Inclusion-Exclusion Principle: Proof by Mathematical Induction *For Dummies*

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Definition (Discrete Interval)**.** $[n] := \{1, 2, 3, \ldots, n\}$

Theorem (Inclusion-Exclusion Principle)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|-1} \left|\bigcap_{i \in J} A_i\right|$$

Proof (induction on n). The theorem holds for $n = 1$:

$$\left|\bigcup_{i=1}^{1} A_i\right| = |A_1| \tag{1}$$

$$\sum_{\substack{J \subseteq [1] \\ J \neq \emptyset}} (-1)^{|J|-1} \left|\bigcap_{i \in J} A_i\right| = (-1)^0 \left|\bigcap_{i \in \{1\}} A_i\right| = |A_1| \tag{2}$$

For the induction step, let us suppose the theorem holds for $n - 1$.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \left|\left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n\right| = \tag{3}$$

We can use the formula $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

$$= \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n\right| = \tag{4}$$

Intersection of unions can be rewritten as a union of intersections.

$$= \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right| = \tag{5}$$

Let us define the substitution $B_i := A_i \cap A_n$.

$$= \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\bigcup_{i=1}^{n-1} B_i\right| = \tag{6}$$

We can now use the induction hypothesis on both $\bigcup A_i$ and $\bigcup B_i$.

$$= \sum_{\substack{J \subseteq [n-1] \\ J \neq \emptyset}} (-1)^{|J|-1} \left|\bigcap_{i \in J} A_i\right| + |A_n| - \sum_{\substack{K \subseteq [n-1] \\ K \neq \emptyset}} (-1)^{|K|-1} \left|\bigcap_{i \in K} B_i\right| = \tag{7}$$

*http://ze.phyr.us

The expression $(-1)\sum(-1)^{|K|-1}\left|\bigcap B_i\right|$ is equivalent to $\sum(-1)(-1)^{|K|-1}\left|\bigcap B_i\right| = \sum(-1)^{|K|}\left|\bigcap B_i\right|$.

$$= \sum_{\substack{J\subseteq[n-1]\\J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + |A_n| + \sum_{\substack{K\subseteq[n-1]\\K\neq\emptyset}} (-1)^{|K|}\left|\bigcap_{i\in K} B_i\right| \tag{8}$$

Since $K \neq \emptyset$, we can revert the substitution:

$$\bigcap_{i\in K} B_i = \bigcap_{i\in K}(A_i \cap A_n) = \left(\bigcap_{i\in K} A_i\right) \cap A_n = \bigcap_{i\in K\cup\{n\}} A_i$$

(8) now becomes

$$\sum_{\substack{J\subseteq[n-1]\\J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + |A_n| + \sum_{\substack{K\subseteq[n-1]\\K\neq\emptyset}} (-1)^{|K|}\left|\bigcap_{i\in K\cup\{n\}} A_i\right| \tag{9}$$

Let us substitute $K \cup \{n\}$ with J. The expression $|K|$ thus becomes $|J| - 1$ (with K defined as a subset of $[n-1]$, K cannot contain n and thus $|K \cup \{n\}| = |K| + 1$).

To replace the expression $i \in K \cup \{n\}$ with $i \in J$, we must impose several conditions on J as the summation index:

- $J \neq \emptyset$ (the same condition that was imposed on K),
- $J \subseteq [n] \wedge n \in J$ (n must be contained in every J, since J replaces $K \cup \{n\}$),
- $J \neq \{n\}$ ($K \cup \{n\} \neq \{n\}$, since $K \neq \emptyset$ and $n \notin K \subseteq [n-1]$).

(9) now becomes

$$\sum_{\substack{J\subseteq[n-1]\\J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + |A_n| + \sum_{\substack{J\subseteq[n]\\n\in J\\J\neq\emptyset,\{n\}}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| \tag{10}$$

The Inclusion-Exclusion Principle can be used on A_n alone (we have already shown that the theorem holds for one set):

$$\sum_{\substack{J\subseteq\{n\}\\J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| = (-1)^{|\{n\}|-1}\left|\bigcap_{i\in\{n\}} A_i\right| = |A_n|$$

(10) now becomes

$$\sum_{\substack{J\subseteq[n-1]\\J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + \sum_{\substack{J\subseteq\{n\}\\J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + \sum_{\substack{J\subseteq[n]\\n\in J\\J\neq\emptyset,\{n\}}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| = \tag{11}$$

For better readability, let us define

$$P_1 := \mathcal{P}([n-1]) \setminus \{\emptyset\}$$
$$P_2 := \mathcal{P}(\{n\}) \setminus \{\emptyset\} = \{\{n\}\}$$
$$P_3 := \mathcal{P}([n]) \setminus \mathcal{P}([n-1]) \setminus \{\emptyset, \{n\}\}$$

and rewrite (11) in this way:

$$= \sum_{J\in P_1} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + \sum_{J\in P_2} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + \sum_{J\in P_3} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| = \tag{12}$$

Since the three sums consist of the same terms, we can combine them into one. As the sets P_1, P_2, P_3 are disjoint, the summation condition now becomes

$$J \in (P_1 \cup P_2 \cup P_3) = (\mathcal{P}([n-1]) \setminus \{\emptyset\}) \cup \{\{n\}\} \cup (\mathcal{P}([n]) \setminus \mathcal{P}([n-1]) \setminus \{\emptyset, \{n\}\}) =$$
$$= (\mathcal{P}([n-1]) \cup \{\{n\}\} \cup (\mathcal{P}([n]) \setminus \mathcal{P}([n-1]) \setminus \{\{n\}\}) \setminus \{\emptyset\} = \mathcal{P}([n]) \setminus \{\emptyset\}$$

Finally, we can replace the logical condition $J \in \mathcal{P}([n]) \setminus \{\emptyset\}$ by the equivalent statement $J \subseteq [n], J \neq \emptyset$. The resulting formula is an instance of the Inclusion-Exclusion Theorem for n sets:

$$= \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|-1} \left| \bigcap_{i \in J} A_i \right| \tag{13}$$

$\square$

Remark. It can be easily seen that every possible value of J is covered *exactly once* by the new summation condition $(J \subseteq [n], J \neq \emptyset)$:

$$\forall J \subseteq [n], J \neq \emptyset \begin{cases} n \notin J & (\Leftrightarrow J \subseteq [n-1]) \\ n \in J & \begin{cases} J = \{n\} & (\Leftrightarrow J \subseteq \{n\}) \\ J \neq \{n\} & (\Leftrightarrow J \subseteq [n], n \in J, J \neq \{n\}) \end{cases} \end{cases}$$