Rotations / Direction Cosine Matrix and Euler Angles

Rotations Matrices

What they Do?!

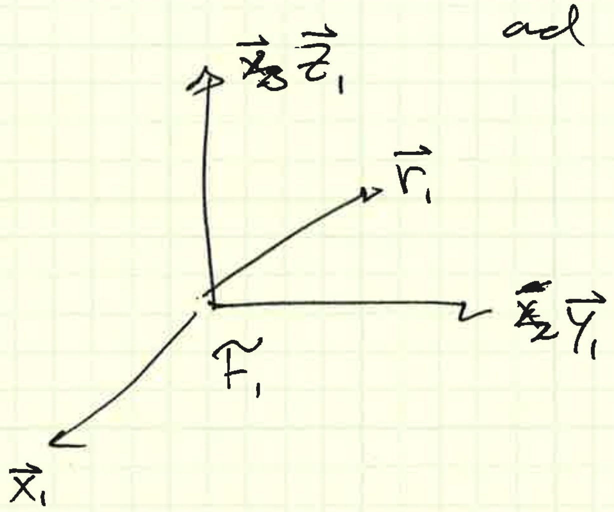
1. Describe the orientation of one reference frame w.r.t. another
2. Transform the cordinates of a vector from one reference frame to another.

Def: Vectrix

$$\vec{F}_1 = \begin{bmatrix} \vec{x}_1 \\ \vec{y}_1 \\ \vec{z}_1 \end{bmatrix}$$

describes the reference frame F_1 and vector in F_1

i.e.



$$\cancel{\vec{r}_1 = r_{x,1}\,\vec{x}_1 + r_{y,1}\,x_2}$$

$$\vec{r}_1 = r_{1x}\,\vec{x}_1 + r_{1y}\,\vec{y}_1 + r_{1z}\,\vec{z}_1$$

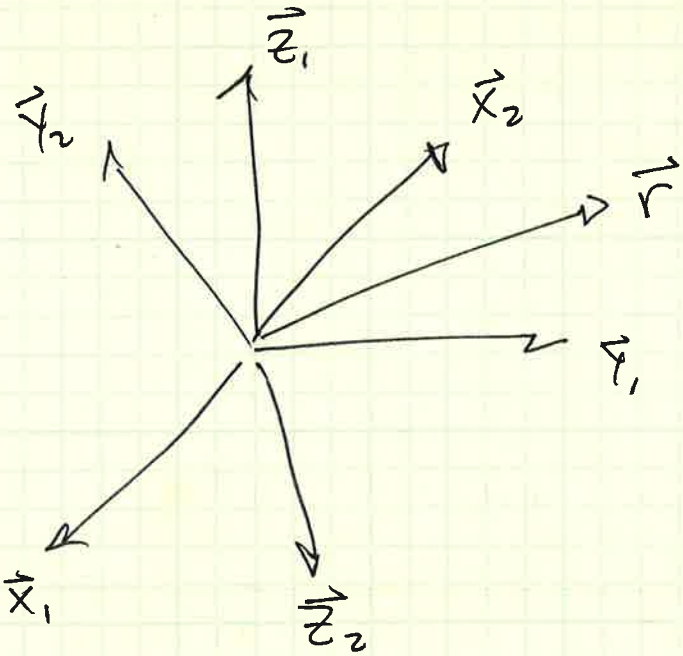
i.e. $\vec{r}_1 = \left[\vec{x}_1 \ \ \vec{y}_1 \ \ \vec{z}_1 \right] \begin{bmatrix} r_{1x} \\ r_{1y} \\ r_{1z} \end{bmatrix}$

$= \vec{\mathcal{F}}_1^{T} \, r_1$

vectrix of ref frame

components of r_1 in $\mathcal{F}_1$ frame

given two reference frames, $\vec{\mathcal{F}}_1$ and $\mathcal{F}_2$



$\Rightarrow \vec{r} = \vec{\mathcal{F}}_1^{T} r_1 = \vec{\mathcal{F}}_2^{T} r_2$

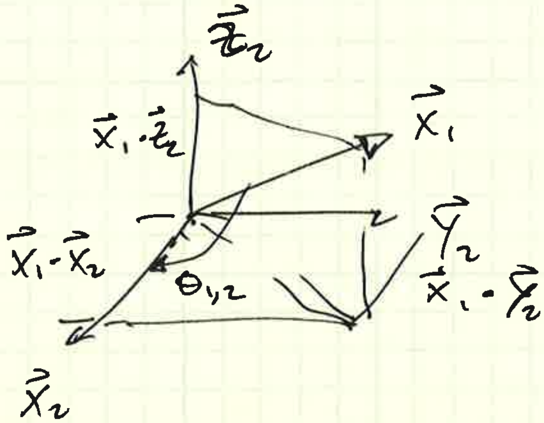
frame 2

components of r in frame $\mathcal{F}_2$

same vector

rotation relates the two

Q: How to express $\vec{x}_1$ in $\vec{\mathcal{F}}_2$?



i.e. $\vec{x}_1 = \vec{\mathcal{F}}_2^T \, x_{1,2}$

$$x_{1,2} = \begin{bmatrix} \vec{x}_1 \cdot \vec{x}_2 \\ \vec{x}_1 \cdot \vec{y}_2 \\ \vec{x}_1 \cdot \vec{z}_2 \end{bmatrix}$$

recall

$\vec{x}_1 \cdot \vec{x}_2 = \|\vec{x}_1\| \, \|\vec{x}_2\| \cos\theta_{1,2}$

and $\|\vec{x}_1\| = \|\vec{x}_2\| = 1$

$\Rightarrow \vec{x}_1 \cdot \vec{x}_2 = \cos\theta_{1,2}$.

Similarly

$\vec{y}_1 \cancel{\vec{x}_2} = \vec{\mathcal{F}}_2^T \, y_{1,2}$

$\cancel{z_2} \, \vec{z}_1 = \vec{\mathcal{F}}_2^T \, z_{1,2}$

Now, Point #2

since $\vec{r} = \vec{\mathcal{F}}_1^T r_1 = \vec{\mathcal{F}}_2^T r_2$

$$\Rightarrow \vec{\mathcal{F}}_2^T r_2 = \vec{\mathcal{F}}_1^T r_1$$

$$\Rightarrow \underbrace{\vec{\mathcal{F}}_2 \cdot \vec{\mathcal{F}}_2^T}_{= \mathbb{1} \text{ - you do!}} r_2 = \vec{\mathcal{F}}_2 \cdot \vec{\mathcal{F}}_1^T r_1$$

$$\Rightarrow r_2 = \vec{\mathcal{F}}_2 \cdot \vec{\mathcal{F}}_1^T r_1$$

Define $\quad C_{21} = \vec{\mathcal{F}}_2 \cdot \vec{\mathcal{F}}_1^T$

$$= \begin{bmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{bmatrix} \cdot \begin{bmatrix} \vec{x}_1 & \vec{y}_1 & \vec{z}_1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vec{x}_2 \cdot \vec{x}_1 & \vec{x}_2 \cdot \vec{y}_1 & \vec{x}_2 \cdot \vec{z}_1 \\ \vec{y}_2 \cdot \vec{x}_1 & \vec{y}_2 \cdot \vec{y}_1 & \vec{y}_2 \cdot \vec{z}_1 \\ \vec{z}_2 \cdot \vec{x}_1 & \vec{z}_2 \cdot \vec{y}_1 & \vec{z}_2 \cdot \vec{z}_1 \end{bmatrix}}_{\text{Direction Cosine Matrix}}$$

C_{21} addresses both 1 and 2

since it 1. describes the orientation between two frames and

2. relates ~~determines~~ ~~coor~~ components of $\vec{r}$ in both frames

Notice

$$\cancel{C_{21}^T C_{21}} =$$

$$C_{21} = \begin{bmatrix} x_{1,2} & y_{1,2} & z_{1,2} \end{bmatrix}$$

$$\Rightarrow C_{21}^T C_{21} = \begin{bmatrix} x_{1,2}^T \\ y_{1,2}^T \\ z_{1,2}^T \end{bmatrix} \begin{bmatrix} x_{1,2} & y_{1,2} & z_{1,2} \end{bmatrix}$$

$$= \begin{bmatrix} x_{1,2}^T x_{1,2} & x_{1,2}^T y_{1,2} & x_{1,2}^T z_{1,2} \\ & \text{etc} & \end{bmatrix}$$

$$x_{1,2}^T x_{1,2} = \begin{bmatrix} (\vec{x}_1 \cdot \vec{x}_2)^T & (\vec{x}_1 \cdot \vec{y}_2)^T & (\vec{x}_1 \cdot \vec{z}_2)^T \end{bmatrix} \begin{bmatrix} \vec{x}_1 \cdot \vec{x}_2 \\ \vec{x}_1 \cdot \vec{y}_2 \\ \vec{x}_1 \cdot \vec{z}_2 \end{bmatrix}$$

$$= (\vec{x}_1 \cdot \vec{x}_2)^2 + (\vec{x}_1 \cdot \vec{y}_2)^2 + (\vec{x}_1 \cdot \vec{z}_2)^2$$

sum of ~~ea~~ squares of $\vec{x}_1$ in frame $\mathcal{F}_2$ but $\vec{x}_1$ is a unit vector

$$\Rightarrow x_{1,2}^T x_{1,2} = 1$$

likewise $x_{1,2}^T y_{1,2} = 0$

$$\Rightarrow C_{21}^T C_{21} = \mathbb{1} \Rightarrow \underline{C_{21}^T = C_{21}^{-1}}$$

unitary

Therefore

$$r_2 = C_{21}\, r_1$$

$$\Rightarrow C_{21}^{-1}\, r_2 = C_{21}^{-1}\, C_{21}\, r_1 = \mathbb{1} r_1 = r_1$$

$$\Rightarrow r_1 = C_{21}^{-1}\, r_2 = C_{21}^{T}\, r_2$$

$$\underline{r_1 = C_{21}^{T}\, r_2}$$

Now given three frames $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$

$$\Rightarrow r_3 = C_{31}\, r_1$$

and

$$r_3 = C_{32} r_2 = C_{32}\, C_{21}\, r_1$$

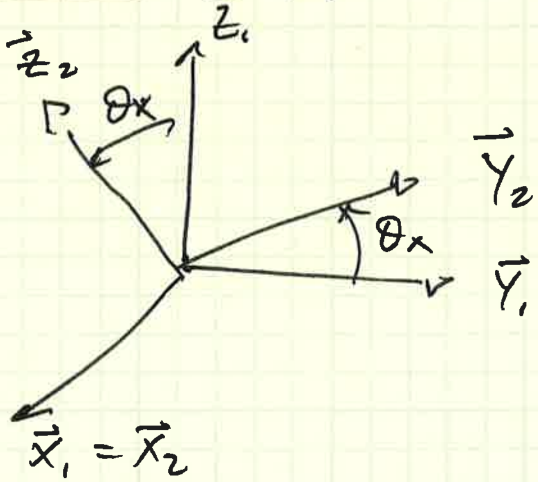
$$\Rightarrow C_{31} = C_{32}\, C_{21}$$

successive Rotations

Principal Rotations

- Goal: Simplify the general DCM into a rotation about a basis vector and then use in a sequence of rotations.

About $\vec{X}_1$



$\vec{Y}_1$ and $\vec{z}_1$ stay in same plane and rotate to $\vec{Y}_2$ and $\vec{z}_2$

$\Rightarrow (\vec{x}_1 = \vec{x}_2) \perp (\vec{Y}_2 \text{ and } \vec{z}_2)$

Define
$$\Rightarrow C_x(\theta_x) = \begin{bmatrix} \vec{x}_2 \cdot \vec{x}_1 & \vec{x}_2 \cdot \vec{Y}_1 & \vec{x}_2 \cdot \vec{z}_1 \\ \vec{Y}_2 \cdot \vec{x}_1 & \vec{Y}_2 \cdot \vec{Y}_1 & \vec{Y}_2 \cdot \vec{z}_1 \\ \vec{z}_2 \cdot \vec{x}_1 & \vec{z}_2 \cdot \vec{Y}_1 & \vec{z}_2 \cdot \vec{z}_1 \end{bmatrix}$$

clearly $\vec{x}_2 \cdot \vec{x}_1 = 1$ and $\vec{x}_2 \cdot \vec{Y}_1 = \vec{x}_2 \cdot \vec{z}_1 = \vec{Y}_2 \cdot \vec{x}_1 = \vec{z}_2 \cdot \vec{x}_1 = 0$

$\vec{Y}_2 \cdot \vec{Y}_1 = \cos\theta_x$ $\quad\quad$ $\vec{Y}_2 \cdot \vec{z}_1 = \cos(\pi/2 - \theta_x) = \sin\theta_x$

$\vec{z}_2 \cdot \vec{z}_1 = \cos\theta_x$ $\quad\quad$ $\vec{z}_2 \cdot \vec{Y}_1 = \cos(\pi/2 + \theta_x) = -\sin\theta_x$

$$\Rightarrow C_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}$$

likewise

$$C_y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

and

$$C_z(\theta_z) = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$