Assignment 1

Morgan Yost

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1 Problem 1

1.1 Problem Description

Let e = (2, -5, 3, 4), find:

a.$\|e\|_\infty$ b. $\|e\|_1$ c. $\|e\|_2$

1.2 Problem Solution

a.

$$\|e\|_\infty = max(|e|) = 5$$

b.

$$\|e\|_1 = \sum_{m=1}^{N} |e| = 14$$

c.

$$\|e\|_2 = \sqrt{\sum_{m=1}^{N} e^2} = \sqrt{54}$$

2 Problem 2

2.1 Problem Description

Let A =

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

find the following:

a.$\|A\|_\infty$ b. $\|A\|_1$ c. $\|A\|_2$

2.2 Problem Solution

a.

$$\|A\|_\infty = \max_{1\le i\le s} \sum_{j=1}^{s} |a_i j| = 58$$

b.

$$\|A\|_1 = \max_{1\le j\le s} \sum_{i=1}^{s} |a_i j| = 40$$

c.

$$\|A\|_2 = \sqrt{\rho(A^T A)} = 38.6227$$

3 Problem 3

3.1 Problem Description

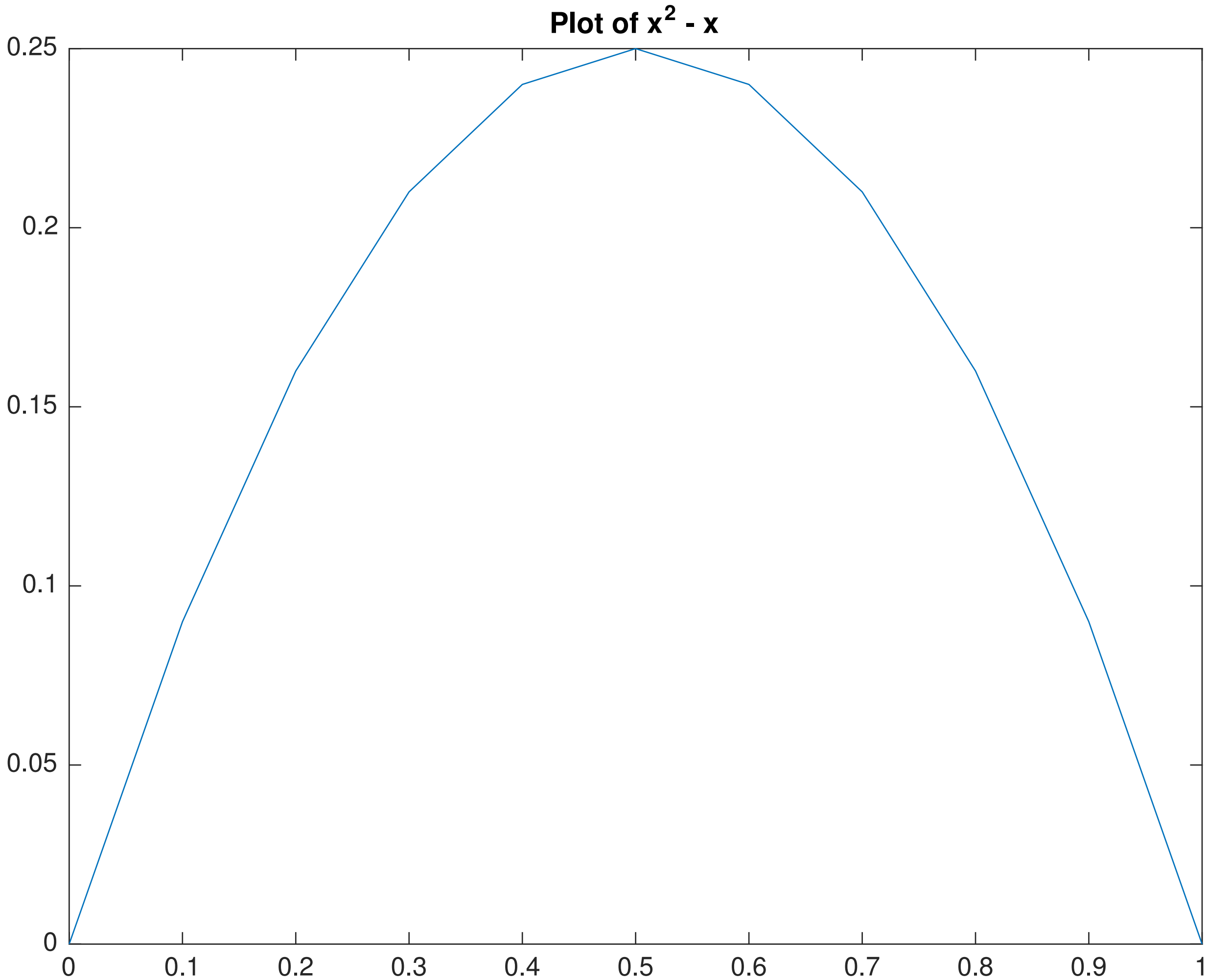
Let $u(x) = x^2 - x$, find: a.$\|u\|_\infty$ b. $\|u\|_1$ c. $\|u\|_2$

3.2 Problem Solution

a.

$$\|u\|_\infty = \max_{1 \le 0 \le 1} |u| = 1/4$$

b.



$$\|u\|_1 = \int_0^1 |x^2 - x| dx = \int 0^1 - x^2 - x = \frac{-x^3}{3} + \frac{x^2}{2}\bigg|_0^1 = 1/6$$

c.

$$\|u\|_2 = \sqrt{\int_0^1 |x^2 - x|^2 dx} = \sqrt{\int 0^1 x^4 - 2x^3 + x^2} = \sqrt{\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3}\bigg|_0^1} = \sqrt{-.633}$$

4 Problem 4

4.1 Problem Description

Let U be defined by $U_i = x_i^1 - x_1$ where $x_1 = .2$, $x_2 = .4$, $x_3 = .6$, $x_4 = .8$, $x_5 = 1$. Find the following: a.$\|U\|_\infty$ b. $\|U\|_1$ c. $\|U\|_2$

4.2 Problem Solution

Evaluating U_i for all values of x we find: U = (-.16, -.24, -.24, -.16, 0). Our grid spacing, h, is .2. a.

$$\|U\|_\infty = \max_{1 \leq 0 \leq 1} |u| = .24$$

b.

$$\|U\|_1 = h \sum_{m=1}^{N} |e| = .16$$

c.

$$\|U\|_2 = \sqrt{h \sum_{m=1}^{N} e^2} = .1824$$

5 Exercise 1.2

5.1 Problem Description

a. Use method of undetermined coefficients to set up the 5x5 Vandermonde system that would define the fourth-order accurate finite difference approximation to $u''(x)$ based on 5 equally spaced points.

$$u''(x) = c_{-2}u(x-2h) + c_{-1}u(x-h) + c_0u(x) + c_1u(x+h) + c_2u(x+2h) + O(h^4)$$

b. Use fdstencil.m to do part a.
c. Use this finite difference formula to approximate $u''(1)$ for $u(x) = sin(2x)$.

5.2 Problem Solution

a. Taylor expansion of the previous equation gives the following:

$$= u(\bar{x})(c_{-2} + c_{-1} + c_0 + c_1 + c_2) +$$

$$u'(\bar{x})h(-2c_{-2} - c_{-1} + c_1 + 2c_2) +$$

$$u''(\bar{x})h^2(2c_{-2} + \frac{1}{2}c_{-1} + \frac{1}{2}c_1 + 2c_2) +$$

$$u'''(\bar{x})h^3(\frac{-4}{3}c_{-2} - \frac{1}{6}c_{-1} + \frac{1}{6}c_1 + \frac{4}{3}c_2) +$$

$$u''''(\bar{x})h^4(\frac{2}{3}c_{-2} + \frac{1}{24}c_{-1} + \frac{1}{24}c_1 + \frac{2}{3}c_2)$$

Choosing to solve for $u''(x)$ and putting into matrix form we have:
Let A =

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & .5 & 0 & .5 & 2 \\ -4/3 & -1/6 & 0 & 1/6 & 4/3 \\ 2/3 & 1/24 & 0 & 1/24 & 2/3 \end{pmatrix}$$

Let x =

$$\begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix}$$

Let b =

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Solving the equation $Ax = b$, we find that coefficients we are looking for are as follows:
x =

$$\begin{pmatrix} -.0833 \\ 1.3333 \\ -2.5000 \\ 1.3333 \\ -.0833 \end{pmatrix}$$

b. Using the fdstencil MATLAB routine resulted in the same answer as seen in part a. The MATALB code is provided in appendix A.
c. The errors were calculated in accordance with the equations below:
Part A error:

$$error = |4 * sin(2) - B|$$

where B is the result of the estimation scheme used in part A.
Part B error:

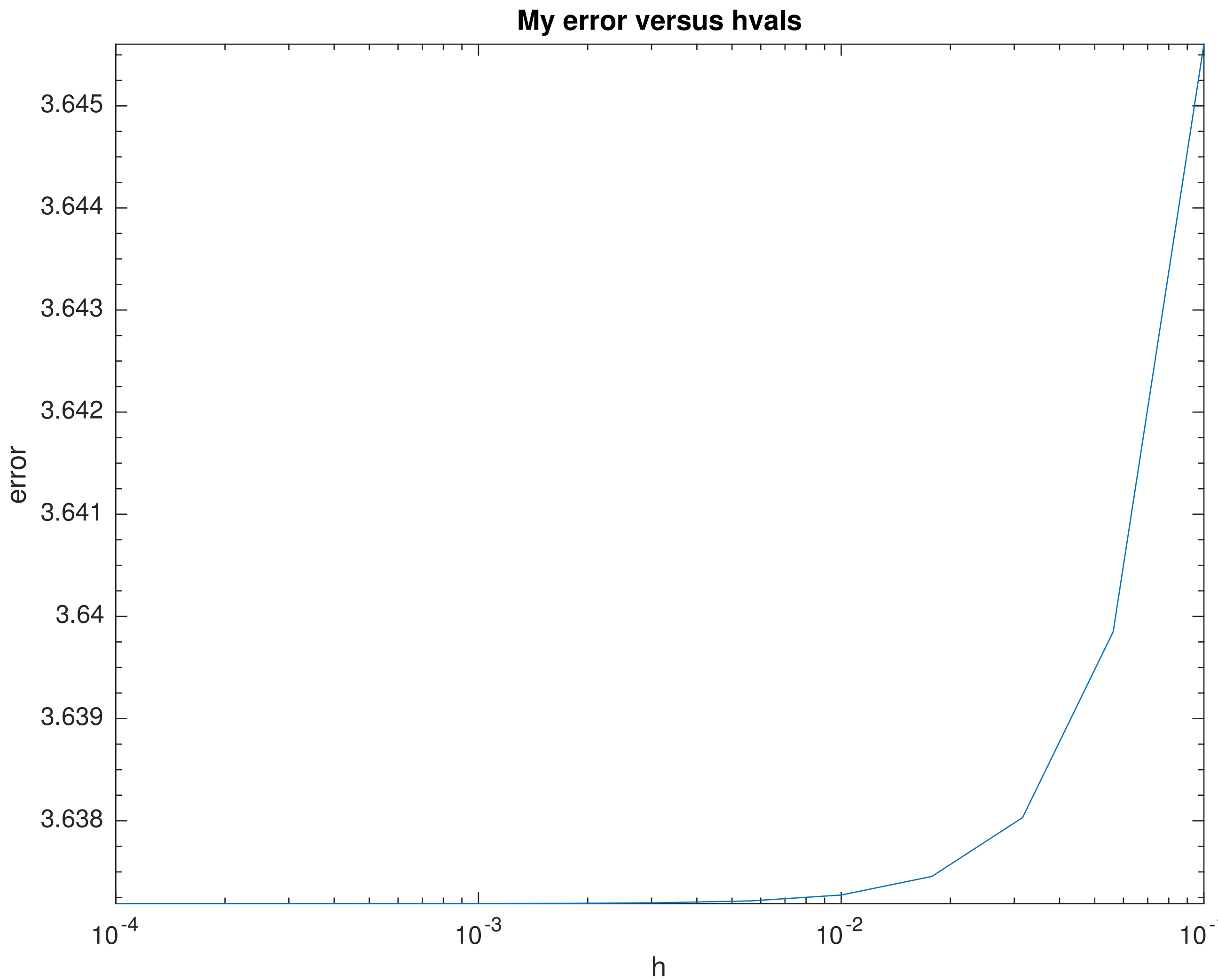
$$error = err_0 * h^3 * u''''' + err_1 * h^4 * u''''''$$

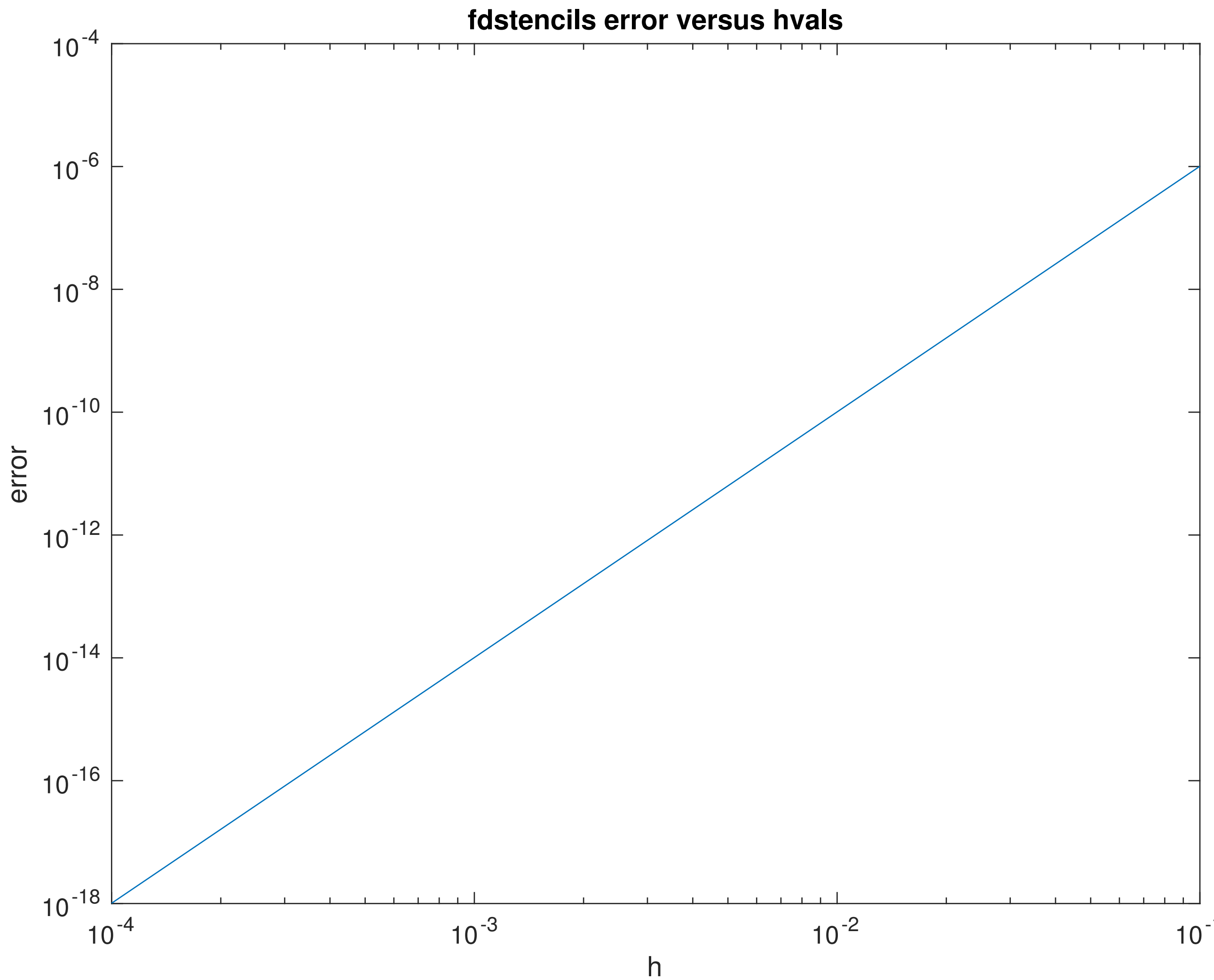
where err_0 and err_1 are the coefficients of the 5th and 6th derivatives returned from fdstencil.
The comparison for error versus h values is represented by the table below:

hval	Part A error	Part B error
0.1000000000000000	3.6456044078094720	0.000001010330474251
0.0562341325190349	3.6398506719040853	0.000000101033047425
0.0316227766016838	3.6380311782781858	0.000000010103304743
0.0177827941003892	3.6374558037921250	0.000000001010330474
0.0100000000000000	3.6372738544011982	0.000000000101033047
0.0056234132519035	3.6372163169516960	0.000000000010103305
0.0031622776601684	3.6371981220125749	0.000000000001010330
0.0017782794100389	3.6371923682676237	0.000000000000101033
0.0010000000000000	3.6371905487737117	0.000000000000010103
0.0005623413251903	3.6371899733992166	0.000000000000001010
0.0003162277660168	3.6371897914498255	0.000000000000000101
0.0001778279410039	3.6371897339123755	0.000000000000000010
0.0001000000000000	3.6371897157174367	0.000000000000000001

The plots below provide a graphical representation of this data:

The trend in both figures shows that the error increases as the h values increase. It makes sense that the estimation scheme would get less accurate as the grid spacing increases. Because the error in part A was calculated by subtracting the actual value from the estimated value, we can see that the truncation error is negligible where the grid spacing is small, then increases drastically as the grid spacing increases. The error calculated by fdstencil shows that the error appears to increase linearly. This is because the error calculation is based on one term, not all of the truncated terms.





A MATLAB Code

Table of Contents


```
clear all
close all
```

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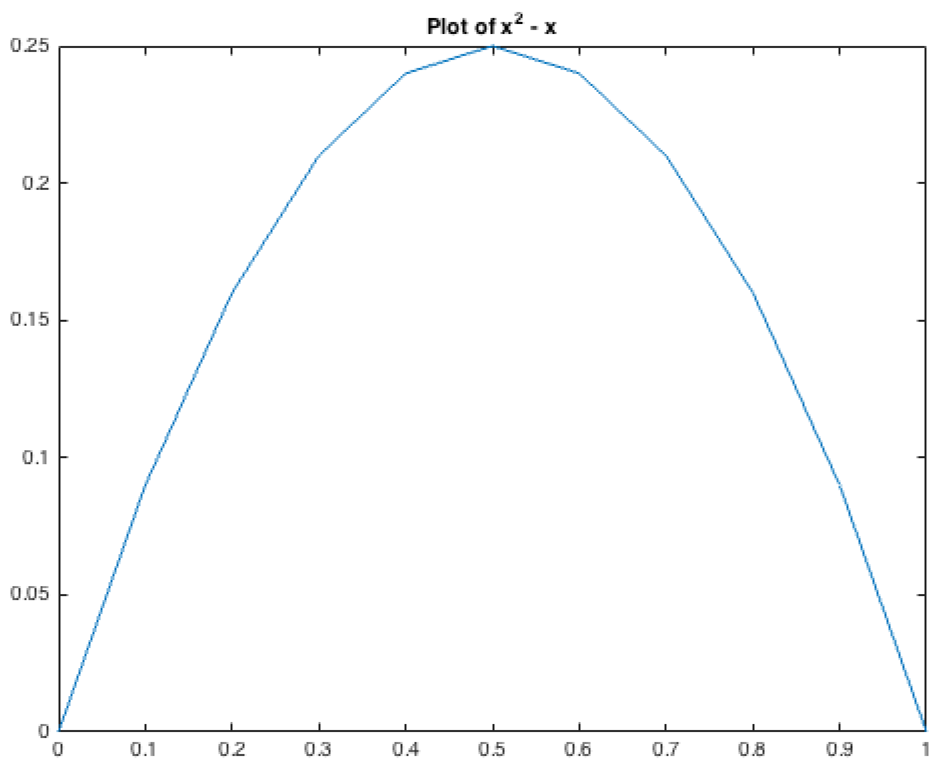
Problem 3

```
x = 0:.1:1;
figure(1)
plot(x, abs(x.^2-x))
title('Plot of x^2 - x')
```



Problem 4

```
x = [.2 .4 .6 .8 1];
h = .2;
U = x.^2-x;
%Part a
U_inf = max(abs(U));
%Part b
U_1 = h*sum(abs(U));
%Part c
U_2 = sqrt(h*sum(U.^2));
```

Exercise 1.2

```
%Part a
u = [1 1 1 1 1;
     -2 -1 0 1 2;
     2 .5 0 .5 2;
     -4/3 -1/6 0 1/6 4/3;
     2/3 1/24 0 1/24 2/3];
v = [0; 0; 1; 0; 0];
C = inv(u)*v;
fprintf('Using the method of undetermined coefficients, \n')
fprintf('I found the coeffiecients to be:\n')
fprintf('%f \n\n', C);
%Part b
fprintf('Using fdstencil:\n')
[c, err0, err1] = fdstencil(2, -2:2);
if(u*c' ~= v)
    fprintf('Error: mismatched results \n')
else
    fprintf('Sucessful Match!\n\n')
end
%Part c
x0 = ones(length(c),1);
hCoeff = [-2 -1 0 1 2]';
hvals = logspace(-1, -4, 13);
count = 1;
for h = hvals
    result(count) = sum(c'.*sin(x0+h.*hCoeff));
    err(count) = abs(4*sin(2)-result(count));
    exptError(count) = abs(err0*h^3*cos(2) + err1*h^4*sin(2));
    count = count+1;
end

%plots
figure(2)
loglog(hvals, err)
title('Part A error versus hvals')
xlabel('h')
ylabel('error')
figure(3)
```

```
loglog(hvals, exptError)
title('Part B error versus hvals')
xlabel('h')
ylabel('error')

%make table for latex
fprintf('Error table:')
for i = 1:length(hvals)
    fprintf('%.16f & %.16f & %.18f \\\\ \n', hvals(i), err(i),
 exptError(i));
end
```

```
Using the method of undetermined coefficients,
I found the coeffiecients to be:
-0.083333

1.333333

-2.500000

1.333333

-0.083333

Using fdstencil:

The derivative u^(2) of u at x0 is approximated by

          1/h^2 * [
                     -8.333333333333333e-02 * u(x0-2*h) +
                      1.333333333333333e+00 * u(x0-1*h) +
                     -2.500000000000000e+00 * u(x0) +
                      1.333333333333333e+00 * u(x0+1*h) +
                     -8.333333333333333e-02 * u(x0+2*h) ]

For smooth u,
       Error = 0 * h^3*u^(5) + -0.0111111 * h^4*u^(6) + ...

Sucessful Match!

Error table:0.1000000000000000 & 3.6456044078094720 &
 0.000001010330474251 \\
0.0562341325190349 & 3.6398506719040853 & 0.000000101033047425 \\
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0.0001000000000000 & 3.6371897157174367 & 0.000000000000000001 \\
```
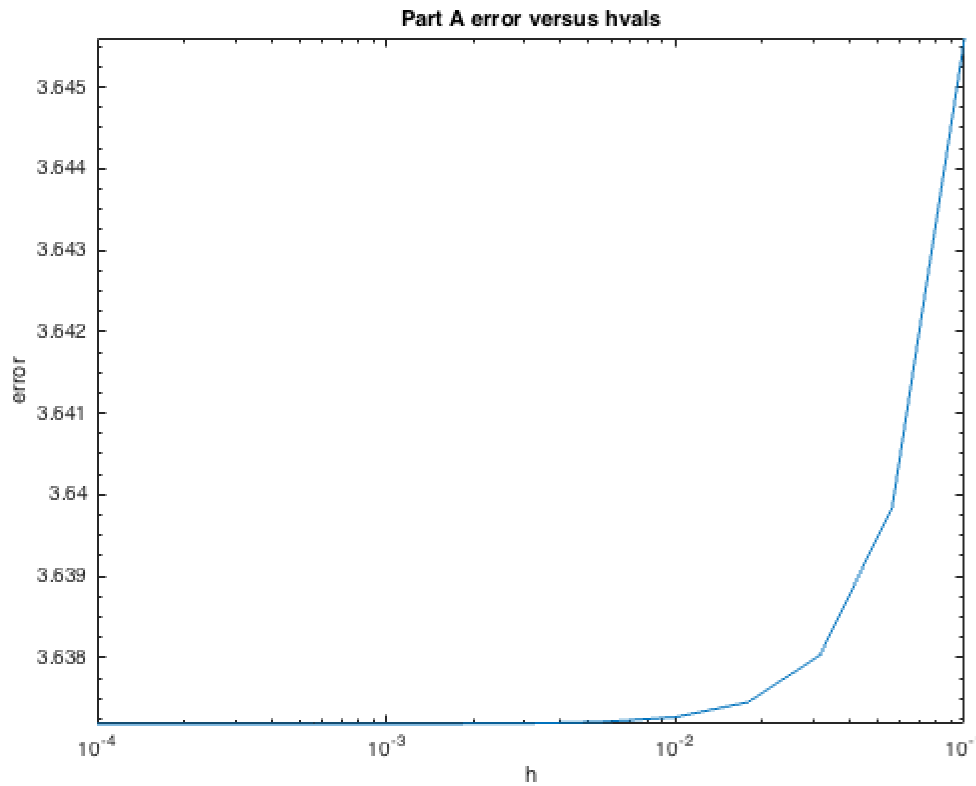
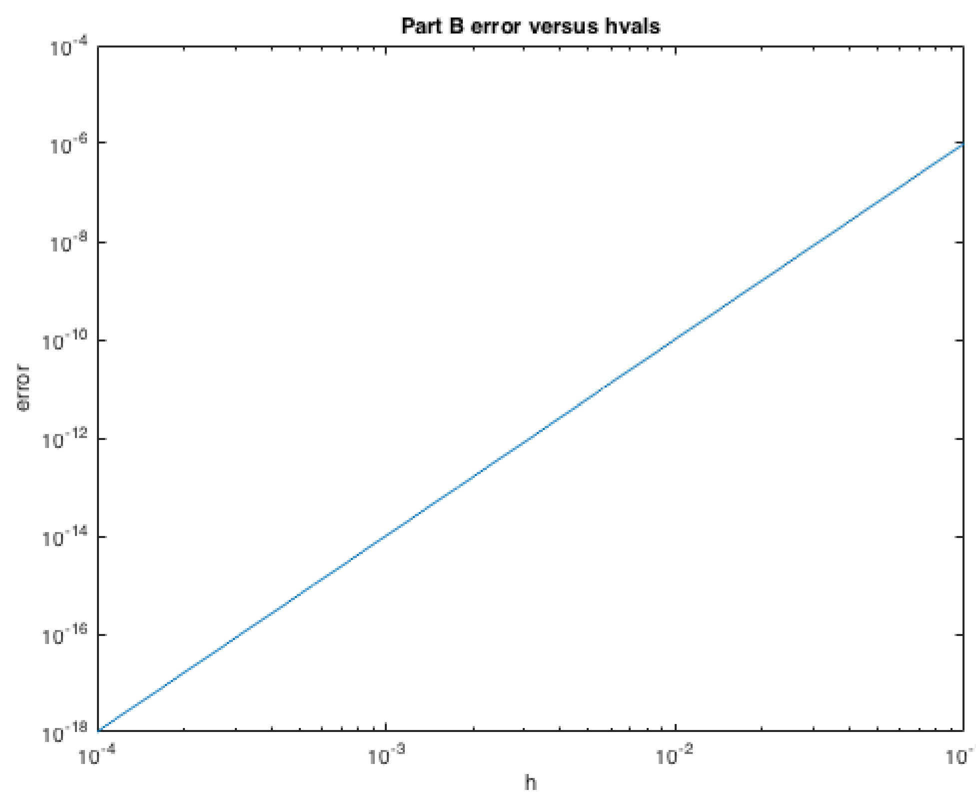


Part A error versus hvals
error
3.645
3.644
3.643
3.642
3.641
3.64
3.639
3.638
h
10^{-4}
10^{-3}
10^{-2}
10^{-1}


Part B error versus hvals
error
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
h
10^{-4}
10^{-3}
10^{-2}
10^{-1}