Appendix

We give 9 classes of elliptic curves with complex multiplications. Each elliptic curve is associated to a quadratic imaginary field of class number 1. There are a total of 13 quadratic imaginary fields of class number 1, and we complete the calculation of the remaining 9 cases. In addition to the 4 cases shown in the introduction, these 13 classes of elliptic curves all have complex multiplications over quadratic imaginary fields.

A N=-1

$E_{1,2}$ is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2 + 3xy = x^3 - x/98 - 1/392,$$

where $p > 3$ is a prime such that -1 is a quadratic residue modulo p. There is a map Φ as an endomorphism of $E_{1,2}$, and Φ satisfies the equation

$$\Phi^2 + 4 = 0.$$

The expression of its x-coordinate is

$$x(\Phi(x,y)) = \frac{x^4 + \omega_3 x^3 + \omega_2 x^2 + \omega_1 x + \omega_0}{\tau_3 x^3 + \tau_2 x^2 + \tau_1 x + \tau_0},$$

where

$$\omega_3 = \frac{20}{7},\ \omega_2 = \frac{96}{49},\ \omega_1 = \frac{107}{343},\ \omega_0 = \frac{193}{19208},$$

$$\tau_3 = -4,\ \tau_2 = \frac{25}{7},\ \tau_1 = -\frac{6}{7},\ \tau_0 = \frac{9}{343}.$$

Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z} + 2\sqrt{-1}\mathbb{Z} \subseteq \mathbb{Q}(\sqrt{-1})$.

B N=-3

$E_{3,2}$ is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2 + 2xy = x^3 - 4x/363 - 4/3267,$$

where $p > 3$ is a prime such that -3 is a quadratic residue modulo p. There is a map Φ as an endomorphism of $E_{3,2}$, and Φ satisfies the equation

$$\Phi^2 + 3 = 0.$$

The expression of its x-coordinate is

$$x(\Phi(x,y)) = \frac{x^3 + \omega_2 x^2 + \omega_1 x + \omega_0}{\tau_2 x^2 + \tau_1 x + \tau_0},$$

where

$$\omega_2 = \frac{12}{11},\ \omega_1 = \frac{8}{33},\ \omega_0 = \frac{304}{35937},$$
$$\tau_2 = -3,\ \tau_1 = -\frac{8}{11},\ \tau_0 = -\frac{16}{363}.$$

Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\sqrt{-3}] \subseteq \mathbb{Q}(\sqrt{-3})$.

$E_{3,3}$ is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2 + 2xy = x^3 + 3x/64009 + 1/192027,$$

where $p > 3$ is a prime such that -3 is a quadratic residue modulo p. There is a map Φ as an endomorphism of $E_{3,3}$, and Φ satisfies the equation

$$\Phi^2 - \Phi + 7 = 0.$$

The expression of its x-coordinate is a rational function over $\mathbb{Q}[\sqrt{-3}]$ of degree 7, and the maximum height of the expression is about 10^{17}. Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\frac{1+3\sqrt{-3}}{2}] \subseteq \mathbb{Q}(\sqrt{-3})$.

C N=-7

$E_{7,2}$ is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2 + 3xy = x^3 - 4x/22743 - 1/22743,$$

where $p > 3$ is a prime such that -7 is a quadratic residue modulo p. There is a map Φ as an endomorphism of $E_{7,2}$, and Φ satisfies the equation

$$\Phi^2 + 7 = 0.$$

The expression of its x-coordinate is a rational function over $\mathbb{Q}[\sqrt{-7}]$ of degree 7, and the maximum height of the expression is about $6 \cdot 10^{14}$. Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\sqrt{-7}] \subseteq \mathbb{Q}(\sqrt{-7})$.

D N=-11

E_{11} is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2 + 2xy = x^3 + 9x/539 + 1/539,$$

where $p > 3$ is a prime such that -11 is a quadratic residue modulo p. There is a map Φ as an endomorphism of E_{11}, and Φ satisfies the equation

$$\Phi^2 - \Phi + 3 = 0.$$

The expression of its x-coordinate is

$$x(\Phi(x,y)) = \frac{x^3+\omega_2 x^2+\omega_1 x+\omega_0}{\tau_1(x+\tau_0)^2},$$

where

$$\omega_2 = \frac{15}{14}(1+\frac{1}{\sqrt{-11}}), \omega_1 = \frac{1}{7}(\frac{8}{7}+\frac{5}{\sqrt{-11}}), \omega_0 = \frac{1}{1078}(-\frac{47}{7}+\frac{43}{\sqrt{-11}}),$$

$$\tau_1 = -\frac{5}{2}+\frac{1}{2}\sqrt{-11}, \tau_0 = -\frac{1}{21}(1+\frac{8}{\sqrt{-11}}).$$

Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\frac{1+\sqrt{-11}}{2}] \subseteq \mathbb{Q}(\sqrt{-11})$.

E N=-19

E_{19} is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2+6xy = x^3+x/19+1/19,$$

where $p>3$ is a prime such that -19 is a quadratic residue modulo p. There is a map Φ as an endomorphism of E_{19}, and Φ satisfies the equation

$$\Phi^2-\Phi+5=0.$$

The expression of its x-coordinate is

$$x(\Phi(x,y)) = \frac{x^5+\omega_4 x^4+\omega_3 x^3+\omega_2 x^2+\omega_1 x+\omega_0}{\tau_2(x^2+\tau_1 x+\tau_0)^2},$$

where

$$\omega_4 = \frac{25}{2}(1-\frac{1}{\sqrt{-19}}), \omega_3 = \frac{2}{19}(346+59\sqrt{-19}), \omega_2 = \frac{1}{19}(197-\frac{3897}{\sqrt{-19}}),$$

$$\omega_1 = \frac{3}{361}(-1227+262\sqrt{-19}), \omega_0 = \frac{1}{722}(-583+\frac{1583}{\sqrt{-19}}),$$

$$\tau_2 = -\frac{9}{2}-\frac{1}{2}\sqrt{-19}, \tau_1 = -2+\frac{8}{\sqrt{-19}}, \tau_0 = \frac{1}{95}(-17+8\sqrt{-19}).$$

Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}] \subseteq \mathbb{Q}(\sqrt{-19})$.

F N=-43

E_{43} is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2+6xy = x^3+x/18963+1/18963,$$

where $p > 3$ is a prime such that -43 is a quadratic residue modulo p. There is a map Φ as an endomorphism of E_{43}, and Φ satisfies the equation

$$\Phi^2 - \Phi + 11 = 0.$$

The expression of its x-coordinate is

$$x(\Phi(x,y)) = \frac{x^{11} + \omega_{10}x^{10} + \cdots + \omega_1 x + \omega_0}{\tau_5(x^5 + \cdots + \tau_1 x + \tau_0)^2},$$

where

$$\begin{aligned}
\omega_{10} &= \frac{263}{14} + \frac{263}{602}\sqrt{-43}, \omega_9 = \frac{2199520}{18963} + \frac{130775}{18963}\sqrt{-43},\\
\omega_8 &= \frac{30067825}{113778} + \frac{1148653115}{34247178}\sqrt{-43},\\
\omega_7 &= \frac{60040482310}{359595369} + \frac{757448660}{13318347}\sqrt{-43},\\
\omega_6 &= -\frac{284663286931}{7551502749} + \frac{230282009695}{7551502749}\sqrt{-43},\\
\omega_5 &= -\frac{228492419340116}{6819006982347} + \frac{24972694394458}{6819006982347}\sqrt{-43},\\
\omega_4 &= -\frac{456831669840485}{143199146629287} - \frac{904232713800025}{6157563305059341}\sqrt{-43},\\
\omega_3 &= -\frac{2938587221712575}{129308829406246161} - \frac{2047708723069060}{129308829406246161}\sqrt{-43},\\
\omega_2 &= \frac{5995904195552465}{5430970835062338762} - \frac{26014665317061535}{233531745907680566766}\sqrt{-43},\\
\omega_1 &= \frac{3740165573715604}{8173611106768819836 81} + \frac{36364879086965}{570251937681545557001}\sqrt{-43},\\
\omega_0 &= -\frac{172145695940399}{343291666484290433146 02} + \frac{726502439497649}{6326374996639066553690 94}\sqrt{-43}.
\end{aligned}$$

$$\begin{aligned}
\tau_5 &= -\frac{21}{2} - \frac{1}{2}\sqrt{-43}, \tau_4 = -\frac{55}{7} - \frac{160}{301}\sqrt{-43},\\
\tau_3 &= \frac{8510}{2107} + \frac{19840}{18963}\sqrt{-43}, \tau_2 = \frac{140050}{398223} - \frac{2599360}{17123589}\sqrt{-43},\\
\tau_1 &= -\frac{7606115}{359595369} + \frac{100480}{359595369}\sqrt{-43},\\
\tau_0 &= \frac{187051}{11866647177} + \frac{33232480}{3571860800277}\sqrt{-43}.
\end{aligned}$$

Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\frac{1+\sqrt{-43}}{2}] \subseteq \mathbb{Q}(\sqrt{-43})$.

G N=-67

E_{67} is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2 + 6xy = x^3 + x/315496 + 1/315496,$$

where $p > 3$ is a prime such that -67 is a quadratic residue modulo p. There is a map Φ as an endomorphism of E_{67}, and Φ satisfies the equation

$$\Phi^2 - \Phi + 17 = 0.$$

The expression of its x-coordinate is

$$x(\Phi(x,y)) = \frac{x^{17} + \omega_{16}x^{16} + \cdots + \omega_1 x + \omega_0}{\tau_8(x^8 + \cdots + \tau_1 x + \tau_0)^2},$$

where

$$\begin{aligned}
\omega_{16} &= \frac{10177}{15624} - \frac{-10177}{1046808}\sqrt{-67}, \\
\omega_{15} &= \frac{1882495}{12169143} - \frac{-2822527}{511104006}\sqrt{-67}, \\
\omega_{14} &= \frac{18716450215}{1140784141392} - \frac{197024968895}{178342587437616}\sqrt{-67}, \\
\omega_{13} &= \frac{1062978534070715}{1393212293062656192} - \frac{199399500045415}{2089818439593984288}\sqrt{-67}, \\
\omega_{12} &= \frac{113929571430443393}{93289495143475458616 32} - \frac{2256850154487549545}{6250396174612855727 29344}\sqrt{-67}, \\
\omega_{11} &= -\frac{86274156514910793383}{21362291525783087661957 15456} - \frac{481622501544544846919}{854491661031323506478 2861824}\sqrt{-67}, \\
\omega_{10} &= -\frac{268809676521587983832105}{2^{12}\cdot 3^{14}\cdot 7^{7}\cdot 31^{7}\cdot 67^{3}} - \frac{1035737221957854967131 13}{2^{12}\cdot 3^{11}\cdot 7^{7}\cdot 31^{7}\cdot 67^{4}}\sqrt{-67}, \\
\omega_{9} &= -\frac{149306000362000801046902 5865}{2^{15}\cdot 3^{16}\cdot 7^{8}\cdot 31^{8}\cdot 67^{4}} - \frac{1044914266338378540495 3415}{2^{13}\cdot 3^{16}\cdot 7^{8}\cdot 31^{8}\cdot 67^{4}}\sqrt{-67}, \\
\omega_{8} &= -\frac{3255605708054190132971524 5085}{2^{18}\cdot 3^{18}\cdot 7^{9}\cdot 31^{9}\cdot 67^{4}} + \frac{1638967629254402607632920 93805}{2^{18}\cdot 3^{18}\cdot 7^{9}\cdot 31^{9}\cdot 67^{5}}\sqrt{-67}, \\
\omega_{7} &= -\frac{13786072854730032339520129 4209}{2^{16}\cdot 3^{20}\cdot 7^{10}\cdot 31^{10}\cdot 67^{5}} + \frac{227815053803163918343117 379195}{2^{17}\cdot 3^{20}\cdot 7^{10}\cdot 31^{10}\cdot 67^{5}}\sqrt{-67}, \\
\omega_{6} &= \frac{764242260879686114638417 1451959}{2^{20}\cdot 3^{22}\cdot 7^{11}\cdot 31^{11}\cdot 67^{5}} + \frac{648153308535564349150390 3149629}{2^{20}\cdot 3^{20}\cdot 7^{11}\cdot 31^{11}\cdot 67^{6}}\sqrt{-67}, \\
\omega_{5} &= \frac{86843489527010518196559042 0535765}{2^{22}\cdot 3^{24}\cdot 7^{12}\cdot 31^{12}\cdot 67^{6}} - \frac{169895707737237935267888 60137997}{2^{21}\cdot 3^{24}\cdot 7^{12}\cdot 31^{12}\cdot 67^{6}}\sqrt{-67},
\end{aligned}$$

$$
\begin{aligned}
\omega_4 &= \frac{24632331456873410798586472800 55}{2^{25}\cdot 3^{26}\cdot 7^{11}\cdot 31^{13}\cdot 67^{6}} - \frac{30225027782729787796954078592 91205}{2^{25}\cdot 3^{25}\cdot 7^{13}\cdot 31^{13}\cdot 67^{7}}\sqrt{-67},\\
\omega_3 &= -\frac{2009816680108174844097509789 11495}{2^{22}\cdot 3^{27}\cdot 7^{14}\cdot 31^{14}\cdot 67^{7}} - \frac{29671351468383725986940987 90585}{2^{25}\cdot 3^{27}\cdot 7^{14}\cdot 31^{13}\cdot 67^{7}}\sqrt{-67},\\
\omega_2 &= -\frac{88336274331413394870527704555 9183}{2^{28}\cdot 3^{30}\cdot 7^{15}\cdot 31^{15}\cdot 67^{7}} + \frac{88590326538507732083894750445 84235}{2^{28}\cdot 3^{30}\cdot 7^{15}\cdot 31^{15}\cdot 67^{8}}\sqrt{-67},\\
\omega_1 &= \frac{66155955079307130317450012269 2181}{2^{32}\cdot 3^{31}\cdot 7^{15}\cdot 31^{16}\cdot 67^{8}} + \frac{82176395907063655924192740428 393}{2^{29}\cdot 3^{31}\cdot 7^{16}\cdot 31^{16}\cdot 67^{8}}\sqrt{-67},\\
\omega_0 &= \frac{69969441965367858040309942136 3905}{2^{35}\cdot 3^{34}\cdot 7^{17}\cdot 31^{17}\cdot 67^{8}} - \frac{65293837341178996806642924 4523}{2^{35}\cdot 3^{31}\cdot 7^{16}\cdot 31^{16}\cdot 67^{9}}\sqrt{-67}.\\
\tau_8 &= -\frac{33}{2} + \frac{1}{2}\sqrt{-67},\\
\tau_7 &= -\frac{788}{1953} + \frac{2090}{130851}\sqrt{-67},\\
\tau_6 &= \frac{18509959}{1022208012} - \frac{116215}{56789334}\sqrt{-67},\\
\tau_5 &= -\frac{66551449}{998186123718} + \frac{8789447315}{267513881156424}\sqrt{-67},\\
\tau_4 &= -\frac{54660888665}{170597423640325248} - \frac{147602383225}{2089818439593984288}\sqrt{-67},\\
\tau_3 &= \frac{309166783111}{906981202783789180992} + \frac{2101147971725}{3646064435190832 50758784}\sqrt{-67},\\
\tau_2 &= -\frac{379380114532829}{2^{10}\cdot 3^{12}\cdot 7^{6}\cdot 31^{6}\cdot 67^{3}} + \frac{1257520612865}{2^{9}\cdot 3^{9}\cdot 7^{6}\cdot 31^{6}\cdot 67^{3}}\sqrt{-67},\\
\tau_1 &= -\frac{5249111975179}{2^{8}\cdot 3^{14}\cdot 7^{7}\cdot 31^{7}\cdot 67^{3}} - \frac{415957234149655}{2^{11}\cdot 3^{14}\cdot 7^{7}\cdot 31^{7}\cdot 67^{4}}\sqrt{-67},\\
\tau_0 &= \frac{20140520885524937}{2^{16}\cdot 3^{16}\cdot 7^{8}\cdot 17\cdot 31^{8}\cdot 67^{4}} - \frac{64934358740185}{2^{13}\cdot 3^{15}\cdot 7^{8}\cdot 17\cdot 31^{8}\cdot 67^{4}}\sqrt{-67}.
\end{aligned}
$$

Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\frac{1+\sqrt{-67}}{2}] \subseteq \mathbb{Q}(\sqrt{-67})$.

H N=-163

E_{163} is an elliptic curve over $\mathbb{F}_p$ defined by

$$y^2 + 6xy = x^3 + x/5627087890963 + 1/5627087890963,$$

where $p > 3$ is a prime such that -163 is a quadratic residue modulo p. There is a map Φ as an endomorphism of E_{163}, and Φ satisfies the equation

$$\Phi^2 - \Phi + 41 = 0.$$

The expression of its x-coordinate is

$$x(\Phi(x, y)) = \frac{x^{41} + \omega_{40}x^{40} + \cdots + \omega_1 x + \omega_0}{\tau_{20}(x^{20} + \cdots + \tau_1 x + \tau_0)^2},$$

where

$$
\begin{aligned}
\omega_{40} =&(-13591409 \cdot \sqrt{-163} + 2215399667)/(2^{1} \cdot 7^{1} \cdot 11^{1} \cdot 19^{1} \cdot 127^{1} \cdot 163^{1}),\\
\omega_{39} =&(-42393653575900 \cdot \sqrt{-163} + 3123274361464280)/(7^{2} \cdot 11^{2} \cdot 19^{2} \cdot 127^{2} \cdot 163^{1}),\\
\omega_{38} =&(-17684210945738711298530 \cdot \sqrt{-163} + 7741626909936947214975 50)/(7^{3} \cdot 11^{3} \cdot 19^{3} \cdot 127^{3} \cdot 163^{2}),\\
\omega_{37} =&(-2413128857059911765618 8393580 \cdot \sqrt{-163}\\
&+ 6922558504827701657602365 85490)/(7^{4} \cdot 11^{4} \cdot 19^{4} \cdot 127^{4} \cdot 163^{2}),\\
\omega_{36} =&(-3126978434608739012677523546346733015 \cdot \sqrt{-163}\\
&+ 6082686779892163498276183723693395 1053)/(7^{5} \cdot 11^{5} \cdot 19^{5} \cdot 127^{5} \cdot 163^{3}),\\
\omega_{35} =&(-1495012811701605304902134751082457035795652 \cdot \sqrt{-163}\\
&+ 1951449125233089725552984443528220586443 8736)/(7^{6} \cdot 11^{6} \cdot 19^{6} \cdot 127^{6} \cdot 163^{3}),\\
\omega_{34} =&(-704135619017787745539119570709657599303338513 31650 \cdot \sqrt{-163}\\
&+ 57332704929168388956389397381899503051045523 2730910)/(7^{7} \cdot 11^{7} \cdot 19^{7} \cdot 127^{7} \cdot 163^{4}),\\
\omega_{33} =&(-12165431374402757537383249818002380817125442811707605620 \cdot \sqrt{-163}\\
&+ 481766673904355025635670037763063687813782806762193 01625)/(7^{8} \cdot 11^{8} \cdot 19^{8} \cdot 127^{8} \cdot 163^{4}),\\
\omega_{32} =&(-400504710464575701260272249252008191388737580218337254808751245 \cdot \sqrt{-163}\\
&- 3217724291568060424799009578178082614383255616481156 7864493065)\\
&/(2^{1} \cdot 7^{9} \cdot 11^{9} \cdot 19^{9} \cdot 127^{9} \cdot 163^{5}),\\
\omega_{31} =&(-1137801346493050676339870471054921194034409169200865466722519 7032560 \cdot \sqrt{-163}\\
&- 53759348426012610825616545134753954877176218911045471997620264704864)\\
&/(7^{10} \cdot 11^{10} \cdot 19^{10} \cdot 127^{10} \cdot 163^{5}),\\
\omega_{30} =&(-5502599815691912770053888081457389946783814061954637597534675214468 6047912 \cdot \sqrt{-163}\\
&- 6465821085731074963964915826773045855536875301977056226161859104344680 18024)\\
&/(7^{11} \cdot 11^{11} \cdot 19^{11} \cdot 127^{11}, 163^{6}),\\
\omega_{29} =&(-677776837273550012275378094366962057828414351517481198463965733220513493390000 \cdot \sqrt{-163}\\
&- 206873481678570761896904305452284941506422673491819982655057854417984731832008 40)\\
&/(7^{12} \cdot 11^{12} \cdot 19^{12}, 127^{12} \cdot 163^{6}),\\
\omega_{28} =&(312739032403867099070377502138434883418072005825063617217387276834731196363714914540 \cdot \sqrt{-163}\\
&- 529578250212364823750641004542311423419235001645188585998657012892773932622501216 40100)\\
&/(7^{13} \cdot 11^{13}, 19^{13} \cdot 127^{13} \cdot 163^{7}),\\
\omega_{27} =&(20649889367070372261463707416442744571800218347698588225365361819451848909948690515198640\cdot\\
&\sqrt{-163} - 396350427306695415945083431088502452631312941615135413463494643316693446876679739464\backslash\\
&158720)/(7^{14}, 11^{14} \cdot 19^{14} \cdot 127^{14} \cdot 163^{7}),\\
\omega_{26} =&(30452869780902885342191573742782219342534013045951988265943996335531248003856311376314205174 5\backslash\\
&20 \cdot \sqrt{-163} - 179382891291148299914880241341471606589029884024381866456171700636803717820901 68\backslash\\
&6753709231882248)/(7^{15} \cdot 11^{15} \cdot 19^{15} \cdot 127^{15} \cdot 163^{8}),\\
\omega_{25} =&(1123563231840760965500673078431698007740867889807672194626722847637481771956450839055485032 78\backslash\\
&830832 \cdot \sqrt{-163} + 229635294184089989348465402159194246381053014492780745990175955870001263755 5\backslash\\
&68303575148635864145154)/(7^{16} \cdot 11^{16} \cdot 19^{16} \cdot 127^{16} \cdot 163^{8}),\\
\omega_{24} =&(27235306729792856993346444894338662721807786673412529226539862997419324232376352711553253194\backslash\\
&200974558375 \cdot \sqrt{-163} + 35016471478940277905040968722449054235629458866701391266733827886071 38\backslash\\
&35475242883885062554723275018542475)/(7^{17} \cdot 11^{17} \cdot 19^{17} \cdot 127^{17} \cdot 163^{9}),\\
\omega_{23} =&(5566045631244876719044386870275374259032346376585377521806003681673084405227535755152543137 03\backslash\\
&9002237599800 \cdot \sqrt{-163} + 62728664230773631782261043700032874383116882984247033205262975264705 7\backslash\\
&861829068140859550701484982799329226000)/(7^{18} \cdot 11^{18} \cdot 19^{18} \cdot 127^{18} \cdot 163^{9}),
\end{aligned}
$$

$$
\begin{aligned}
\omega_{22} ={}&(-35463669218404024830792158249387930356057735985402984562129325517100843497591700445888197700\backslash\\
&381116979240478763100 \cdot \sqrt{-163} + 66893126664986036584305787280825714728076545894349311414473240\backslash\\
&51695055199532664112644744980078732439343136472473000)/(7^{19} \cdot 11^{19} \cdot 19^{19} \cdot 127^{19} \cdot 163^{10}),\\
\omega_{21} ={}&(-29668649996745911861870926189228140208688780370976785939196096022619246567593615380032287320\backslash\\
&672151172158040464722600 \cdot \sqrt{-163} + 884230536961191834180129225248875704129289063559879735889\backslash\\
&900866458712113839707138026867614767122845208335196886158 0)/(7^{20} \cdot 11^{20} \cdot 19^{20} \cdot 127^{20} \cdot 163^{10}),\\
\omega_{20} ={}&(-12460653114393894109189466939627081926220055404598053543634353635574546588148807725469061260\backslash\\
&94247467041891745703319253610 \cdot \sqrt{-163} - 98773348643447806131632055522719005740843006395982957\backslash\\
&22246713645087813805609431664584475718805520966914230248474685563 70)\\
&/(7^{21} \cdot 11^{21} \cdot 19^{21} \cdot 127^{21} \cdot 163^{11}),\\
\omega_{19} ={}&(-50629630938346591436918131397610603027602960171331554801191358892674532670138865551580692000\backslash\\
&302831616211078628153069139000 \cdot \sqrt{-163} - 33295850283499975159799728505393803567933577624052 3\backslash\\
&6031139234495856944312331036319036249268432345634032545235424342009048 00)\\
&/(7^{22} \cdot 11^{22} \cdot 19^{22} \cdot 127^{22} \cdot 163^{11}),\\
\omega_{18} ={}&(29871017357443961413897399821614962307845233246202252248074849471610376702055918076118034925 2\backslash\\
&3143494973238886091284792124553 00 \cdot \sqrt{-163} - 48129758060234650179811991535133670053573661296 01\backslash\\
&2387045692239287669607482729019730256796634155225455854423476692997033241735 00)\\
&/(7^{23} \cdot 11^{23} \cdot 19^{23} \cdot 127^{23} \cdot 163^{12}),\\
\omega_{17} ={}&(32759804453535425013316696432192019686069481925766846443549064884332259807363381972315909659 6\backslash\\
&240621243745672131328735546765338 00 \cdot \sqrt{-163} - 21735469399687377523520283732853741481287597706\backslash\\
&66412413480871662312079071649911158133979371976128971035854823591262043312621 50)\\
&/(7^{24} \cdot 11^{24} \cdot 19^{24} \cdot 127^{24} \cdot 163^{12}),\\
\omega_{16} ={}&(13188416459806530863673769779328568711440579729282944253572873330784116985619556775922466650 00\backslash\\
&2013909643797812624398680666537703077 5 \cdot \sqrt{-163} + 22960306569781087854767632110318320319020948 0\backslash\\
&90055472658315467583961870555639427223351642355796448820552023323583073804031221950362 99)\\
&/(7^{25} \cdot 11^{25} \cdot 19^{25} \cdot 127^{25} \cdot 163^{13}),\\
\omega_{15} ={}&(-19268717695742749266849348534533919718051474831574408831914055968379226960930051505596913609\backslash\\
&5425692979517226339778436219340313519121 6 \cdot \sqrt{-163} + 686392276762813443530088724056034640535 94\backslash\\
&75162885096719713911455563707971626561302697476301039857738093931328242970494064653331044060 8)\\
&/(7^{26} \cdot 11^{26} \cdot 19^{26} \cdot 127^{26} \cdot 163^{13}),\\
\omega_{14} ={}&(-14292024097876871207634901522809549503440580896674282526897613315635572512750053851388425520\backslash\\
&4328979917228514518062123869137093921500566600 \cdot \sqrt{-163} + 3916343868999507191094649874922715 39\backslash\\
&94943350242736139218758778774860487335793708151115776454482810740584142808745392187009597639268\backslash\\
&1718520)/(7^{27} \cdot 11^{27} \cdot 19^{27} \cdot 127^{27} \cdot 163^{14}),\\
\omega_{13} ={}&(-93714669852177694617392703171037416015913845422431225360782576180043078626382541802428289884\backslash\\
&284189638232893327445829255425197684861421040 \cdot \sqrt{-163} - 1340541947202568637232095148658911769\backslash\\
&62459821390780200264670570100090574974306164259605311378363149583778091702587443741959686583 42\backslash\\
&53918600)/(7^{28} \cdot 11^{28} \cdot 19^{28} \cdot 127^{28} \cdot 163^{14}),\\
\omega_{12} ={}&(43737902060964059666611814394935501529777998230769853780888286924711982107320722001369098781 6\backslash\\
&2121009036269366128554494973502636454976854980 \cdot \sqrt{-163} - 146931369526812438893931251840012494\backslash\\
&76639129001907388250097462788821381409291065828449294103660412801071714157409423292385792392 98\backslash\\
&30588664940)/(7^{29} \cdot 11^{28} \cdot 19^{29} \cdot 127^{29} \cdot 163^{15}),\\
\omega_{11} ={}&(48873713496032097318394550790854925442979488497323598312494652112377397494021393853925609836 2\backslash\\
&76210077604011723393896825628528073032178431448 0 \cdot \sqrt{-163} - 48429505391196471660902750966469 55\backslash\\
&12499070489753624949461394651448224818492631901225117790460779080101232186882157210748465179 50\backslash\\
&9666893439616)/(7^{30} \cdot 11^{30} \cdot 19^{30} \cdot 127^{30} \cdot 163^{15}),\\
\omega_{10} ={}&(81863436920762401927841734752438416006227302296444917414209169659951923366442939882468096041 3\backslash\\
&94565085583612119809994626253479970289415742680472 \cdot \sqrt{-163} + 2008303178100090198457777880023 8\backslash\\
&94867071335495241628193914655724579204098323080636357659296886051486650184839898279194665414 09\backslash\\
&835536603311723970 4)/(7^{31} \cdot 11^{31} \cdot 19^{31} \cdot 127^{31} \cdot 163^{16}),
\end{aligned}
$$

$$
\begin{aligned}
\omega_9 =&(-16030341664666587708255749474220911723059285506522545964439326473361650911398182139170742199\backslash\\
&716280512740598465124847717453608291021109326035720 0 \cdot \sqrt{-163} + 2302009156608973996536269708141\backslash\\
&527457365747035388845236537054381425912315480700920053491939552995903211678220383005289493228 06\backslash\\
&6772155090141365405)/(7^{32} \cdot 11^{32} \cdot 19^{32} \cdot 127^{32} \cdot 163^{16}),\\
\omega_8 =&(-603932704087292282325629063115161921224443820899375754976735668719875528517689460112121951 36\backslash\\
&3227641896436709974294799980610457471153874840602770 85 \cdot \sqrt{-163} - 3264993377442947113690689346 0\backslash\\
&66329676304032467316966152448077789932648631106680132953600087608335561458300623260661358668717\backslash\\
&9188302697762180257 5625)/(2^{1} \cdot 7^{33} \cdot 11^{33} \cdot 19^{33} \cdot 127^{33} \cdot 163^{17}),\\
\omega_7 =&(187349967933476470945959914635350317715023105463922651910269090412955172438517292468846888154 3\backslash\\
&344350106796509818288644493398967028553948631278020 \cdot \sqrt{-163} - 2124863236621356859862922275830 9\backslash\\
&565091452149715390067218405322218486308195755414309257276700278640649492515113983607937848488 39\backslash\\
&7272334382924905476 0)/(7^{34} \cdot 11^{34} \cdot 19^{34} \cdot 127^{34} \cdot 163^{17}),\\
\omega_6 =&(844461981138402116645598416796180191511303432502850061761438262935074115979900285965712430379 1\backslash\\
&6902939536831797104799711005328606451810969964118251 0 \cdot \sqrt{-163} - 1510847811554398967459532302 17\backslash\\
&519515477776482643496773360022724574269185379083030081466657954616838980785288844751555945422 22\backslash\\
&8881461519371411717 0466)/(7^{35} \cdot 11^{35} \cdot 19^{35} \cdot 127^{35} \cdot 163^{18}),\\
\omega_5 =&(563398430722510098360989103963785434640762313153774422313040991158038053489233484828065070974 9\backslash\\
&329622811036029191449698884429805058915133745580024 4 \cdot \sqrt{-163} + 1943449862084074161190663693520\backslash\\
&830252249606971906064688736785292387201117250942914101695504768077381712844010772571708769987 55\backslash\\
&4631134975130333695538)/(7^{36} \cdot 11^{36} \cdot 19^{36} \cdot 127^{36} \cdot 163^{18}),\\
\omega_4 =&(-374076887869613557085726086099291792165383301295167524010170908966781090711353778463845702 97\backslash\\
&620762043182443046847282831086222388142881020461177662 5 \cdot \sqrt{-163} + 2833223467050537298893217429\backslash\\
&182041067055359075862776211988274158962728003000325358159804992533151480387228587853023388527 51\backslash\\
&6373468823493809333737995)/(7^{36} \cdot 11^{37} \cdot 19^{37} \cdot 127^{37} \cdot 163^{19}),\\
\omega_3 =&(-131617359717040167344321745186549369851603071207108185451263544272417201833813489763246197 59\backslash\\
&878641095897259753598044733790892155887433052482750058 0 \cdot \sqrt{-163} - 2057895524785840945484907404\backslash\\
&599246199940830128151698729955094057343166848075071389460208723173404762762725106701981553728 38\backslash\\
&7063384364591191343618800)/(7^{38} \cdot 11^{38} \cdot 19^{38} \cdot 127^{38} \cdot 163^{19}),\\
\omega_2 =&(916641539523693925419938387755023216803448211227531006942103972607041300457655883401476149444 5\backslash\\
&230211795264995432364012725222801946245146446113521 10 \cdot \sqrt{-163} - 121965472273326194590278230128\backslash\\
&574675747248611440489182316056250986684153482937407127870521798941578880127973073012643166310 62\backslash\\
&4105911936053802011 96530)/(7^{39} \cdot 11^{39} \cdot 19^{39} \cdot 127^{39}, 163^{20}),\\
\omega_1 =&(211529643909572685923012239804290690166733285075680088610942088256531420568482542827939208488 0\backslash\\
&793775088571348400933846449054798580540242375479646 0 \cdot \sqrt{-163} + 2127370180334118541568682373094\backslash\\
&607689080366328585490020592903831055367789154735218189655302238710448507543810584673096211767 3\backslash\\
&9069675089543905737261)/(7^{40} \cdot 11^{40} \cdot 19^{40} \cdot 127^{40} \cdot 163^{20}),\\
\omega_0 =&(-397720474284275289254044208042139443448420597920835343144640989300000902503268850989865253 93\backslash\\
&828595197262124359116746729794689989797195041176154849 \cdot \sqrt{-163} + 7599400442237718983852265258 9\backslash\\
&362204044617510907808489184784905452012210440783591350200018935594183775647964714668477027038 84\backslash\\
&2572719394449814772 7347)/(2^{1} \cdot 7^{41} \cdot 11^{41} \cdot 19^{41} \cdot 127^{41} \cdot 163^{21}),\\
\tau_{20} =& -\frac{81}{2} + \frac{1}{2}\sqrt{-163},\\
\tau_{19} =&(19316320 \cdot \sqrt{-163} - 1331426380)/(7 \cdot 11 \cdot 19 \cdot 127 \cdot 163),\\
\tau_{18} =&(-65994021907360 \cdot \sqrt{-163} + 1945594373502410)/(7^{2} \cdot 11^{2} \cdot 19^{2} \cdot 127^{2} \cdot 163),\\
\tau_{17} =&(7632830161066999603040 \cdot \sqrt{-163} - 119347157393388490521700)/(7^{3} \cdot 11^{3} \cdot 19^{3} \cdot 127^{3} \cdot 163^{2}),\\
\tau_{16} =&(-15207425427090738323359195 20 \cdot \sqrt{-163}\\
&+ 1195256578316254156677284496 5)/(7^{4} \cdot 11^{4} \cdot 19^{4} \cdot 127^{4} \cdot 163^{2}),\\
\tau_{15} =&(92759142547808143704934847199360 \cdot \sqrt{-163}\\
&- 18263435517578853492343997372411 2)/(7^{5} \cdot 11^{5} \cdot 19^{5} \cdot 127^{5} \cdot 163^{2}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{14} =&(-280128488615906120885142930795224151680 \cdot \sqrt{-163} \\
&- 1165917490653872180160835661753940750280)/(7^{6} \cdot 11^{6} \cdot 19^{6} \cdot 127^{6} \cdot 163^{3}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{13} =&(225434813994988931907267510921346008082664320 \cdot \sqrt{-163} \\
&+ 3314830355523451351214932888380149068692586480)/(7^{7} \cdot 11^{7} \cdot 19^{7} \cdot 127^{7} \cdot 163^{4}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{12} =&(-91874749106778592874101191088636766635455352960 \cdot \sqrt{-163} \\
&- 11701247859930560778438633571891314012414753179150)/(7^{8} \cdot 11^{8} \cdot 19^{8} \cdot 127^{8} \cdot 163^{4}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{11} =&(-84046473530828853509017628497596195737658067463661760 \cdot \sqrt{-163} \\
&+ 1600173524480130782452065446665750457872535107819340120)/(7^{9} \cdot 11^{9} \cdot 19^{9} \cdot 127^{9} \cdot 163^{5}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{10} =&(55685294412714457927785374911079603299095091729633743680 \cdot \sqrt{-163} \\
&- 154056533716303095184639519594582169383375624519084954948) \\
&/(7^{10} \cdot 11^{10} \cdot 19^{10} \cdot 127^{10} \cdot 163^{5}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{9} =&(-1065167664769743681354792306953534893004087797786012605529280 \cdot \sqrt{-163} \\
&- 9908630972683074196680675999102875122804047237160514031492280) \\
&/(7^{11} \cdot 11^{11} \cdot 19^{11} \cdot 127^{11} \cdot 163^{6}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{8} =&(1514040034778829258844020756733406675881703399901362969349440 \cdot \sqrt{-163} \\
&+ 897360483932112708795122333976708140335966801587263166030370210) \\
&/(7^{12} \cdot 11^{12} \cdot 19^{12} \cdot 127^{12} \cdot 163^{6}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{7} =&(190163249539059945305891505092266264259825451892976829890473939840 \cdot \sqrt{-163} \\
&- 1647400028087123031533685651778799652053626526170401034702910999600) \\
&/(7^{13} \cdot 11^{13} \cdot 19^{13} \cdot 127^{13} \cdot 163^{7}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{6} =&(-1539560111478977814025093340680428879644195336136100552612153633920 \cdot \sqrt{-163} \\
&- 10147427550054067695975771878195185082586806163188960446421347487560) \\
&/(7^{14} \cdot 11^{14} \cdot 19^{14} \cdot 127^{14} \cdot 163^{7}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{5} =&(-44088934796976430704063277905495928792807066440914161830847930158720 \cdot \sqrt{-163} \\
&+ 10204434419450981030841290519646942062584213194931285222728066609612848) \\
&/(7^{15} \cdot 11^{15} \cdot 19^{15} \cdot 127^{15} \cdot 163^{8}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{4} =&(817281634276103787030193615971648420588773376518336903717167459358080 \cdot \sqrt{-163} \\
&- 3112505422848909771217695254098823907458301273777680127507094393033395) \\
&/(7^{16} \cdot 11^{16} \cdot 19^{16} \cdot 127^{16} \cdot 163^{8}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{3} =&(-29496553340784762603906736722751784627796150655562820214374310839861920 \cdot \sqrt{-163} \\
&- 602015795715358381386016236416321100999029370225208909900180125610211180) \\
&/(7^{17} \cdot 11^{17} \cdot 19^{17} \cdot 127^{17} \cdot 163^{9}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{2} =&(-6052509608481091415214298023445826817343287123020162130190013683246240 \cdot \sqrt{-163} \\
&+ 72634175408266832661992555200814800947231108722533450936429417798311050) \\
&/(7^{18} \cdot 11^{18} \cdot 19^{18} \cdot 127^{18} \cdot 163^{9}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{1} =&(56461681096769027344051665230697061001212363550426634743617772554174560 \cdot \sqrt{-163} \\
&+ 591703006064316675573691948688826616303934929493139081383890143779760380) \\
&/(7^{19} \cdot 11^{19} \cdot 19^{19} \cdot 127^{19} \cdot 163^{10}),
\end{aligned}
$$

$$
\begin{aligned}
\tau_{0} =&(27947381934141069970701855982718079657799394207785185804341922200026720 \cdot \sqrt{-163} \\
&- 511575290280779028431329493403062960403807717810436046221468899318721951) \\
&/(7^{20} \cdot 11^{20} \cdot 19^{20} \cdot 41 \cdot 127^{20} \cdot 163^{10}).
\end{aligned}
$$

Moreover, $\mathbb{Z}[\Phi] = \mathbb{Z}[\frac{1+\sqrt{-163}}{2}] \subseteq \mathbb{Q}(\sqrt{-163})$.