Chapter 1
One-Dimensional Problems

2025/3/24

主題一

One-Dimensional General Problem

CH1 ONE-DIMENSIONAL PROBLEMS

◆ General Problem

- Becker書中所描述的一維問題:

$$\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u=f,\qquad x\in[0,L]$$

- 廣義的二階微分方程通式
  - ✓ 拉壓桿件問題

    $$a(x)=EA,\,c(x)=0$$

  - ✓ 熱傳導(Heat Conduction)

    $$a(x)=kA,\,c(x)=0$$

  - ✓ 對流擴散(Convection-Diffusion)

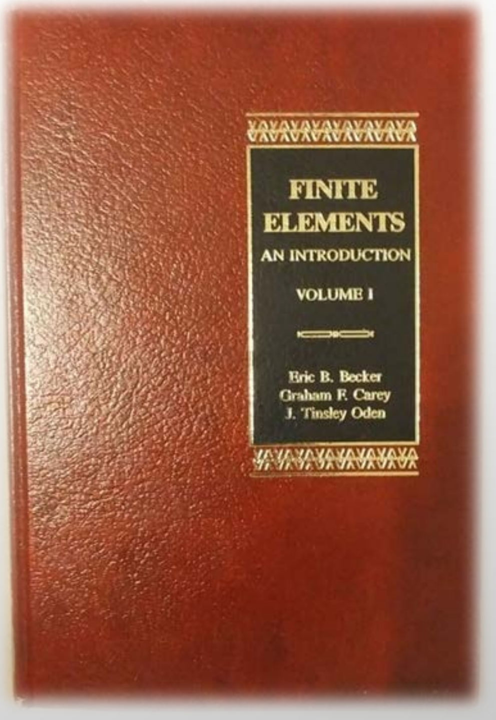
    $$c(x)\neq 0$$

- Ex:

$$\frac{d}{dx}\left(EA(x)\frac{du}{dx}\right)+f(x)=0$$

$$-\frac{d}{dx}\left(k(x)\frac{dT}{dx}\right)=q(x)$$

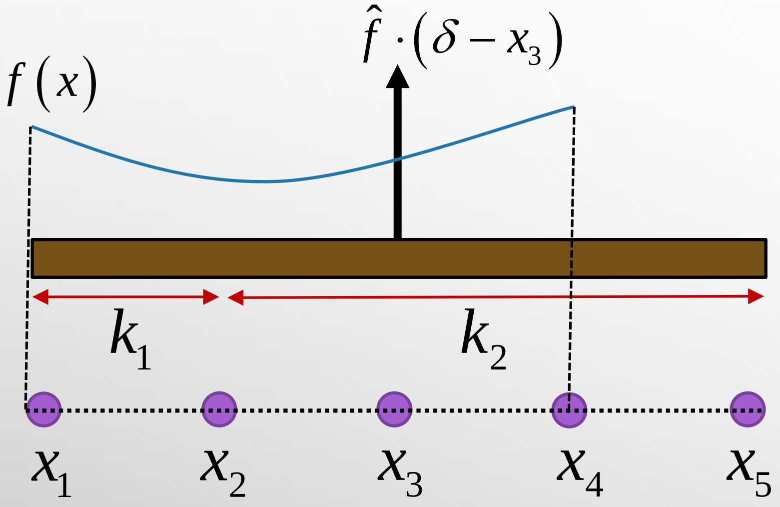
$$-D\frac{d^2u}{dx^2}+v\frac{du}{dx}=s(x)$$



Finite Elements
Eric B Becker, 1981

General Problem

- Discontinuous boundary conditions ($0 < x < L$)



- ✓ Incompressible
- ✓ Conservation of mass

- Derivation of the governing equations

$$\sigma(b) - \sigma(a) = \int_a^b f(x)\,dx \qquad \frac{d\sigma(x)}{dx} = f(x)$$

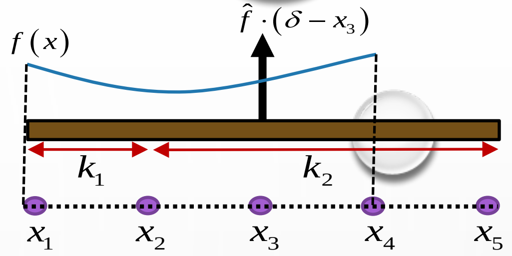
- 需轉成主要(primary)變數

$$\sigma(x) = -k(x) \cdot \frac{du}{dx}$$

結構$\sigma(x) = E \cdot \dfrac{du}{dx} = E \cdot \varepsilon$

流力$\sigma(x) = -\dfrac{d\Phi}{dx}$

CH1 ONE-DIMENSIONAL PROBLEMS



General Problem

- Derivation of the governing equations

$$\frac{d}{dx}\left[-k(x)\cdot\frac{du}{dx}\right]=f(x)$$

- 若從靜態平衡擴充至動態與阻尼系統

$$\frac{d\sigma(x)}{dx}=f(x)-b(x)u(x)-\frac{du(x)}{dt}$$

其中

$b(x)u(x)$:阻尼力

$\frac{du}{dt}$:動態項

- 處理時間項 $u=u(x,t)$

$$\frac{du(x)}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}$$

$$=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}c(x)$$

其中

$\frac{\partial x}{\partial t}=c(x)$:表示觀察點在空間上移動的速度

- 考慮Steady state $\left(\frac{\partial u}{\partial t}=0\right)$

$$\frac{d\sigma(x)}{dx}=f(x)-b(x)u(x)-c(x)\frac{du(x)}{dt}$$

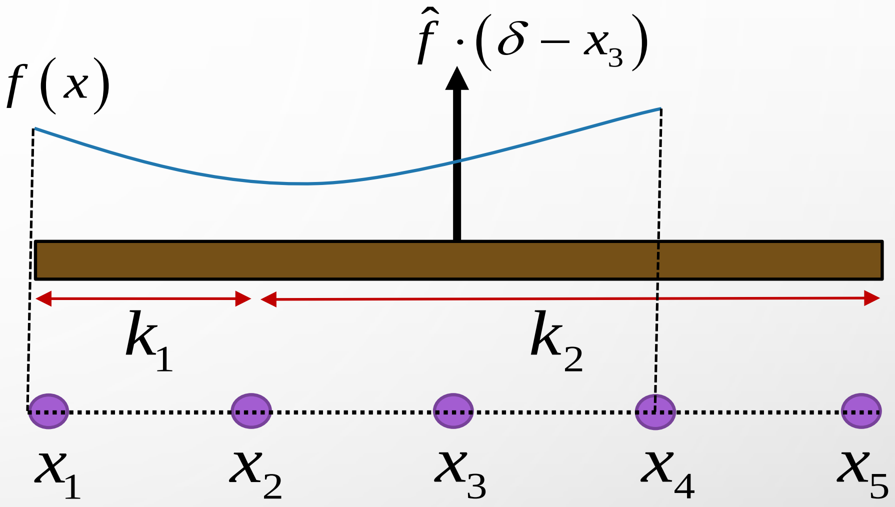
CH1 ONE-DIMENSIONAL PROBLEMS

General Problem

- Derivation of the governing equations

$$\frac{d}{dx}\left[-k(x)\cdot\frac{du}{dx}\right]+c(x)\frac{du(x)}{dx}+b(x)u(x)=f(x)$$

此為一條一維二階常微分方程式(2nd-order P.D.E.)
(包含二次微分、一次微分、不微分項)



- 分計算領域，再看不連續位置的條件表示式

1. 不連續位置的條件

由flux守恆

$$\sigma(b)-\sigma(a)=\int_{a\to x_2^-}^{b\to x_2^+} f(x)dx$$

$$\sigma(x_2^+)-\sigma(x_2^-)=0 \quad or \quad [\sigma(x_2)]=0$$

同理 $[\sigma(x_4)]=0$

但是在 $x=x_3$

令 $f(x)=\bar{f}(x)+\hat{f}(x)\cdot\delta(x-x_2)$

$$\sigma(x_3^+)-\sigma(x_3^-)=\int_{x_3^-}^{x_3^+}\bar{f}(x)dx+\int_{x_3^-}^{x_3^+}\hat{f}(x)\cdot\delta(x-x_2)dx$$

$$=>[\sigma(x_3)]=\hat{f}$$

CH1 ONE-DIMENSIONAL PROBLEMS

General Problem

- Derivation of the governing equations

2. 兩點邊界值問題(Two-point B.V.P.)

有限元素法中:

✓ 必要邊界條件(Essential B.C.)

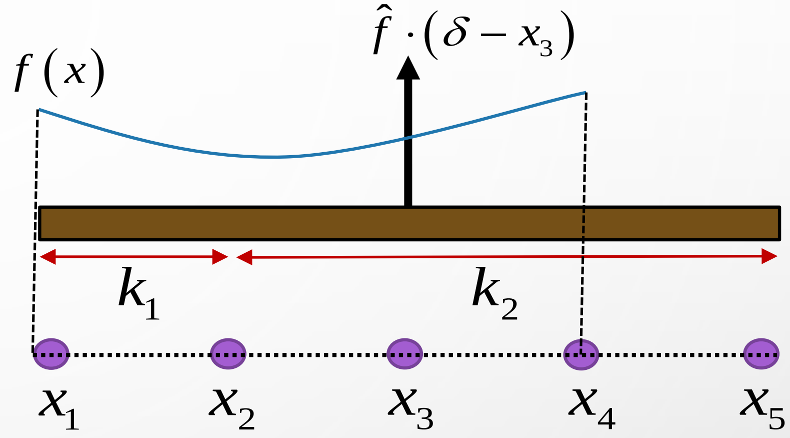
指定未知變數值

$$u = \bar{u}$$

✓ 自然邊界條件(Natural B.C.)

指定導數/應力/通量

$$\sigma = \bar{\sigma} = \frac{du}{dx}$$



Boundary-Value Problem

$$-\frac{d}{dx}\left[k(x)\cdot\frac{du}{dx}\right]+c(x)\frac{du(x)}{dx}+b(x)u(x)=f(x)$$

Discontinuity Condition

$$[\sigma(x_2)]=0,\quad [\sigma(x_3)]=\hat{f},\quad [\sigma(x_4)]=0,$$

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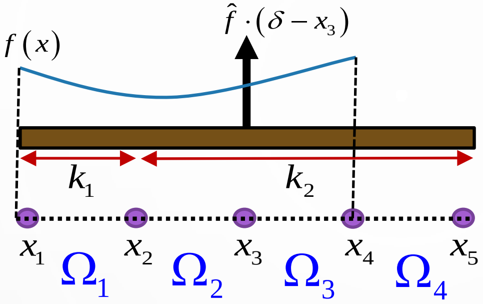
General Problem

➢ 加權殘插弱滿足表示式



Boundary-Value Problem

$$-\frac{d}{dx}\left[k(x)\cdot\frac{du}{dx}\right]+c(x)\frac{du(x)}{dx}+b(x)u(x)=f(x)$$

Discontinuity Condition

$$[\sigma(x_2)]=0,\quad [\sigma(x_3)]=\hat{f},\quad [\sigma(x_4)]=0,$$

Residual Function

$$r(x)=-\left[k(x)\cdot u'(x)\right]'+c(x)u'(x)+b(x)u(x)-f(x)$$

Weighting Function

$$w(x)$$

Approximate Function

$$u(x)=\sum_{j=1}^{N}\alpha_j\phi_j(x)$$

- Weighted Residual Method

$$\int_{\Omega_i} r(x)\cdot w(x)\,dx=0$$

➢ Higher-order term

$$-\int_{\Omega_i}(ku')'\,w\,dx=-\int_{\Omega_i}\left[(ku'w)'-ku'w'\right]dx$$

$$=\int_{\Omega_i}ku'w'\,dx-ku'w\Big|_{x_i}^{x_{i+1}}$$

$$=\int_{\Omega_i}ku'w'\,dx-\left(ku'w\Big|_{x_{i+1}}-ku'w\Big|_{x_i}\right)$$

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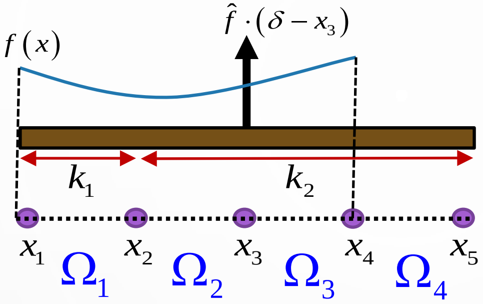

Boundary-Value Problem

$$-\frac{d}{dx}\left[k(x)\cdot\frac{du}{dx}\right]+c(x)\frac{du(x)}{dx}+b(x)u(x)=f(x)$$

Discontinuity Condition

$$[\sigma(x_2)]=0,\quad [\sigma(x_3)]=\hat{f},\quad [\sigma(x_4)]=0,$$

- 加權殘插弱滿足表示式改寫後

$$\int_{\Omega_i} rwdx = \int_{\Omega_i}(ku'w' + cu'w + buw)dx - \int_{\Omega_i}(fu)dx - ku'w\big|_{x_{i+1}} + ku'w\big|_{x_i}$$

- Element (i=1,2,3,4)

$$i=1 \qquad \int_{\Omega_1} rwdx = \int_{\Omega_1}(ku'w' + cu'w + buw)dx - \int_{\Omega_1}(fu)dx - ku'w\big|_{x_2^-} + ku'w\big|_{x_1}$$

$$i=2 \qquad \int_{\Omega_1} rwdx = \int_{\Omega_1}(ku'w' + cu'w + buw)dx - \int_{\Omega_1}(fu)dx - ku'w\big|_{x_3^-} + ku'w\big|_{x_2^+}$$

$$i=3 \qquad \int_{\Omega_1} rwdx = \int_{\Omega_1}(ku'w' + cu'w + buw)dx - \int_{\Omega_1}(fu)dx - ku'w\big|_{x_4^-} + ku'w\big|_{x_3^+}$$

$$i=4 \qquad \int_{\Omega_1} rwdx = \int_{\Omega_1}(ku'w' + cu'w + buw)dx - \int_{\Omega_1}(fu)dx - ku'w\big|_{x_5} + ku'w\big|_{x_4^+}$$

- For this problem

$$\sum_{i=1}^{4}\int_{\Omega_i} r\cdot wdx = \int_0^l r\cdot wdx = 0$$

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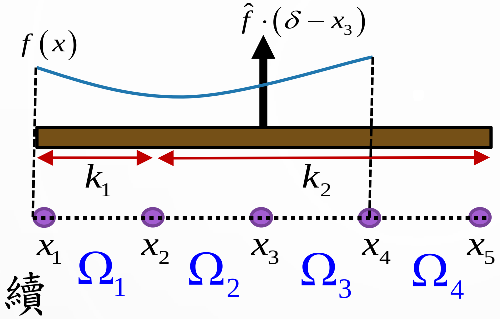

Boundary-Value Problem

$$-\frac{d}{dx}\left[k(x)\cdot\frac{du}{dx}\right]+c(x)\frac{du(x)}{dx}+b(x)u(x)=f(x)$$

Discontinuity Condition

$$[\sigma(x_2)]=0,\quad [\sigma(x_3)]=\hat{f},\quad [\sigma(x_4)]=0,$$

✓ 其中w函數在任意節點上左右連續

$$ku'w\big|_{x_2^+}+ku'w\big|_{x_2^-}=\left[ku'(x_2)\right]w(x_2)$$

➢ 整理可得

$$\int_0^l(ku'w'+cu'w+buw)\,dx=\int_0^l\bar{f}w\,dx+\left[ku'(x_2)\right]w(x_2)+\left[ku'(x_3)\right]w(x_3)+\left[ku'(x_4)\right]w(x_4)+ku'w\big|_{x=l}-ku'w\big|_{x=0}=0$$

➢ 代入 x_2, x_3, x_4 不連續條件

$$u'(x_2)=0$$

$$u'(x_3)=\hat{f}$$

$$u'(x_4)=0$$

➢ 改寫成

$$\int_0^l(ku'w'+cu'w+buw)\,dx=\int_0^l\bar{f}w\,dx+\hat{f}w(x_3)+ku'w\big|_{x=l}-ku'w\big|_{x=0}=0$$

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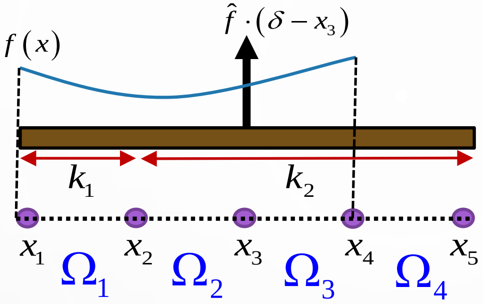
- Galerkin Method ?



Boundary-Value Problem

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