Exercise 7

(2) $f_c = 1 - \frac{1}{\kappa - 1}$, $\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle}$

a) Erdös - Rényi

$\langle k \rangle = p(N-1)$

$\langle k^2 \rangle = p(1-p)(N-1) + p^2(N-1)^2 =$
$= p(N-1)\left((1-p) + p(N-1)\right) =$
$= \langle k \rangle (1 + \langle k \rangle)$

$$f_c^{ER} = 1 - \frac{1}{\frac{\langle k \rangle(1+\langle k \rangle)}{\langle k \rangle} - 1} = 1 - \frac{1}{1 + \langle k \rangle - 1} =$$

$$= 1 - \frac{1}{\langle k \rangle} = 1 - \frac{1}{p(N-1)}$$

for $N \to \infty$, $f_c^{ER} \to 1$
we need to remove all nodes
in order to disconnect the grap.

b) Scale - free

From the lectures we have that:

$$f_c = \begin{cases} 1 - \left(\frac{\gamma - 2}{3 - \gamma} k_{min}^{\gamma - 2} k_{max}^{\gamma - 3} - 1 \right)^{-1} & 2 < \gamma < 3 \\ 1 - \left(\frac{\gamma - 2}{\gamma - 3} k_{min} - 1 \right)^{-1} & \gamma > 3 \end{cases}$$

We observe that for $\gamma > 3$, there is no dependence on N.
Removing nodes leads to network fragmentation as
in random graps.

For $2 < \gamma < 3$, if we apply : $k_{max} = k_{min} N^{\frac{1}{\gamma - 1}}$:

$$1 - \left(\frac{\gamma - 2}{3 - \gamma} k_{min}^{\gamma - 2} k_{min}^{\gamma - 3} \cdot N^{\frac{\gamma - 3}{\gamma - 1}} - 1 \right)^{-1}$$

If we look at $N^{\frac{\gamma - 3}{\gamma - 1}}$ for $\gamma \in (2,3)$ it has a value $\frac{\gamma - 3}{\gamma - 1} < 0$.
which means that $N^{\frac{\gamma - 3}{\gamma - 1}} \longrightarrow 0$ for $N \to \infty$

$\Rightarrow f_c \to 1$
We need to remove all nodes to
disconnect the network.