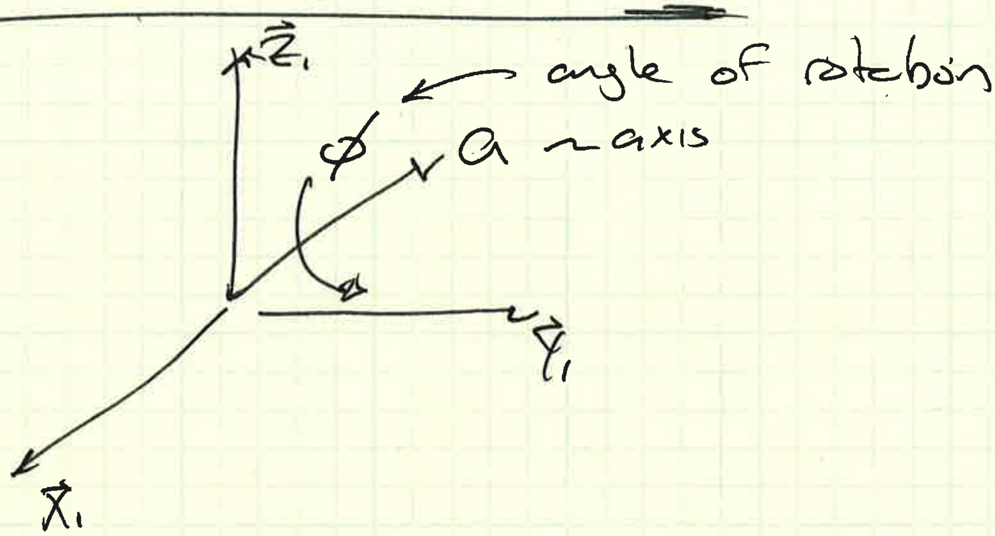
Euler's Thm (~~why a quaternion~~ Eigen axis of rotation)
or the axis of invariance

: When a sphere is moved around its center it is always possible to find a diameter whose direction in the displaces position is the same as in the initial position.

- The diameter defines the axis of said rotation and is invariant. ie. that diameter ~~dose~~ does not move.

Define ~~that~~ axis as $\underline{a}$



How to find a and ϕ?

Let C be any rotation

$\Rightarrow \quad C C^T = C^T C = \underline{I}$

Notice

$$\det(I) = 1 = \det(CC^T) = \det(C)\det(C^T)$$
$$= \det(C)\det(C) = \det(C)^2$$
$$\Rightarrow \det(C) = \pm 1$$

You Do: $\det(C_x(\theta_x)) = 1$

~~what~~ when $\det(C) = 1 \Rightarrow$ proper rotation

$\det(C) = -1 \rightarrow$ improper rotation

$\rightarrow$ a proper rotation followed by a reflection through the plane perpindicular to the axis of rotation

So, if C is a proper rotation
$$\Rightarrow \det(C) = 1$$

Now $\det(I) = \det(CC^{-1}) = \det(C)\det(C^{-1}) = 1$

since $\det(C) = 1$ and $\det(C^{-1}) = \dfrac{1}{\det(C)} = 1$

and $\det(-C) = (-1)^3\det(C) = -\det(C) = -1$

So, $\det(C-I) = \det\left[(C-I)^T\right]$

$$= \det\left[C^T - I\right]$$

$$= \det\left[C^{-1} - I\right] = \det\left[-C^{-1}(C-I)\right]$$

$$= -\det(C^{-1})\det(C-I) = -\det(C-I)$$

or ~~wthe~~ we just showed

$$\det(C-I) = -\det(C-I)\ !$$

$$\Rightarrow \det(C-I) = -\det(C-I) = 0$$

or $\det(C-I) = 0$

$\Leftrightarrow$ $C-I$ is singular

or, $\cancel{C - \eta I = 0}$

$$\det(C - \eta I) = 0 \quad \text{for } \eta = 1$$

i.e. $C - \eta I$ is singular

$$\Rightarrow (C - \eta I)a = 0 \quad \text{for some vector } a$$

$$\Rightarrow (C - I)a = 0$$

or $Ca = a$

TOPS

Summary

- 1 is always an eigenvalue of a rotation and the associated vector, a, is the invariant axis of rotation.
- a is known as the eigenaxis of C

Quaternions (Kuipers and de Ruiters)

Define $q = \eta + \epsilon$

with $\epsilon = \epsilon_1 i + \epsilon_2 j + \epsilon_3 k$

with $i^2 = j^2 = k^2 = ijk = -1$

and

$$\begin{tikzcd} & i \arrow[dr] & \\ k \arrow[ur] & + & j \arrow[ll] \end{tikzcd}$$

i.e. $ij = k$

and $ji = -k$

Addition

$$p + q = (\eta_p + \epsilon_p) + (\eta_q + \epsilon_q)$$
$$= (\eta_p + \eta_q) + (\epsilon_p + \epsilon_q)$$
$$= \epsilon_{1p} + \epsilon_{1q})i$$
$$+ (\epsilon_{2p} + \epsilon_{2q})j$$
$$+ (\epsilon_{3p} + \epsilon_{3q})k$$

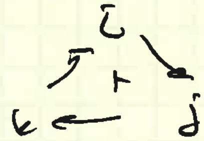
multiplication (scalar)

$$\cancel{p} \quad cq = c(\eta + \epsilon) = c\eta + c\epsilon$$

Multiplication (quaternion)

$$
\begin{aligned}
p\,q &= (\cancel{\epsilon_p}\,\eta_p + \epsilon_p)(\eta_q + \epsilon_q) \\
&= \eta_p \eta_q + \eta_p \epsilon_q + \epsilon_p \eta_q + \epsilon_p \epsilon_q
\end{aligned}
$$

with $\quad \eta_p \epsilon_q = \eta_p (\epsilon_{1q} i + \epsilon_{2q} j + \epsilon_{3q} k)$

now use 

result

$$
\boxed{
\begin{aligned}
p\,q &= (\eta_p \eta_q - \epsilon_p^T \epsilon_q) \\
&\quad + \eta_p \epsilon_q + \eta_q \epsilon_p + \epsilon_p^{\times} \epsilon_q
\end{aligned}
}
$$

w/ $\quad \epsilon_p^{\times} = \begin{bmatrix} 0 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 0 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 0 \end{bmatrix}$

Complex Conjugate

$$q^* = (\eta + \epsilon)^* = \eta - \epsilon$$

$$(p\,q)^* = q^* p^*$$

$$q + q^* = 2\eta$$

Norm

$$|q| = \sqrt{q^* q} = \sqrt{q q^*}$$
$$= \sqrt{\eta^2 + |\epsilon|^2}$$

Inverse

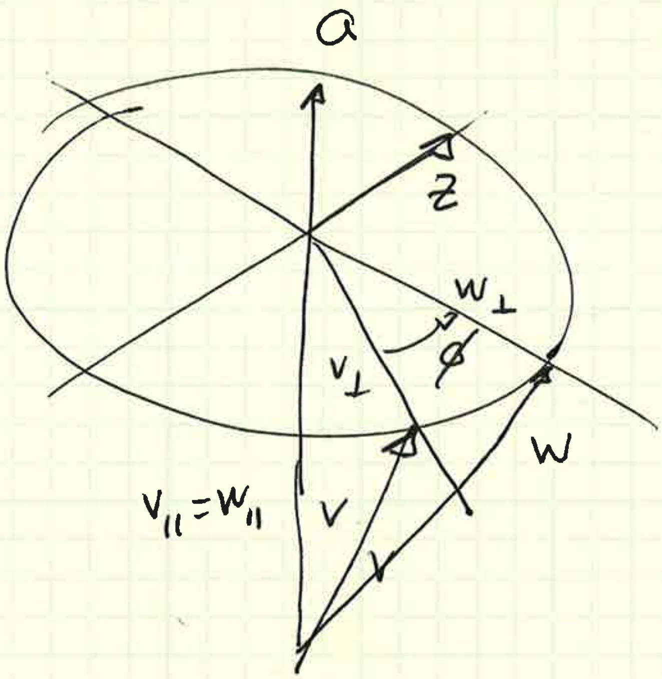
$$q^{-1} q = q q^{-1} = 1$$

$$\Rightarrow q^{-1} q q^* = q^* q q^{-1} = q^*$$

$$\Rightarrow q^{-1} = \frac{q^*}{q^* q} = \frac{q^*}{|q|}$$

A Quaternion ~~as~~ is a Rotation (Kuipers)

If the quaternion product, $q\vec{v}q^* = \vec{w}$ is a rotation of the vector $\vec{v}$ to $\vec{w}$ about a through ϕ, then the following picture is the case under consideration:



<u>notice</u>

$V = V_\parallel + V_\perp$

$W = W_\parallel + W_\perp = V_\parallel + W_\perp$

<u>Define</u> $q = \eta + \epsilon = \cos\frac{\phi}{2} + a\sin\frac{\phi}{2}$

i.e.

<u>notice</u> $|q| = 1$ when $|a| = 1$

You Do

i.e. q is a "unit quaternion"

now $V = V_{\parallel} + V_{\perp}$

and $q(V)q^* = q(V_{\parallel} + V_{\perp})q^*$

$$= q V_{\parallel} q^* + q V_{\perp} q^*$$

look @ $q V_{\parallel} q^* = (\eta^2 - |\epsilon|^2)V_{\parallel}$

$$+ 2(\epsilon \cdot V_{\parallel})\epsilon + 2\eta(\epsilon \times V_{\parallel})$$

but $\epsilon \times V_{\parallel} = 0$ since ϵ is in the $a/V_{\parallel}$ direction

since $V_{\parallel}$ is in same direction as ϵ

$\Rightarrow V_{\parallel} = k\epsilon$

$\Rightarrow \epsilon \cdot V_{\parallel} = \epsilon \cdot k\epsilon = k|\epsilon|^2$

$\Rightarrow q V_{\parallel} q^* = (\eta^2 - |\epsilon|^2)k\epsilon + 2k|\epsilon|^2\epsilon$

$$= k(\eta^2 - \cancel{|\epsilon|^2} + \cancel{2}|\epsilon|^2)\epsilon$$

$$= k\underbrace{(\eta^2 + |\epsilon|^2)}_{=1}\epsilon$$

$$= k\epsilon = V_{\parallel}$$

i.e. $q V_{\parallel} q^* = V_{\parallel}$

Now $q v_\perp q^* = (\eta - |\epsilon|^2) v_\perp$

$$+ 2(\epsilon \cdot v_\perp)\epsilon + 2\eta(\epsilon \times v_\perp)$$

$$\epsilon \cdot v_\perp = 0 \quad \text{and} \quad \epsilon = |\epsilon| a$$

$$\Rightarrow q v_\perp q^* = (\eta^2 - |\epsilon|^2) v_\perp + 2\eta |\epsilon| (a \times v_\perp)$$

let $a \times v_\perp = z$

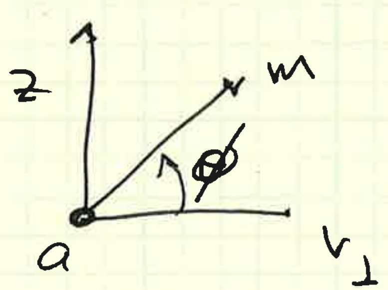
$$\Rightarrow q v_\perp q^* = (\eta^2 - |\epsilon|^2) v_\perp + 2\eta |\epsilon| z$$

notice $|z| = |a \times v_\perp| = |a||v_\perp| \sin \pi/2 = |v_\perp|$

$$\Rightarrow q v_\perp q^* = (\cos^2 \phi/2 - \sin^2 \phi/2) v_\perp + 2 \cos\phi/2 \sin\phi/2 \, z$$

$$= \cos\phi \, v_\perp + \sin\theta \, z = m$$

looking down on a



so, if $|m| = |v_\perp|$

$$\Rightarrow m = v_\perp$$

$$|m|^2 = m \cdot m = \cos^2\phi \, |v_\perp|^2 + \sin^2\phi \, |z|^2$$

$$= \left(\cos^2\phi + \sin^2\phi\right) |v_\perp|^2 = |v_\perp|^2$$

$$\Rightarrow \quad |m| = |v_\perp|$$

$$\Rightarrow \quad m = w_\perp$$

$$\Rightarrow \quad g v_\perp g^* = w_\perp$$

$$\Rightarrow g v g^* = g v_\parallel g^* + g v_\perp g^*$$

$$= w_\parallel + w_\perp = w$$

Relation between q and C

* <u>Recall</u> $r_2 = C_{21} r_1$

and $r_1 = C_{12} r_2$

<u>now</u> $q r_1 q^* = r_2$ ← i.e. rotate the vector

$$\Rightarrow q r_1 q^* = (\eta + \epsilon)(0 + r_1)(\eta - \epsilon)$$

$$= (\eta^2 - |\epsilon|^2) r_1 + 2(\epsilon^T r_1)\epsilon + 2\eta(\epsilon^\times r_1)$$

$$|\epsilon|^2 = 1 - \eta^2$$

⇒ and

$(\epsilon \cdot r_1)\epsilon = \epsilon\epsilon^T r_1$ ← You Do

$$\Rightarrow r_2 = \underbrace{\left[(2\eta^2 - 1)\mathbb{1} + 2\epsilon\epsilon^T + 2\eta\epsilon^\times\right]}_{= C_{12}} r_1$$

$$\Rightarrow C_{21} = C_{12}^T \Rightarrow$$

$$\boxed{C_{21} = (2\eta^2 - 1)\mathbb{1} + 2\epsilon\epsilon^T - 2\eta\epsilon^\times}$$

and $r_2 = q^* r_1 q$

Finally

given C

$$\Rightarrow \eta = \frac{1}{2}\sqrt{1 + \mathrm{trace}(C)}$$

$$\epsilon_1 = \frac{1}{4}\frac{(C_{23} - C_{32})}{\eta}$$

$$\epsilon_2 = \frac{1}{4}\frac{(C_{31} - C_{13})}{\eta}$$

$$\epsilon_3 = \frac{1}{4}\frac{(C_{12} - C_{21})}{\eta}$$

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if ~~$r_2 = q_{12} r_1 q_{12}^*$~~ $r_2 = q_{21}^* r_1 q_{21}$

and $r_3 = q_{32}^* r_2 q_{32}$

$$\Rightarrow r_3 = q_{32}^* q_{21}^* r_1 q_{21} q_{32}$$

$$= (q_{21} q_{32})^* r_1 \underbrace{(q_{21} q_{32})}_{= q_{31}}$$

$$= q_{31}^* r_1 q_{31}$$