诺特定理

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1 有限自由度体系

1.1 一般讨论

设有限自由度体系有广义坐标 $q_\alpha(t)$, $\alpha = 1, 2, \ldots, r$，r 为体系的自由度，下面为了简化讨论将 $q_\alpha(t)$ 简记为 $q(t)$，下标 α 很容易还原到最后的结果中．体系的作用量可以写为

$$S[q; t_1, t_2] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\,\mathrm{d}t$$

由最小作用量原理：给定初始时刻 t_1 和终止时刻 t_2 的广义坐标取值 $q(t_1), q(t_2)$，体系从 $q(t_1)$ 运动到 $q(t_2)$ 所经历的真实路径是所有路径之中使得作用量取驻值的，即满足：

$$\frac{\delta S}{\delta q} = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0$$

这就是体系的运动方程，称为欧拉－拉格朗日方程．

考虑体系的一个变换

$$\begin{aligned} t \to t' &= f_\varepsilon(t) \\ q(t) \to q'(t') &\quad (q' = F_\varepsilon[q]) \end{aligned} \tag{1.1}$$

其中 ε 为变换的参数，并且 f_0, F_0 为恒等函数，中括号表示 $q(t)$ 函数本身（而不是函数值）作为自变量．变换后系统的运动规律一般对应一个新的作用量（即在 S 中将 (1.1) 式的逆代入）

$$\begin{aligned} S'[q'; t_1', t_2'] &= S[q; t_1, t_2] \\ &= \int_{t_1}^{t_2} L(q[q'](t), \dot{q}[q'](t), t)\,\mathrm{d}t \\ &= \int_{t_1'}^{t_2'} L'[q'; t']\,\mathrm{d}t' \end{aligned}$$

上面 S' 与所对应的 S 的值相同，但与 S 的表达式不一定相同．如果 S' 与 S 的表达式相同，即

$$\begin{aligned} S'[q'; t_1', t_2'] &= \int_{t_1'}^{t_2'} L(q'(t'), \dot{q}'(t), t')\,\mathrm{d}t' \\ &= S[q'; t_1', t_2'] \end{aligned}$$

或者等价地说有

$$S\left[q^{\prime};t_1^{\prime},t_2^{\prime}\right]=S\left[q;t_1,t_2\right] \tag{1.2}$$

那么称变换 (1.1) 是体系的一种**对称性**.

设变换 (1.1) 是对称性，接着我们来推导与之对应的守恒律. 已知

$$\begin{aligned}0=\Delta S&=S\left[q^{\prime};t_1^{\prime},t_2^{\prime}\right]-S\left[q;t_1,t_2\right]\\&=\int_{t_1^{\prime}}^{t_2^{\prime}}L\left(q^{\prime}\left(t^{\prime}\right),\dot{q}^{\prime}\left(t^{\prime}\right),t^{\prime}\right)\mathrm{d}t^{\prime}-\int_{t_1}^{t_2}L\left(q\left(t\right),\dot{q}\left(t\right),t\right)\mathrm{d}t\end{aligned}$$

有两种等效的方法处理上面的等式，第一种方法是将第一项积分中的 $t^{\prime}$ 直接写成 t，然后两式相减，这时由于积分限的不同会有边界项剩余；第二种方法是使用 (1.1) 式将 $t^{\prime}$ 代换回 t，这时积分限相同，但换元积分会产生一个 $f^{\prime}$ 因子. 下面进行具体计算.

第一种方法 考虑无穷小变换（$\varepsilon\ll1$，忽略 ε 的高阶无穷小）

$$\begin{aligned}t^{\prime}&=t+\delta t\\q^{\prime}\left(t\right)&=q\left(t\right)+\bar{\delta}q\end{aligned}$$

（$\bar{\delta}$ 表示同一 t 处两函数的差）有

$$\begin{aligned}0=\Delta S&=\int_{t_1^{\prime}}^{t_2^{\prime}}L\left(q^{\prime}\left(t\right),\dot{q}^{\prime}\left(t\right),t\right)\mathrm{d}t-\int_{t_1}^{t_2}L\left(q\left(t\right),\dot{q}\left(t\right),t\right)\mathrm{d}t\\&=\int_{t_1}^{t_2}\left\{L\left(q^{\prime}\left(t\right),\dot{q}^{\prime}\left(t\right),t\right)-L\left(q\left(t\right),\dot{q}\left(t\right),t\right)\right\}\mathrm{d}t+\left.L\right|_{t_2}\delta t_2-\left.L\right|_{t_1}\delta t_1\\&=\int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q}\bar{\delta}q+\frac{\partial L}{\partial\dot{q}}\bar{\delta}\dot{q}+\frac{\mathrm{d}}{\mathrm{d}t}\left(L\delta t\right)\right\}\mathrm{d}t\end{aligned} \tag{1.3}$$

因为 $\bar{\delta}$ 表示同一 t 处两函数的差，所以 $\bar{\delta}\dot{q}=\dfrac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}q$，(1.3) 式变成

$$\begin{aligned}0&=\int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q}\bar{\delta}q+\frac{\partial L}{\partial\dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}q+\frac{\mathrm{d}}{\mathrm{d}t}\left(L\delta t\right)\right\}\mathrm{d}t\\&=\int_{t_1}^{t_2}\left\{\left(\frac{\partial L}{\partial q}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}}\right)\bar{\delta}q+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\bar{\delta}q+L\delta t\right)\right\}\mathrm{d}t\end{aligned}$$

对于真实的运动路径，满足欧拉−拉格朗日方程，上式积分中第一项等于零，又由于上式对任意 t_1,t_2 都成立，所以有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\bar{\delta}q+L\delta t\right)=0$$

即得到了一个守恒量

$$\frac{\partial L}{\partial\dot{q}}\frac{\bar{\delta}q}{\varepsilon}+L\frac{\delta t}{\varepsilon}=\text{const. in time}$$

第二种方法 考虑无穷小变换（$\varepsilon\ll1$，忽略 ε 的高阶无穷小）

$$\begin{aligned}t^{\prime}&=t+\delta t\\q^{\prime}\left(t^{\prime}\right)&=q\left(t\right)+\delta q\end{aligned}$$

于是

$$
\begin{aligned}
0=\Delta S &= \int_{t_1}^{t_2} L\left(q'\left(t'\right),\dot{q}'\left(t'\right),t'\right)\left(1+\frac{\mathrm{d}}{\mathrm{d}t}\delta t\right)\mathrm{d}t-\int_{t_1}^{t_2} L\left(q\left(t\right),\dot{q}\left(t\right),t\right)\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left\{L\left(q'\left(t'\right),\dot{q}'\left(t'\right),t'\right)-L\left(q\left(t\right),\dot{q}\left(t\right),t\right)+L\frac{\mathrm{d}}{\mathrm{d}t}\delta t\right\}\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot{q}}\delta\dot{q}+\frac{\partial L}{\partial t}\delta t+L\frac{\mathrm{d}}{\mathrm{d}t}\delta t\right\}\mathrm{d}t
\end{aligned}
\tag{1.4}
$$

这里因为 δ 表示两个时刻两函数的差，对任意函数 $h\left(t\right)$ 有

$$
\begin{aligned}
\delta h\left(t\right) &= h'\left(t'\right)-h\left(t\right)=h'\left(t'\right)-h'\left(t\right)+h'\left(t\right)-h\left(t\right)\\
&= \dot{h}\left(t\right)\delta t+\bar{\delta}h\quad\left(+O\left(\varepsilon^2\right)\right)
\end{aligned}
$$

所以 (1.4) 式变成

$$
\begin{aligned}
0 &= \int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q}\left(\bar{\delta}q+\underline{\dot{q}\delta t}\right)+\frac{\partial L}{\partial \dot{q}}\left(\bar{\delta}\dot{q}+\underline{\ddot{q}\delta t}\right)+\underline{\frac{\partial L}{\partial t}\delta t}+L\frac{\mathrm{d}}{\mathrm{d}t}\delta t\right\}\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left\{\left(\frac{\partial L}{\partial q}\bar{\delta}q+\frac{\partial L}{\partial \dot{q}}\bar{\delta}\dot{q}\right)+\underline{\left(\frac{\partial L}{\partial q}\dot{q}+\frac{\partial L}{\partial \dot{q}}\ddot{q}+\frac{\partial L}{\partial t}\right)\delta t}+L\frac{\mathrm{d}}{\mathrm{d}t}\delta t\right\}\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q}\bar{\delta}q+\frac{\partial L}{\partial \dot{q}}\bar{\delta}\dot{q}+\frac{\mathrm{d}}{\mathrm{d}t}\left(L\delta t\right)\right\}\mathrm{d}t
\end{aligned}
$$

与 (1.3) 式完全相同，所以接下来重复第一种方法剩下的步骤，可得

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\frac{\bar{\delta}q}{\varepsilon}+L\frac{\delta t}{\varepsilon}\right)=0
$$

1.2 短程线与 Killing 矢量场

考虑在相对论情形下，粒子沿短程线的运动，作用量可写为[1]

$$
\begin{aligned}
S\left[x^\mu;\lambda_1,\lambda_2\right]=\tau &= \int_{\lambda 1}^{\lambda_2}\sqrt{g_{\mu\nu}\left(x^\alpha\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda\\
&= \int_{\lambda 1}^{\lambda_2} L\left(x^\mu,\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda
\end{aligned}
$$

其中 λ 为参数，注意上式中度规 $g_{\mu\nu}$ 是 x^μ 的函数．现考虑变换

$$
x^\mu\left(\lambda\right)\to x'^\mu\left(\lambda\right)=f^\mu_\varepsilon\left(x^\nu\left(\lambda\right)\right)
\tag{1.5}
$$

由 (1.2) 式可知，变换 (1.5) 是一种对称性的条件为

$$
S\left[x'^\mu;\lambda_1,\lambda_2\right]=S\left[x^\mu;\lambda_1,\lambda_2\right]
$$

即

$$
\int_{\lambda 1}^{\lambda_2}\sqrt{g_{\mu\nu}\left(x'^\alpha\right)\frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x'^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda=\int_{\lambda 1}^{\lambda_2}\sqrt{g_{\mu\nu}\left(x^\alpha\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda
\tag{1.6}
$$

[1]这里使用 $+---$ 度规，并且这里的 L 与相对论力学中的拉氏量有所区别．

显然，若

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}g_{\mu\nu}\left(f_{\varepsilon}^{\alpha}\left(x^{\beta}\right)\right)=g_{\rho\sigma}\left(x^{\beta}\right)$$

则 (1.6) 式成立．上式也可以写成

$$f_{\varepsilon}^{*}\left(g\left(f_{\varepsilon}\left(x\right)\right)\right)=g\left(x\right)$$

其中 f_{ε}^{*} 为拉回（pullback）映射，按照李导数的定义还可以写成

$$\mathfrak{L}_{X}g=\lim_{\varepsilon\to 0}\frac{f_{\varepsilon}^{*}\left(g\left(f_{\varepsilon}\left(x\right)\right)\right)-g\left(x\right)}{\varepsilon}=0 \tag{1.7}$$

其中 $X^{\mu}=\frac{\mathrm{d}f^{\mu}}{\mathrm{d}\varepsilon}=\frac{\bar{\delta}x^{\mu}}{\varepsilon}$ 为 (1.5) 式的无穷小形式所对应的矢量场．换句话说，满足 (1.7) 式的无穷小变换是体系的一种对称性，由诺特定理，这种对称性对应体系的一个守恒方程

$$\begin{aligned}0=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial\left(\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\lambda}\right)}\frac{\bar{\delta}x^{\rho}}{\varepsilon}\right)&=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{g_{\rho\mu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}}{\sqrt{g_{\mu\nu}\left(x^{\alpha}\right)\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda}}}X^{\rho}\right)\\&=\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{g_{\rho\mu}X^{\rho}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}}{\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}}\end{aligned}$$

即有

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(X_{\mu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\right)=0$$

满足 (1.7) 式的矢量场 X^{μ} 称为 **Killing 矢量场**.

2 场论

2.1 一般讨论

场论中的诺特定理的推导与有限自由度体系是相似的，上述讨论可以很容易地类比到场论的情形.

考虑 n 维空间中的场 $\varphi\left(x^{\mu}\right)$，其中 φ 可能包含有多个场以及它们的各个分量，$\mu=0,\ldots,n-1$或者$1,\ldots n$．φ 的运动方程可以从作用量

$$S\left[\varphi;\Omega\right]=\int_{\Omega}\mathcal{L}\left(\varphi,\partial_{\mu}\varphi,x^{\mu}\right)\mathrm{d}^{n}x$$

的具体形式得出．由最小作用量原理，对于给定的边界条件 $\varphi|_{\partial\Omega}=h\left(x\right)$，真实的场 φ_{re} 使得作用量取驻值，即满足

$$\frac{\delta S}{\delta\varphi}=\frac{\partial\mathcal{L}}{\partial\varphi}-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\varphi\right)}\right)=0$$

现在考虑体系的一个变换，在此变换下：

$$\begin{aligned}x^{\mu}\to x'^{\mu}=f_{\varepsilon}^{\mu}\left(x^{\nu}\right)\\ \varphi\left(x^{\mu}\right)\to\varphi'\left(x'^{\mu}\right)\quad\left(\varphi'=F_{\varepsilon}\left[\varphi\right]\right)\end{aligned} \tag{2.1}$$

（ε 是变换的参数，且当 $\varepsilon=0$ 时为恒等变换）．变换 (2.1) 是一种对称性的条件与上一节的 (1.2) 式类似，为

$$S\left[\varphi';\Omega'\right]=S\left[\varphi;\Omega\right]$$

设 (2.1) 是一种对称性，那么有

$$\Delta S=\int_{\Omega'}\mathcal{L}\left(\varphi',\partial'_\mu\varphi',x'^\mu\right)\mathrm{d}^nx'-\int_\Omega\mathcal{L}\left(\varphi,\partial_\mu\varphi,x^\mu\right)\mathrm{d}^nx=0$$

为了导出与之相应守恒律，依然有两种等效的处理方法：一种是将上式左边的 x' 直接换为 x，然后相减，由于积分区域 Ω' 与 Ω 不一致，会导致多出一个边界项；另一种是将 $x'^\mu=f^\mu_\varepsilon\left(x^\nu\right)$ 代回上式左边，然后再相减，这时积分区域一致，但积分微元在从 d^nx' 变换为 d^nx 时出现一个雅可比行列式，也会多出一项．下面分别按这两种方法进行推导．

第一种方法　考虑无穷小变换（$\varepsilon\ll1$，忽略 ε 的高阶无穷小）

$$x'^\mu=x^\mu+\delta x^\mu$$
$$\varphi'\left(x^\mu\right)=\varphi\left(x^\mu\right)+\bar{\delta}\varphi$$

（$\bar{\delta}$ 表示同一 x^μ 处两函数的差），有[2]

$$\begin{aligned}0=\Delta S&=\int_{\Omega'}\mathcal{L}\left(\varphi',\partial_\mu\varphi',x^\mu\right)\mathrm{d}^nx-\int_\Omega\mathcal{L}\left(\varphi,\partial_\mu\varphi,x^\mu\right)\mathrm{d}^nx\\&=\int_\Omega\left\{\mathcal{L}\left(\varphi',\partial_\mu\varphi',x^\mu\right)-\mathcal{L}\left(\varphi,\partial_\mu\varphi,x^\mu\right)\right\}\mathrm{d}^nx+\int_{\partial\Omega}\mathcal{L}\delta x^\mu\mathrm{d}s_\mu\\&=\int_\Omega\left\{\frac{\partial\mathcal{L}}{\partial\varphi}\bar{\delta}\varphi+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\bar{\delta}\left(\partial_\mu\varphi\right)\right\}\mathrm{d}^nx+\int_\Omega\partial_\mu\left(\mathcal{L}\delta x^\mu\right)\mathrm{d}^nx\quad\text{（斯托克斯定理）}\end{aligned}\tag{2.2}$$

因为 $\bar{\delta}$ 表示同一 x^μ 处两函数的差，所以 $\bar{\delta}$ 与 ∂_μ 可交换次序．再考虑到真实场 φ 满足欧拉－拉格朗日方程，(2.2) 式可化成

$$\begin{aligned}0&=\int_\Omega\left\{\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\right)\bar{\delta}\varphi+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\partial_\mu\left(\bar{\delta}\varphi\right)+\partial_\mu\left(\mathcal{L}\delta x^\mu\right)\right\}\mathrm{d}^nx\\&=\int_\Omega\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\bar{\delta}\varphi+\mathcal{L}\delta x^\mu\right)\mathrm{d}^nx\end{aligned}\tag{2.3}$$

上式对于任意区域 Ω 都成立．因此有守恒流方程

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\bar{\delta}\varphi+\mathcal{L}\delta x^\mu\right)=0$$

可以记

$$j^\mu\equiv\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\frac{\bar{\delta}\varphi}{\varepsilon}+\mathcal{L}\frac{\delta x^\mu}{\varepsilon}\tag{2.4}$$
$$\partial_\mu j^\mu=0$$

第二种方法　考虑无穷小变换（$\varepsilon\ll1$，忽略 ε 的高阶无穷小）：

$$x'^\mu=x^\mu+\delta x^\mu$$

[2]这里 $\mathrm{d}s_\mu=(-1)^{\mu-1}\,\mathrm{d}x^1\wedge\cdots\wedge\cancel{\mathrm{d}x^\mu}\wedge\cdots\wedge\mathrm{d}x^n$ 是边界的微分形式．

$$\varphi'(x'^\mu) = \varphi(x^\mu) + \delta\varphi$$

$$\begin{aligned}
\mathrm{d}^n x' &= \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| \mathrm{d}^n x = \left|1 + \frac{\partial \delta x^\mu}{\partial x^\nu}\right| \mathrm{d}^n x \\
&= \left\{1 + \mathrm{tr}\left(\frac{\partial \delta x^\mu}{\partial x^\nu}\right)\right\} \mathrm{d}^n x \\
&= \{1 + \partial_\mu(\delta x^\mu)\} \mathrm{d}^n x
\end{aligned}$$

于是

$$\begin{aligned}
0 = \Delta S &= \int_\Omega \mathcal{L}(\varphi', \partial'_\mu\varphi', x'^\mu)\{1 + \partial_\mu(\delta x^\mu)\} \mathrm{d}^n x - \int_\Omega \mathcal{L}(\varphi, \partial_\mu\varphi, x^\mu) \mathrm{d}^n x \\
&= \int_\Omega \{\mathcal{L}(\varphi', \partial'_\mu\varphi', x'^\mu) - \mathcal{L}(\varphi, \partial_\mu\varphi, x^\mu)\} \mathrm{d}^n x + \int_\Omega \mathcal{L}\partial_\mu(\delta x^\mu) \mathrm{d}^n x \\
&= \int_\Omega \left\{\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu)\right\} \mathrm{d}^n x
\end{aligned}$$

（这里用 $\dfrac{\partial\mathcal{L}}{\partial x^\mu}$ 表示函数 $\mathcal{L}(\bullet, \bullet, x^\mu)$ 对 x^μ 求导，$\partial_\mu\mathcal{L}$ 表示 $\mathcal{L}(\varphi, \partial_\mu\varphi, x^\mu)$ 作为复合函数对 x^μ 求导）．由

$$\begin{aligned}
\delta h &= h'(x'^\mu) - h(x) = h'(x'^\mu) - h'(x^\mu) + h'(x^\mu) - h(x^\mu) \\
&= \delta x^\mu \partial_\mu h(x^\nu) + \bar{\delta} h \quad (+O(\varepsilon^2))
\end{aligned}$$

可得

$$\begin{aligned}
0 = &\int_\Omega \Bigg\{\frac{\partial\mathcal{L}}{\partial\varphi}\left(\bar{\delta}\varphi + \underline{\delta x^\mu\partial_\mu\varphi}\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\left(\bar{\delta}(\partial_\mu\varphi) + \underline{\delta x^\nu\partial_\nu(\partial_\mu\varphi)}\right) \\
&+ \underline{\frac{\partial\mathcal{L}}{\partial x^\mu}}\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu)\Bigg\} \mathrm{d}^n x \\
= &\int_\Omega \Bigg\{\frac{\partial\mathcal{L}}{\partial\varphi}\bar{\delta}\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\bar{\delta}(\partial_\mu\varphi) + \Bigg(\underline{\frac{\partial\mathcal{L}}{\partial\varphi}\partial_\mu\varphi} + \underline{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)}\partial_\mu(\partial_\nu\varphi)} \\
&+ \underline{\frac{\partial\mathcal{L}}{\partial x^\mu}}\Bigg)\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu)\Bigg\} \mathrm{d}^n x \\
= &\int_\Omega \left\{\frac{\partial\mathcal{L}}{\partial\varphi}\bar{\delta}\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\bar{\delta}(\partial_\mu\varphi) + \underline{\partial_\mu\mathcal{L}}\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu)\right\} \mathrm{d}^n x \\
= &\int_\Omega \left\{\frac{\partial\mathcal{L}}{\partial\varphi}\bar{\delta}\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\bar{\delta}(\partial_\mu\varphi) + \partial_\mu(\mathcal{L}\delta x^\mu)\right\} \mathrm{d}^n x
\end{aligned}$$

与 (2.2) 式相同．因此与之前一样可以得到守恒流方程

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\frac{\bar{\delta}\varphi}{\varepsilon} + \mathcal{L}\frac{\delta x^\mu}{\varepsilon}\right) = 0$$

2.2 拓展

2.2.1 推广

从上述推导过程可以看出，要想得到守恒流方程，只需 $\Delta S=0$ 在 $\varphi=\varphi_{\mathrm{re}}$ 处成立即可，将这样的变换称为在壳（on-shell）对称性；而对于任意的 φ 都有 $\Delta S=0$ 的变换称为离壳（off-shell）对称性．除此以外，对称性的概念还可以进一步推广，变换只需满足

$$\begin{aligned}\Delta S&=\int_{\partial\Omega}\varepsilon f^{\mu}\left(\varphi,\partial\varphi,\partial\partial\varphi,\ldots\right)\mathrm{d}s_{\mu}\\&=\int_{\Omega}\varepsilon\partial_{\mu}f^{\mu}\left(\varphi,\partial\varphi,\partial\partial\varphi,\ldots\right)\mathrm{d}^{n}x\end{aligned}$$

即可称为一种对称性．此时只需修改 (2.3) 式：

$$\int_{\Omega}\varepsilon\partial_{\mu}f^{\mu}\left(\varphi,\partial\varphi,\partial\partial\varphi,\ldots\right)\mathrm{d}^{n}x=\int_{\Omega}\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\varphi\right)}\bar{\delta}\varphi+\mathcal{L}\delta x^{\mu}\right)\mathrm{d}^{n}x$$

得到守恒流方程

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\varphi\right)}\frac{\bar{\delta}\varphi}{\varepsilon}+\mathcal{L}\frac{\delta x^{\mu}}{\varepsilon}-f^{\mu}\right)=0$$

2.2.2 李导数记法

通常，对于所研究的场 φ 我们都认为：在 (2.1) 式的坐标变换

$$x^{\mu}\rightarrow x'^{\mu}=f_{\varepsilon}^{\mu}\left(x^{\nu}\right)$$

下，φ 同时会有一个伴随的变换，例如矢量场在转动变换下除了场在空间中移动外每一点处场矢量的箭头指向也会跟随着转动．我们将这种伴随的变换记为

$$\varphi\left(x^{\mu}\right)\rightarrow\varphi'\left(x'^{\mu}\right)=f_{\varepsilon*}\left(\varphi\left(f_{\varepsilon}^{\nu-1}\left(x'^{\mu}\right)\right)\right)\tag{2.5}$$

$f_{\varepsilon*}$ 称为推前（pushforward）映射．考虑 $\varepsilon\ll1$ 的无穷小变换，由坐标变换的伴随变换导致的场的变化为

$$\bar{\delta}_{\mathrm{c}}\varphi=f_{\varepsilon*}\left(\varphi\left(f_{\varepsilon}^{\nu-1}\left(x^{\mu}\right)\right)\right)-\varphi\left(x^{\mu}\right)=-\varepsilon\left(\mathfrak{L}_{X}\varphi\right)\left(x^{\mu}\right)$$

其中 $\mathfrak{L}_{X}\varphi$ 为场的李导数，$X^{\mu}=\dfrac{1}{\varepsilon}\delta_{\mathrm{c}}x^{\mu}$ 为无穷小坐标变换对应的矢量场．如果除坐标变换引起的场的伴随变换以外，还有内禀变换

$$\varphi\left(x^{\mu}\right)\rightarrow\varphi'\left(x^{\mu}\right)=F_{\varepsilon}\left(\varphi\left(x^{\mu}\right)\right)\tag{2.6}$$

$$\begin{aligned}\delta_{\mathrm{i}}x^{\mu}&=0\\\bar{\delta}_{\mathrm{i}}\varphi&=\delta_{\mathrm{i}}\varphi=\varphi'\left(x^{\mu}\right)-\varphi\left(x^{\mu}\right)\\&=\varepsilon\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}F_{\varepsilon}\left(\varphi\right)\equiv\varepsilon\Phi\end{aligned}$$

那么对 (2.5)(2.6) 两变换的复合，有

$$\bar{\delta}\varphi = \bar{\delta}_{\mathrm{c}}\varphi + \bar{\delta}_{\mathrm{i}}\varphi = -\varepsilon \mathfrak{L}_X \varphi + \varepsilon \Phi$$

若变换是对称性，这时诺特流 (2.4) 可写成

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)} \frac{\bar{\delta}\varphi}{\varepsilon} + \mathcal{L}\frac{\delta x^\mu}{\varepsilon} \\ &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)}\left(-\mathfrak{L}_X \varphi + \Phi\right) + \mathcal{L}X^\mu \\ &= \left(-\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)} \mathfrak{L}_X \varphi + \mathcal{L}X^\mu\right) + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)}\Phi \end{aligned}$$

上式中的第一项对应坐标变换对称性，第二项对应场的内禀对称性.