Chapter 2 Probability:

2.1 Sample Spaces and Events:

A random experiment is any activity or process where the outcome has uncertainty.

Ex: Flip a coin
Roll a die
Play the Lottery

The Sample Space S of an experiment is the set of all possible outcomes of that experiment.

Ex: 1 coin flip $S = \{\text{Heads}, \text{Tails}\}$

roll a die 1 time $S = \{1, 2, 3, 4, 5, 6\}$

play the lottery $S = \{\text{win big prize, win small prize, lose}\}$

flip a coin twice: $S = \{HH, HT, TH, TT\}$

Events: An event is any collection (subset) of outcomes contained in the sample space.

An event A occurs if the experiment's outcome is contained in A.

Ex: Flip a coin twice, $S = \{HH, HT, TH, TT\}$

event A is that we get same number of heads and tails
$A = \{HT, TH\}$

Roll a die once, $S = \{1, 2, 3, 4, 5, 6\}$
event A is roll an even number, $A = \{2, 4, 6\}$

Some Set Theory: An event is a set.

Set operations:

1) Complement: The complement of A is the event A' which contains all outcomes of S which are not in A. "A does not occur"

A: roll even #
A': roll odd #

2) Union: The union of events A and B, written $A \cup B$, is the event consisting of all outcomes in A or B (or both). "A or B (or both) occurs"

A: roll even # $\{2, 4, 6\}$
B: roll prime # $\{2, 3, 5\}$

$\underbrace{A \cup B = \{2, 3, 4, 5, 6\}}_{\text{even or prime}}$

3) Intersection: The intersection of events A and B, written $A \cap B$, is the event consisting of all outcomes in A and B. "A and B occur"

$\underbrace{A \cap B = \{2\}}_{\text{even and prime}}$

Example 1: We roll a 6-sided die. 

Find
- the sample space S
- the event A for rolling an even number
- the event B for rolling at least 4
- the events A', $A' \cup B$, $A \cap B$

$S = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 4, 6\}$, $B = \{4, 5, 6\}$

$\underbrace{A' = \{1, 3, 5\}}_{\text{odd \#}}$, $\underbrace{A' \cup B = \{1, 3, 4, 5, 6\}}_{\text{odd \# or at least 4}}$, $\underbrace{A \cap B = \{4, 6\}}_{\text{even and at least 4}}$

The null event or null set is written $\overset{\{\}}{\varnothing}$ and contains no outcomes.

If $A \cap B = \varnothing$, then A and B are mutually exclusive or disjoint.

Ex: Rolling a die, $S = \{1, 2, 3, 4, 5, 6\}$

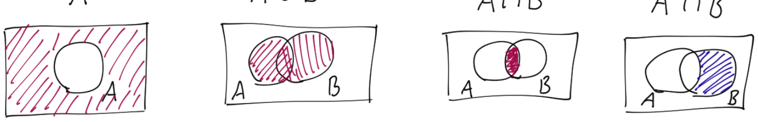
$A = \{2, 4, 6\}$ $B = \{1, 3, 5\}$ $A \cap B = \varnothing$ A and B are disjoint	$A = \{1, 2\}$ $B = \{3, 4, 5, 6\}$ A and B are disjoint

We can extend these ideas to 3+ events as well.

For 1, 2, or 3 events, Venn Diagrams can be helpful.

Example 2: Draw a Venn Diagram for A', $A \cup B$, $A \cap B$, and $A' \cap B$

S A^c $A \cup B$ $A \cap B$ $A' \cap B$



2.2 Axioms of Probability:

Ex: $S = \{1,2,3,4,5,6\}$  event $A = \{2,4,6\}$

Given a sample space S, we want to assign probabilities $P(A)$ to the various events.

We require these axioms to be satisfied:

(i) For any event A, $P(A) \geq 0$

(ii) $P(S) = 1$, "probability that something happens is one"

(iii) If $A_1, A_2, A_3, \ldots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

(Also works for just $2, 3, 4, \ldots$ disjoint events $P(A \cup B) = P(A) + P(B)$)

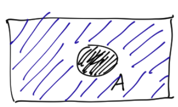
We can use these axioms to derive properties.

Prop. 1: $P(\varnothing) = 0$. Therefore axiom (iii) also works for finitely many events.

Pf:
$$P(\varnothing) = P(\varnothing \cup \varnothing \cup \varnothing \cup \cdots) = \sum_{i=1}^{\infty} P(\varnothing)$$

$$P(\varnothing) = 0 \qquad (0 + 0 + 0 + 0 + 0 + \ldots = 0)$$

Prop. 2: For any event A, $P(A) + P(A') = 1$



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Prop 3: For any event A, $P(A) \leq 1$.

$$0 \leq P(A) \leq 1$$

Example 1: Let A and B be disjoint events with $P(A) = 0.3$ and $P(B') = 0.6$.

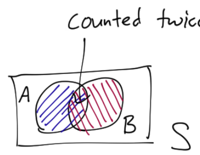
Find $P(A \cup B)$.

$$P(B) = 1 - P(B') = 1 - 0.6 = 0.4$$

$$\underbrace{P(A \cup B) = P(A) + P(B)}_{\text{disjoint}} = 0.3 + 0.4 = \boxed{0.7}$$

Inclusion-Exclusion Principle:

For any two events A, B

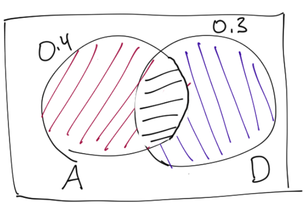


$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 2: At a certain restaurant, 40% of customers order an appetizer, 30% order a dessert, and 15% order both.

What is the probability a randomly selected customer orders an appetizer or a dessert or both? What is the probability the customer orders neither?



$P(A \cup D)$

$= 0.4 + 0.3 - 0.15 = \boxed{0.55}$

$P(\text{neither}) = 1 - 0.55 = \boxed{0.45}$

$P((A \cup D)')$

de Morgan's Law $= P(A' \cap D')$

We can also apply Inclusion-Exclusion for 3 events:

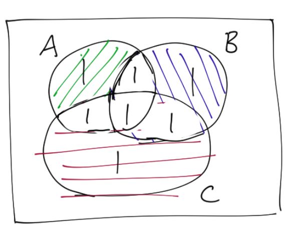
For any events A, B, C

$P(A \cup B \cup C) =$

$P(A) + P(B) + P(C)$

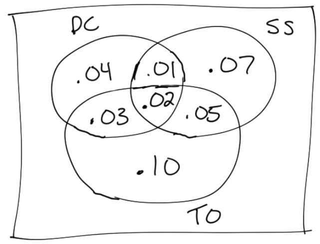
$- P(A \cap B) - P(A \cap C) - P(B \cap C)$

$+ P(A \cap B \cap C)$



Example 3: At a certain CPU manufacturing facility, for a specific CPU model,

- 10% of the CPUs have a defective core (DC)
- 15% fail to meet the speed standards (SS)
- 20% fail to meet the thermal output standards (TO)
- 3% have DC and SS
- 5% have DC and TO
- 7% have SS and TO
- 2% have DC, SS, and TO.



Find the probability a randomly selected CPU has

(a) DC, SS, or TO :

$$P(DC \cup SS \cup TO) = 0.04 + .01 + .03 + .02 + .1 + .05 + .07 = \boxed{0.32}$$

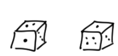
(b) DC but not TO. $P(DC \cap TO') = 0.04 + 0.01 = \boxed{0.05}$

Equally Likely Outcomes:

For many experiments, each of the N outcomes are equally likely.

For each outcome, the probability is $\frac{1}{N}$

For any event A, $P(A) = \dfrac{\#\text{ outcomes in } A}{N}$

Example 4: We roll 2 fair 6-sided dice. 

What is the probability the sum of the dice is 7 ?

$$P(\text{sum } 7) = \frac{\#\text{ ways for sum to be } 7}{\text{total } \#\text{ of roles}}$$

$$= \frac{6}{36} = \boxed{\frac{1}{6}}$$

2.3 Counting Techniques

When an experiment has N equally likely outcomes,

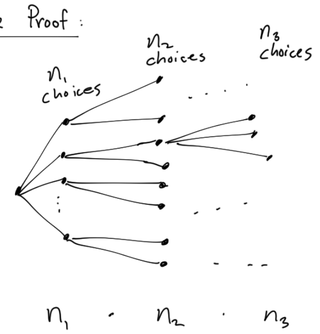
$$P(A) = \frac{\#\text{ of outcomes in } A}{N}$$

We will discuss three techniques to help us count the outcomes in an event.

1) Multiplication Rule: Suppose we are counting the ways to pick objects in a list. There are n_1 choices for the first position. Then regardless of the first position, there are n_2 choices for the second position. So on, until regardless of the previous positions, there are n_k choices for the k^{th} position.

Then there are $n_1 n_2 \cdots n_k$ ways to do this.

Picture Proof:



Example 1: At a certain restaurant, you order an entre, a side, and a drink. There are 6 entres, 5 sides, and 10 drinks to pick from. How many different orders are possible?

Multiplication Rule

$6 \cdot 5 \cdot 10 = 300$ total orders.

2) Permutations:

Say we have n (distinct) objects and we select k of them where order matters. Each of these ordered subsets is called a permutation. The number of permutations is $P_{k,n} = n(n-1)(n-2)\cdot \ldots \cdot(n-k+1) = \dfrac{n!}{(n-k)!}$

factorial

$n! = n(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1$

$$\frac{n!}{(n-k)!} = \frac{n(n-1)(n-2)\cdot \ldots \cdot \cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-k)}\,\cancel{(n-k-1)}\cdot \ldots \cdot \cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = n(n-1)\cdot \ldots \cdot(n-k+1)$$

Ex: $\{A, B, C, D, E\}$ order 3 of them

$$P_{3,5} = \overset{n}{\underset{B}{5}} \cdot \overset{n-1}{\underset{A}{4}} \cdot \overset{n-2}{\underset{E}{3}}$$

Example 2: If a musician knows 10 pieces that can be included in a performance, how many different ways can 5 pieces be selected for the performance?

Permutation

$$P_{5,10} = \boxed{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} \qquad \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \boxed{\frac{10!}{(10-5)!}}$$

3) Combinations: Say we have n objects and want to pick an unordered subset of k (distinct) of them. These unordered subsets are called combinations. The number of combinations is

$$\binom{n}{k} = \frac{n(n-1)\cdot \ldots \cdot (n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

read "n choose k"

Ex:

$\{A, B, C, D, E\}$

Orderings: ABC, BAC, CBA, ⋮ $\Big\}$ $3! = 6$ $\qquad \binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!}$

Example 3: Out of 20 students in a class, 3 are randomly picked to form a committee. If the committee positions are identical, what is the probability that student A is on the committee and student T is not?

Equally Likely Outcomes

$$P(\text{student } A \text{ is on committee, student } T \text{ is not}) = \frac{\#\text{ ways to include } A \text{ but not } T}{\text{total } \#\text{ ways to form committee}}$$

$$\text{total } \#\text{ ways} = \binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3!} \qquad = \boxed{\frac{\binom{18}{2}}{\binom{20}{3}}}$$

\# ways to include A but not T

$$\binom{18}{2} = \frac{18 \cdot 17}{2!}$$

Example 4: You pick a 6-digit pin number at random. Ex: 287230

What is the probability there are no repeated digits in the pin? $\qquad P_{6,10} \quad \frac{10!}{(10-6)!}$

$$P(\text{no repeated digits}) = \frac{\#\text{ pins with no repeated digits}}{\text{total } \#\text{ pins}} = \boxed{\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10}}$$

Permutations

Multiplication Rule

Multiplication Rule: decisions do not impact later choices, order matters

Permutation: no repeats, order matters

Combination: no repeats, order doesn't matter

2.4 Conditional Probability:

How does $P(A)$ change when we learn additional information that event B happened?

Example 1: A store sells 2 models of ovens, and some units are defective.

	working	defective	
model A	55	5	60
model B	30	10	40
	85	15	100

What is the probability a randomly selected oven is defective? $\boxed{\frac{15}{100}}$

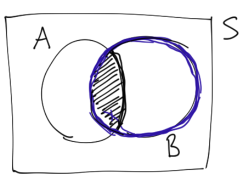
What is the conditional probability the oven is defective given that it is a model B oven? $\boxed{\frac{10}{40}}$

For two events A and B, with $P(B)>0$, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

A "given" B

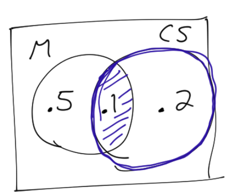
Picture:



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

know

Example 2: In a certain class, 60% of the students are math majors, 30% are CS majors, and 10% are both.



What is the probability a randomly selected student is a math major given they are a CS major?

$$P(M \mid CS) = \frac{P(M \cap CS)}{P(CS)} = \frac{0.1}{0.3} = \boxed{\frac{1}{3}}$$

We can rearrange $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ to get the

Multiplication Rule for $P(A \cap B)$:

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$
$$P(A) \cdot P(B|A)$$

Example 3: A hat contains papers with the numbers 1, 2, 3, 4, 5 on them.

We draw two numbers randomly from the hat without replacing them.

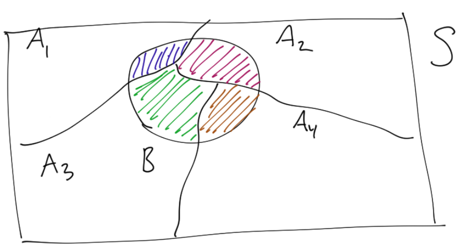
What is the probability the first number picked is odd and the second is even?

$$P(1^{st} \text{ odd} \cap 2^{nd} \text{ even}) = P(1^{st} \text{ odd}) \cdot P(2^{nd} \text{ even} \mid 1^{st} \text{ odd}) = \frac{3}{5} \cdot \frac{2}{4} = \boxed{\frac{3}{10}}$$

Law of Total Probability: Let $A_1, \ldots, A_k$ be disjoint events that make up the entire sample space, $A_1 \cup \cdots \cup A_k = S$. Then for any event B,

$$P(B) = P(B|A_1)P(A_1) + \ldots + P(B|A_k)P(A_k).$$

Picture:



LoTP: $P(B) = P(B|A_1)P(A_1) + \ldots + P(B|A_k) \cdot P(A_k)$

Example 4: A hat contains papers with the numbers 1, 2, 3, 4, 5 on them.

We draw two numbers randomly from the hat without replacing them.

What is the probability the second number is even?

$A_1 = \{1^{st} \text{ odd}\}, \quad A_2 = \{1^{st} \text{ even}\}$

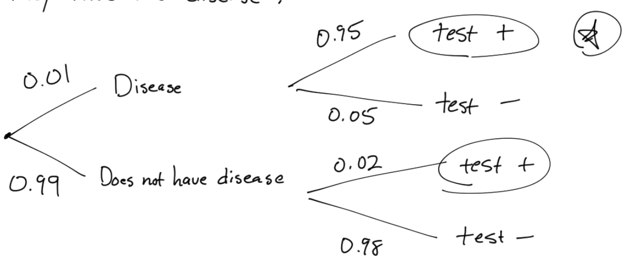
$$P(2^{nd} \text{ even}) = P(2^{nd} \text{ even} \mid 1^{st} \text{ odd}) \cdot P(1^{st} \text{ odd}) + P(2^{nd} \text{ even} \mid 1^{st} \text{ even}) \cdot P(1^{st} \text{ even})$$

$$= \boxed{\frac{2}{4} \cdot \frac{3}{5} + \frac{1}{4} \cdot \frac{2}{5}} = \frac{3}{10} + \frac{1}{10} = \boxed{\frac{2}{5}}$$

Bayes Theorem: Let $A_1, \ldots, A_k$ be disjoint and make up the entire sample space.

Then
$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)\, P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i)\, P(A_i)}$$

Example 5: In a certain population, 1 in 100 people have a certain disease. Given someone has the disease, the diagnostic test has a 95% chance of being positive. Given some one does not have the disease, the test has a 98% of being negative. If a random person received a positive test result, what is the probability they have the disease?



$$P(\text{disease} \mid \text{positive}) = \frac{0.01 \cdot 0.95}{0.01 \cdot 0.95 + 0.99 \cdot 0.02} = \frac{.0095}{.0095 + .0198} = \frac{.0095}{.0293} = \boxed{.324}$$

2.5 Independence:

A and B are <u>independent</u> events if $P(A|B) = P(A)$. Otherwise they are <u>dependent</u>.

<u>Interpretation</u>: B happening has no effect on whether or not A happens.

<u>Example 1</u>: We roll a fair 6-sided die.  Let A be the event that the roll is even. Let B be the event that the roll is at least 4.

Are A and B independent?

$A = \{2,4,6\}$ $\qquad P(A|B) = \frac{2}{3}$

$B = \{4,5,6\}$ $\qquad P(A) = \frac{3}{6}$

So A and B are <u>not</u> independent.

<u>Proposition</u>:
Two events A and B are independent iff $P(A \cap B) = P(A) \cdot P(B)$.

<u>Proof</u>:

$$P(A \cap B) \underset{\substack{\uparrow \\ \text{Multiplication} \\ \text{Rule}}}{=} P(A|B) \cdot P(B) \underset{\substack{\uparrow \\ \text{definition} \\ \text{of independent events}}}{=} P(A) \cdot P(B).$$

<u>Example 2</u>: There are 2 computers in an office. Assume each has a 10% chance of needing repairs this month, independently of each other. What is the probability both computers will need repairs this month?

$$P(\text{both need repairs}) = P(1^{st}\text{ needs repair}) \cdot P(2^{nd}\text{ needs repair})$$
$$= 0.1 \cdot 0.1 = \boxed{0.01}$$

Independence for more than 2 events:

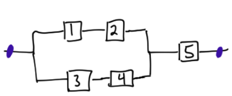
Events $A_1, \dots, A_n$ are mutually independent if for every $k = 2, 3, \dots, n$,

and every subset of indices $i_1, \dots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \dots \cdot P(A_{i_k})$$

Ex: $P(A_2 \cap A_4 \cap A_5 \cap A_{10}) = P(A_2) \cdot P(A_4) \cdot P(A_5) \cdot P(A_{10})$

Example 3: Consider 5 electrical components in this network:



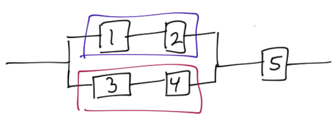
Assume each component has a 0.8 probability of working when the network is powered on, independently of each other.

For the system to work, either a path of working components must go from one end to the other. (In this case, either 1,2,5 all work, OR 3,4,5 all work.)

What is the probability that the system works?

Many possible approaches

Method 1:



Idea: we need at least one of the groups $\{1,2\}$ and $\{3,4\}$ to work.

Then we also need 5 to work no matter what.

$$P(1 \text{ and } 2 \text{ work}) = 0.8 \cdot 0.8 = 0.64$$
$$P(3 \text{ and } 4 \text{ work}) = 0.8 \cdot 0.8 = 0.64$$

$$P(\text{at least one of } \{1,2\} \text{ and } \{3,4\} \text{ work}) = 1 - P(\{1,2\} \text{ and } \{3,4\} \text{ both don't work})$$

$$= 1 - P(\{1,2\} \text{ doesn't work}) \cdot P(\{3,4\} \text{ doesn't work}) = 1 - (1-0.64)(1-0.64)$$
$$= 1 - (0.36)^2$$

Lastly, we also need $\boxed{5}$ to work:

$$P(\text{system works}) = \boxed{(1-(0.36)^2) \cdot 0.8} \approx \boxed{0.696}$$

Method 2: We need either path A: $\{1,2,5\}$ to all work, or path B: $\{3,4,5\}$ to all work.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{(inclusion-exclusion)}$$
$$= (0.8)^3 + (0.8)^3 - (0.8)^5 \approx \boxed{.696}$$

This method is nice with 2-3 paths, but inclusion-exclusion gets tricky with 4+ paths.