7.

a. $\hat{\mu} = \bar{x} = \dfrac{\sum x_i}{n} = \dfrac{1206}{10} = 120.6.$

b. $\hat{\tau} = 10{,}000\ \hat{\mu} = 1{,}206{,}000.$

c. 8 of 10 houses in the sample used at least 100 therms (the "successes"), so $\hat{p} = \frac{8}{10} = .80.$

d. The ordered sample values are 89, 99, 103, 109, 118, 122, 125, 138, 147, 156, from which the two middle values are 118 and 122, so $\hat{\tilde{\mu}} = \tilde{x} = \dfrac{118+122}{2} = 120.0.$

9.

a. $E(\bar{X}) = \mu = E(X)$, so $\bar{X}$ is an unbiased estimator for the Poisson parameter μ. Since $n = 150$,
$\hat{\mu} = \bar{x} = \dfrac{\Sigma x_i}{n} = \dfrac{(0)(18)+(1)(37)+\ldots+(7)(1)}{150} = \dfrac{317}{150} = 2.11.$

b. $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{\sqrt{\mu}}{\sqrt{n}}$, so the estimated standard error is $\sqrt{\dfrac{\hat{\mu}}{n}} = \dfrac{\sqrt{2.11}}{\sqrt{150}} = .119.$

11.

a. $E\left(\dfrac{X_1}{n_1} - \dfrac{X_2}{n_2}\right) = \dfrac{1}{n_1}E(X_1) - \dfrac{1}{n_2}E(X_2) = \dfrac{1}{n_1}(n_1 p_1) - \dfrac{1}{n_2}(n_2 p_2) = p_1 - p_2.$

b. $V\left(\dfrac{X_1}{n_1} - \dfrac{X_2}{n_2}\right) = V\left(\dfrac{X_1}{n_1}\right) + V\left(\dfrac{X_2}{n_2}\right) = \left(\dfrac{1}{n_1}\right)^2 V(X_1) + \left(\dfrac{1}{n_2}\right)^2 V(X_2) =$
$\dfrac{1}{n_1^2}(n_1 p_1 q_1) + \dfrac{1}{n_2^2}(n_2 p_2 q_2) = \dfrac{p_1 q_1}{n_1} + \dfrac{p_2 q_2}{n_2}$, and the standard error is the square root of this quantity.

c. With $\hat{p}_1 = \dfrac{x_1}{n_1}$, $\hat{q}_1 = 1 - \hat{p}_1$, $\hat{p}_2 = \dfrac{x_2}{n_2}$, $\hat{q}_2 = 1 - \hat{p}_2$, the estimated standard error is $\sqrt{\dfrac{\hat{p}_1\hat{q}_1}{n_1} + \dfrac{\hat{p}_2\hat{q}_2}{n_2}}$.

d. $(\hat{p}_1 - \hat{p}_2) = \dfrac{127}{200} - \dfrac{176}{200} = .635 - .880 = -.245$

e. $\sqrt{\dfrac{(.635)(.365)}{200} + \dfrac{(.880)(.120)}{200}} = .041$

13. $\mu = E(X) = \int_{-1}^{1} x \cdot \frac{1}{2}(1+\theta x)dx = \left.\dfrac{x^2}{4} + \dfrac{\theta x^3}{6}\right|_{-1}^{1} = \dfrac{1}{3}\theta \Rightarrow \theta = 3\mu \Rightarrow$

$\hat{\theta} = 3\bar{X} \Rightarrow E(\hat{\theta}) = E(3\bar{X}) = 3E(\bar{X}) = 3\mu = 3\left(\dfrac{1}{3}\right)\theta = \theta.$

15.

a. $E(X^2) = 2\theta$ implies that $E\left(\frac{X^2}{2}\right) = \theta$. Consider $\hat{\theta} = \frac{\sum X_i^2}{2n}$. Then

$$E\left(\hat{\theta}\right) = E\left(\frac{\sum X_i^2}{2n}\right) = \frac{\sum E\left(X_i^2\right)}{2n} = \frac{\sum 2\theta}{2n} = \frac{2n\theta}{2n} = \theta$$, implying that $\hat{\theta}$ is an unbiased estimator for θ.

b. $\sum x_i^2 = 1490.1058$, so $\hat{\theta} = \frac{1490.1058}{20} = 74.505$.

17.

a. $E(\hat{p}) = \sum_{x=0}^{\infty} \frac{r-1}{x+r-1} \cdot \binom{x+r-1}{x} \cdot p^r \cdot (1-p)^x$

$$= p\sum_{x=0}^{\infty} \frac{(x+r-2)!}{x!(r-2)!} \cdot p^{r-1} \cdot (1-p)^x = p\sum_{x=0}^{\infty} \binom{x+r-2}{x} p^{r-1}(1-p)^x = p\sum_{x=0}^{\infty} nb(x; r-1, p) = p.$$

b. For the given sequence, $x = 5$, so $\hat{p} = \frac{5-1}{5+5-1} = \frac{4}{9} = .444$.

19.

a. $\lambda = .5p + .15 \Rightarrow 2\lambda = p + .3$, so $p = 2\lambda - .3$ and $\hat{p} = 2\hat{\lambda} - .3 = 2\left(\frac{Y}{n}\right) - .3$; the estimate is $2\left(\frac{20}{80}\right) - .3 = .2$.

b. $E(\hat{p}) = E\left(2\hat{\lambda} - .3\right) = 2E\left(\hat{\lambda}\right) - .3 = 2\lambda - .3 = p$, as desired.

c. Here $\lambda = .7p + (.3)(.3)$, so $p = \frac{10}{7}\lambda - \frac{9}{70}$ and $\hat{p} = \frac{10}{7}\left(\frac{Y}{n}\right) - \frac{9}{70}$.

Section 6.2

21.

a. $E(X) = \beta \cdot \Gamma\left(1+\frac{1}{\alpha}\right)$ and $E(X^2) = V(X) + [E(X)]^2 = \beta^2\Gamma\left(1+\frac{2}{\alpha}\right)$, so the moment estimators $\hat{\alpha}$ and $\hat{\beta}$ are the solution to $\overline{X} = \hat{\beta} \cdot \Gamma\left(1+\frac{1}{\hat{\alpha}}\right)$, $\frac{1}{n}\sum X_i^2 = \hat{\beta}^2\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)$. Thus $\hat{\beta} = \dfrac{\overline{X}}{\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)}$, so once $\hat{\alpha}$ has been determined $\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)$ is evaluated and $\hat{\beta}$ then computed. Since $\overline{X}^2 = \hat{\beta}^2 \cdot \Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)$,

$\frac{1}{n}\sum \frac{X_i^2}{\overline{X}^2} = \dfrac{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}$, so this equation must be solved to obtain $\hat{\alpha}$.

b. From **a**, $\frac{1}{20}\left(\frac{16{,}500}{28.0^2}\right) = 1.05 = \dfrac{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}$, so $\frac{1}{1.05} = .95 = \dfrac{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}$, and from the hint,

$\frac{1}{\hat{\alpha}} = .2 \Rightarrow \hat{\alpha} = 5$. Then $\hat{\beta} = \dfrac{\overline{x}}{\Gamma(1.2)} = \dfrac{28.0}{\Gamma(1.2)}$.

23. Determine the joint pdf (aka the likelihood function), take a logarithm, and then use calculus:

$$f(x_1,\ldots,x_n \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-x_i^2/2\theta} = (2\pi\theta)^{-n/2} e^{-\sum x_i^2/2\theta}$$

$$\ell(\theta) = \ln[f(x_1,\ldots,x_n \mid \theta)] = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \sum x_i^2/2\theta$$

$$\ell'(\theta) = 0 - \frac{n}{2\theta} + \sum x_i^2/2\theta^2 = 0 \Rightarrow -n\theta + \sum x_i^2 = 0$$

Solving for θ, the maximum likelihood estimator is $\hat{\theta} = \frac{1}{n}\sum X_i^2$.

25.

a. $\hat{\mu} = \overline{x} = 384.4; s^2 = 395.16$, so $\frac{1}{n}\sum(x_i - \overline{x})^2 = \hat{\sigma}^2 = \frac{9}{10}(395.16) = 355.64$ and $\hat{\sigma} = \sqrt{355.64} = 18.86$ (this is <u>not</u> s).

b. The 95[th] percentile is $\mu + 1.645\sigma$, so the mle of this is (by the invariance principle) $\hat{\mu} + 1.645\hat{\sigma} = 415.42$.

c. The mle of $P(X \le 400)$ is, by the invariance principle, $\Phi\left(\frac{400-\hat{\mu}}{\hat{\sigma}}\right) = \Phi\left(\frac{400-384.4}{18.86}\right) = \Phi(0.83) =$ .7967.

27.

a. $f(x_1,\ldots,x_n;\alpha,\beta)=\dfrac{(x_1x_2\ldots x_n)^{\alpha-1}e^{-\Sigma x_i/\beta}}{\beta^{n\alpha}\Gamma^n(\alpha)}$, so the log likelihood is

$(\alpha-1)\sum\ln(x_i)-\dfrac{\sum x_i}{\beta}-n\alpha\ln(\beta)-n\ln\Gamma(\alpha)$. Equating both $\dfrac{d}{d\alpha}$ and $\dfrac{d}{d\beta}$ to 0 yields

$\sum\ln(x_i)-n\ln(\beta)-n\dfrac{d}{d\alpha}\Gamma(\alpha)=0$ and $\dfrac{\sum x_i}{\beta^2}=\dfrac{n\alpha}{\beta}=0$, a very difficult system of equations to solve.

b. From the second equation in **a**, $\dfrac{\sum x_i}{\beta}=n\alpha\Rightarrow\overline{x}=\alpha\beta=\mu$, so the mle of μ is $\hat{\mu}=\overline{X}$.

29.

a. The joint pdf (likelihood function) is

$$f(x_1,\ldots,x_n;\lambda,\theta)=\begin{cases}\lambda^n e^{-\lambda\Sigma(x_i-\theta)} & x_1\ge\theta,\ldots,x_n\ge\theta\\ 0 & \text{otherwise}\end{cases}$$

Notice that $x_1\ge\theta,\ldots,x_n\ge\theta$ iff $\min(x_i)\ge\theta$, and that $-\lambda\Sigma(x_i-\theta)=-\lambda\Sigma x_i+n\lambda\theta$.

Thus likelihood $=\begin{cases}\lambda^n\exp(-\lambda\Sigma x_i)\exp(n\lambda\theta) & \min(x_i)\ge\theta\\ 0 & \min(x_i)<\theta\end{cases}$

Consider maximization with respect to θ. Because the exponent $n\lambda\theta$ is positive, increasing θ will increase the likelihood provided that $\min(x_i)\ge\theta$; if we make θ larger than $\min(x_i)$, the likelihood drops to 0. This implies that the mle of θ is $\hat{\theta}=\min(x_i)$. The log likelihood is now $n\ln(\lambda)-\lambda\Sigma(x_i-\hat{\theta})$. Equating the derivative w.r.t. λ to 0 and solving yields $\hat{\lambda}=\dfrac{n}{\Sigma(x_i-\hat{\theta})}=\dfrac{n}{\Sigma x_i-n\hat{\theta}}$.

b. $\hat{\theta}=\min(x_i)=.64$, and $\Sigma x_i=55.80$, so $\hat{\lambda}=\dfrac{10}{55.80-6.4}=.202$

Supplementary Exercises

31. Substitute $k=\varepsilon/\sigma_Y$ into Chebyshev's inequality to write $P(|Y-\mu_Y|\ge\varepsilon)\le 1/(\varepsilon/\sigma_Y)^2=V(Y)/\varepsilon^2$. Since $E(\overline{X})=\mu$ and $V(\overline{X})=\sigma^2/n$, we may then write $P(|\overline{X}-\mu|\ge\varepsilon)\le\dfrac{\sigma^2/n}{\varepsilon^2}$. As $n\to\infty$, this fraction converges to 0, hence $P(|\overline{X}-\mu|\ge\varepsilon)\to 0$, as desired.

33. Let x_1 = the time until the first birth, x_2 = the elapsed time between the first and second births, and so on. Then $f(x_1,\ldots,x_n;\lambda)=\lambda e^{-\lambda x_1}\cdot(2\lambda)e^{-2\lambda x_2}\ldots(n\lambda)e^{-n\lambda x_n}=n!\lambda^n e^{-\lambda\Sigma kx_k}$. Thus the log likelihood is

$\ln(n!)+n\ln(\lambda)-\lambda\Sigma kx_k$. Taking $\dfrac{d}{d\lambda}$ and equating to 0 yields $\hat{\lambda}=\dfrac{n}{\Sigma kx_k}$.

For the given sample, $n=6$, $x_1=25.2$, $x_2=41.7-25.2=16.5$, $x_3=9.5$, $x_4=4.3$, $x_5=4.0$, $x_6=2.3$; so $\sum_{k=1}^{6}kx_k=(1)(25.2)+(2)(16.5)+\ldots+(6)(2.3)=137.7$ and $\hat{\lambda}=\dfrac{6}{137.7}=.0436$.