HW 7 Even Solutions

Required problems: Ch 5: 12, 20a, 32

12.

a. $P(X > 3) = \int_3^\infty \int_0^\infty xe^{-x(1+y)}dydx = \int_3^\infty e^{-x}dx = .050.$

b. The marginal pdf of X is $f_X(x) = \int_0^\infty xe^{-x(1+y)}dy = e^{-x}$ for $x \ge 0$. The marginal pdf of Y is $f_Y(y) = \int_3^\infty xe^{-x(1+y)}dx = \frac{1}{(1+y)^2}$ for $y \ge 0$. It is now clear that $f(x,y)$ is not the product of the marginal pdfs, so the two rvs are not independent.

c. $P(\text{at least one exceeds } 3) = P(X > 3 \text{ or } Y > 3) = 1 - P(X \le 3 \text{ and } Y \le 3)$
$= 1 - \int_0^3 \int_0^3 xe^{-x(1+y)}dydx = 1 - \int_0^3 \int_0^3 xe^{-x}e^{-xy}dy$
$= 1 - \int_0^3 e^{-x}(1 - e^{-3x})dx = e^{-3} + .25 - .25e^{-12} = .300.$

20.

a. $P(X_1 = 2, \ldots, X_6 = 2) = \frac{12!}{2!2!2!2!2!2!}(.24)^2(.13)^2(.16)^2(.20)^2(.13)^2(.14)^2 = .00247.$

32. $E(XY) = \int_0^\infty \int_0^\infty xy \cdot xe^{-x(1+y)}dydx = \cdots = 1$. Yet, since the marginal pdf of Y is $f_Y(y) = \frac{1}{(1-y)^2}$ for $y \ge 0$, $E(Y) = \int_0^\infty \frac{y}{(1+y)^2}dy = \infty$. Therefore, Cov($X$, Y) and Corr(X, Y) do not exist, since they require this integral (among others) to be convergent.

The full steps for E$[XY]$ are on the next page:

$$
\begin{aligned}
\mathrm{E}[XY] &= \int_0^\infty \int_0^\infty xy \cdot xe^{-x(1+y)}\,dy\,dx \\
&= \int_0^\infty x^2 \left(\int_0^\infty ye^{-x(1+y)}\,dy \right) dx \qquad \text{integrate by parts with } u = y, \qquad dv = e^{-x(1+y)}\,dy \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad du = dy, \qquad v = -\frac{1}{x}e^{-x(1+y)} \\
&= \int_0^\infty x^2 \left(-\frac{y}{x}e^{-x(1+y)}\Big|_{y=0}^{y=\infty} + \int_0^\infty \frac{1}{x}e^{-x(1+y)}\,dy \right) dx \\
&= \int_0^\infty x^2 \left(-0 + 0 + \int_0^\infty \frac{1}{x}e^{-x(1+y)}\,dy \right) dx, \qquad \text{when evaluating at } y = \infty \text{ use L'Hopital's Rule,} \\
&\qquad\qquad \text{or recall that the exponential decay will win against the linear part } y \\
&= \int_0^\infty x^2 \left(\int_0^\infty \frac{1}{x}e^{-x(1+y)}\,dy \right) dx \\
&= \int_0^\infty x^2 \left(-\frac{1}{x^2}e^{-x(1+y)}\Big|_{y=0}^{y=\infty} \right) dx \\
&= \int_0^\infty x^2 \cdot \frac{1}{x^2}e^{-x}\,dx \\
&= \int_0^\infty e^{-x}\,dx \\
&= -e^{-x}\Big|_{x=0}^{x=\infty} \\
&= 0 + e^{-0} = 1
\end{aligned}
$$