Exercise 2

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ii) Prove $f(x)$ is smooth:

$$f(x) = \frac{1}{2} x^T Q x + b^T x + c$$

is smooth with parameter $\|Q\|$.

To prove this we use the lemma:

→ If $\nabla f(x)$ is L-Lipschitz, then it is also L-smooth.

$\|\nabla f(x) - \nabla f(y)\| \le L \|x - y\|$.

Compute $\nabla f(x)$:

$$\nabla f(x) = \frac{\partial}{\partial x}\left(\frac{1}{2} x^T Q x + b^T x + c\right) =$$

$$= \frac{1}{2}(x^T Q^T + x^T Q) + b^T =$$

$$= \frac{1}{2} x^T (Q^T + Q) + b^T = \frac{1}{\cancel{2}} x^T \cdot \cancel{2} Q + b^T = \boxed{x^T Q + b^T}$$

We want to show that $\|\nabla f(x) - \nabla f(y)\| \le L \|x - y\|$, where $L = \|Q\|$.

$$\|\nabla f(x) - \nabla f(y)\| = \|x^T Q + \cancel{b^T} - y^T Q - \cancel{b^T}\| =$$

$$= \|x^T Q - y^T Q\| =$$

$$= \|(x^T Q - y^T Q)^T\| =$$

$$= \|(x^T Q)^T - (y^T Q)^T\| =$$

$$= \|Q^T x - Q^T y\| =$$

$$= \|Q^T (x - y)\| \le \|Q^T\| \|x - y\| = \|Q\| \|x - y\|$$

lemma: $\|Qa\| \le \|Q\| \|a\|$

⇒ ∇f is L-Lipschitz ⇒ f is smooth with parameter $\|Q\|$.

iii) Suppose we have observations (x_i, y_i), which are centered, meaning that $\sum_{i=1}^n x_i = 0 = \sum_{i=1}^n y_i$. Let (b^*, w^*) be the global minimum of the least squares objective

$$f(b,w) = \sum_{i=1}^{n} (b + w^T x_i - y_i)^2.$$

Prove that $b^* = 0$.

(b^*, w^*) are the global minimum of $f(b,w)$

$\Rightarrow \nabla f(b^*, w^*) = 0$

Computing $\nabla f(b,w)$:

$$\nabla f(b,w) = \sum_{i=1}^{n} 2(b + w^T x_i - y_i) =$$

$\frac{\partial}{\partial b} f(b,w)$

$$= \sum_{i=1}^{n} (2b + 2w^T x_i - 2y_i) =$$

$$= 2\sum_{i=1}^{n} b + 2w^T \underbrace{\sum_{i=1}^{n} x_i}_{=0} - 2\underbrace{\sum_{i=1}^{n} y_i}_{=0} =$$

$$= 2nb = 0$$

$$\Rightarrow b^* = 0.$$