Chapter 4 Continuous Random Variables:

4.1 Probability Density Functions

A random variable is continuous if the possible values are an interval of the number line (or a union of several intervals), and $P(X=c)=0$ for all possible values c.

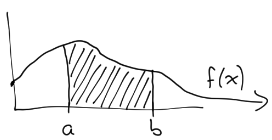
Ex: Temperature
Time
Mass

Probability Density Functions:

(Motivation in R)

Let X be a continuous RV. Then a probability density function (pdf) is a function $f(x)$ such that for any real numbers a and b, $a \le b$,

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$



For $f(x)$ to be a legitimate pdf, it must satisfy these conditions:

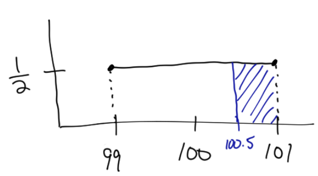
1. $f(x) \geq 0$ for all x (No negative probabilities)
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (All probabilities add up to 1)

Example 1: A factory makes string in 100 m lengths. Each string's actual length X is a continuous RV with pdf

$$f(x) = \begin{cases} \frac{1}{2}, & 99 \le x \le 101 \\ 0, & \text{otherwise} \end{cases}$$

What is $P(X \geq 100.5)$? What is $P(99.25 \le X \le 100)$?

$$P(X \geq 100.5) = \int_{100.5}^{101} \tfrac{1}{2}\,dy$$



$$= \boxed{\tfrac{1}{4}}$$

$$P(99.25 \le X \le 100) = \int_{99.25}^{100} \tfrac{1}{2}\,dy = \tfrac{1}{2}y \Big|_{99.25}^{100} = \boxed{\tfrac{3}{8}} = 0.375$$

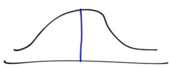
<u>Uniform Distribution</u>: A continuous RV X has a Uniform distribution on the interval $[A, B]$ if the pdf of X is

$$f(x) = \begin{cases} \frac{1}{B-A} & \text{if } A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

<u>Continuous RVs and endpoints</u>: $f(x)$

For any continuous RV X and any number c

$$P(X=c) = \int_c^c f(x)\,dx = 0$$



Because of this, the endpoints don't matter: (Only for Continuous RV)

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b).$$

<u>Example 2</u>: Let X be the lifetime in years of a certain electrical component. Assume X is a continuous RV with pdf

$$f(x) = \begin{cases} \frac{1}{9}x^2, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

What is the probability the component lasts for at least 1 year?

$$P(X \ge 1) = \int_1^3 \frac{1}{9}x^2\,dx = \frac{1}{9}\cdot\frac{x^3}{3}\Big|_1^3 = \frac{27}{27} - \frac{1}{27} = \boxed{\frac{26}{27}}.$$

4.2 Cumulative Distribution Functions and Expected Values

The Cumulative Distribution Function (CDF) for a continuous RV X is defined for every number x by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$$

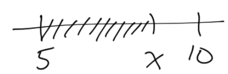
Example 1: Let X be the thickness of a metal sheet in mm.

Assume $X \sim \text{Uniform}(5, 10)$. Find the CDF for X.

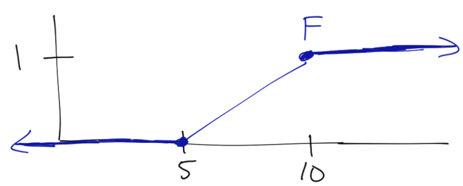
pdf for X is $\quad f(x) = \begin{cases} \frac{1}{10-5} = \frac{1}{5}, & 5 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$

Let $5 \le x \le 10$.

$$F(x) = P(X \le x) = \int_5^x \frac{1}{5}\,dy = \frac{1}{5}y \Big|_5^x = \frac{1}{5}x - 1$$



$$\boxed{F(x) = P(X \le x) = \begin{cases} 0 & x < 5 \\ \frac{1}{5}x - 1 & 5 \le x \le 10 \\ 1 & x > 10 \end{cases}}$$



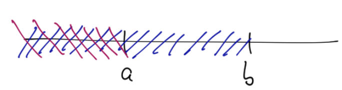
Using the CDF to find probabilities:

Let X be a continuous RV with pdf $f(x)$ and CDF $F(x)$ [$P(X \le x)$]. For any number a,

$$P(X > a) = P(X \ge a) = 1 - F(a)$$

For any real numbers a and b,

$$P(a \le X \le b) = P(a < X < b) = F(b) - F(a)$$



Ex:

$f(x) = 2x, \quad 0 < x < 1$

$F(x) = x^2, \quad 0 < x < 1$

$P(X \le \frac{1}{2}) = \int_0^{\frac{1}{2}} 2x\,dx$

$= F(\frac{1}{2})$

Example 2: Suppose the pdf of the magnitude X of a dynamic load on a bridge (in newtons) is given by

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

For any $0 \le x \le 2$, $\quad F(x) = \int_{-\infty}^{x} f(y)\,dy = \frac{x}{8} + \frac{3}{16}x^2$

Find $P(X > 1)$ and $P(0.5 < X \le 1)$.

$$P(X > 1) = 1 - F(1) = 1 - \left(\frac{1}{8} + \frac{3}{16}\right) = 1 - \frac{5}{16} = \boxed{\frac{11}{16}}$$

$$P(0.5 < X \le 1) = F(1) - F\left(\frac{1}{2}\right) = \frac{5}{16} - \left(\frac{1}{16} + \frac{3}{64}\right) = \frac{20}{64} - \frac{7}{64} = \boxed{\frac{13}{64}}$$

Obtaining $f(x)$ from $F(x)$:

If X is a continuous RV with pdf $f(x)$ and CDF $F(x)$, then for every x where $F'(x)$ exists, $F'(x) = f(x)$.

Example 3: Let X be a continuous RV with CDF

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^3 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Find the pdf of X.

$$f(x) = F'(x) = \frac{d}{dx} x^3 = 3x^2, \quad 0 < x < 1$$

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Percentiles and the Median

Let X be a continuous RV with pdf $f(x)$ and CDF $F(x)$.

The median $\tilde{\mu}$ of the distribution is the 50^{th} percentile:

$$0.5 = P(X \le \tilde{\mu}) = \int_{-\infty}^{\tilde{\mu}} f(y)\, dy = F(\tilde{\mu})$$

Similarly, the $(100p)^{th}$ percentile of the distribution of X, denoted by $\eta(p)$ is defined by

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)\, dy$$

Example 4: Let X be a continuous RV with pdf

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

What is the median $\tilde{\mu}$ of the distribution of X?

$$F(x) = \int_0^x f(t)\, dt = \int_0^x 2t\, dt = x^2, \quad 0 < x < 1$$

$$0.5 = F(\tilde{\mu}) = \tilde{\mu}^2, \quad \boxed{\tilde{\mu} = \sqrt{0.5}}$$

ALT:

$$0.5 = \int_0^{\tilde{\mu}} f(t)\, dt = \int_0^{\tilde{\mu}} 2t\, dt = t^2 \Big|_0^{\tilde{\mu}} = (\tilde{\mu})^2$$

$$(\tilde{\mu})^2 = 0.5$$

$$\tilde{\mu} = \sqrt{0.5}$$

Expected Value for Continuous RVs:

Let X be a continuous RV with pdf $f(x)$.

The expected value or mean of X is

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Mean for Example 4:

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

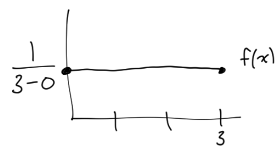
$$\mu_X = E[X] = \int_0^1 x \cdot f(x)\, dx = \int_0^1 x \cdot 2x\, dx = \frac{2}{3} x^3 \Big|_0^1 = \boxed{\frac{2}{3}}$$

For a function h,

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x)\, f(x)\, dx.$$

Example 5: Let $X \sim Unif(0,3)$ (Uniform dist.). Find $E[X]$ and $E\left[\frac{1}{X+1}\right]$.

X has pdf $f(x) = \begin{cases} \frac{1}{3}, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$



$$E[X] = \int_0^3 t \cdot \frac{1}{3}\, dt = \frac{t^2}{6}\Big|_0^3 = \frac{9}{6} = \boxed{\frac{3}{2}}$$

$$E\left[\frac{1}{X+1}\right] = \int_0^3 \frac{1}{t+1} \cdot \frac{1}{3}\, dt = \frac{1}{3} \ln|t+1| \Big|_0^3 = \frac{1}{3}\ln 4 - \frac{1}{3}\ln 1 = \boxed{\frac{1}{3}\ln 4}$$

Variance and Standard Deviation

Let X be a continuous RV with pdf $f(x)$. The variance of X is

$$Var(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = E[(X-\mu)^2]$$

The standard deviation of X is

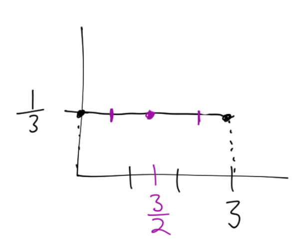
$$SD(X) = \sigma_X = \sqrt{Var(X)}$$

Usually, we use the shortcut formula for variance:

$$Var(X) = E[X^2] - (E[X])^2$$

Example 6: Let $X \sim Unif(0,3)$. We saw above $E[X] = \frac{3}{2}$. Find $Var(X)$.

$$f(x) = \begin{cases} \frac{1}{3}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$



$$E[X^2] = \int_0^3 x^2 \cdot \frac{1}{3}\, dx = \frac{x^3}{9}\Big|_0^3 = \frac{3^3}{9} - \frac{0^3}{9} = 3$$

$$Var(X) = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \boxed{\frac{3}{4}}$$

$$SD(X) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = .866$$