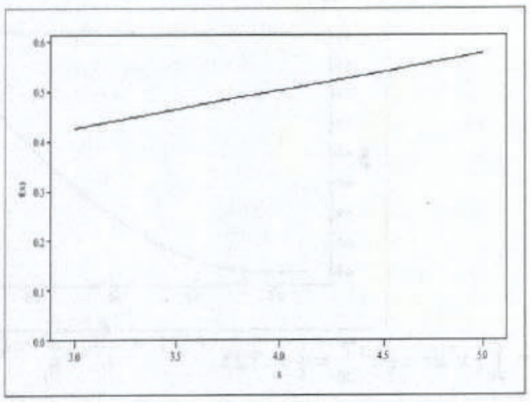
CHAPTER 4

Section 4.1

1.

a. The pdf is the straight-line function graphed below on [3, 5]. The function is clearly non-negative; to verify its integral equals 1, compute:

$$\int_3^5 (.075x+.2)\,dx = .0375x^2+.2x\Big]_3^5 = (.0375(5)^2+.2(5))-(.0375(3)^2+.2(3))$$
$$= 1.9375 - .9375 = 1$$



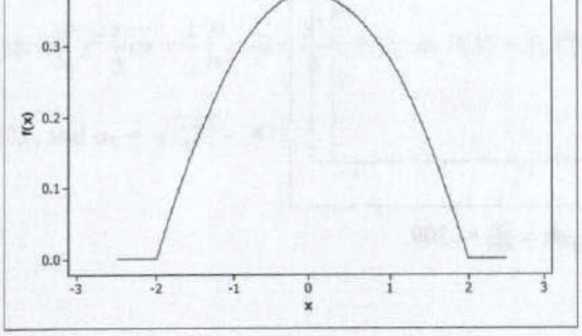
b. $P(X \le 4) = \int_3^4 (.075x+.2)\,dx = .0375x^2+.2x\Big]_3^4 = (.0375(4)^2+.2(4))-(.0375(3)^2+.2(3))$ $= 1.4 - .9375 = .4625$. Since X is a continuous rv, $P(X < 4) = P(X \le 4) = .4625$ as well.

c. $P(3.5 \le X \le 4.5) = \int_{3.5}^{4.5} (.075x+.2)\,dx = .0375x^2+.2x\Big]_{3.5}^{4.5} = \cdots = .5$.

$P(4.5 < X) = P(4.5 \le X) = \int_{4.5}^{5} (.075x+.2)\,dx = .0375x^2+.2x\Big]_{4.5}^{5} = \cdots = .278125$.

3.

a.



c. $P(\mu - 1 \le X \le \mu + 1) = \int_{\mu-1}^{\mu+1} \frac{1}{4.05}\,dx = \frac{2}{4.05} = .494$. (We don't actually need to know μ here, but it's clearly the midpoint of 2.225 mm by symmetry.)

d. $P(a \le X \le a + 1) = \int_a^{a+1} \frac{1}{4.05}\,dx = \frac{1}{4.05} = .247$.

9.

a. $P(X \le 5) = \int_1^5 .15e^{-.15(x-1)}dx = .15\int_0^4 e^{-.15u}du$ (after the substitution $u = x - 1$)

$= -e^{-.15u}\Big]_0^4 = 1 - e^{-.6} \approx .451$. $P(X > 5) = 1 - P(X \le 5) = 1 - .451 = .549$.

b. $P(2 \le X \le 5) = \int_2^5 .15e^{-.15(x-1)}\,dx = \int_1^4 .15e^{-.15u}\,du = -e^{-.15u}\Big]_1^4 = .312$.

Section 4.2

11.

a. $P(X \le 1) = F(1) = \frac{1^2}{4} = .25$.

b. $P(.5 \le X \le 1) = F(1) - F(.5) = \frac{1^2}{4} - \frac{.5^2}{4} = .1875$.

c. $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - F(1.5) = 1 - \frac{1.5^2}{4} = .4375$.

d. $.5 = F(\tilde{\mu}) = \frac{\tilde{\mu}^2}{4} \Rightarrow \tilde{\mu}^2 = 2 \Rightarrow \tilde{\mu} = \sqrt{2} \approx 1.414$.

e. $f(x) = F'(x) = \frac{x}{2}$ for $0 \le x < 2$, and $= 0$ otherwise.

f. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^2 x \cdot \frac{x}{2}dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{x^3}{6}\Big]_0^2 = \frac{8}{6} \approx 1.333$.

g. $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2 \frac{x}{2}dx = \frac{1}{2}\int_0^2 x^3 dx = \frac{x^4}{8}\Big]_0^2 = 2$, so $V(X) = E(X^2) - [E(X)]^2 =$

$2 - \left(\frac{8}{6}\right)^2 = \frac{8}{36} \approx .222$, and $\sigma_X = \sqrt{.222} = .471$.

h. From g, $E(X^2) = 2$.

13.

a. $1=\int_1^\infty \frac{k}{x^4}dx = k\int_1^\infty x^{-4}dx = \frac{k}{-3}x^{-3}\Big]_1^\infty = 0-\left(\frac{k}{-3}\right)(1)^{-3} = \frac{k}{3} \Rightarrow k=3$.

b. For $x \ge 1$, $F(x)= \int_{-\infty}^{x} f(y)dy = \int_1^x \frac{3}{y^4}dy = -y^{-3}\Big|_1^x = -x^{-3}+1 = 1-\frac{1}{x^3}$. For $x < 1$, $F(x) = 0$ since the distribution begins at 1. Put together, $F(x)=\begin{cases} 0 & x<1 \\ 1-\frac{1}{x^3} & 1\le x \end{cases}$.

c. $P(X > 2) = 1 - F(2) = 1 - \frac{7}{8} = \frac{1}{8}$ or .125;
$P(2 < X < 3) = F(3) - F(2) = (1-\frac{1}{27}) - (1-\frac{1}{8}) = .963 - .875 = .088$.

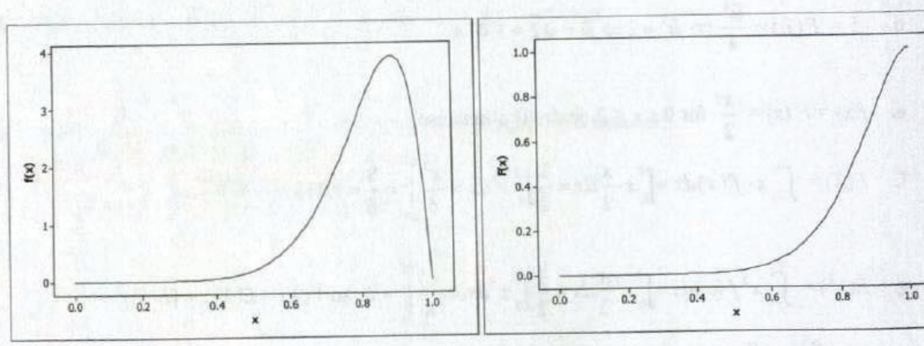
d. The mean is $E(X)=\int_1^\infty x\left(\frac{3}{x^4}\right)dx = \int_1^\infty\left(\frac{3}{x^3}\right)dx = -\frac{3}{2}x^{-2}\Big|_1^\infty = 0+\frac{3}{2}=\frac{3}{2} = 1.5$. Next,

$E(X^2)=\int_1^\infty x^2\left(\frac{3}{x^4}\right)dx = \int_1^\infty\left(\frac{3}{x^2}\right)dx = -3x^{-1}\Big|_1^\infty = 0+3=3$, so $V(X) = 3-(1.5)^2 = .75$. Finally, the standard deviation of X is $\sigma = \sqrt{.75} = .866$.

e. $P(1.5-.866 < X < 1.5+.866) = P(.634 < X < 2.366) = F(2.366) - F(.634) = .9245 - 0 = .9245$.

15.

a. Since X is limited to the interval (0, 1), $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$.
For $0 < x < 1$,
$F(x)=\int_{-\infty}^{x} f(y)dy = \int_0^x 90y^8(1-y)dy = \int_0^x (90y^8-90y^9)dy = 10y^9-9y^{10}\Big]_0^x = 10x^9-9x^{10}$.
The graphs of the pdf and cdf of X appear below.



b. $F(.5) = 10(.5)^9 - 9(.5)^{10} = .0107$.

c. $P(.25 < X \le .5) = F(.5) - F(.25) = .0107 - [10(.25)^9 - 9(.25)^{10}] = .0107 - .0000 = .0107$.
Since X is continuous, $P(.25 \le X \le .5) = P(.25 < X \le .5) = .0107$.

d. The 75[th] percentile is the value of x for which $F(x) = .75$: $10x^9 - 9x^{10} = .75 \Rightarrow x = .9036$ using software.

e. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x \cdot 90x^8(1-x)dx = \int_0^1 (90x^9 - 90x^{10})dx = 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = 9 - \frac{90}{11} = \frac{9}{11} = .8182.$

Similarly, $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_0^1 x^2 \cdot 90x^8(1-x)dx = \ldots = .6818$, from which $V(X) = .6818 - (.8182)^2 = .0124$ and $\sigma_X = .11134$.

f. $\mu \pm \sigma = (.7068, .9295)$. Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068) = .8465 - .1602 = .6863$, and the probability X is more than 1 standard deviation from its mean value equals $1 - .6863 = 3137$.

17.

a. To find the $(100p)$th percentile, set $F(x) = p$ and solve for x:

$\frac{x-A}{B-A} = p \Rightarrow x = A + (B-A)p.$

b. $E(X) = \int_A^B x \cdot \frac{1}{B-A}dx = \frac{A+B}{2}$, the midpoint of the interval. Also,

$E(X^2) = \frac{A^2 + AB + B^2}{3}$, from which $V(X) = E(X^2) - [E(X)]^2 = \ldots = \frac{(B-A)^2}{12}$. Finally,

$\sigma_X = \sqrt{V(X)} = \frac{B-A}{\sqrt{12}}$.

c. $E(X^n) = \int_A^B x^n \cdot \frac{1}{B-A}dx = \frac{1}{B-A}\frac{x^{n+1}}{n+1}\Big]_A^B = \frac{B^{n+1} - A^{n+1}}{(n+1)(B-A)}$.

19.

a. $P(X \le 1) = F(1) = .25[1 + \ln(4)] = .597.$

b. $P(1 \le X \le 3) = F(3) - F(1) = .966 - .597 = .369.$

c. For $x < 0$ or $x > 4$, the pdf is $f(x) = 0$ since X is restricted to $(0, 4)$. For $0 < x < 4$, take the first derivative of the cdf:

$F(x) = \frac{x}{4}\left[1 + \ln\left(\frac{4}{x}\right)\right] = \frac{1}{4}x + \frac{\ln(4)}{4}x - \frac{1}{4}x\ln(x) \Rightarrow$

$f(x) = F'(x) = \frac{1}{4} + \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) - \frac{1}{4}x\frac{1}{x} = \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) = .3466 - .25\ln(x)$

21. $E(\text{area}) = E(\pi R^2) = \int_{-\infty}^{\infty} \pi r^2 f(r)dr = \int_9^{11} \pi r^2 \frac{3}{4}\left(1 - (10-r)^2\right)dr = \cdots = \frac{501}{5}\pi = 314.79 \text{ m}^2.$

23. With X = temperature in °C, the temperature in °F equals $1.8X + 32$, so the mean and standard deviation in °F are $1.8\mu_X + 32 = 1.8(120) + 32 = 248$°F and $|1.8|\sigma_X = 1.8(2) = 3.6$°F. Notice that the additive constant, 32, affects the mean but does not affect the standard deviation.