Exercise 5

Introduction to Complex Network Analysis

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1 Friendship paradox

- 1) Average number of friends of friends

  Let N be the total number of nodes (people) in the network. A person i has k_i friends. If we want to sum all the friends of friends, then we think about how many times will k_i shows in the final sum. That means that each of the k_i friends of person i will contribute the term k_i in the final sum. So we have the total number of friends of friends: $\sum_{i=1}^{N} k_i^2$.

  We now divide this by the total number of friends, which is given by: $\langle k \rangle = \sum_{i=1}^{N} k_i$.

  Therefore, the mean number of friends of friends is:

$$\langle k_{\mathrm{nn}} \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

- 2) Calculate an explicit expression for scale-free networks

  Scale-free networks follow a power-law degree distribution: $P(k) = Ck^{-\gamma}$, where $C = (k_{min}, k_{max})$. For a normalized power-law distribution with degrees between k_{min} and k_{max} the following holds:

$$\int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} Ck^{-\gamma} dk = 1$$

  By solving this equation we have that:

$$C = \frac{1-\gamma}{k_{\mathrm{max}}^{1-\gamma} - k_{\mathrm{min}}^{1-\gamma}}$$

  To obtain the average degree we use the formula for scale free networks:

$$\langle k\rangle = \int_{k_{\min}}^{k_{\max}} kP(k)dk = \int_{k_{\min}}^{k_{\max}} kCk^{-\gamma}dk = \int_{k_{\min}}^{k_{\max}} k\frac{1-\gamma}{k_{\max}^{1-\gamma}-k_{\min}^{1-\gamma}}k^{-\gamma}dk$$

from which we obtain the following equation:

$$\langle k\rangle = \left(\frac{\gamma-1}{\gamma-2}\right)\left(\frac{k_{\max}^{2-\gamma}-k_{\min}^{2-\gamma}}{k_{\max}^{1-\gamma}-k_{\min}^{1-\gamma}}\right)$$

Then for $\langle k^2\rangle$ we have:

$$\langle k^2\rangle = \left(\frac{\gamma-1}{\gamma-3}\right)\left(\frac{k_{\max}^{3-\gamma}-k_{\min}^{3-\gamma}}{k_{\max}^{1-\gamma}-k_{\min}^{1-\gamma}}\right)$$

We can plug this in the formula in 1) to obtain the equation for $\langle k_{nn}\rangle$:

$$\langle k_{nn}\rangle = \left(\frac{\gamma-2}{\gamma-3}\right)\left(\frac{k_{\max}^{3-\gamma}-k_{\min}^{3-\gamma}}{k_{\max}^{2-\gamma}-k_{\min}^{2-\gamma}}\right)$$

- 3) Limiting cases $\gamma\to 2$ and $\gamma\to 3$

  It is not hard to see that for the values 2 and 3 for γ we have undetermined forms, hence we apply the L'Hopital's rule and obtain:

$$\lim_{\gamma\to 2}\langle k_{nn}\rangle = \lim_{\gamma\to 2}\frac{1(k_{max}^{3-\gamma}-k_{min}^{3-\gamma})+0}{1(k_{max}^{2-\gamma}-k_{min}^{2-\gamma})+(-1)(1ln(k_{max}-1ln(k_{min}))} = \frac{k_{\max}-k_{\min}}{\ln\left(\frac{k_{\max}}{k_{\min}}\right)}$$

$$\lim_{\gamma\to 3}\langle k_{nn}\rangle = \lim_{\gamma\to 3}\frac{1(k_{max}^{3-\gamma}-k_{min}^{3-\gamma})+1(ln(k_{max})-1ln(k_{min}))}{1(k_{max}^{2-\gamma}-k_{min}^{2-\gamma})+0} = \frac{\ln\left(\frac{k_{\max}}{k_{\min}}\right)}{k_{\max}^{-1}-k_{\min}^{-1}}$$

- 4) Compare with random network with Poisson distribution

  In random networks that follow a Poisson distribution it holds:

$$P(k) = \frac{\langle k\rangle^k}{k!}\mathrm{e}^{-\langle k\rangle}$$

  For this we look at the first moment: λ, and the second moment: $\lambda^2+\lambda$. We use the formula from 1) to compute

$$\langle k_{nn} \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\lambda^2 + \lambda}{\lambda} = \lambda + 1$$

where $\lambda = \langle k \rangle$ in this case.

We can observe that the friendship paradox exists in random networks but it's weak in the sense that $\langle k_{nn} \rangle$ is just $\langle k \rangle + 1$. In scale free networks, on the otherhand, it is really strong: it diverges either logarithmic (γ=3) or linearly (γ=2) as the network gets large.

- 5) Discuss how $< k_{nn} >$ behaves for increasing network size.

Here we use the formula:

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

If we substitute this in the formula for $< k_{nn} >$, we have:

$$\langle k_{nn} \rangle = \left(\frac{\gamma - 2}{\gamma - 3} \right) \left(\frac{k_{\min N}^{\frac{3-\gamma}{\gamma-1}} - k_{\min}^{3-\gamma}}{k_{\min N}^{\frac{2-\gamma}{\gamma-1}} - k_{\min}^{2-\gamma}} \right) = \left(\frac{\gamma - 2}{\gamma - 3} \right) k_{\min}$$

The size of the network increases linearly.