Warped Product 相关公式的推导

2017 年 11 月 14 日

给定两个装备了度规的微分流形 (M_1, g),(M_2, h)，流形的 warped product 定义为：$M = M_1 \otimes M_2$，且 M 上的度规可以写为

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu + \mathrm{e}^{\Phi(x)} h_{ab} dy^a dy^b \\
&= \hat{g}_{ij} d\hat{x}^i d\hat{x}^j
\end{aligned}
$$

1 联络

由联络的公式

$$
\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right)
$$

$$
\begin{aligned}
\hat{\Gamma}^c_{ab} &= \frac{1}{2} \hat{g}^{cd} \left(\partial_a \hat{g}_{bd} + \partial_b \hat{g}_{da} - \partial_d \hat{g}_{ab} \right) \\
&= \frac{1}{2} \mathrm{e}^{-\Phi} h^{cd} \mathrm{e}^{\Phi} \left(\partial_a h_{bd} + \partial_b h_{da} - \partial_d h_{ab} \right) \\
&= \frac{1}{2} h^{cd} \left(\partial_a h_{bd} + \partial_b h_{da} - \partial_d h_{ab} \right) \\
&= \Gamma^c_{ab} \qquad (1.1)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Gamma}^\mu_{ab} &= \frac{1}{2} \hat{g}^{\mu\lambda} \left(\partial_a \hat{g}_{b\lambda} + \partial_b \hat{g}_{\lambda a} - \partial_\lambda \hat{g}_{ab} \right) \\
&= -\frac{1}{2} g^{\mu\lambda} \partial_\lambda \left(\mathrm{e}^{\Phi} h_{ab} \right) \\
&= -\frac{1}{2} \left(\nabla^\mu \Phi \right) \mathrm{e}^{\Phi} h_{ab} \\
&= -\frac{1}{2} \left(\nabla^\mu \Phi \right) \hat{g}_{ab} \qquad (1.2)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Gamma}^b_{a\mu} &= \frac{1}{2} \hat{g}^{bc} \left(\partial_a \hat{g}_{\mu c} + \partial_\mu \hat{g}_{ca} - \partial_c \hat{g}_{a\mu} \right) \\
&= \frac{1}{2} \mathrm{e}^{-\Phi} h^{bc} \partial_\mu \left(\mathrm{e}^{\Phi} h_{ca} \right) \\
&= \frac{1}{2} \mathrm{e}^{-\Phi} h^{bc} h_{ca} \mathrm{e}^{\Phi} \nabla_\mu \Phi \\
&= \frac{1}{2} \delta^b_a \nabla_\mu \Phi \qquad (1.3)
\end{aligned}
$$

$$\hat{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} \tag{1.4}$$

$$\hat{\Gamma}^{a}_{\mu\nu} = \frac{1}{2}\hat{g}^{ab}\left(\partial_\mu \hat{g}_{\nu b} + \partial_\nu \hat{g}_{b\mu} - \partial_b \hat{g}_{\mu\nu}\right) = 0 \tag{1.5}$$

$$\hat{\Gamma}^{\rho}_{\mu a} = \frac{1}{2}\hat{g}^{\rho\lambda}\left(\partial_\mu \hat{g}_{a\lambda} + \partial_a \hat{g}_{\lambda\mu} - \partial_\lambda \hat{g}_{\mu a}\right) = 0 \tag{1.6}$$

2 黎曼曲率张量

由公式

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$$\begin{aligned}
\hat{R}^{d}{}_{cab} &= R^{d}{}_{cab} + \hat{\Gamma}^{d}_{a\lambda}\hat{\Gamma}^{\lambda}_{bc} - \hat{\Gamma}^{d}_{b\lambda}\hat{\Gamma}^{\lambda}_{ac} \\
&= R^{d}{}_{cab} - \frac{1}{4}\delta^{d}_{a}\nabla_\lambda \Phi \nabla^\lambda \Phi \hat{g}_{bc} + \frac{1}{4}\delta^{d}_{b}\nabla_\lambda \Phi \nabla^\lambda \Phi \hat{g}_{ac} \\
&= R^{d}{}_{cab} - \frac{1}{4}\left(\hat{g}_{cb}\delta^{d}_{a} - \hat{g}_{ca}\delta^{d}_{b}\right)\nabla_\lambda \Phi \nabla^\lambda \Phi
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\hat{R}^{\mu}{}_{a\nu b} &= \partial_\nu \hat{\Gamma}^{\mu}_{ba} - \partial_b \hat{\Gamma}^{\mu}_{\nu a} + \hat{\Gamma}^{\mu}_{\nu\lambda}\hat{\Gamma}^{\lambda}_{ba} - \hat{\Gamma}^{\mu}_{bc}\hat{\Gamma}^{c}_{\nu a} \\
&= \left(\partial_\nu \hat{\Gamma}^{\mu}_{ba} + \hat{\Gamma}^{\mu}_{\nu\lambda}\hat{\Gamma}^{\lambda}_{ba}\right) + \frac{1}{4}\left(\nabla^\mu \Phi\right)\hat{g}_{bc}\delta^{c}_{a}\nabla_\nu \Phi \\
&= -\frac{1}{2}\nabla_\nu\left(\hat{g}_{ab}\nabla^\mu \Phi\right) + \frac{1}{4}\nabla^\mu \Phi \nabla_\nu \Phi \hat{g}_{ba} \\
&= -\frac{1}{2}\left(\nabla_\nu \nabla^\mu \Phi\right)\hat{g}_{ab} - \frac{1}{2}\nabla^\mu \Phi \nabla_\nu \Phi \hat{g}_{ab} + \frac{1}{4}\nabla^\mu \Phi \nabla_\nu \Phi \hat{g}_{ab} \\
&= -\frac{1}{2}\left(\nabla_\nu \nabla^\mu \Phi + \frac{1}{2}\nabla^\mu \Phi \nabla_\nu \Phi\right)\hat{g}_{ab}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\hat{R}^{a}{}_{\mu b\nu} &= \partial_b \hat{\Gamma}^{a}_{\nu\mu} - \partial_\nu \hat{\Gamma}^{a}_{b\mu} + \hat{\Gamma}^{a}_{b\lambda}\hat{\Gamma}^{\lambda}_{\nu\mu} - \hat{\Gamma}^{a}_{\nu c}\hat{\Gamma}^{c}_{b\mu} \\
&= -\left(\partial_\nu \hat{\Gamma}^{a}_{b\mu} - \hat{\Gamma}^{\lambda}_{\nu\mu}\hat{\Gamma}^{a}_{b\lambda}\right) - \hat{\Gamma}^{a}_{\nu c}\hat{\Gamma}^{c}_{b\mu} \\
&= -\frac{1}{2}\nabla_\nu\left(\delta^{a}_{b}\nabla_\mu \Phi\right) - \frac{1}{4}\delta^{a}_{c}\nabla_\nu \Phi \delta^{c}_{b}\nabla_\mu \Phi \\
&= -\frac{1}{2}\delta^{a}_{b}\left(\nabla_\nu \nabla_\mu \Phi + \frac{1}{2}\nabla_\nu \Phi \nabla_\mu \Phi\right)
\end{aligned} \tag{2.3}$$

$$\hat{R}^{\sigma}{}_{\rho\mu\nu} = R^{\sigma}{}_{\rho\mu\nu} \tag{2.4}$$

$$\begin{aligned}
\hat{R}^{\mu}{}_{abc} &= \partial_b \hat{\Gamma}^{\mu}_{ca} - \partial_c \hat{\Gamma}^{\mu}_{ba} + \hat{\Gamma}^{\mu}_{bd}\hat{\Gamma}^{d}_{ca} - \hat{\Gamma}^{\mu}_{cd}\hat{\Gamma}^{d}_{ba} \\
&= \left(\partial_b \hat{\Gamma}^{\mu}_{ca} - \hat{\Gamma}^{d}_{ba}\hat{\Gamma}^{\mu}_{cd} - \hat{\Gamma}^{d}_{bc}\hat{\Gamma}^{\mu}_{da}\right) - \left(\partial_c \hat{\Gamma}^{\mu}_{ba} - \hat{\Gamma}^{d}_{ca}\hat{\Gamma}^{\mu}_{bd} - \hat{\Gamma}^{d}_{cb}\hat{\Gamma}^{\mu}_{da}\right) \\
&= -\frac{1}{2}\left(\nabla^\mu \Phi\right)\nabla_b \hat{g}_{ca} + \frac{1}{2}\left(\nabla^\mu \Phi\right)\nabla_c \hat{g}_{ba}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left(\nabla^{\mu}\Phi\right)\mathrm{e}^{\Phi(x)}\left(\nabla_b h_{ca} - \nabla_c h_{ba}\right) \\
&= 0
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
\hat{R}^{\mu}{}_{\nu ab} &= \partial_a \hat{\Gamma}^{\mu}_{b\nu} - \partial_b \hat{\Gamma}^{\mu}_{a\nu} + \hat{\Gamma}^{\mu}_{ac}\hat{\Gamma}^{c}_{b\nu} - \hat{\Gamma}^{\mu}_{bc}\hat{\Gamma}^{c}_{a\nu} \\
&= \hat{\Gamma}^{\mu}_{ac}\hat{\Gamma}^{c}_{b\nu} - \hat{\Gamma}^{\mu}_{bc}\hat{\Gamma}^{c}_{a\nu} \\
&= -\frac{1}{4}\left(\nabla^{\mu}\Phi\right)\hat{g}_{ac}\delta^{c}_{b}\nabla_{\nu}\Phi + \frac{1}{4}\left(\nabla^{\mu}\Phi\right)\hat{g}_{bc}\delta^{c}_{a}\nabla_{\nu}\Phi \\
&= -\frac{1}{4}\nabla^{\mu}\Phi\nabla_{\nu}\Phi\left(\hat{g}_{ab} - \hat{g}_{ba}\right) \\
&= 0
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\hat{R}^{a}{}_{\rho\mu\nu} &= \partial_{\mu}\hat{\Gamma}^{a}_{\nu\rho} - \partial_{\nu}\hat{\Gamma}^{a}_{\mu\rho} + \hat{\Gamma}^{a}_{\mu\lambda}\hat{\Gamma}^{\lambda}_{\nu\rho} - \hat{\Gamma}^{a}_{\nu\lambda}\hat{\Gamma}^{\lambda}_{\mu\rho} \\
&= 0
\end{aligned}
\tag{2.7}
$$

3 里奇张量

由定义

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

$$
\begin{aligned}
\hat{R}_{\mu\nu} &= \hat{R}^{\lambda}{}_{\mu\lambda\nu} + \hat{R}^{a}{}_{\mu a\nu} \\
&= R_{\mu\nu} - \frac{1}{2}\delta^{a}_{a}\left(\nabla_{\nu}\nabla_{\mu}\Phi + \frac{1}{2}\nabla_{\nu}\Phi\nabla_{\mu}\Phi\right) \\
&= R_{\mu\nu} + nA_{\mu\nu}
\end{aligned}
\tag{3.1}
$$

其中

$$n = \dim M_2, \quad A_{\mu\nu} = -\frac{1}{2}\left(\nabla_{\nu}\nabla_{\mu}\Phi + \frac{1}{2}\nabla_{\nu}\Phi\nabla_{\mu}\Phi\right)$$

$$
\begin{aligned}
\hat{R}_{ab} &= \hat{R}^{c}{}_{acb} + \hat{R}^{\lambda}{}_{a\lambda b} \\
&= R_{ab} - \frac{1}{4}\left(\hat{g}_{ab}\delta^{c}_{c} - \hat{g}_{ac}\delta^{c}_{b}\right)\nabla_{\lambda}\Phi\nabla^{\lambda}\Phi + \hat{R}^{\lambda}{}_{a\lambda b} \\
&= R_{ab} - \frac{n-1}{4}\hat{g}_{ab}\nabla_{\lambda}\Phi\nabla^{\lambda}\Phi - \frac{1}{2}\left(\nabla_{\lambda}\nabla^{\lambda}\Phi + \frac{1}{2}\nabla^{\lambda}\Phi\nabla_{\lambda}\Phi\right)\hat{g}_{ab} \\
&= R_{ab} - \frac{1}{2}\left(\nabla_{\lambda}\nabla^{\lambda}\Phi + \frac{n}{2}\nabla^{\lambda}\Phi\nabla_{\lambda}\Phi\right)\hat{g}_{ab} \\
&= R_{ab} + B\hat{g}_{ab}
\end{aligned}
\tag{3.2}
$$

其中

$$B = -\frac{1}{2}\left(\nabla_{\lambda}\nabla^{\lambda}\Phi + \frac{n}{2}\nabla^{\lambda}\Phi\nabla_{\lambda}\Phi\right)$$

$$\hat{R}_{\mu a} = 0 \tag{3.3}$$

4 曲率标量

由定义

$$R = R^{\mu}{}_{\mu}$$

$$\begin{aligned}
\hat{R} &= \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + \hat{g}^{ab}\hat{R}_{ab} \\
&= \hat{g}^{\mu\nu}R_{\mu\nu} + n\hat{g}^{\mu\nu}A_{\mu\nu} + \hat{g}^{ab}R_{ab} + B\hat{g}^{ab}\hat{g}_{ab} \\
&= g^{\mu\nu}R_{\mu\nu} + nA + \mathrm{e}^{-\Phi}h^{ab}R_{ab} + nB \\
&= R^{(g)} + \mathrm{e}^{-\Phi}R^{(h)} + n\left(A + B\right)
\end{aligned} \tag{4.1}$$

其中

$$A = A^{\mu}{}_{\mu}$$