插值函數通式

$$\phi_1^e(\bar{x}) = \frac{(\bar{x}-\bar{x}_2)(\bar{x}-\bar{x}_3)}{(\bar{x}_1-\bar{x}_2)(\bar{x}_1-\bar{x}_3)} = \left(1-\frac{\bar{x}}{h_e}\right)\left(1-\frac{1}{\alpha}\frac{\bar{x}}{h_e}\right)$$

$$\phi_2^e(\bar{x}) = \frac{(\bar{x}-\bar{x}_1)(\bar{x}-\bar{x}_3)}{(\bar{x}_2-\bar{x}_1)(\bar{x}_2-\bar{x}_3)} = \frac{1}{\alpha(1-\alpha)}\frac{\bar{x}}{h_e}\left(1-\frac{\bar{x}}{h_e}\right)$$

$$\phi_3^e(\bar{x}) = \frac{(\bar{x}-\bar{x}_1)(\bar{x}-\bar{x}_2)}{(\bar{x}_3-\bar{x}_1)(\bar{x}_3-\bar{x}_2)} = \frac{\alpha}{(1-\alpha)}\frac{\bar{x}}{h_e}\left(1-\frac{1}{\alpha}\frac{\bar{x}}{h_e}\right)$$



CH1 ONE-DIMENSIONAL PROBLEMS

Lagrange Interpolation Functions

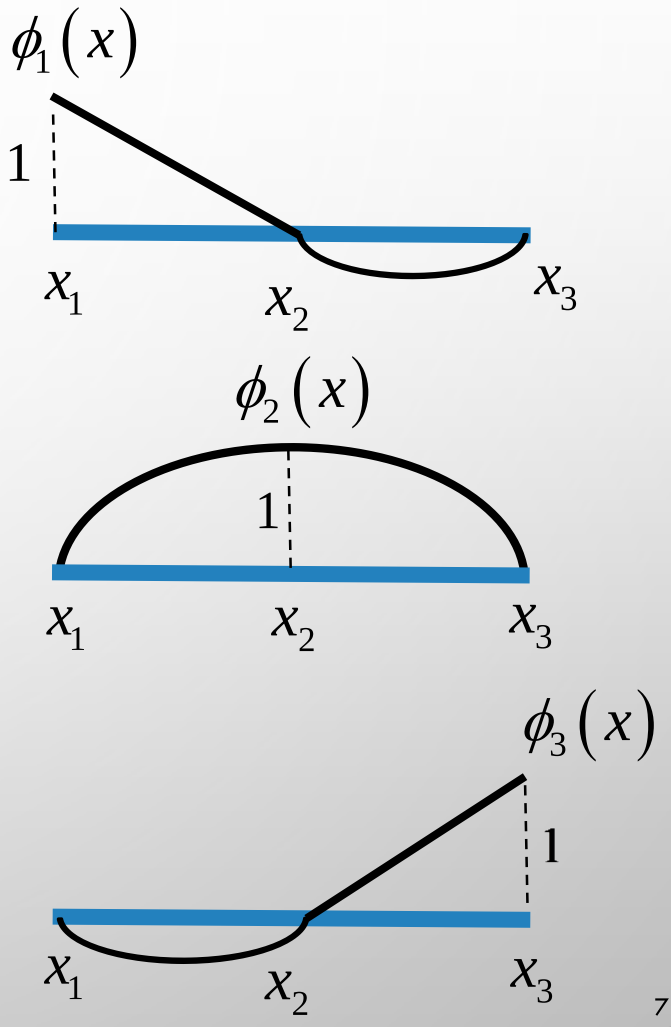
- Quadratic Largrange Interpolation Function

$$\alpha = \frac{1}{2} \quad \text{(等距分佈)}$$

$$\phi_1^e(\bar{x}) = \left(1-\frac{\bar{x}}{h_e}\right)\left(1-\frac{1}{\alpha}\frac{\bar{x}}{h_e}\right) \rightarrow \left(1-\frac{\bar{x}}{h_e}\right)\left(1-\frac{2\bar{x}}{h_e}\right)$$

$$\phi_2^e(\bar{x}) = \frac{1}{\alpha(1-\alpha)}\frac{\bar{x}}{h_e}\left(1-\frac{\bar{x}}{h_e}\right) \rightarrow 4\frac{\bar{x}}{h_e}\left(1-\frac{\bar{x}}{h_e}\right)$$

$$\phi_3^e(\bar{x}) = \frac{\alpha}{(1-\alpha)}\frac{\bar{x}}{h_e}\left(1-\frac{1}{\alpha}\frac{\bar{x}}{h_e}\right) \rightarrow \frac{\bar{x}}{h_e}\left(1-\frac{2\bar{x}}{h_e}\right)$$



主題二

Derivation of Element Equation

CH1 ONE-DIMENSIONAL PROBLEMS

Derivation of Element Equation

- For example by Ritz method

Consider the P.D.E. problem

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \quad \text{for } 0<x<L$$

Boundary conditions:

$$u(0)=u_0,\ \left(a\frac{\partial u}{\partial x}\right)\Bigg|_{x=L}=Q_0$$

- Assume the polynomial approximation solution

$$U_N=\sum_{j=1}^{N}c_j\phi_j+\phi_0$$

$c_j\phi_j$ is replace $u_j^e\psi_j^e$ $(\phi_0=0)$

$$u_h^e=\sum_{j=1}^{n}u_j^e\cdot\psi_j^e(x)$$

Derivation of Element Equation

- For example by Ritz method

Weighted-integral form

$$\int_{x_a}^{x_b}\left(a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right)dx-\left[wa\frac{du}{dx}\right]\Bigg|_{x_a}^{x_b}=0$$

代入邊界條件: $u(0)=u_0,\ \left(a\frac{\partial u}{\partial x}\right)\Bigg|_{x=L}=Q_0$

$$\int_{x_a}^{x_b}\left(a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right)dx-w(x_a)Q_1-w(x_b)Q_2=0$$

- Principles of minimum potential energy (MPE)

$$\delta I^e=B^e(w,u)-l^e(w)=0\ ,w=\delta u$$

$$B^e(w,u)=\int_{x_a}^{x_b}\left(a\frac{dw}{dx}\frac{du}{dx}+cwu\right)dx=0$$

$$l^e(w)=\int_{x_a}^{x_b}w\,f\,dx+w(x_a)Q_1+w(x_b)Q_2$$

Derivation of Element Equation

- For example by Ritz method

The *i*th algebraic equation of system of *n* equations:

$$0 = \sum_{j=1}^{n} K_{ij}^{e} u_{j}^{e} - f_{i}^{e} - Q_{i}^{e} \quad (i = 1, 2, \ldots, n)$$

where

$$K_{ij}^{e} = B^{e}(\psi_{i}^{e}, \psi_{j}^{e}) = \int_{x_a}^{x_b} \left(a \frac{d\psi_{i}^{e}}{dx} \frac{d\psi_{j}^{e}}{dx} + c\psi_{i}^{e}\psi_{j}^{e} \right) dx$$

$$f_{i}^{e} = \int_{x_a}^{x_b} f \psi_{i}^{e} \, dx$$

$$Q_{i}^{e} = \sum_{j=1}^{n} \psi_{i}^{e}\left(x_{j}^{e}\right) \cdot Q_{j}^{e}$$

- In matrix notation

$$\left[K^{e}\right]\left\{u^{e}\right\} = \left\{f^{e}\right\} + \left\{Q^{e}\right\}$$

or

$$\mathbf{K}^{e}\mathbf{u}^{e} = \mathbf{f}^{e} + \mathbf{Q}^{e}$$

Derivation of Element Equation

- Linear Element

$$K_{ij}^{e}=\int_{x_e}^{x_{e+1}}\left(a\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}+c\psi_i^e\psi_j^e\right)dx$$

$$f_i^{e}=\int_{x_e}^{x_{e+1}} f\psi_i^e\,dx$$

- In the local coordinate: $x=x_1^e+\bar{x}$

$$K_{ij}^{e}=\int_{0}^{h_e}\left(a\frac{d\psi_i^e}{d\bar{x}}\frac{d\psi_j^e}{d\bar{x}}+c\psi_i^e\psi_j^e\right)d\bar{x}$$

$$f_i^{e}=\int_{0}^{h_e} f\psi_i^e\,d\bar{x}$$

- Linear Largrange Interpolation Function

$$\psi_1^e(\bar{x})=1-\frac{\bar{x}}{h_e}$$

$$\psi_2^e(\bar{x})=\frac{\bar{x}}{h_e}$$

Derivation of Element Equation

- Linear Element

$$\psi_1^e = 1 - \frac{\bar{x}}{h_e}, \quad \psi_2^e = \frac{\bar{x}}{h_e}$$

$$\frac{d\psi_1^e}{d\bar{x}} = -\frac{1}{h_e}, \quad \frac{d\psi_2^e}{d\bar{x}} = \frac{1}{h_e}$$

$$K_{11}^e = \int_0^{h_e} \left(a \frac{d\psi_1^e}{d\bar{x}} \frac{d\psi_1^e}{d\bar{x}} + c\psi_1^e \psi_1^e \right) d\bar{x}$$

$$= \int_0^{h_e} \left[a\left(-\frac{1}{h_e}\right)\left(-\frac{1}{h_e}\right) + c\left(1 - \frac{\bar{x}}{h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right) \right] d\bar{x}$$

$$K_{12}^e = \int_0^{h_e} \left(a \frac{d\psi_1^e}{d\bar{x}} \frac{d\psi_2^e}{d\bar{x}} + c\psi_1^e \psi_2^e \right) d\bar{x}$$

$$= \int_0^{h_e} \left[a\left(-\frac{1}{h_e}\right)\left(\frac{1}{h_e}\right) + c\left(1 - \frac{\bar{x}}{h_e}\right)\left(\frac{\bar{x}}{h_e}\right) \right] d\bar{x} = K_{21}^e$$

$$K_{22}^e = \int_0^{h_e} \left(a \frac{d\psi_2^e}{d\bar{x}} \frac{d\psi_2^e}{d\bar{x}} + c\psi_2^e \psi_2^e \right) d\bar{x}$$

$$= \int_0^{h_e} \left[a\left(\frac{1}{h_e}\right)\left(\frac{1}{h_e}\right) + c\left(\frac{\bar{x}}{h_e}\right)\left(\frac{\bar{x}}{h_e}\right) \right] d\bar{x}$$

Derivation of Element Equation

- Linear Element

G.E.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \quad \text{for } 0<x<L$$

Stiffness Matrix

$$K_{11}^e=\frac{a_e}{h_e}+\frac{1}{3}c_e h_e$$

$$K_{12}^e=-\frac{a_e}{h_e}+\frac{1}{6}c_e h_e=K_{21}^e$$

$$K_{22}^e=\frac{a_e}{h_e}+\frac{1}{3}c_e h_e$$

$$f_1^e=\int_0^{h_e} f_e\left(1-\frac{\bar{x}}{h_e}\right)dx=\frac{1}{2}f_e h_e$$

$$f_2^e=\int_0^{h_e} f_e\left(\frac{\bar{x}}{h_e}\right)dx=\frac{1}{2}f_e h_e$$

✓ The coefficient matrix and column vector

$$\left[K^e\right]=\frac{a_e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{c_e h_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\left\{f^e\right\}=\frac{f_e h_e}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

Derivation of Element Equation

- Quadratic Element

$$\psi_1^e(\bar{x}) = \left(1-\frac{\bar{x}}{h_e}\right)\left(1-\frac{2\bar{x}}{h_e}\right) = 1-3\frac{\bar{x}}{h_e}+2\left(\frac{\bar{x}}{h_e}\right)^2$$

$$\psi_2^e(\bar{x}) = 4\frac{\bar{x}}{h_e}\left(1-\frac{\bar{x}}{h_e}\right) = 4\frac{\bar{x}}{h_e}-4\left(\frac{\bar{x}}{h_e}\right)^2$$

$$\psi_3^e(\bar{x}) = \frac{\bar{x}}{h_e}\left(1-\frac{2\bar{x}}{h_e}\right) = \frac{\bar{x}}{h_e}-2\left(\frac{\bar{x}}{h_e}\right)^2$$

$$\frac{\partial\psi_1^e}{\partial\bar{x}} = -\frac{3}{h_e}+\frac{4\bar{x}}{h_e^2}$$

$$\frac{\partial\psi_2^e}{\partial\bar{x}} = \frac{4}{h_e}-\frac{8\bar{x}}{h_e^2}$$

$$\frac{\partial\psi_3^e}{\partial\bar{x}} = \frac{1}{h_e}-\frac{4\bar{x}}{h_e^2}$$

$$K_{11}^e = \int_0^{h_e}\left(a\frac{d\psi_1^e}{d\bar{x}}\frac{d\psi_1^e}{d\bar{x}}+c\psi_1^e\psi_1^e\right)d\bar{x}$$

$$= \int_0^{h_e}\left\{\begin{matrix} a_e\left(-\frac{3}{h_e}+\frac{4\bar{x}}{h_e^2}\right)\left(-\frac{3}{h_e}+\frac{4\bar{x}}{h_e^2}\right) \\ +c_e\left[1-3\frac{\bar{x}}{h_e}+2\left(\frac{\bar{x}}{h_e}\right)^2\right]\left[1-3\frac{\bar{x}}{h_e}+2\left(\frac{\bar{x}}{h_e}\right)^2\right]\end{matrix}\right\}d\bar{x} = \frac{7}{3}\frac{a_e}{h_e}+\frac{2}{15}c_e h_e$$

$$K_{12}^e = \int_0^{h_e}\left(a\frac{d\psi_1^e}{d\bar{x}}\frac{d\psi_1^e}{d\bar{x}}+c\psi_1^e\psi_1^e\right)d\bar{x}$$

$$= \int_0^{h_e}\left\{\begin{matrix} a_e\left(-\frac{3}{h_e}+\frac{4\bar{x}}{h_e^2}\right)\left(\frac{4}{h_e}-\frac{8\bar{x}}{h_e^2}\right) \\ +c_e\left[1-3\frac{\bar{x}}{h_e}+2\left(\frac{\bar{x}}{h_e}\right)^2\right]\left[4\frac{\bar{x}}{h_e}\left(1-\frac{\bar{x}}{h_e}\right)\right]\end{matrix}\right\}d\bar{x} = -\frac{8}{3}\frac{a_e}{h_e}+\frac{2}{30}c_e h_e = K_{21}^e$$

Derivation of Element Equation

- Quadratic Element

G.E.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \quad \text{for } 0<x<L$$

Stiffness Matrix

K_{11}^{e}

$K_{12}^{e}=K_{21}^{e}$

$K_{13}^{e}=K_{31}^{e}$

K_{22}^{e}

$K_{23}^{e}=K_{32}^{e}$

K_{33}^{e}

$$f_1^e=\int_0^{h_e} f_e\left[1-3\frac{\bar{x}}{h_e}+2\left(\frac{\bar{x}}{h_e}\right)^2\right]dx=\frac{1}{6}f_e h_e$$

$$f_2^e=\int_0^{h_e} f_e\left[4\frac{\bar{x}}{h_e}-4\left(\frac{\bar{x}}{h_e}\right)^2\right]dx=\frac{4}{6}f_e h_e$$

$f_3^e=f_1^e$ (by symmetry)

✓ The coefficient matrix and column vector

$$\left[K^e\right]=\frac{a_e}{3h_e}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}+\frac{c_e h_e}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}$$

$$\left\{f^e\right\}=\frac{f_e h_e}{6}\begin{bmatrix}1\\4\\1\end{bmatrix}$$

CH1 ONE-DIMENSIONAL PROBLEMS

Derivation of Element Equation

G.E.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \quad \text{for } 0<x<L$$

- Linear Element

$$\left[K^e\right]=\frac{a_e}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{c_e h_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\left\{f^e\right\}=\frac{f_e h_e}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

- Quadratic Element

$$\left[K^e\right]=\frac{a_e}{3h_e}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}+\frac{c_e h_e}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}$$

$$\left\{f^e\right\}=\frac{f_e h_e}{6}\begin{bmatrix}1\\ 4\\ 1\end{bmatrix}$$

主題三

Example

CH1 ONE-DIMENSIONAL PROBLEMS

For example

G.E.

$$-\left(\frac{d^2u}{dx^2}\right)-u+x^2=0 \quad \text{for } \; 0<x<L$$

Boundary conditions:

$$u(0)=u_0,\ u(1)=0$$

Exact solution:

$$u_{exact}=\frac{\sin(x)+2\sin(1-x)}{\sin 1}+x^2-2$$

已知係數矩陣形式

$$K_{ij}^e=\int_{x_a}^{x_b}\left(\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}-\psi_i^e\psi_j^e\right)dx$$

$$f_i^e=\int_{x_a}^{x_b}\left(-x^2\right)\psi_i^e\,dx$$

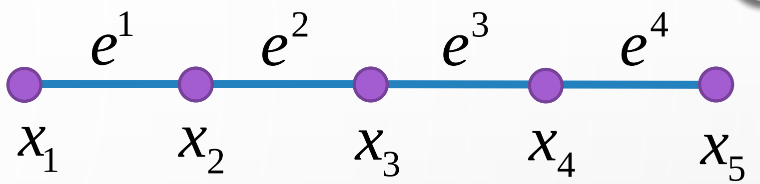
分別使用線性及二次函數求解，並比較其誤差

CH1 ONE-DIMENSIONAL PROBLEMS

For example

G.E. $-\left(\frac{d^2u}{dx^2}\right) - u + x^2 = 0 \quad \text{for } 0 < x < L$

B.C. $u(0) = u_0,\ u(1) = 0 \quad u_{exact} = \frac{\sin(x) + 2\sin(1-x)}{\sin 1} + x^2 - 2$



- Linear Element

$$a = 1,\quad c = -1,\quad h_e = \frac{1}{4}$$

$$\phi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e},\quad \phi_2^e(\bar{x}) = \frac{\bar{x}}{h_e}$$

$$\left[K^e\right] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\left\{f^e\right\} = \frac{f_e h_e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- The element coefficient matrix

$$\left[K^e\right] = \frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} = \begin{bmatrix} 3.9167 & -4.0417 \\ -4.0417 & 3.9167 \end{bmatrix}$$

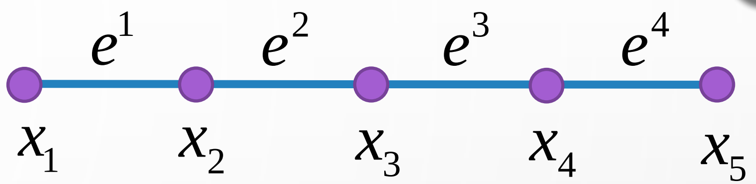
$$f_1^e = \int_{x_a}^{x_b} (-\bar{x}^2)\left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x} = -\frac{1}{h_e}\left[\frac{x_b}{3}\left(x_b^3 - x_a^3\right) - \frac{1}{4}\left(x_b^4 - x_a^4\right)\right]$$

$$f_2^e = \int_{x_a}^{x_b} (-\bar{x}^2)\left(\frac{\bar{x}}{h_e}\right) d\bar{x} = -\frac{1}{h_e}\left[\frac{1}{4}\left(x_b^4 - x_a^4\right) - \frac{x_a}{3}\left(x_b^3 - x_a^3\right)\right]$$

CH1 ONE-DIMENSIONAL PROBLEMS

- **For example** G.E. $-\left(\frac{d^2u}{dx^2}\right) - u + x^2 = 0 \quad \text{for } 0 < x < L$

  - Linear Element B.C. $u(0) = u_0,\ u(1) = 0 \quad u_{exact} = \frac{\sin(x) + 2\sin(1-x)}{\sin 1} + x^2 - 2$



$$f_1^e = \int_{x_a}^{x_b} (-\bar{x}^2)\left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x} = -\frac{1}{h_e}\left[\frac{x_b}{3}\left(x_b^{\ 3} - x_a^{\ 3}\right) - \frac{1}{4}\left(x_b^{\ 4} - x_a^{\ 4}\right)\right]$$

$$f_2^e = \int_{x_a}^{x_b} (-\bar{x}^2)\left(\frac{\bar{x}}{h_e}\right) d\bar{x} = -\frac{1}{h_e}\left[\frac{1}{4}\left(x_b^{\ 4} - x_a^{\ 4}\right) - \frac{x_a}{3}\left(x_b^{\ 3} - x_a^{\ 3}\right)\right]$$

- Element 1.

$$f_1^1 = -0.001302, \qquad f_2^1 = -0.003906$$

- Element 2.

$$f_1^2 = -0.014323, \qquad f_2^2 = -0.022135$$

- Element 3.

$$f_1^3 = -0.042969, \qquad f_2^3 = -0.05599$$

- Element 4.

$$f_1^4 = -0.08724, \qquad f_2^4 = -0.10547$$

$$[F]_{5\times1} = \sum_{e=1}^{4}\left[f^e\right]_{2\times1}$$

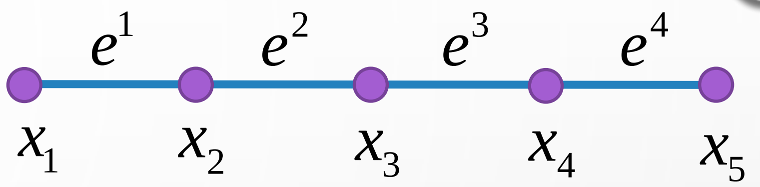
$$= -\begin{bmatrix} 0.00130 \\ 0.00391 + 0.01432 \\ 0.02213 + 0.04297 \\ 0.05599 + 0.08724 \\ 0.10547 \end{bmatrix} = -\begin{bmatrix} 0.00130 \\ 0.01823 \\ 0.06510 \\ 0.14323 \\ 0.10547 \end{bmatrix}$$

CH1 ONE-DIMENSIONAL PROBLEMS

- **For example**
  - Linear Element

G.E. $-\left(\dfrac{d^2u}{dx^2}\right)-u+x^2=0 \quad \text{for } 0<x<L$

B.C. $u(0)=u_0,\ u(1)=0 \quad u_{exact}=\dfrac{\sin(x)+2\sin(1-x)}{\sin 1}+x^2-2$



$$[K]_{5\times5}\{u\}_{5\times1}=\{F\}_{5\times1}$$

$$\begin{bmatrix} 3.9167 & -4.0417 & 0 & 0 & 0 \\ -4.0417 & 7.8333 & -4.0417 & 0 & 0 \\ 0 & -4.0417 & 7.8333 & -4.0417 & 0 \\ 0 & 0 & -4.0417 & 7.8333 & -4.0417 \\ 0 & 0 & 0 & -4.0417 & 3.9167 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}=-\begin{bmatrix} 0.00130 \\ 0.01823 \\ 0.06510 \\ 0.14323 \\ 0.10547 \end{bmatrix}+\begin{bmatrix} Q_1^1 \\ Q_2^1+Q_1^2 \\ Q_2^2+Q_1^3 \\ Q_2^3+Q_1^4 \\ Q_2^4 \end{bmatrix}$$

解矩陣:

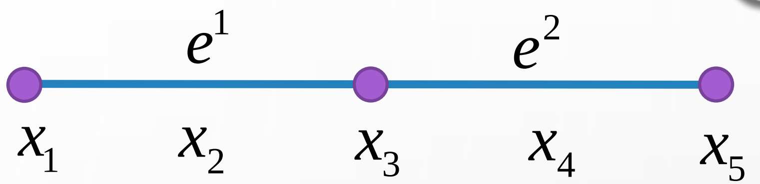
$$u_2=-0.02323,\ u_3=-0.04052,\ u_4=-0.03919$$

CH1 ONE-DIMENSIONAL PROBLEMS

For example

G.E. $-\left(\frac{d^2u}{dx^2}\right)-u+x^2=0 \quad \text{for } 0<x<L$

B.C. $u(0)=u_0,\ u(1)=0 \quad u_{exact}=\frac{\sin(x)+2\sin(1-x)}{\sin 1}+x^2-2$



- Quadratic Element?

$$a=1,\quad c=-1,\quad h_e=\frac{1}{2}$$

$$\left[K^e\right]=\frac{a_e}{3h_e}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}+\frac{c_e h_e}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}$$

$$\left\{f^e\right\}=\frac{f_e h_e}{6}\begin{bmatrix}1\\ 4\\ 1\end{bmatrix}$$

- The element coefficient matrix

$$\left[K^e\right]=\frac{1}{60}\begin{bmatrix}276 & -322 & 41\\ -322 & 624 & -322\\ 41 & -322 & 276\end{bmatrix}=\begin{bmatrix}4.6000 & -5.3667 & 0.6833\\ -5.3667 & 10.4000 & -5.3667\\ 0.6833 & -5.3667 & 4.6000\end{bmatrix}$$

$$f_1^e=-\frac{h_e}{60}\left(-h_e^{\ 2}+10x_a^2\right)$$

$$f_2^e=-\frac{h_e}{15}\left(3h_e^{\ 2}+10x_a^2+10x_a h_e\right)$$

$$f_3^e=-\frac{h_e}{60}\left(9h_e^{\ 2}+10x_a^2+20x_a h_e\right)$$

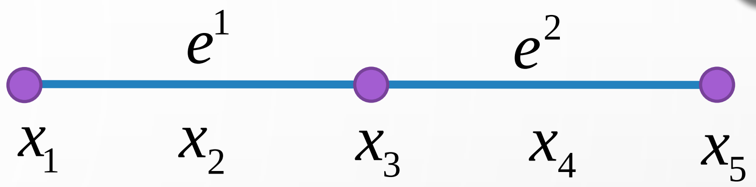
CH1 ONE-DIMENSIONAL PROBLEMS

For example

- Quadratic Element?

G.E. $-\left(\frac{d^2u}{dx^2}\right) - u + x^2 = 0 \quad \text{for } 0 < x < L$

B.C. $u(0) = u_0,\ u(1) = 0 \quad u_{exact} = \frac{\sin(x) + 2\sin(1-x)}{\sin 1} + x^2 - 2$



$$[K]_{5\times5}\{u\}_{5\times1} = \{F\}_{5\times1}$$

- Assemble

$$[?]_{5\times5}\{u\}_{5\times1} = \{?\}_{5\times1}$$

- 代入邊界條件

$$\begin{bmatrix} 10.4000 & -5.3667 & 0 \\ -5.3667 & 9.2000 & -5.3667 \\ 0 & -5.3667 & 10.4000 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = -\begin{Bmatrix} 0.02500 \\ 0.03750 \\ 0.19167 \end{Bmatrix}$$

解矩陣:

$$u_2 = -0.02345,\ u_3 = -0.04078,\ u_4 = -0.03947$$