Beyond Black-Litterman: Views on Non-Normal Markets

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Abstract

We extend the Black-Litterman methodology to generic non-normal market distributions and non-normal views. We draw on the copula and opinion pooling literature to express views directly on the market realizations, instead of the market parameters as in the Black-Litterman case. We compare the two approaches and we show an application to a thick-tailed, skewed and highly dependent market, where the views are expressed as uncertainty ranges.

Keywords: opinion pooling, copula, views, fat tails, Bayesian prior, posterior, Monte Carlo,quantitative portfolio management, asset allocation, skew t distribution, CVaR, expected shortfall

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The pathbreaking technique by Black and Litterman (1990) (BL in the sequel) allows a portfolio manager to smoothly blend his subjective views on the market with a prior market distribution. The popularity of the BL methodology is due to its intuitiveness, flexibility and ease of implementation: the portfolio manager can express an arbitrary number of views on generic linear combinations of market expected values and immediately obtain the market distribution that reflects these views in the most consistent way.

Nevertheless, the BL approach suffers from two drawbacks.

The first problem is of practical nature: in the BL approach both the market prior and the manager's views are normally distributed. For most markets the normal assumption is too strong: fat tails, skewness and high dependence among extreme events characterize the joint distribution of market risk factors in many contexts. As far as the views are concerned, the manager might think for instance in terms of ranges within whose boundaries all market realizations are equally likely: this corresponds to uniformly distributed views, as opposed to the "alpha + Gaussian noise" specification in the BL approach.

The normality assumption on the market and the views can be overcome within the BL framework, if one is willing to accept numerical results instead of closed analytical formulas.

The second problem in the BL approach is conceptual: due to its Bayesian nature, in this context the manager is supposed to express views on the *parameters* that determine the market distribution. In reality, in general the manager expresses views on the possible *realizations* of the market. This is not evident under the normal assumption, because there exists a clear relationship between the parameters and the realizations of the market: indeed in this case the parameters on which the manager expresses the views (the "mus") represent the expected values of the market realizations. Under different distributional assumptions, such as the classical lognormal Black-Scholes hypothesis, the relationship between the parameters (the "mus" of the lognormal distribution) and the realizations of the market is not transparent and the practitioner would not be able to express views on the parameters in an intuitive way.

As far as this problem is concerned, it is possible to reformulate the BL approach in terms of views on the market instead of the market parameters, see Meucci (2005). Nevertheless, this reformulation prescribes that the manager's views be inputted in a rather counterintuitive way as statements conditioned on the realization of the market. Instead, it is more natural to solve the dichotomy between the practitioner's "subjective" views and the "official" prior market distribution in terms of opinion pooling. In this paper we draw on this interpretation to improve on the BL approach, providing a posterior market distribution that smoothly blends an arbitrarily distributed market prior with arbitrarily distributed manager's views on the realization of the market.

The intuition behind our approach is the following: we use opinion pooling criteria to determine the marginal distribution of each view separately, whereas the joint co-dependence, i.e. the copula, among the views is directly inherited from the prior market structure. Finally, a suitable change of coordinates allows us to translate the joint distribution of the views into a joint posterior distribution for the market.

We remark that the purpose of the proposed methodology is to determine one market distribution (the "posterior") that cogently, in a sense defined in the paper, incorporates two different distributions (the "views" and the "prior"). In the special case where the market describes the returns on a set of securities, the most natural application is in the context of portfolio management, where linear coefficients (exposures) map the market distribution into the portfolio distribution. On the other hand, the market distribution can represent any source of randomness, not necessarily asset returns. Therefore, the proposed methodology can find applications, among others, on the trading floor: in this case the prior can be set for instance in terms of quantities implied by the market prices and the views can be provided by some technical indicator, thereby yielding a rich-cheap analysis.

In Section 1 we present the general theory. In Section 2 we compare our approach with the traditional BL methodology in a simulated normal market. In Section 3 we apply the general theory to a more realistic case: we model the thick-tailedness, skewness and high dependence among extreme events in the market by means of a multivariate skew t distribution and we model the manager's views as uniformly distributed on arbitrary ranges; we compute the highly non-symmetrical posterior market distribution; in order to pursue the optimal allocation, instead of the classical mean-variance approach, which is suitable in elliptical markets, we use the expected return - expected shortfall ("CVaR") approach, which accounts for asymmetries and tail risk. In Section 4 we conclude.

1 The general theory

Consider an N-dimensional market vector $\mathbf{M}$, i.e. a set of random variables that determines the randomness in the market. For instance, $\mathbf{M}$ could represent the returns on a set of securities, or a collection of risk-factors in an APT-like model, as well as the times-to-default of a basket of bonds in a CDO tranche.

The statistical properties of the market are described by an "official" prior distribution, which is determined by means of backward-looking estimation techniques, forward-looking market-implied values, equilibrium arguments, etc. We represent this distribution equivalently in terms of its probability density function (pdf), or its cumulative distribution function (cdf), or its characteristic function (cf) respectively:

$$\mathbf{M} \sim (f_{\mathbf{M}}, F_{\mathbf{M}}, \phi_{\mathbf{M}}) . \tag{1}$$

As in the BL framework, the views are a set of $K \leq N$ statements on generic linear combinations of the market. Unlike in the BL framework, where the portfolio manager expresses views on the market *parameters,* the portfolio manager expresses views directly on the market *realizations* $\mathbf{M}$. The above linear combinations are represented by a $K \times N$ dimensional "pick" matrix $\mathbf{P}$.

Therefore the views are the following K-variate random vector:

$$\mathbf{V} \equiv \mathbf{PM}. \tag{2}$$

In the most general setting, the manager expresses the generic k-th view in terms of a subjective univariate distribution, which is represented by its pdf, cdf, or cf:

$$f_{\widehat{V}_k}, F_{\widehat{V}_k}, \phi_{\widehat{V}_k}. \tag{3}$$

On the other hand, the prior market distribution (1) induces a structure on the distribution of the views. Indeed, the marginal distribution of the generic k-th view reads as follows in terms of its cf:

$$V_k \sim \phi_{V_k}(\omega) = \phi_{\mathbf{M}}(\mathbf{p}_k' \omega), \tag{4}$$

where $\mathbf{p}_k$ denotes the k-th row of the pick matrix $\mathbf{P}$.

The K market-induced distributions (4) clash with the manager's subjective views (3). Our aim is to determine a distribution for the market, which we call the posterior market distribution, that solves this dichotomy.

In order to determine the posterior market distribution we adopt a "bottom-up" approach: first, we determine the posterior marginal distribution of each view separately by drawing on the literature on opinion pooling; then we determine the joint posterior distribution of the views by using the dependence structure inherited from the market prior; finally, we determine the joint posterior distribution of the market by embedding the views in a suitable set of market coordinates.

According to the plan outlined above, we first define the posterior pdf of the generic k-th view as a weighted average of the market-induced prior pdf and the subjective pdf that expresses the view:

$$\widetilde{V}_k \sim f_{\widetilde{V}_k} \equiv (1 - c_k) f_{V_k} + c_k f_{\widehat{V}_k}, \tag{5}$$

where the weight c_k spans the interval $[0, 1]$. There exists a vast literature on the choice of the "best" opinion pooling method, see Genest and Zidek (1986) for a review. As discussed in Clemen and Winkler (1997), the simple rule (5) is as effective as more complex pooling techniques and it has the advantage of being intuitive. In particular, the weight c_k can be interpreted as the confidence in the manager's views. If the confidence is null, the posterior distribution results in the original market-induced distribution; if the confidence is full, the posterior distribution coincides with the manager's statement. The confidence c_k can be inputted directly by the manager. Alternatively, it can be imposed exogenously, based on the manager's track record: for instance, one could link the confidence to the correlation between past views and actual market realizations, see Grinold and Kahn (1999). Furthermore, we see that generalizations of (5) to a multi-manager context are immediate.

As a second step in our approach we determine the joint distribution of the views. In order to determine a joint distribution that is consistent with the posterior marginal distributions of the views (5) and reflects the co-dependence structure of the original market prior (1) we first compute the copula of the views induced through (2) by the market prior:

$$\mathbf{C} \stackrel{d}{=} \left(F_{V_1}\left(V_1\right), \ldots, F_{V_K}\left(V_K\right)\right)'. \tag{6}$$

Then we impose the marginal structure (5) obtaining the following joint posterior distribution for the views:

$$\widetilde{\mathbf{V}} \stackrel{d}{\equiv} \left(F_{\widetilde{V}_1}^{-1}\left(C_1\right), \ldots, F_{\widetilde{V}_K}^{-1}\left(C_K\right)\right)', \tag{7}$$

where F_X^{-1} denotes the quantile function relative to the cdf F_X.

The final step in our approach is the determination of a posterior distribution for the market. In order to determine a market posterior distribution that is consistent with the joint posterior distribution of the views (7) we notice that the market vector can be expressed equivalently in a set of view-adjusted coordinates:

$$\mathbf{M} \Leftrightarrow \left(\begin{array}{c} \mathbf{V} \equiv \mathbf{PM} \\ \mathbf{W} \equiv \mathbf{P}^{\perp}\mathbf{M} \end{array}\right), \tag{8}$$

where $\mathbf{P}^{\perp}$ is any $(N-K) \times N$ matrix whose rows span the null space of $\mathbf{P}$[1]. In the view-adjusted coordinates, the posterior distribution of the first entries was determined in (7), whereas the posterior distribution of the remaining entries has been left unaltered. The posterior market distribution is then defined naturally by reverting back to the original market coordinates:

$$\widetilde{\mathbf{M}} \stackrel{d}{=} \left(\begin{array}{c} \mathbf{P} \\ \mathbf{P}^{\perp} \end{array}\right)^{-1} \left(\begin{array}{c} \widetilde{\mathbf{V}} \\ \mathbf{W} \end{array}\right). \tag{9}$$

The posterior distribution of the market displays a few intuitive properties. For instance, views that coincide with the market prior leave the market prior unaltered: this follows immediately from (5). Furthermore, as in the BL framework, the whole market distribution is affected by the views. We can see this clearly in the limit case where the manager's uncertainty range relative to the generic k-th view collapses to a specific value v_k and where the confidence on that view is full, for all the views. As we show in the technical appendix, in this circumstance we obtain the intuitive result that the posterior market distribution is represented by the prior market distribution conditioned on the realization of the views:

$$\widetilde{\mathbf{M}} \stackrel{d}{=} \mathbf{M}|_{\mathbf{PM} \equiv \mathbf{v}}. \tag{10}$$

This is the case also in the BL approach, where the posterior normal distribution becomes the conditional normal distribution in the limit of total confidence, see Meucci (2005).

[1] As we show in the technical appendix, the specific choice of $\mathbf{P}^{\perp}$ does not influence the final result. Nevertheless, in order to improve the efficiency of the calculations we impose that the rows of $\mathbf{P}^{\perp}$ be orthogonal.

2 A comparison with the BL methodology

In this section we compare the opinion pooling approach discussed in Section 1 with the BL methodology. In order to meet the assumptions of the BL approach we consider a simulated normal market, where the views are inputted according to the "alpha + normal noise" prescription.

We perform our comparison directly on the posterior distributions ensuing from the BL approach and the opinion pooling approach. Alternatively, we could compare the two approaches on the basis of the optimal allocations to which they give rise. However, we do not pursue this route for two reasons. In the first place, this paper proposes a methodology to compute a posterior distribution that can be used in any context, such as a rich-cheap analysis for the trading desk, and not necessarily only for portfolio management purposes. Secondly, the outcome of a portfolio optimization depends on the market distribution as much as it depends on the optimization technique (mean-variance, full grid-search, genetic algorithms, etc.): this would add a spurious degree of subjectivity in the comparison. For an explicit application of the theory to portfolio management refer to Section 3.

We consider an artificial "prior" market, which is normally distributed:

$$\mathbf{M} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{11}$$

This market is composed of $N \equiv 25$ securities. The parameters of this market are set as follows. The standard deviations read:

$$\sigma_1 \equiv 5\%, \quad \sigma_2 \equiv \sigma_1 + \Delta, \ldots, \quad \sigma_N \equiv 25\% \equiv \sigma_1 + (N-1)\,\Delta.$$

The correlations are assumed homogeneous. In particular, we compare the two approaches in a highly correlated environment:

$$\rho_{mn} \equiv 0.9, \quad m \neq n.$$

The expected values are determined by an equilibrium argument:

$$\boldsymbol{\mu} \equiv \frac{\delta}{N}\boldsymbol{\Sigma}\mathbf{1},$$

where the equally weighted portfolio $\mathbf{1}/N$ is assumed to represent equilibrium and $\delta \approx 2.5$ as in BL. The investor expresses views on the most volatile security: therefore the "pick" matrix reads $\mathbf{P} \equiv (0, 0, \ldots, 1)$.

In the opinion pooling framework, the views are expressed directly on the market. For consistency with the BL framework, we express them as normally distributed. In particular, the dispersion is inherited from the market (11) and the mean is bearish:

$$\widehat{V} \equiv M_N \sim \mathrm{N}\left(-\mu_N, \sigma_N^2\right). \tag{12}$$

According to (5) the posterior is a mixture of normal distributions:

$$\widetilde{V} \sim f_{\widetilde{V}} \equiv (1-c)\, f^{\mathrm{N}}_{\mu_N, \sigma_N^2} + c f^{\mathrm{N}}_{-\mu_N, \sigma_N^2}, \tag{13}$$

where c is the confidence level. With the above ingredients we can compute the opinion pooling posterior distribution of the market $\widetilde{\mathbf{M}}_{\mathrm{OP}}(c)$ as represented by its pdf $f_{\widetilde{\mathbf{M}}_{\mathrm{OP}}(c)}$, which depends on the confidence level c. The details for this computation are provided in the technical appendix.

In the BL framework, the views are expressed on the market parameters:

$$\mathbf{P}\boldsymbol{\mu} \sim \mathrm{N}(\mathbf{q}, \boldsymbol{\Omega}), \tag{14}$$

where the mean is the single view $\mathbf{q} \equiv -\mu_N$ and where the covariance among the views is inherited from the market $\boldsymbol{\Omega} \equiv \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'$.

As discussed in He and Litterman (2002), the market is then distributed as follows:

$$\widetilde{\mathbf{M}}_{\mathrm{BL}}(c) \sim \mathrm{N}\left(\overline{\boldsymbol{\mu}}_c, \overline{\boldsymbol{\Sigma}}_c\right), \tag{15}$$

where

$$\overline{\boldsymbol{\mu}}_c \equiv \mathbf{D}_c\left(\left(\tau_c\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\mu} + \mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}\mathbf{q}\right), \quad \overline{\boldsymbol{\Sigma}}_c \equiv \boldsymbol{\Sigma} + \mathbf{D}_c.$$

In this expression the matrix $\mathbf{D}_c$ is defined in terms of its inverse: $\mathbf{D}_c^{-1} \equiv (\tau_c\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}\mathbf{P}$ and the confidence in the prior can be linked to the confidence c that appears in (13) by the relationship $\tau_c \equiv 1/(1-c) - 1$.

We measure the distance between the pooling posterior distribution $\widetilde{\mathbf{M}}_{\mathrm{OP}}(c)$ and the BL posterior distribution $\widetilde{\mathbf{M}}_{\mathrm{BL}}(c)$ according to the following measure of divergence:

$$d_c(\mathrm{OP}, \mathrm{BL}) \equiv \mathrm{E}\left\{\left|\ln f_{\widetilde{\mathbf{M}}_{\mathrm{OP}}(c)}(\mathbf{M}) - \ln f_{\widetilde{\mathbf{M}}_{\mathrm{BL}}(c)}(\mathbf{M})\right|\right\}, \tag{16}$$

where the expectation is computed according to the prior distribution (11). This distance is reminiscent of the Kullback-Leibler relative entropy, except that it hinges on the natural benchmark, namely the prior distribution. We can easily compute this distance as a function of the overall level of confidence c by means of simulations, see the technical appendix for details. Similarly, we compute the distance between the prior and the BL posterior, as well as the distance between the opinion-pooling posterior and the prior.

In Figure 1 we plot the results. As expected, at zero confidence in the views all the distances are null because both the BL posterior and the opinion pooling posterior are equal to the prior. As the confidence in the views increases, the two posteriors depart from the prior and from each other. The opinion pooling posterior represents a more gentle modification of the prior, more in line with the Bayesian learning model, see Zellner (2002). This should result in a less pronounced twisting of the optimal allocations in a portfolio optimization context. On the other hand, we stress that a more gentle modification of the prior can also be achieved in the BL framework by choosing a lower confidence level in the views. We tested several alternative specifications for the number of assets, the number of views and the market correlations: the results are qualitatively similar to Figure 1.

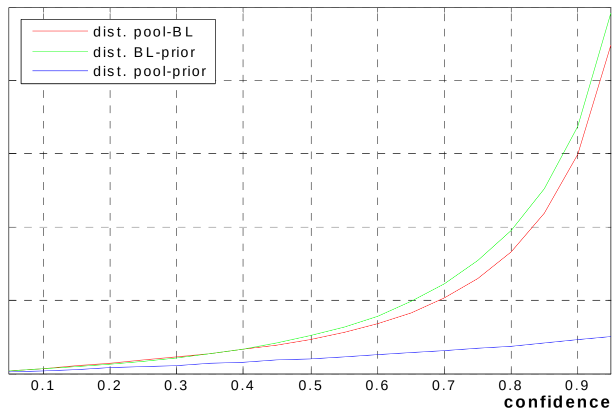

Figure 1: Comparison of Black-Litterman and opinion pooling posterior distributions

3 Applications to portfolio management

To show an application of the above general theory we model a simplified, yet non-trivial, portfolio allocation problem, although we stress that the applications of the theory go beyond the portfolio management context.

We consider an international-equity fund manager, whose investment horizon in one week and whose market $\mathbf{M}$ is represented by the returns on $N \equiv 4$ international stock indices: the American S&P 500, the British FTSE 100, the French CAC 40 and the German DAX.

In Figure 2 we plot the weekly returns of the American index versus the British index (the plot is qualitatively similar for other combinations). The market $\mathbf{M}$ is not normally distributed: the marginal distributions of the above indices display fat tails, potentially some degree of skewness, and the copula of the above indices displays larger than normal dependence among extreme events, see also Mashal and Zeevi (2002).

We can model these features by means of the skew t distribution, see Azzalini and Capitanio (2003):

$$\mathbf{M} \sim \mathrm{SkT}\left(\psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\alpha}\right). \tag{17}$$

In this expression ψ is a positive number; $\boldsymbol{\mu}$ is an N-dimensional vector; $\boldsymbol{\Sigma}$ is an $N \times N$ symmetric and positive matrix; and $\boldsymbol{\alpha}$ is an N-dimensional vector. We report in the technical appendix the pdf $f^{\mathrm{SkT}}_{\psi,\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}}$ of this distribution. The skew t distribution coincides with the Student t distribution when the shape parameter $\boldsymbol{\alpha}$ is null. In this case $\boldsymbol{\mu}$ is the expected value and $\boldsymbol{\Sigma}$ is the (re-scaled) covariance matrix. In particular, in the limit $\psi \to \infty$ we recover the normal distribution. As the degrees of freedom ψ decrease, this distribution displays heavier tails and larger dependence among extreme events, see Embrechts, Lindskog, and McNeil

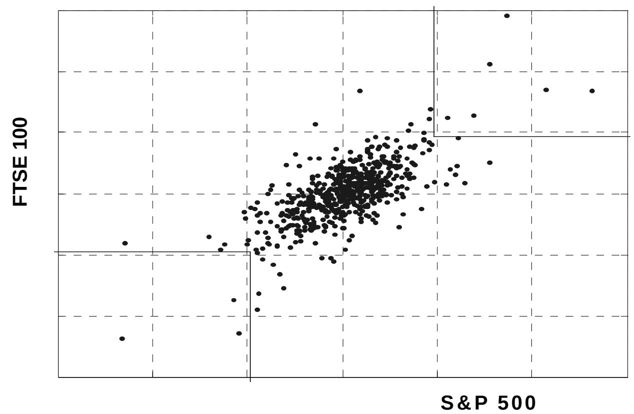

Figure 2: Co-dependence of extreme events in the markets

(2003). For non-null values of the shape parameter $\boldsymbol{\alpha}$ the skew t distribution also displays skewness.

We estimate the parameters of the distribution in the example by maximum likelihood, see Azzalini (2005) for details. We report below the results:

$$\begin{aligned} \psi &\approx 5, \quad \boldsymbol{\alpha} \approx \mathbf{0} \\ \boldsymbol{\Sigma} &\approx 10^{-3} \times \begin{pmatrix} .376 & .253 & .333 & .397 \\ \cdot & .360 & .360 & .396 \\ \cdot & \cdot & .600 & .578 \\ \cdot & \cdot & \cdot & .775 \end{pmatrix} \end{aligned}$$

We replace the estimate of the parameter $\boldsymbol{\mu}$ with an equilibrium argument as in BL:

$$\boldsymbol{\mu} \equiv \delta \boldsymbol{\Sigma} \mathbf{w}_{eq}, \tag{18}$$

where $\mathbf{w}_{eq}$ is the relative capitalization of the four above indices and $\delta \approx 2.5$.

Notice that in this context the specification (18) is apparently nonsensical. Indeed, a sufficient condition for a CAPM-like specification such as (18) to be viable is that the returns be elliptically distributed, see Ingersoll (1987). The skew t in general is inconsistent with a CAPM-like equilibrium specification, because it is not elliptical. Furthermore, in general (18) is not even a CAPM equilibrium specification because $\boldsymbol{\mu}$ does not represent the expected value of the skew t distribution. However, since $\boldsymbol{\alpha} \approx \mathbf{0}$ the skew t distribution reduces to a regular t distribution, which is elliptical and thus CAPM-like equilibrium specifications are viable. Furthermore, when $\boldsymbol{\alpha} \approx \mathbf{0}$, $\boldsymbol{\mu}$ represents the expected value of the returns: therefore (18) is a CAPM-like equilibrium specification. To show the computations in their full generality, we continue the discussion under the more general assumption of a general skew t distribution. We stress that in the more general case $\boldsymbol{\alpha} \neq \mathbf{0}$ the most suitable estimation/modelling methodology should be tailored to the needs of the specific investor and market.

As far as the manager's views are concerned, we model them as uniformly distributed on given ranges. Any other specification is equally feasible, although parsimonious specifications should be favored. Indeed, one of the criticisms to the (yet very parsimonious) "alpha plus normal noise" specification of BL is that it already has too many inputs. In formulas, the cdf of the generic k-th view reads as follows:

$$F_{\widehat{V}_k}(v) \equiv F^{\mathrm{U}}_{[a_k,b_k]}(v) \equiv \begin{cases} 0 & v \leq a_k \\ \frac{v-a_k}{b_k-a_k} & v \in [a_k, b_k] \\ 1 & v \geq b_k. \end{cases} \tag{19}$$

In particular, we consider the case where the manager expresses one view, although for the reader's convenience in future applications, in the sequel we use the multi-views notation. The manager is bearish on the DAX, which he believes will realize between zero and -2% within a week. In other words, the pick matrix and the ranges read respectively in this case:

$$\mathbf{P} \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{a} \equiv -.02, \quad \mathbf{b} \equiv 0.$$

As discussed in Azzalini (2005) the skew t distribution (17) induces a skew t structure on the joint distribution of the views, and in particular on the marginal distribution of each single view:

$$f_{V_k} = f^{\mathrm{SkT}}_{\psi,\xi_k,\Phi_{kk},\beta_k}, \tag{20}$$

where

$$\boldsymbol{\xi} \equiv \mathbf{P}\boldsymbol{\mu}, \quad \boldsymbol{\Phi} \equiv \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}';$$

and

$$\begin{aligned} \boldsymbol{\beta} &\equiv \frac{\operatorname{diag}(\boldsymbol{\Phi})^{\frac{1}{2}}\, \boldsymbol{\Phi}^{-1}\mathbf{H}'\boldsymbol{\alpha}}{\left(1 + \boldsymbol{\alpha}'\left(\operatorname{diag}(\boldsymbol{\Sigma})^{-\frac{1}{2}}\,\boldsymbol{\Sigma}\operatorname{diag}(\boldsymbol{\Sigma})^{-\frac{1}{2}} - \mathbf{H}\boldsymbol{\Phi}^{-1}\mathbf{H}'\right)\boldsymbol{\alpha}\right)^{\frac{1}{2}}} \\ \mathbf{H} &\equiv \operatorname{diag}(\boldsymbol{\Sigma})^{-\frac{1}{2}}\,\boldsymbol{\Sigma}\mathbf{P}'. \end{aligned}$$

The pdf's of each view (20) can then be integrated by means of efficient numerical techniques similar to those that yield the error function in the case of the normal distribution:

$$F_{V_k} = F^{\mathrm{SkT}}_{\psi,\xi_k,\Phi_{kk},\beta_k}. \tag{21}$$

By applying the opinion pooling technique (5) we immediately obtain the posterior cdf of the generic k-th view:

$$F_{\widetilde{V}_k} \equiv (1 - c_k)\, F^{\mathrm{SkT}}_{\psi,\xi_k,\Phi_{kk},\beta_k} + c_k F^{\mathrm{U}}_{[a_k,b_k]}, \tag{22}$$

where we set a 20% confidence in the views: $c_k \equiv 0.2$. In Figure 3 we display the cdf of the prior, the view and the posterior.

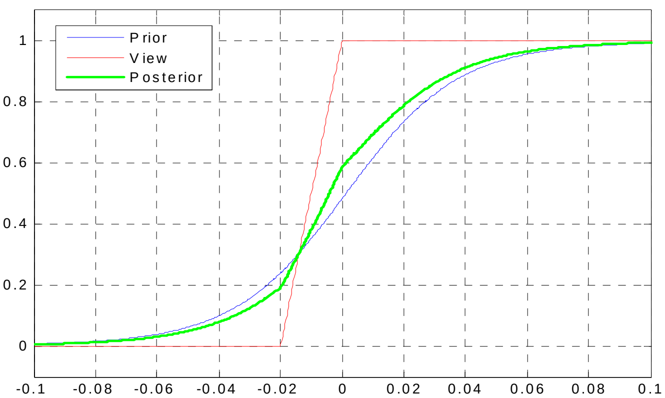

Figure 3: Cdf of the prior, the view, and the posterior

The posterior quantile functions of the views are easily obtained by linear interpolation of their respective cdf's:

$$F_{\widetilde{V}_k} \mapsto F^{-1}_{\widetilde{V}_k}. \tag{23}$$

To derive the posterior distribution of the market, we first generate a large number J of Monte Carlo scenarios of the prior market distribution (17) using its equivalent stochastic representation, see Azzalini (2005):

$$\mathbf{M}^{(j)} \equiv \boldsymbol{\mu} + \frac{\mathbf{g}\left|Y^{(j)}\right| + \mathbf{B}\mathbf{X}^{(j)}}{\sqrt{s^{(j)}_{\psi}}}, \quad j = 1, \ldots, J. \tag{24}$$

In this expression each $\left(\mathbf{X}^{(j)}, Y^{(j)}\right)$ is a drawing from the standard $(N+1)$-variate normal distribution; each $s^{(j)}_{\psi}$ is an independent random drawing from the chi-square distribution with ψ degrees of freedom; $\mathbf{g}$ is a suitable conformable vector and $\mathbf{B}$ is a suitable conformable matrix.

From the Monte Carlo scenarios of the prior market distribution (24), the prior marginal cdf's (21) and the posterior quantiles (23) we immediately obtain Monte Carlo scenarios of the posterior market distribution (9):

$$\widetilde{\mathbf{M}}^{(j)} \equiv \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}^{-1} \begin{pmatrix} F^{-1}_{\widetilde{V}_k}\left(F_{V_1}\left(\mathbf{p}_1 \mathbf{M}^{(j)}\right)\right) \\ \vdots \\ F^{-1}_{\widetilde{V}_k}\left(F_{V_K}\left(\mathbf{p}_K \mathbf{M}^{(j)}\right)\right) \\ \mathbf{P}^{\perp}\mathbf{M}^{(j)} \end{pmatrix}. \tag{25}$$

We can now proceed to determine the optimal asset allocation. In theory, the manager evaluates a portfolio $\mathbf{w}$ in a given market $\widetilde{\mathbf{M}}$ according to multiple indices of satisfaction or dissatisfaction such as value at risk, coherent measures of outperformance with respect to a benchmark, expected utility, see Meucci (2005) for a thorough discussion of these issues. In practice, it is very common to optimize the risk-reward profile of a portfolio in terms of its mean-variance coordinates, because in most cases this optimization can be solved by quadratic programming and yields close-to-optimal results.

Nevertheless, in the present context the posterior market distribution can potentially be heavily twisted by the manager's views. Unlike in BL, where the "twisted" posterior distribution is again normal, in the proposed opinion-pooling approach the posterior is not even elliptical: asymmetries and tail risk can come to play a major role and therefore the mean-variance approach can potentially become highly sub-optimal.

Therefore, as in Rockafellar and Uryasev (2000) we evaluate the riskiness of an allocation in terms of its expected shortfall[2], with a confidence, say, $\gamma \equiv 95\%$. The manager minimizes the expected shortfall subject to the standard long-only and full-investment constraints, as well as the constraint of a minimum target expected return r. In formulas, the optimal portfolio reads[3]:

$$\mathbf{w}(r) \equiv \underset{\substack{\mathbf{w}'\mathbf{m} \geq r \\ \mathbf{w}'\mathbf{1} \equiv 1, \mathbf{w} \geq \mathbf{0}}}{\operatorname{argmax}} \left(\mathrm{E}\left\{ \widetilde{R} | \widetilde{R} \leq F_{\widetilde{R}}^{-1}(1-\gamma) \right\} \right), \tag{26}$$

where $\widetilde{R} \equiv \mathbf{w}'\widetilde{\mathbf{M}}$ is the portfolio return; $F_X^{-1}(\alpha)$ denotes the α-quantile of the generic random variable X; and $\mathbf{m}$ is the expected value of the market $\widetilde{\mathbf{M}}$. As the target expected return r varies, the optimal portfolios (26) span an efficient frontier in the risk-reward space defined by expected shortfall and target return. These optimal allocations account for the non-normality of the market.

In order to determine the optimal allocations in our scenario-based posterior market distribution (25) we make use of a result in Rockafellar and Uryasev (2000), according to which (26) can be quickly solved by linear programming. We provide more details in the technical appendix.

In Figure 4 we display the efficient prior and posterior frontiers. The posterior frontier is lower than the prior frontier because the bearish view on the DAX shifts the investment opportunities in a less favorable region.

Posterior allocation

asset*target return* ($\times 10^3$)	*.96*	*.99*	*1.02*	*1.05*
S&P	47	45	39	35
FTSE	36	25	17	8
CAC	17	30	44	57
DAX	0	0	0	0

(27)

[2] The expected shortfall is also known as conditional value at risk (CVaR), see Meucci (2005) for issues regarding the terminology.

[3] This expression is exact for continuous distributions and somewhat different for discrete distributions, see Rockafellar and Uryasev (2002) for more details.

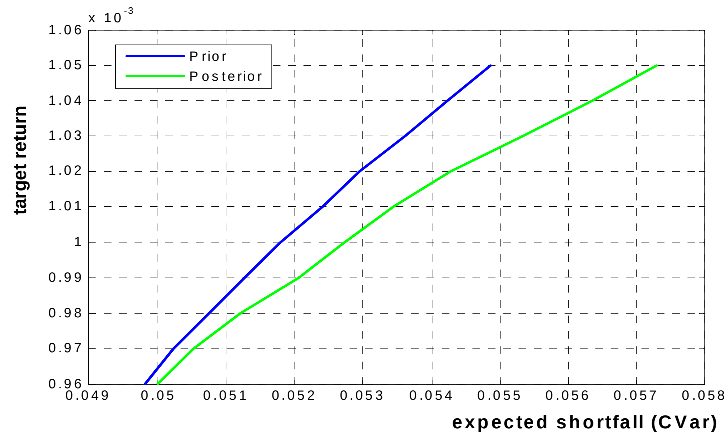

Figure 4: Prior and posterior efficient frontiers

Prior allocation

asset*target return* ($\times 10^3$)	*.96*	*.99*	*1.02*	*1.05*
S&P	45	42	39	37
FTSE	43	37	32	26
CAC	5	8	11	13
DAX	7	13	18	24

(28)

In Table 27 we display the optimal posterior portfolio compositions (26) as functions of the target return and in Table 28 we display the respective optimal prior portfolio compositions.

First of all, we notice that the bearish view on the DAX drops this asset class from the portfolio. Furthermore, as the minimum required target return varies, i.e. as the risk-aversion level changes, the investment is reallocated in different proportions among the remaining securities. As in the BL framework, the posterior relative weights of the remaining securities are not proportional to the respective prior relative weights. This process reflects the non-linear interactions between the views, the market prior and the market copula.

4 Conclusions

We present a new approach to smoothly blend the views of the portfolio manager with a prior market distribution based on opinion pooling principles.

The proposed methodology is arguably more intuitive than the BL approach, because the practitioner can input views directly on the market, instead of resorting to the market parameters. Furthermore, this methodology is very general, as it applies in principle to any market distribution and any distributional shape for the views.

We apply the general theory to a realistic example where one manager expresses views on a skewed, fat-tailed, highly codependent market: the manager formulates views in terms of ranges and easily obtains the posterior market distribution numerically by means of Monte Carlo simulations.

The opinion pooling principles underlying the proposed methodology can be extended in a straightforward way to a multi-manager setting, where priority is given to the senior members who more consistently have made successful bets in the past. Furthermore, the possible applications of the proposed methodology go beyond portfolio management. For instance, this methodology can find applications on the trading floor, where market-implied priors can be blended with views based on technical indicators.

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5 Technical Appendix

5.1 Null space generator of the "pick" matrix

All possible choices of the basis for the null space of $\mathbf{P}$ are obtained from the specific choice $\mathbf{P}^{\perp}$ by means of the following transformation:

$$\mathbf{P}^{\perp} \mapsto \mathbf{AP}^{\perp}, \tag{29}$$

where $\mathbf{A}$ is a suitable invertible $(N-K)\times(N-K)$ matrix. Consider the market posterior (9) stemming from the new specification for the basis:

$$\widetilde{\mathbf{M}}_{\mathbf{A}} \overset{d}{=} \begin{pmatrix} \mathbf{P} \\ \mathbf{AP}^{\perp} \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{\mathbf{V}} \\ \mathbf{AP}^{\perp}\mathbf{M} \end{pmatrix} \tag{30}$$

Notice that we can write:

$$\begin{pmatrix} \mathbf{P} \\ \mathbf{AP}^{\perp} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{I}_{K\times K} & \mathbf{0}_{K\times(N-K)} \\ \mathbf{0}_{(N-K)\times K} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix} \tag{31}$$

Therefore:

$$\begin{aligned} \widetilde{\mathbf{M}}_{\mathbf{A}} \quad & \overset{d}{=} \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I}_{K\times K} & \mathbf{0}_{K\times(N-K)} \\ \mathbf{0}_{(N-K)\times K} & \mathbf{A}^{-1} \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{V}} \\ \mathbf{AP}^{\perp}\mathbf{M} \end{pmatrix} \\ & = \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{\mathbf{V}} \\ \mathbf{P}^{\perp}\mathbf{M} \end{pmatrix} \overset{d}{=} \widetilde{\mathbf{M}} \end{aligned} \tag{32}$$

5.2 Limit of infinite confidence in the prior

In general, from the steps that lead to (9) the distribution of the market reads:

$$\widetilde{\mathbf{M}} \overset{d}{=} \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}^{-1} \begin{pmatrix} F_{\widetilde{V}_k}^{-1}\left(F_{V_1}\left(\mathbf{p}_1\mathbf{M}\right)\right) \\ \vdots \\ F_{\widetilde{V}_k}^{-1}\left(F_{V_K}\left(\mathbf{p}_K\mathbf{M}\right)\right) \\ \mathbf{P}^{\perp}\mathbf{M} \end{pmatrix}. \tag{33}$$

In the case of infinite confidence in one specific value $\mathbf{PM} \equiv \mathbf{v}$ the above expression reads:

$$\widetilde{\mathbf{M}} \overset{d}{=} \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{v} \\ \mathbf{P}^{\perp}\mathbf{M} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix} \mathbf{M} \Big|_{\mathbf{PM}\equiv\mathbf{v}}. \tag{34}$$

Therefore:

$$\widetilde{\mathbf{M}} \overset{d}{=} \mathbf{M}|_{\mathbf{PM}\equiv\mathbf{v}}, \tag{35}$$

i.e. the posterior market distribution is the prior distribution conditioned on the realization of the views.

5.3 Comparison between BL posterior and pooling posterior

We need to compute

$$\rho_c(\mathrm{OP},\mathrm{BL}) \equiv \mathrm{E}\left\{\left|\ln f_{\widetilde{\mathbf{M}}_{\mathrm{OP}(c)}}(\mathbf{M}) - \ln f_{\widetilde{\mathbf{M}}_{\mathrm{BL}(c)}}(\mathbf{M})\right|\right\}, \tag{36}$$

where

$$\mathbf{M} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{37}$$

To compute the first function in (36), we first derive the distribution of the market in the views coordinates. Define

$$\mathbf{R} \equiv \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}. \tag{38}$$

Then

$$\mathbf{V} \equiv \mathbf{RM} \sim \mathrm{N}\left(\boldsymbol{\nu} \equiv \mathbf{R}\boldsymbol{\mu}, \boldsymbol{\Xi} \equiv \mathbf{R}\boldsymbol{\Sigma}\mathbf{R}'\right). \tag{39}$$

From this expression we can compute the pdf of the copula, see e.g. Meucci (2005) p. 41:

$$f_{\mathbf{U}}(u_1, \ldots, u_N) = \frac{f^{\mathrm{N}}_{\boldsymbol{\nu},\boldsymbol{\Xi}}\left(Q^{\mathrm{N}}_{\nu_1,\xi_{11}}(u_1), \ldots, Q^{\mathrm{N}}_{\nu_N,\xi_{NN}}(u_N)\right)}{f^{\mathrm{N}}_{\nu_1,\xi_{11}}\left(Q^{\mathrm{N}}_{\nu_1,\xi_{11}}(u_1)\right)\cdots f^{\mathrm{N}}_{\nu_N,\xi_{NN}}\left(Q^{\mathrm{N}}_{\nu_N,\xi_{NN}}(u_N)\right)}, \tag{40}$$

where $f^{\mathrm{N}}_{\nu_n,\xi_{nn}}$ and $Q^{\mathrm{N}}_{\nu_n,\xi_{nn}}$ are the normal quantile and the normal pdf of the generic n-th marginal entry of $\mathbf{V}$ respectively with parameters provided by (39); and $f^{\mathrm{N}}_{\boldsymbol{\nu},\boldsymbol{\Xi}}$ is the joint normal pdf with parameters provided by (39). The posterior pdf in the views coordinates then reads:

$$\begin{aligned} f_{\widetilde{\mathbf{V}}}(v_1, \ldots, v_N) &= f_{\mathbf{U}}\Big(F_{\widetilde{V}_1}(v_1), \ldots, F_{\widetilde{V}_K}(v_K), \\ &\qquad F^{\mathrm{N}}_{\nu_{K+1},\xi_{K+1,K+1}}(v_{K+1}), \ldots, F^{\mathrm{N}}_{\nu_N,\xi_{NN}}(v_N)\Big) \\ &\qquad \prod_{n=1}^{K}\left[f_{\widetilde{V}_n}(v_n)\right]\prod_{n=K+1}^{N}\left[f^{\mathrm{N}}_{\nu_n,\xi_{nn}}(v_n)\right], \end{aligned} \tag{41}$$

see e.g. Meucci (2005) p. 42. In this expression $F^{\mathrm{N}}_{\nu_n,\xi_{nn}}$ is the normal cdf with parameters provided by (39); $f_{\widetilde{V}_n}$ and $F_{\widetilde{V}_n}$ follows from (5). In this case we obtain:

$$f_{\widetilde{V}_n} \equiv (1-c)\, f^{\mathrm{N}}_{\nu_n,\xi_{nn}} + c f^{\mathrm{N}}_{q_n,S_{nn}}. \tag{42}$$

Finally we represent the posterior in the market coordinates:

$$f_{\widetilde{\mathbf{M}}_{\mathrm{OP}(c)}}(\mathbf{m}) \equiv \left|\left(\mathbf{R}^{-1}\right)\left(\mathbf{R}^{-1}\right)'\right|^{-\frac{1}{2}} f_{\widetilde{\mathbf{V}}}(\mathbf{Rm}). \tag{43}$$

The first term in (36) follows by computing the logarithm of (43).

It is very easy to compute the second term in (36). Indeed from (15) we obtain:

$$\ln f_{\widetilde{\mathbf{M}}_{\mathrm{BL}(c)}}(\mathbf{m}) = -\frac{1}{2}\ln\left|\overline{\mathbf{\Sigma}}_c\right| - \frac{N}{2}\ln(2\pi) - \frac{1}{2}(\mathbf{m}-\overline{\boldsymbol{\mu}}_c)'\,\overline{\mathbf{\Sigma}}_c^{-1}(\mathbf{m}-\overline{\boldsymbol{\mu}}_c). \quad (44)$$

To compute the distance in (36) we generate according to (37) a large number J of Monte Carlo scenarios of the prior market distribution:

$$\mathbf{M}^{(j)}, \quad j = 1,\ldots J. \quad (45)$$

Then we evaluate the distance as follows:

$$\rho_c(\mathrm{OP},\mathrm{BL}) \approx \frac{1}{J}\sum_{j=1}^{J}\left|\ln f_{\widetilde{\mathbf{M}}_{\mathrm{OP}(c)}}\left(\mathbf{M}^{(j)}\right) - \ln f_{\widetilde{\mathbf{M}}_{\mathrm{BL}(c)}}\left(\mathbf{M}^{(j)}\right)\right|. \quad (46)$$

Expressions for other combinations of distances follow similarly to (44).

5.4 The skew t distribution

The pdf of the skew t distribution (17) as defined in Azzalini (2005) reads:

$$f^{\mathrm{SkT}}_{\psi,\boldsymbol{\mu},\mathbf{\Sigma},\boldsymbol{\alpha}}(\mathbf{m}) \equiv 2 f^{\mathrm{St}}_{\psi,\boldsymbol{\mu},\mathbf{\Sigma}}(\mathbf{m})\, F^{\mathrm{St}}_{\psi+N}\left(\sqrt{\frac{\psi+N}{\mathrm{Ma}^2_{\boldsymbol{\mu},\mathbf{\Sigma}}(\mathbf{m})+\psi}}\,\boldsymbol{\alpha}'\operatorname{diag}(\mathbf{\Sigma})^{-\frac{1}{2}}(\mathbf{m}-\boldsymbol{\mu})\right). \quad (47)$$

In this expression, Ma is the Mahalanobis distance:

$$\mathrm{Ma}^2_{\boldsymbol{\mu},\mathbf{\Sigma}}(\mathbf{m}) \equiv (\mathbf{m}-\boldsymbol{\mu})'\,\mathbf{\Sigma}^{-1}(\mathbf{m}-\boldsymbol{\mu}); \quad (48)$$

$f^{\mathrm{St}}_{\psi,\boldsymbol{\mu},\mathbf{\Sigma}}$ is the multivariate Student t pdf with location $\boldsymbol{\mu}$, scatter $\mathbf{\Sigma}$ and ψ degrees of freedom:

$$f^{\mathrm{St}}_{\psi,\boldsymbol{\mu},\mathbf{\Sigma}}(\mathbf{m}) \equiv \frac{\Gamma\left(\frac{1}{2}(\psi+N)\right)}{|\mathbf{\Sigma}|^{\frac{1}{2}}(\pi\psi)^{\frac{N}{2}}\,\Gamma\left(\frac{1}{2}\psi\right)}\left(1+\frac{\mathrm{Ma}^2_{\boldsymbol{\mu},\mathbf{\Sigma}}(\mathbf{m})}{\psi}\right)^{-\frac{\psi+N}{2}}; \quad (49)$$

F^{St}_{ψ} is the standard univariate Student t cdf with ψ degrees of freedom:

$$F^{\mathrm{St}}_{\psi}(x) \equiv \int_{-\infty}^{x} f^{\mathrm{St}}_{\psi,0,1}(u)\,du. \quad (50)$$

5.5 CVaR optimization as linear programming

Rockafellar and Uryasev (2000) show that the problem (26) can be restated equivalently as follows:

$$\mathbf{w}(r) = \underset{\substack{\alpha\in\mathbb{R},\mathbf{w}'\mathbf{m}\geq r\\ \mathbf{w}'\mathbf{1}\equiv 1,\mathbf{w}\geq\mathbf{0}}}{\operatorname{argmin}}\left\{\alpha + \frac{1}{(1-\gamma)}\int\left[-\mathbf{w}'\mathbf{x}-\alpha\right]^{+} f_{\widetilde{\mathbf{M}}}(\mathbf{x})\,d\mathbf{x}\right\}, \quad (51)$$

In our case the market distribution is discrete, i.e.

$$f_{\widetilde{\mathbf{M}}} = \frac{1}{J}\sum_{j=1}^{J}\delta^{\left(\widetilde{\mathbf{M}}^{(j)}\right)}, \tag{52}$$

where $\delta^{(\mathbf{y})}$ is the Dirac delta with point mass in $\mathbf{y}$. Therefore (51) can be solved by linear programming (LP). Indeed, introducing a J-dimensional vector of auxiliary variables $\mathbf{u}$, the optimization (51) can be restated equivalently as follows:

$$\mathbf{w}(r) = \underset{\{\mathbf{w},\mathbf{u},\alpha\}\in\mathcal{C}(r)}{\operatorname{argmin}} \left\{\alpha + \frac{1}{J(1-\gamma)}\mathbf{u}'\mathbf{1}\right\}, \tag{53}$$

where the set of constraints reads:

$$\begin{aligned}\mathcal{C}(r): \qquad & \Big\{\mathbf{w}'\overline{\mathbf{m}} \geq r, -\mathbf{w}'\widetilde{\mathbf{M}}^{(j)} - \alpha - u^{(j)} \leq 0 \\ & \mathbf{w} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{0}, \mathbf{w}'\mathbf{1} \equiv 1\Big\};\end{aligned} \tag{54}$$

and where

$$\overline{\mathbf{m}} \equiv \frac{1}{J}\sum_{j=1}^{J}\widetilde{\mathbf{M}}^{(j)}. \tag{55}$$

Introducing the variable

$$\mathbf{x}' \equiv (\mathbf{w}', \alpha, \mathbf{u}'), \tag{56}$$

the optimization (53) can be written in LP form:

$$\mathbf{x}(r) = \underset{\substack{\mathbf{A}\mathbf{x}\leq\mathbf{b}(r) \\ \mathbf{c}'\mathbf{x}=1}}{\operatorname{argmin}} \{\mathbf{q}'\mathbf{x}\}, \tag{57}$$

where

$$\begin{aligned}\mathbf{A} \quad &\equiv \quad \begin{pmatrix} -\mathbf{1}_{1\times J}\widetilde{\mathcal{M}}/J & 0 & \mathbf{0}_{1\times J} \\ -\widetilde{\mathcal{M}} & -\mathbf{1}_{J\times 1} & -\mathbf{I}_{J\times J} \\ -\mathbf{I}_{N\times N} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times J} \\ \mathbf{0}_{J\times N} & \mathbf{0}_{J\times 1} & -\mathbf{I}_{J\times J} \end{pmatrix}, \quad \mathbf{b}(r) \equiv \begin{pmatrix} -r \\ \mathbf{0}_{J\times 1} \\ \mathbf{0}_{N\times 1} \\ \mathbf{0}_{J\times 1} \end{pmatrix} \\ \mathbf{q}' \quad &\equiv \quad \left(\mathbf{0}_{1\times N}, 1, \frac{\mathbf{1}_{1\times J}}{J(1-\gamma)}\right), \quad \mathbf{c}' \equiv (\mathbf{1}_{1\times N}, 0, \mathbf{0}_{1\times J}),\end{aligned} \tag{58}$$

and $\widetilde{\mathcal{M}}$ is the $J \times N$ matrix of the market scenarios $\widetilde{\mathbf{M}}^{(j)}$. As r varies, we obtain the efficient frontier. Stan Uryasev's team computed the efficient frontier of thirty portfolios displayed in the main text based on 10,000 scenarios using CPLEX® in a couple of seconds.