Discrete Mathematics

Counting

Myles Moylan

6.1 The Basics of Counting

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

By the product rule, $18 \cdot 325 = 5850$ way to pick the two representatives.

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major.

By the sum rule, $18 + 325 = 343$ ways to pick the representative.

2. An office building contains 27 floors and had 37 offices on each floor. How many offices are in the building?

$27 \cdot 37 = 999$ offices in the building.

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

$10 \cdot 4 = 40$ ways to answer the questions on the test.

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

$10 \cdot (4 + 1) = 50$ ways to answer the questions if leaving blank answers is an option.

4. A particular brand of shirt comes in 12 colors, has a male and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

$(3 \cdot 12) + (3 \cdot 12) = 72$ different types of shirt.

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How may different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?

$6 \cdot 7 = 42$ differently possible flight plans.

6. There are four major auto routes from Boston to Detroit and six from Detroit to Las Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

$4 \cdot 6 = 24$ major auto routes from Boston To LA via Detroit.

7. How many different three-letter initials can people have?

$26 \cdot 26 \cdot 26 = 26^3 = 17576$ possible three-letter initials.

8. How many different three-letter initials with none of the letters repeated can people have?

$26 \cdot 25 \cdot 24 = 15600$ different possible three-letter initials without repeating letters.

9. How many different three-letter initials are there that begin with an A?

$1 \cdot 26 \cdot 26 = 26^2 = 676$ possible three-letter initials beginning with A.

10. How many bit strings are there of length eight?

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256$ possible bit strings of length eight.

11. How many bit strings of length ten both begin and end with a 1?

$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 2^8 = 256$ possible strings of length ten that begin and end with a 1.

12. How many bit strings are there of length six or less, not counting the empty string?

$2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 = 126$ different bit strings of length six or less, not counting the empty string.

13. How many bit strings of length not exceeding n, where n is a positive integer, consist entirely of 1s, not counting the empty string?

Since the string is given to consist entirely of 1's, there is nothing to choose except the length. Since there are $n + 1$ possible lengths not exceeding n (if we include the empty string, of length 0), the answer is simply $n + 1$. Note that the empty string consists—vacuously—entirely of 1's.

14. How many bit strings of length n, where n is a positive integers, start and end with 1s?

2^{n-2} different bit strings of length n which start and end with 1s.

15. How many strings are there of lowercase letters of length four or less, not counting the empty string?

By the sum rule we can count the number of strings of length 4 or less by counting the number of strings of length i, for $0 \leq i \leq 4$, and then adding the results.

$$\sum_{i=0}^{4} 26^i = 1 + 26 + 676 + 17576 + 456976 = 475255$$

16. How many strings are there of four lowercase letters that have the letter x in them?

Here, for example, we use the product rule to count the number of possible strings: $26 \cdot 26 \cdot 26 \cdot 1$, where the 1 is the location of the x. And since the x can be in any of four places we multiply the product by 4.

$26^3 \cdot 4 = 70304$ possible strings of four lowercase letters that have the letter x in them.

17. How many strings of five ASCII characters contain the character @ ("at" sign) at least once? [*Note:* There are 128 different ASCII characters.]

An easy way to count this is to find the total number of ASCII strings of length five and then subtract off the number of such strings that do not contain the @ character. Since there are 128 characters to choose from in each location in the string, the answer is $128^5 - 127^5 = 34359738368 - 33038369407 = 1321368961$.

18. How many 6-element RNA sequences

Recall that an RNA sequence is a sequence of letters, each of which is one of A, C, G, or U. Thus by the product rule there are 4^6 RNA sequences of length six if we impose no restrictions.

a) do not contain U?

If U is excluded, then each position can be chosen from among three letters, rather than four. Therefore the answer is $3^6 = 729$.

b) end with GU?

If the last two letters are specified, then we get to choose only four letters, rather than six, so the answer is $4^4 = 256$

c) start with C?

If the first letter is specified, then we get to choose only five letters, rather than six, so the answer is $4^5 = 1024$.

d) contain only A or U?

If only A or U is allowed in each position, then there are just two choices at each of six stages, so the answer is $2^6 = 64$.

19. How many positive integers between 50 and 100

Because neither 100 nor 50 is divisible by either 7 or 11, whether the ranges are meant to be inclusive or exclusive of their endpoints is moot.

a) are divisible by 7? Which integers are these?

There are $\lfloor 100/7 \rfloor = 14$ integers less than 100 that are divisible by 7, and $\lfloor 50/7 \rfloor = 7$ of them are less than 50 as well. This leaves $14 - 7 = 7$ numbers between 50 and 100 that are divisible by 7. They are 56, 63, 70, 77, 84, 91, and 98.

b) are divisible by 11? Which integers are these?

There are $\lfloor 100/11 \rfloor = 9$ integers less than 100 that are divisible by 11, and $\lfloor 50/11 \rfloor = 4$ of them are less than 50 as well. This leaves $9 - 4 = 5$ numbers between 50 and 100 that are divisible by 11. They are 55, 66, 77, 88, and 99.

c) are divisible by both 7 and 11? Which integers are these?

A number is divisible by both 7 and 11 if and only if it is divisible by their least common multiple, which is 77. There is only one such number between 50 and 100, namely 77.

20. How many strings of three decimal digits

This problem involves 1000 possible strings, since there is a choice of 10 digits for each of the three positions in the string.

a) do not contain the same digit three times?

This is most easily done by subtracting from the total number of strings the number of strings that violate the condition. There are 10 strings that consist of the same digit three times (000, 111, . . ., 999). Therefore there are $1000 - 10 = 990$ strings that do not.

b) begin with an odd digit?

If we must begin our string with an odd digit, then we have only 5 choices for this digit. We still have 10 choices for the remaining digits. Therefore there are $5 \cdot 10 \cdot 10 = 500$ such strings.

c) have exactly two digits that are 4s?

Here we need to choose the position of the digits that is not a 4 (3 ways) and choose that digit (9 ways). Therefore there are $3 \cdot 9 = 27$ such strings.

6.2 The Pigeonhole Principle

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

There are six classes: these are the pigeons. There are five days on which classes may meet (Monday through Friday): these are the pigeonholes. Each class must meet on a day (each pigeon must occupy a pigeonhole). By the pigeonhole principle at least one day must contain at least two classes.

2. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains two pigeons. By the pigeonhole principle the answer is 3. If three socks are taken from the drawer, at least two much have the same color. On the other hand two sock are not enough, because one might be brown and the other black.

b) How many socks must he take out to be sure that he has at least two black socks?

He needs to take out 14 socks in order to insure at least two black socks. If he does so, then at most 12 of them are brown, so at least two are black. On the other hand, if he removes 13 or fewer socks, then 12 of them could be brown, and he might not get his pair of black socks.

3. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

There are 26 letters in the English alphabet (these are the pigeons) and 30 students in the class (these are the pigeonholes). By the pigeonhole principle, actually, there only need be 27 students in the class for there to be two students with last names the begin with the same letter.

4. Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

In this question, the students are pigeons. The boxes or pigeonholes are less clear: since the question is asking about students expected to graduate in the same year with the same major, we think of a year-major pair as being a box. More formally, if Y is the set of 4 years and M is the set of 21 different majors, then $Y \times M$ is the set of boxes. This means that there are $4 \times 21 = 84$ boxes, and thus $84 + 1 = 85$ students that are needed to ensure that there are two in the same box.

5. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

There are four possible remainders when an integer is divided by 4 (these are the pigeonholes here): 0, 1, 2, or 3. Therefore, by the pigeonhole principle at least two of the five given remainders (these are the pigeons) must be the same.

6. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

Let the n consecutive integers be denoted $x + 1, x + 2, \ldots, x + n$, where x is some integer. We want to show that exactly one of these is divisible by n. There are n possible remainders when an integer is divided by n, namely $0, 1, 2, \ldots, n - 1$. There are two possibilities for the remainders of our collection of n numbers: either they cover all the possible remainders (in which case exactly one of our numbers has a remainder of 0 and is therefore divisible by n), or they do not. If they do not, then by the pigeonhole principle, since there are then fewer than n pigeonholes (remainders) for n pigeons (the numbers in our collection), at least one remainder must occur twice. In other words, it must be the case that $x + i$ and $x + j$ have the same remainder when divided by n for some pair of numbers i and j with $0 < i < j \leq n$. Since $x + i$ and $x + j$ have the same remainder when divided by n, if we subtract $x + i$ from $x + j$, then we will get a number divisible by n. This means that $j - i$ is divisible by n. But this is impossible, since $j - i$ is a positive integer strictly less than n. Therefore the first possibility must hold, that exactly one of the numbers in our collection is divisible by n.

7. What is the minimum number of students, each of whom come from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

The generalized pigeonhole principle applies here. The pigeons are the students, and the pigeonholes are the states, 50 in number. By the generalized pigeonhole principle if we want there to be at least 100 pigeons in at least one of the pigeonholes, then we need to have a total of N pigeons so that $\lceil N/50 \rceil \geq 100$. This will be the case as long as $N \geq 99 \cdot 50 + 1 = 4951$. Therefore we need at least 4951 students to guarantee that at least 100 come from a single state.

8. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.

If this statement were not true, then there would be at most 8 from each class standing, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

If this statement were not true, then there would be at most 2 freshmen, at most 18 sophomores, and at most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

9. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.

One way to do this is to have the sequence contain four groups of four numbers each, so that the numbers within each group are decreasing, and so that the numbers between groups are increasing. For example, we could take the sequence to be 4, 3, 2, 1; 8, 7, 6, 5; 12, 11, 10, 9; 16, 15, 14, 13. There can be no increasing subsequence of five terms, because any increasing subsequence can have only one element from each of the four groups. There can be no decreasing subsequence of five terms, because any decreasing subsequence cannot have elements from more than one group.

6.3 Permutations and Combinations

1.