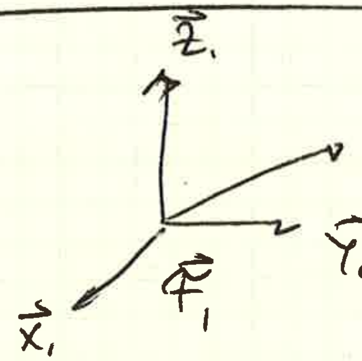
Derivatives of vectors (1.4)



$$\vec{r} = \vec{\mathcal{F}}_1^T r_1$$

evolution of $\vec{r}$ is 100% dependent on view/frame of reference

now define

$$\mathring{\vec{r}} \triangleq \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t}$$

$$\delta \vec{r} \triangleq \vec{\mathcal{F}}_1^T \left(r_1(t+\delta t) - r_1(t) \right)$$

$$\Rightarrow \mathring{\vec{r}} = \vec{\mathcal{F}}_1^T \dot{r}_1 = \vec{\mathcal{F}}_1^T \begin{bmatrix} \dot{r}_{x,1} \\ \dot{r}_{y,1} \\ \dot{r}_{z,1} \end{bmatrix}$$

since $\underbrace{\mathring{\vec{x}}_1, \mathring{\vec{y}}_1, \mathring{\vec{z}}_1 = 0}$ why - inertial

i.e. if frame is moving wrt inertial say on a S/C or A/C, $\mathring{\vec{x}}_1, \mathring{\vec{y}}_1, \mathring{\vec{z}}_1 \neq 0$

Notice $\mathring{(a\vec{r})} = \vec{\mathcal{F}}_1^T \dot{(a r_1)}$

$$= \vec{\mathcal{F}}_1^T (\dot{a} r_1 + a \dot{r}_1)$$

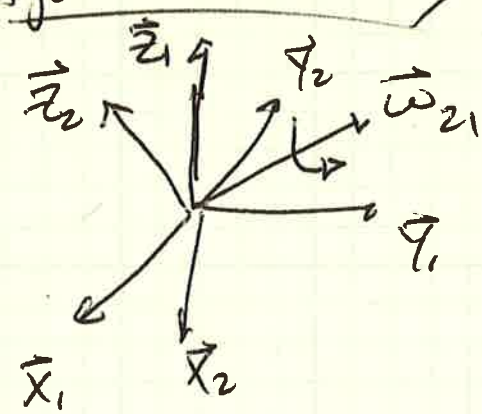
$$= \dot{a}\vec{r} + a\mathring{\vec{r}}$$

$$\dot{(\vec{a}+\vec{b})} = \dot{\vec{a}} + \dot{\vec{b}}$$

$$\dot{(\vec{a}\cdot\vec{b})} = \dot{\vec{a}}\cdot\vec{b} + \vec{a} + \dot{\vec{b}}$$

$$\dot{(\vec{a}\times\vec{b})} = \dot{\vec{a}}\times\vec{b} + \vec{a}\times\dot{\vec{b}}$$

Angular Velocity (1.4.1)



$\vec{F}_2$ is rotating about $\vec{\omega}_{21}$ at some instant relative to $\vec{F}_1$

$|\vec{\omega}_{21}| = \omega_{21}$ is rotational speed

$\hat{\omega}_{21}$ = rotational direction

} at some instant in time = principal axes

Now $\vec{\omega}_{21} = \lim_{\delta t \to 0} \vec{a}(\delta t)\,\dfrac{\phi(\delta t)}{\delta t}$

since $\vec{\omega}_{21}$ is $\vec{a}$ at that instant between $t = t$ and $t + \delta t$

~~well~~ $\vec{v}_{rot}$

Now, consider some vector $\vec{v}$ in $\vec{\mathcal{F}}_1$

$$\Rightarrow \vec{v}_{rot} = \cancel{C_{21} v}\ \vec{\mathcal{F}}_1^{T} C_{12} v$$

$$= \vec{\mathcal{F}}_1^{T} \left[\cos\phi\, \mathbb{1} + (1-\cos\phi)aa^T + \sin\phi\, a^{\times}\right] v$$

when δt is small $\Rightarrow \phi(\delta t) \approx$ small

$\Rightarrow \sin\phi \approx \phi$ and $\cos\phi \approx 1$

$$\Rightarrow \vec{v}(t+\delta t) = \vec{\mathcal{F}}_1^{T}\left[\mathbb{1} + a^{\times}\phi\right] v(t)$$

or $\vec{v}(t+\delta t) - \vec{v}(t) = \vec{\mathcal{F}}_1^{T}\left[a^{\times}\phi\right] v$

and $$\mathring{\vec{v}} = \lim_{\delta t \to 0} \frac{\vec{v}(t+\delta t) - \vec{v}(t)}{\delta t}$$

$$= \vec{\mathcal{F}}_1^{T} a^{\times} \frac{\phi(\delta t)}{\delta t} v$$

$$= \vec{\mathcal{F}}_1^{T} \omega_{21}^{\times} v = \vec{\omega}_{21} \times \vec{v}$$

true for $\vec{x}_1, \vec{y}_1, \vec{z}_1$

$$\Rightarrow \begin{aligned} \mathring{\vec{x}}_2 &= \vec{\omega}_{21} \times \vec{x}_2 \\ \mathring{\vec{y}}_2 &= \vec{\omega}_{21} \times \vec{y}_2 \quad \text{or} \quad \mathring{\vec{\mathcal{F}}}_2^{T} = \vec{\omega}_{21} \times \vec{\mathcal{F}}_2^{T} \\ \mathring{\vec{z}}_2 &= \vec{\omega}_{21} \times \vec{z}_2 \end{aligned}$$

now $\vec{r} = \vec{\mathcal{F}}_1^T r_1 = \vec{\mathcal{F}}_2^T r_2$

denote

$$\dot{\vec{r}} = \vec{\mathcal{F}}_1^T \dot{r}_1, \qquad \overset{\circ}{\vec{r}} = \vec{\mathcal{F}}_2^T \dot{r}_2$$

$\frac{d}{dt}$ of $\vec{r}$ viewed from $\vec{\mathcal{F}}_1^T$

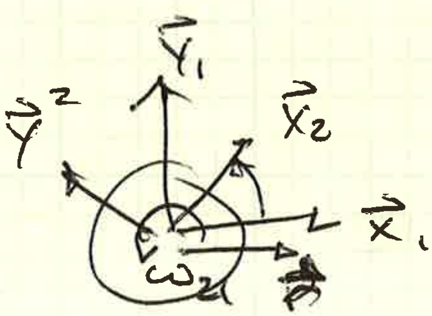
$\frac{d}{dt}$ of $\vec{r}$ viewed from $\vec{\mathcal{F}}_2^T$

$$\Rightarrow \dot{\vec{r}} = \vec{\mathcal{F}}_2^T \dot{r}_2 + \dot{\vec{\mathcal{F}}}_2^T r_2$$

$$= \overset{\circ}{\vec{r}} + \vec{\omega}_{21} \times \vec{\mathcal{F}}_2^T r_2$$

$$\dot{\vec{r}} = \overset{\circ}{\vec{r}} + \underbrace{\vec{\omega}_{21} \times \vec{r}}$$

absolute derivative — relative derivative — apparent motion due to rotating frame



Ant on a record

$\dot{\vec{r}}$ is a spiral

$\overset{\circ}{\vec{r}}$ is going ~~straight~~ straight out $\vec{X}_2$

Now

$$\vec{\omega}_{21} = \vec{\mathcal{F}}_2^T \omega_{21}$$

$\dot{\vec{r}} = \vec{\mathcal{F}}_1^T \dot{r}_1 = \vec{\mathcal{F}}_2^T \dot{r}_2 + \vec{\mathcal{F}}_2^T \omega_{21}^\times r_2$
$= \vec{\mathcal{F}}_2^T (\dot{r}_2 + \omega_{21}^\times r_2)$

$$\dot{\vec{r}} = \vec{\mathcal{F}}_1^T \dot{r}_1 = \vec{\mathcal{F}}_2^T [\dot{r}_2 + \omega_{21}^\times r_2]$$

$\vec{\mathcal{F}}_2^T = \vec{\mathcal{F}}_1^T C_{12}$

$\vec{\mathcal{F}}_2^T C_{21} = \vec{\mathcal{F}}_1^T$

$\vec{\mathcal{F}}_1^T = \cancel{C_{21} \vec{\mathcal{F}}_2^T}$

$$= \vec{\mathcal{F}}_1^T C_{12} [\dot{r}_2 + \omega_{21}^\times r_2]$$

$$\Rightarrow \dot{r}_1 = C_{12} [\dot{r}_2 + \omega_{21}^\times r_2]$$

use this to find $\underset{\sim}{\dot{C}_{21}}$

Let $\vec{r}$ = const. vector in $\vec{\mathcal{F}}_1$

$$\Rightarrow \dot{\vec{r}} = 0 \quad \dot{\vec{r}} = \vec{\mathcal{F}}_1^T \dot{r}_1 = 0$$

$$\Rightarrow \dot{r}_2 + \omega_{21}^\times r_2 = 0$$

$$r_2 = C_{21} r_1$$

$$\Rightarrow \dot{C}_{21} r_1 + \cancel{C_{21} \dot{r}_1}^{\,0} + \omega_{21}^\times C_{21} r_1 = 0$$

$$\Rightarrow [\dot{C}_{21} + \omega_{21}^\times C_{21}] r_1 = 0$$

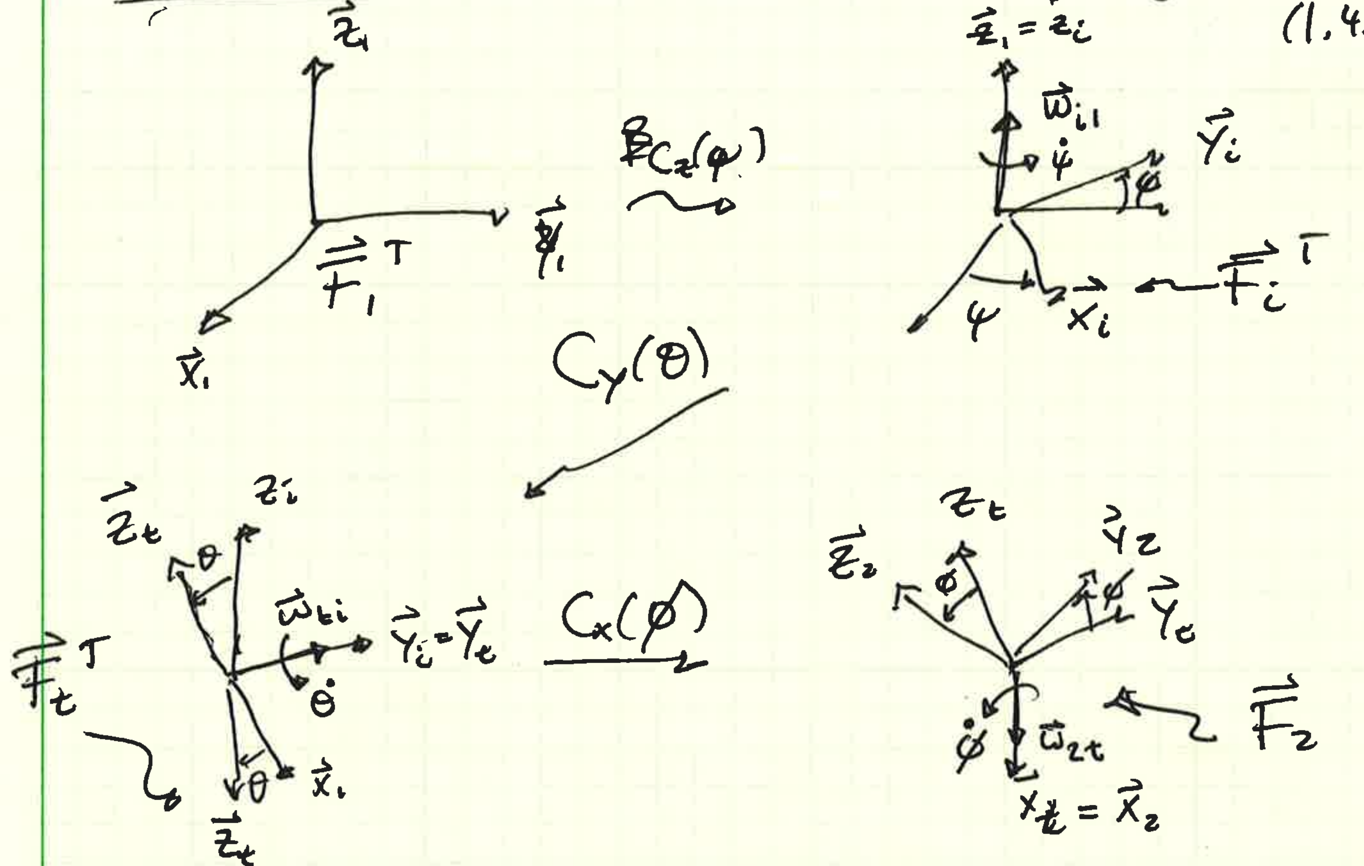
$$\Rightarrow \boxed{\dot{C}_{21} = -\omega_{21}^\times C_{21}}$$

since $\vec{r}$ is arbitrary

and $\boxed{\omega_{21}^\times = -\dot{C}_{21} C_{21}^T = -\dot{C}_{21} C_{12}}$

Angular Velocity in Terms of Euler Angle Rates

(1.4.2)



so, $$\vec{\omega}_{i1} = \vec{F}_i^T \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \vec{F}_2^T C_x(\phi) C_y(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

angular velocity of i frame wrt 1 frame in $\vec{F}_2^T$ coords

$$\vec{\omega}_{ti} = \vec{F}_t^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} = \vec{F}_2 C_x(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$\vec{\omega}_{2t} = \vec{F}_2^T \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

and $$\vec{\omega}_{21} = \cancel{\vec{\omega}_{tt}} + \vec{\omega}_{2t} + \vec{\omega}_{ti} + \omega_{i1}$$

$$\Rightarrow \omega_{21} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + C_x(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + C_x(\phi) C_y(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\Rightarrow \omega_{21} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{c\theta} \begin{bmatrix} c\theta & s\phi s\theta & c\phi s\theta \\ 0 & c\phi c\theta & -s\phi c\theta \\ 0 & s\phi & c\phi \end{bmatrix} \omega_{21}$$

<u>Notice</u> $\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$ is undefined when $\theta = \pm 90^\circ$

Angular Velocity in terms of Quaternion Rates

(1.4.3)

$$\omega_{21}^{\times} = -\dot{C}_{21} C_{21}^{T}$$

$$C_{21} = (2\eta^2 - 1)\mathbf{1} + 2\epsilon\epsilon^T - 2\eta\epsilon^{\times}$$

<u>but</u> $\epsilon^{\times}\epsilon^{\times} = \epsilon\epsilon^T - \epsilon^T\epsilon\mathbf{1}$

and $\epsilon^T\epsilon + \eta^2 = 1 \Rightarrow \eta^2 = 1 - \epsilon^T\epsilon$

$$\Rightarrow \cancel{2(\epsilon^T\epsilon + \eta^2}$$

$$\Rightarrow \cancel{2(1-\epsilon^T\epsilon)\mathbf{1} + 2\epsilon\epsilon^T - 2\eta\epsilon^{\times}}$$

$$\cancel{= 2 - 2\epsilon^T\epsilon\mathbf{1}}$$

$$\cancel{= 2(\mathbf{1} - \underbrace{\epsilon^T\epsilon\mathbf{1} + \epsilon\epsilon^T}_{\epsilon^{\times}\epsilon^{\times}}) - 2\eta\epsilon^{\times}}$$

$$\Rightarrow C_{21} = \left[2(1-\epsilon^T\epsilon) - 1\right]\mathbf{1} + 2\epsilon\epsilon^T - 2\eta\epsilon^{\times}$$

$$= \left[1 - 2\epsilon^T\epsilon\right]\mathbf{1} + 2\epsilon\epsilon^T - 2\eta\epsilon^{\times}$$

$$= \mathbf{1} + 2(\epsilon\epsilon^T - \epsilon^T\epsilon\mathbf{1}) - 2\eta\epsilon^{\times}$$

$$\underline{C_{21} = \mathbf{1} + 2\epsilon^{\times}\epsilon^{\times} - 2\eta\epsilon^{\times}}$$

$$\Rightarrow \dot{C}_{21} = 2\left[\dot{\epsilon}^{\times}\epsilon^{\times} + \epsilon^{\times}\dot{\epsilon}^{\times} - \dot{\eta}\epsilon^{\times} - \eta\dot{\epsilon}^{\times}\right]$$

<u>now</u> $$\underset{\uparrow}{\tfrac{1}{2}}\,\dot{C}_{21}C_{21} = \underset{\uparrow}{(\epsilon^{\times}\dot{\epsilon})^{\times}} - \eta\dot{\epsilon}^{\times} + \dot{\eta}\epsilon^{\times}$$

<u>comes in handy</u> ~~comes in handy~~

lots of algebra...

$$\Rightarrow \omega_{21}^{\times} = -2\left[(\epsilon^{\times}\dot{\epsilon})^{\times} - \eta\dot{\epsilon}^{\times} + \dot{\eta}\epsilon^{\times}\right]$$

or $$\omega_{21} = 2(\eta\mathbb{1} - \epsilon^{\times})\dot{\epsilon} - 2\epsilon\dot{\eta}$$

$$\Rightarrow \begin{bmatrix}\omega_{21}\\ 0\end{bmatrix} = 2\begin{bmatrix}(\eta\mathbb{1} - \epsilon^{\times}) & -\epsilon\\ \underline{\epsilon^{T}} & \underline{\eta}\end{bmatrix}\begin{bmatrix}\dot{\epsilon}\\ \dot{\eta}\end{bmatrix}$$

$\downarrow$

Since $\epsilon^{T}\epsilon + \eta^{2} = 1$

$$\Rightarrow \frac{d}{dt}(\epsilon^{T}\epsilon + \eta^{2}) = 0$$

$$\Rightarrow \underbrace{\dot{\epsilon}^{T}\epsilon + \epsilon^{T}\dot{\epsilon}}_{=\text{ to each other}} + 2\eta\dot{\eta} = 0$$

$$\Rightarrow 2\epsilon^{T}\dot{\epsilon} + 2\eta\dot{\eta} = 0$$

$$\Rightarrow \underline{\epsilon^{T}\dot{\epsilon} + \eta\dot{\eta} = 0}$$

Now inverting

$$\Rightarrow \begin{bmatrix} \dot{\epsilon} \\ \dot{\eta} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (\eta \mathbb{1} + \epsilon^{\times}) & \epsilon \\ -\epsilon^T & \eta \end{bmatrix} \begin{bmatrix} \omega_{21} \\ 0 \end{bmatrix}$$

or

$$\left.\begin{aligned} \dot{\epsilon} &= \frac{1}{2}(\eta \mathbb{1} + \epsilon^{\times})\omega_{21} \\ \dot{\eta} &= -\frac{1}{2}\epsilon^T \omega_{21} \end{aligned}\right\}$$ no trig and no singularity

How will we use this?

Ex given $\vec{\omega}_{21} = \text{const} = \vec{\mathcal{F}}_1^T \begin{bmatrix} .5 \\ +.5 \\ 1 \end{bmatrix} \frac{\text{rad}}{\text{s}}$

and $\psi_0 = 10^\circ \quad \theta_0 = -5^\circ \quad \phi_0 = 20^\circ$ – initial conditions

$$\Rightarrow C_{21_0} = C_x(\phi_0)\, C_y(\theta_0)\, C_z(\psi_0)$$
$$= C_x(20^\circ)\, C_y(-5^\circ)\, C_z(10^\circ)$$
$$=$$

Matlab Code

Find ϵ_o and η_o

$$\eta_o = \sqrt{\frac{\text{trace}(C_{21o}) + 1}{2}}$$

$$\epsilon_{1o} = \frac{C_{23o} - C_{32o}}{4\eta_o}$$

$$\epsilon_{2o} = \frac{C_{31o} - C_{13o}}{4\eta_o} \qquad \epsilon_o = \begin{bmatrix} \epsilon_{1o} \\ \epsilon_{2o} \\ \epsilon_{3o} \end{bmatrix}$$

$$\epsilon_{3o} = \frac{C_{12o} - C_{21o}}{4\eta_o}$$

$\eta_o =$ $\epsilon_o =$

~~Fto~~ Find $\epsilon(t), \eta(t), \psi, \theta, \phi(t)$

<u>need</u> to integrate use ODE 45