Chapter 7 Statistical Intervals Based on a Single Sample

A point estimate by itself gives no information about the precision of the estimation. An alternative is to provide an interval estimate or <u>confidence interval</u> (CI).

7.1 Basic Properties of Confidence Intervals:

For this section, we assume that we are studying the population mean μ, and

1. The population distribution is Normal
2. The population SD σ is known. ⟵ unrealistic

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and SD σ. Then results from Ch. 5 tell us

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

We can standardize $\bar{X}$ to get a standard normal RV Z.

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

Since $P(-1.96 < Z < 1.96) = 0.95$,

$$P\left(-1.96 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.96\right) = 0.95$$

Let's solve the inequality $-1.96 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.96$ for μ.

$$-1.96 \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < 1.96 \frac{\sigma}{\sqrt{n}}$$

$$+1.96 \frac{\sigma}{\sqrt{n}} > \mu - \bar{X} > -1.96 \frac{\sigma}{\sqrt{n}}$$

$$-1.96 \frac{\sigma}{\sqrt{n}} < \mu - \bar{X} < 1.96 \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}$$

So we get

$$P\left(\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}\right) = 0.95.$$

95% CI for Population Mean for Normal Population with Known SD σ:

If $\bar{x}$ is the sample mean we obtained, then

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{\sigma}{\sqrt{n}} \quad \text{with 95\% confidence.}$$

$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$ is a 95% Confidence Interval for μ.

Concise expression: $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$.

Example 1: We obtain a sample $x_1, x_2, \dots, x_n$ of the preferred keyboard height of n randomly selected typists. Assume it is known the preferred height is normally distributed with sd $\sigma = 2.0$ cm. Our sample of $n = 31$ typists gave $\bar{x} = 80.0$ cm. Find a 95% CI for μ, the true average preferred height.

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$$

$$80.0 - 1.96 \cdot \frac{2.0}{\sqrt{31}} < \mu < 80.0 + 1.96\frac{2.0}{\sqrt{31}}$$

$$79.3 < \mu < 80.7$$

$(79.3, 80.7)$ is an 95% CI for μ.

Interpreting a CI:

It is tempting to say there is a 95% probability that μ is captured by a 95% CI.

Issue: There is no randomness anymore.
The randomness was from $\bar{X}$

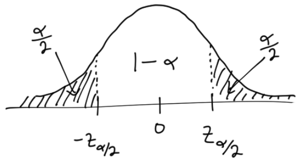
Instead we need to look at the long term behavior of many CIs. Say we repeat the process of obtaining a random sample and generating a 95% CI for μ. In the long run, 95% of these CIs will contain μ.

See Figure 7.3 in our textbook.

Other Levels of Confidence:

We can find CI with other confidence levels, like 90% confidence or 99% confidence.

The only change we have to make is changing 1.96 to the appropriate critical value $z_{\alpha/2}$



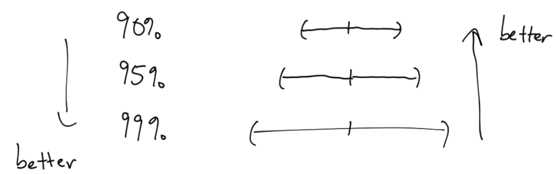
qnorm(.95)

Ex: To get a 90% CI, we use $z_{0.05} =$ qnorm(.05, lower.tail = FALSE) ≈ 1.645

To get a 99% CI, we use $z_{0.005} =$ qnorm(.995) ≈ 2.576

A 100(1−α)% CI for the mean μ of normal population when σ is known is

$$\left(\bar{x} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right).$$



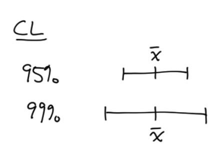
Shorthand: $\bar{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$.

Example 2: Assume the brightness of a certain model of lightbulb is normally distributed with SD $\sigma = 100$ lumens. We obtained a random sample of 20 lightbulbs and got a sample mean of $\bar{x} = 820$ lumens. Find a 99% CI for μ, the true average lumens of this model of lightbulb.

$$820 \pm 2.576 \cdot \frac{100}{\sqrt{20}}$$

(762.4, 877.6) is our 99% CI for μ.

Confidence Level, Precision, and Sample Size:



Higher confidence levels lead to a wider (less precise) CI.

The width of a CI is $w = 2 \cdot z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$. Solving for n,

$n = \left(2 z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{w}\right)^2$. This is the smallest sample size that allows us to have the desired confidence level and have width no more than w.

Example 3: What sample size n is required so in Example 2 we can have a 99% CI with width 20 lumens?

$$n = \left(2 \cdot 2.576 \cdot \frac{100}{20}\right)^2 \approx \boxed{664}$$

663.57

always round up

7.2 Large Sample CIs for a Population Mean and Proportion:

Now we look at a more practical case.

Let $X_1, \dots, X_n$ be a random sample from a population with mean μ and SD σ.

Here we only assume that n is large, $n > 40$.

Since n is large, the CLT says $\bar{X}$ is approximately normal.

$$P\left(-z_{\frac{\alpha}{2}} < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < z_{\frac{\alpha}{2}}\right) \approx 1-\alpha$$

So, like before, $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ is a $100(1-\alpha)\%$ CI for μ.

But, generally we won't know what σ is. So we have to use the sample SD s instead.

Since n is large, this does not add much extra randomness.

<u>Prop.</u> If n is sufficiently large $(n > 40)$, then $\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$ is approximately standard normal. So

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \text{ is a } 100(1-\alpha)\% \text{ CI for } \mu.$$

This is true regardless of the shape of the population distribution.

<u>Example 1</u>: We obtain a random sample from a certain lightbulb model and measure the brightness in lumens of 50 lightbulbs. The sample mean is 960 lumens with sample SD 80 lumens. Find a 95% CI for the true mean brightness μ for this model.

$$960 \pm 1.96 \cdot \frac{80}{\sqrt{50}}$$

$(937.8, 982.2)$ is our 95% CI for μ.

Like before, the width of the CI is $w = 2 \cdot z_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$, so $n = \left(2 \cdot z_{\frac{\alpha}{2}} \cdot \frac{s}{w}\right)^2$. The problem is we won't know s in advance.

Instead, we guess the value of s (try to overestimate it to be safe)

CI for a Population Proportion:

Say we are interested in the proportion p of a population that has some property.

We sample n individuals and count the number of successes X.

As long as n is small compared to the whole population, $X \approx Bin(n, p)$

If also $np \geq 10$ and $n(1-p) \geq 10$, then the CLT says $X \approx N(np, np(1-p))$

In 6.1, we estimated p with $\hat{p} = \frac{X}{n}$

$$E\left[\frac{X}{n}\right] = \frac{E[X]}{n}$$

$$Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X)$$

So $\hat{p} \approx N\left(\frac{np}{n}, \frac{np(1-p)}{n^2}\right) = N\left(p, \frac{p(1-p)}{n}\right)$.

Then we normalize to get

$$P\left(-z_{\frac{\alpha}{2}} \leq \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \leq z_{\frac{\alpha}{2}}\right) \approx 1-\alpha$$

Like in 7.1, we want to solve for the parameter p. This time it is a quadratic equation in p.

We get

$$p = \underbrace{\frac{\hat{p} + (z_{\alpha/2}^2/2n)}{1 + (z_{\alpha/2}^2/n)}}_{\tilde{p}} \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_{\alpha/2}^2/4n^2}}{1 + (z_{\alpha/2}^2/n)}$$

Prop: Let $\tilde{p} = \frac{\hat{p} + (z_{\alpha/2}^2/2n)}{1 + (z_{\alpha/2}^2/n)}$. A $100(1-\alpha)\%$ CI for a population proportion p is

$$\tilde{p} \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_{\alpha/2}^2/4n^2}}{1 + (z_{\alpha/2}^2/n)}.$$

This is the "score CI". ⊛

⊛ Note: There is an approximation when n is very large. $\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{2}}$. But this is not very accurate for small or large values of p, even if n is quite large.

(See Figure 7.6 in book.)

Given all these restrictions, we instead will always use the score CI.

Example 2: Say we are studying the proportion p of electrical components which are defective. We get a random sample of $n=70$ components and find 14 of them have a defect. Find a 90% CI for p.

$\hat{p} = \frac{14}{70} = 0.2$ $\quad z_{\frac{0.10}{2}} = 1.645$ $\quad n = 70$ $\quad \tilde{p} = \frac{0.2 + (1.645^2/140)}{1 + (1.645^2/70)} \approx 0.2112$

$$0.2112 \pm 1.645 \cdot \frac{\sqrt{(.2)(.8)/70 + 1.645^2/(4 \cdot 70^2)}}{1 + (1.645^2/70)}$$

$(0.133, 0.289)$ is our 90% CI for p.

For a CI for a population proportion, we can solve for the size of n required to get a certain width w and confidence level.

We get a formula that involves $\hat{p}$, (7.12) in the book. Since we don't know $\hat{p}$, the safest approach is to use $\hat{p} = 0.5$ which is the worst-case scenario for the width w. Since we are using $\hat{p} = 0.5$, the simpler version of the CI is pretty accurate.

$$n \approx \left(\frac{4 z_{\frac{\alpha}{2}} \overset{0.5}{\overset{\shortparallel}{\hat{p}}}(1-\overset{0.5}{\overset{\shortparallel}{\hat{p}}})}{w} \right)^2 = \left(\frac{z_{\frac{\alpha}{2}}}{w} \right)^2$$

<u>Example 3</u>: What value of n is needed so the width of a 99% CI for p is no more than 0.06 no matter what the value of $\hat{p}$ is?

$$z_{\frac{0.01}{2}} = z_{0.005} = \text{qnorm}(0.995) \approx 2.576$$

$$n \approx \left(\frac{2.576}{0.06} \right)^2 \approx 1843.3 \nearrow \boxed{n = 1844}$$

<u>One-Sided CIs / Confidence Bounds</u>:

So far we have looked at 2-sided CIs, which bound the parameter above and below. Sometimes, we may only be interested in one of these bounds.

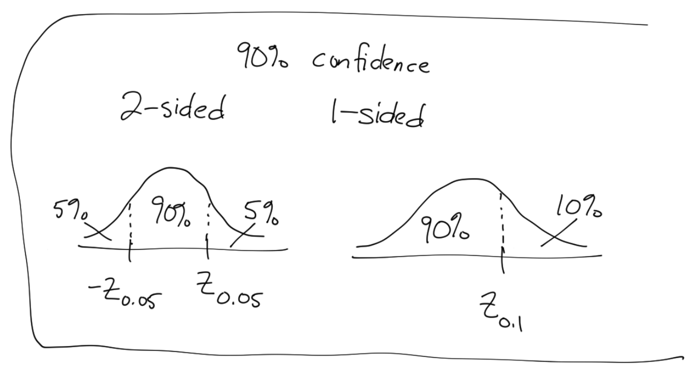
<u>Ex</u>: Proportion of cars that pass inspection is at least 98%
Quality Control (e.g. lifetime is at least 100)

<u>Prop</u>: A large-sample ($n > 40$) upper confidence bound for μ is

$$\mu < \bar{x} + z_\alpha \cdot \frac{s}{\sqrt{n}}$$

and a large-sample lower confidence bound for μ is

$$\mu > \bar{x} - z_\alpha \cdot \frac{s}{\sqrt{n}}.$$



In general, we use the one side of the CI and replace $z_{\frac{\alpha}{2}}$ with z_α.

<u>Example 4</u>: A random sample of $n = 45$ participants reaction times to a certain event yields $\bar{x} = 1.6$ s and $s = 0.2$ s. Find a 95% upper confidence bound for μ.

$$\mu < \bar{x} + z_\alpha \cdot \frac{s}{\sqrt{n}}$$

$$z_\alpha = \text{qnorm}(0.95) \approx 1.645$$

$$\mu < 1.6 + 1.645 \cdot \frac{0.2}{\sqrt{45}}$$

$\mu < 1.649$ with 95% confidence.