方差相关公式

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1　样本方差与总体方差

$$
\begin{aligned}
E\left\{\frac{1}{N}\sum_{i=1}^{N}\left(x_i-\frac{1}{N}\sum_{j=1}^{N}x_j\right)^2\right\} &= E\left\{\frac{1}{N}\sum_{i=1}^{N}\left(x_i^2+\frac{1}{N^2}\sum_{j=1}^{N}x_j^2-2x_i\frac{1}{N}\sum_{j=1}^{N}x_j+\frac{1}{N^2}\sum^{N}x_{j_1}x_{j_2}\right)\right\} \\
&= \frac{1}{N}\Bigg(NE\left[x^2\right]+E\left[x^2\right]-\frac{2}{N}\left(N\left(N-1\right)\left(E\left[x\right]\right)^2+NE\left[x^2\right]\right) \\
&\qquad +\frac{1}{N}N\left(N-1\right)\left(E\left[x\right]\right)^2\Bigg) \\
&= E\left[x^2\right]+\frac{1}{N}E\left[x^2\right]-2\frac{\left(N-1\right)\left(E\left[x\right]\right)^2+E\left[x^2\right]}{N}+\frac{N-1}{N}\left(E\left[x\right]\right)^2 \\
&= \frac{N-1}{N}E\left[x^2\right]-\frac{N-1}{N}\left(E\left[x\right]\right)^2 \\
&= \frac{N-1}{N}\mathrm{Var}\left[x\right]
\end{aligned}
$$

所以定义

$$
s^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i-\frac{1}{N}\sum_{j=1}^{N}x_j\right)^2
$$

$$
E\left[s^2\right]=\mathrm{Var}\left[x\right]
$$

2　均值方差递推式

$$
S_N\equiv\sum_{i=1}^{N}x_i,\quad \bar{x}_N\equiv\frac{S_N}{N},\quad F_N\equiv\sum_{i=1}^{N}\left(x_i-\bar{x}_N\right)^2
$$

$$
\begin{aligned}
F_N &= \sum_{i=1}^{N}x_i^2-N\bar{x}_N^2 \\
&= F_{N-1}+x_N^2+\left(N-1\right)\bar{x}_{N-1}^2-N\bar{x}_N^2 \\
&= F_{N-1}+x_N^2+S_{N-1}\bar{x}_{N-1}-S_N\bar{x}_N \\
&= F_{N-1}+x_N^2+\left(S_N-x_N\right)\bar{x}_{N-1}-\left(S_{N-1}+x_N\right)\bar{x}_N
\end{aligned}
$$

$$
\begin{aligned}
&= F_{N-1} + x_N^2 - x_N\left(\bar{x}_{N-1} + \bar{x}_N\right) + N\bar{x}_N\bar{x}_{N-1} - (N-1)\,\bar{x}_N\bar{x}_{N-1} \\
&= F_{N-1} + \left(x_N - \bar{x}_{N-1}\right)\left(x_N - \bar{x}_N\right)
\end{aligned}
$$

均值递推式

$$
\bar{x}_N = \frac{(N-1)\,\bar{x}_{N-1} + x_N}{N} = \bar{x}_{N-1} + \frac{x_N - \bar{x}_{N-1}}{N}
$$