Chapter 5 Joint Probability Distributions and Random Samples:

5.1 Jointly Distributed RVs:

Two discrete random variables.

Let X and Y be two discrete RVs on the same sample space S.
The joint probability mass function (pmf) $p(x,y)$ is defined

$$p(x,y) = P(X = x \text{ and } Y = y).$$

Like before, $p(x,y) \geq 0$, and $\sum_x \sum_y p(x,y) = 1$.

Example 1: Let X be the number of cats at a randomly selected residence in a particular city. Let Y be the number of dogs at a randomly selected residence in that city. The joint pmf is given in the table:

	$p(x,y)$	$y=0$	$y=1$	$y=2$	$p_X(x)$
x	0	.3	.05	.1	.45
	1	.15	.1	.1	.35
	2	.05	.05	.05	.15
	3	.05	0	0	.05
	$p_Y(y)$	.55	.2	.25	

Find $P(X=1 \text{ and } Y=2) = \boxed{.1}$

$P(X=Y) = .3 + .1 + .05 = \boxed{.45}$ $\quad (P(X=Y=0) + P(X=Y=1) + P(X=Y=2))$

$P(X=0) = .3 + .05 + .1 = \boxed{.45}$ $\quad (P(X=0 \text{ and } Y=0) + P(X=0 \text{ and } Y=1) + P(X=0 \text{ and } Y=2))$

The marginal probability mass function (pmf) of X, $p_X(x)$ is given by

$$p_X(x) = \sum_{y:\, p(x,y)>0} p(x,y) \quad \text{for each possible } x.$$

(add row or column)

Similarly, the marginal pmf of Y is

$$p_Y(y) = \sum_{x:\, p(x,y)>0} p(x,y) \quad \text{for each possible } y.$$

Example 2: Find the marginal pmfs for Example 1.

Two Continuous RVs:

Let X and Y be continuous RVs. A joint probability density function (pdf) is a function $f(x,y)$ such that for any two-dimensional set A,

$$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy$$

In particular

$$P(a\le X\le b \text{ and } c\le Y\le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

As before, $f(x,y)\geq 0$ for all x,y and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Example 3: Let X and Y have joint pdf

$$f(x,y) = \begin{cases} xy, & 0<x<1 \text{ and } 0<y<2 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(\frac{1}{4}\le X\le \frac{1}{2} \text{ and } Y<1)$.

constant

iterated integral

$$P(\tfrac{1}{4}\le X\le \tfrac{1}{2} \text{ and } Y<1) = \int_0^1 \underbrace{\int_{\frac{1}{4}}^{\frac{1}{2}} xy\,dx}_{1^{st}}\,dy$$

$$= \int_0^1 \tfrac{1}{2}yx^2\Big|_{x=\frac{1}{4}}^{x=\frac{1}{2}}\,dy = \int_0^1\left(\tfrac{1}{8}y - \tfrac{1}{32}y\right)dy = \int_0^1 \tfrac{3}{32}y\,dy$$

$$= \tfrac{3}{32}\cdot\tfrac{1}{2}y^2\Big|_0^1 = \tfrac{3}{32}\cdot\tfrac{1}{2} = \boxed{\tfrac{3}{64}}$$

The marginal probability density functions of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \quad \text{for } -\infty<x<\infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx, \quad \text{for } -\infty<y<\infty$$

Example 4: Let X and Y have joint pdf

$$f(x,y) = \begin{cases} xy, & 0<x<1 \text{ and } 0<y<2 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^2 xy\,dy = x\cdot\frac{y^2}{2}\Big|_{y=0}^{y=2} = x\cdot\frac{2^2}{2} - x\cdot\frac{0^2}{2} = 2x, \quad 0<x<1$$

$$f_Y(y) = \int_0^1 xy\,dx = y\cdot\frac{x^2}{2}\Big|_{x=0}^{x=1} = y\cdot\frac{1}{2} - 0 = \frac{y}{2}, \quad 0<y<2$$

Independent Random Variables:

If X, Y are discrete RVs, then X and Y are independent if and only if

$$p(x,y) = p_X(x) \cdot p_Y(y) \quad \text{for all } x, y.$$

If X, Y are continuous, then X and Y are independent if and only if

$$f(x,y) = f_X(x) \cdot f_Y(y) \quad \text{for all } x, y.$$

If X, Y are not independent, then they are dependent.

Example 5: Are X and Y in example 1 independent or dependent? Why?

$\boxed{\text{Dependent}}$ since $p(x,y) \neq p_X(x) \cdot p_Y(y)$. For example

$$P(X=3, Y=2) = 0 \quad \text{but}$$
$$P(X=3) \cdot P(Y=2) = (0.05)(0.25).$$

So $p(3,2) \neq p_X(3) \cdot p_Y(2)$.

Example 6: Let $X \sim Exp(5)$ and $Y \sim Exp(2)$ be independent.

$$\boxed{\begin{array}{l} Exp(\lambda) \\ f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \\ F(x) = 1 - e^{-\lambda x}, \quad x \geq 0 \end{array}}$$

What is their joint pdf $f(x,y)$?

What is $P(X < 3 \text{ and } Y > 4)$?

$$f_X(x) = 5e^{-5x}, \quad x \geq 0$$
$$f_Y(y) = 2e^{-2y}, \quad y \geq 0$$
$$f(x,y) = f_X(x) \cdot f_Y(y) = \boxed{10e^{-5x-2y}}, \quad x \geq 0 \text{ and } y \geq 0$$

$$P(X<3 \text{ and } Y>4) = \int_4^\infty \int_0^3 10e^{-5x} e^{-2y} \, dx \, dy = \dots$$

OR

$$P(X<3 \text{ and } Y>4) \underset{\substack{\uparrow \\ \text{independent}}}{=} P(X<3) \cdot P(Y>4) = F_X(3) \cdot (1 - F_Y(4))$$
$$= (1 - e^{-5 \cdot 3}) \cdot e^{-2 \cdot 4}$$
$$= \boxed{(1 - e^{-15}) \cdot e^{-8}} \approx \boxed{0.000335}$$

More than 2 RVs:

These ideas also work if we have several RVs.

If $X_1, \dots, X_n$ are discrete, their joint pmf is

$$p(x_1, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n).$$

If $X_1, \dots, X_n$ are jointly continuous with pdf $f(x_1, \dots, x_n)$ then

$$P(a_1 \le X_1 \le b_1, \dots, a_n \le X_n \le b_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n)\, dx_n \cdots dx_1$$

The RVs $X_1, \dots, X_n$ are all independent if and only if for every subset of them (every pair, every triple, ...) the joint pmf/pdf is the product of the marginal pmfs/pdfs.

The Multinomial Distribution

Review: If X is a Binomial RU with parameters n and p, this means

n trials with probability of success p

X is counting the number of successes.

For the multinomial distribution, we have n independent trials, each of which has r possible outcomes (e.g. r colors).

Each trial has the same probabilities: p_1 to be type 1, p_2 to be type 2, ... p_r to be type r. $\quad p_1 + p_2 + \dots + p_r = 1$

We count how many of each type we get.

$$p(x_1, x_2, \dots, x_r) = \underbrace{\frac{n!}{x_1!\, x_2! \cdots x_r!}}_{\text{multinomial coefficient}} \; p_1^{x_1} \cdot p_2^{x_2} \cdots p_r^{x_r}$$

Example 7: A grab-bag contains 10 small candies of 3 different types A, B, C.

Each candy has a 40% chance to be type A, a 50% chance to be type B, and a 10% chance to be type C.

What is the probability a randomly selected bag has 3 As, 5 Bs, and 2 Cs?

$$\boxed{\frac{10!}{3!\; 5!\; 2!} \cdot (.4)^3 (.5)^5 (.1)^2}$$

5.2 Expected Values, Covariance, and Correlation:

In Chapters 3-4, we saw how to calculate $E[h(X)]$ for functions h.

We can do similar calculations with multiple RVs.

Let X, Y be discrete with joint pmf $p(x,y)$. Then

$$E[h(X,Y)] = \sum_x \sum_y h(x,y) \cdot p(x,y).$$

If instead X, Y are jointly continuous with joint pdf $f(x,y)$, then

$$E[h(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dx\, dy.$$

Example 1: Two friends play a game. Each takes a turn and scores either 0, 1, or 2 points. Let X be the points scored by the first friend and Y be the points scored by the second friend. The joint pmf for X, Y is given in the table. What is the expected value of the positive difference between their points, $E[|X-Y|]$?

$x \backslash y$	0	1	2
0	.1	.05	.15
1	.2	.1	.1
2	0	.15	.15

$|X-Y|$

$x \backslash y$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

$$E[|X-Y|] = 0(0.1) + 1(0.05) + 2(0.15) + \dots + 0(0.15)$$

$$= 0(0.1 + 0.1 + 0.15) + 1(.2 + .15 + .05 + .1) + 2(0 + 0.15)$$

$$= 0 + 0.5 + 0.3 = \boxed{0.8}$$

Example 2: On a certain exam, two subscores are given. For a randomly selected test-taker, let X and Y be the subscores they receive. The joint pdf for X, Y is

$$f(x,y) = \begin{cases} 4xy, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

What is $E[XY]$? $\quad E[XY] = \int_0^1 \int_0^1 xy \cdot 4xy\, dx\, dy$

$$= 4\int_0^1 y^2 \int_0^1 x^2\, dx\, dy = 4\int_0^1 x^2\, dx \cdot \int_0^1 y^2\, dy = 4 \cdot \frac{1}{3} \cdot \frac{1}{3} = \boxed{\frac{4}{9}}$$

<u>Covariance</u>: When we have multiple RVs, we might wonder how strongly related they are.

If X, Y are RVs with means μ_X and μ_Y, then the <u>Covariance</u> of X and Y is

$$Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$

<u>Note</u>: $Cov(X,X) = E[(X-\mu_x)\cdot(X-\mu_x)] = E[(X-\mu_x)^2] = Var(X)$

Like with variance, there is a shortcut formula for covariance

$$Cov(X,Y) = E[XY] - E[X]E[Y].$$

Most of the time this is easier to use.

<u>Example 3</u>: Let X, Y be as in Example 1:

Find $Cov(X,Y)$.

$x \backslash y$	0	1	2	marginal pmf for X
0	.1	.05	.15	.3
1	.2	.1	.1	.4
2	0	.15	.15	.3
marginal pmf for Y	.3	.3	.4	

$$Cov(X,Y) = E[XY] - E[X]\cdot E[Y]$$

$$E[XY] = 0\cdot0\cdot0.1 + 0\cdot1\cdot0.05 + 0\cdot2\cdot0.15 + 1\cdot0\cdot0.2 + 2\cdot0\cdot0$$
$$+ 1\cdot1\cdot0.1 + 1\cdot2\cdot0.1 + 2\cdot1\cdot0.15 + 2\cdot2\cdot0.15$$
$$= 0.1 + 0.2 + 0.3 + 0.6 = 1.2$$

$$E[X] = 0\cdot0.3 + 1\cdot0.4 + 2\cdot0.3 = 1$$
$$E[Y] = 0\cdot0.3 + 1\cdot0.3 + 2\cdot0.4 = 1.1$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$
$$= 1.2 - 1(1.1) = \boxed{0.1}$$

<u>Example 4</u>: Let X, Y be as in Example 2. Find $Cov(X,Y)$.

The joint pdf for X, Y is $f(x,y) = \begin{cases} 4xy, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

We found $E[XY] = \frac{4}{9}$.

$$Cov(X,Y) = E[XY] - E[X]\cdot E[Y]$$

$$E[X] = \int_0^1\int_0^1 x\cdot 4xy \, dx\, dy \qquad E[XY] = \int_0^1\int_0^1 xy\cdot 4xy\, dx\, dy = \frac{4}{9}$$

$$= \int_0^1 4y\cdot \frac{x^3}{3}\Big|_{x=0}^{x=1} dy = \int_0^1 4y\cdot\frac{1}{3}\, dy = \frac{4}{3}\cdot\frac{y^2}{2}\Big|_{y=0}^{y=1} = \frac{4}{3}\cdot\frac{1}{2} = \frac{2}{3}$$

$$E[Y] = \int_0^1\int_0^1 y\cdot 4xy\, dy\, dx = \frac{2}{3}. \qquad Cov(X,Y) = \frac{4}{9} - \frac{2}{3}\cdot\frac{2}{3} = 0$$

Correlation:

Covariance helps us understand how two random variables are related, but the covariance value depends greatly on the units of measurement.

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

To address this, we use correlation instead.

The correlation coefficient of X and Y is

$$\mathrm{Corr}(X,Y) = \rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

"rho"

Example 5: Say X and Y are RVs with these properties:

$E[X] = 4 \qquad \mathrm{Var}(X) = 25 = \sigma_X^2 \qquad \sigma_X = 5$

$E[Y] = -2 \qquad \mathrm{Var}(Y) = 9 = \sigma_Y^2 \qquad \sigma_Y = 3$

$E[XY] = -1$

What is $\mathrm{Corr}(X,Y)$? $\qquad \mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{-1-(-2)(4)}{5 \cdot 3} = \boxed{\frac{7}{15}}$$

Some Correlation Facts:

- If a and c are both positive or both negative, then

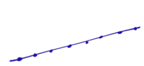
  $$\mathrm{Corr}(aX+b, cY+d) = \mathrm{Corr}(X,Y)$$

- For any RVs X,Y (where $E[X], E[Y], E[XY]$ exist and are finite),

  $$-1 \le \rho_{X,Y} \le 1.$$

- X and Y are uncorrelated if $\rho_{X,Y} = 0$.

- If X,Y are independent, then $\rho_{X,Y} = 0$. But $\rho_{X,Y} = 0$ does not imply X,Y are independent.

 $\rho_{X,Y} = 1$  $\rho_{X,Y} = -1$

- $\rho = 1$ or -1 if and only if $Y = aX + b$ for real numbers a and b with $a \neq 0$.