Exercise 1

① Show that all norms are convex functions.

Definition of convex: function $f: V \to \mathbb{R}$ is convex iff:

for every x, y in a vector space V and $\lambda \in [0,1]$ the following holds:

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

For norms it will be:

$$\|\lambda x + (1-\lambda) y\| \leq \|\lambda x\| + \|(1-\lambda) y\|$$ using the Triangle inequality $\|a+b\| \leq \|a\| + \|b\|$

$\nearrow \lambda \in [0,1] \Rightarrow |\lambda| = \lambda$

$$\|\lambda x\| + \|(1-\lambda)(y)\| = \lambda \|x\| + (1-\lambda)\|y\|$$ using the $|\lambda| \|a\| = \|\lambda a\|$

$$\Leftrightarrow \|\lambda x + (1-\lambda) y\| \leq \lambda \|x\| + (1-\lambda)\|y\|$$

□

② Let $g: \mathbb{R}^m \to \mathbb{R}$ be a convex function and $A \in \mathbb{R}^{m \times d}$ a linear operator (matrix). Show that $f(\cdot) = g(A \cdot)$ is also convex.

g is convex:

$$g(\lambda x + (1-\lambda) y) \leq \lambda g(x) + (1-\lambda) g(y) \quad *$$

for f we take a look at:

$$f(\lambda x + (1-\lambda) y) = g(A(\lambda x + (1-\lambda) y)) =$$
$$= g(\lambda (Ax) + (1-\lambda)(Ay))$$

We use $*$:

$$g(\lambda (Ax) + (1-\lambda)(Ay)) \leq \lambda g(Ax) + (1-\lambda) g(Ay) =$$
$$= \lambda f(x) + (1-\lambda) f(y)$$

Therefore: $f(\lambda x + (1-\lambda) y) = \lambda f(x) + (1-\lambda) f(y)$
meaning f is convex.

□

③ Let x^* be a local minimum of a convex function $f: \mathbb{R}^d \to \mathbb{R}$. Show that x^* is a global minimum.

Since x^* is a local minimum, we can choose $\lambda \in [0,1]$ such that:

$$f(x^*) \leq f(x^* + \lambda(y - x^*)) \qquad y \text{ arbitrary point.}$$

f is convex:

$$f(x^* + \lambda(y-x^*)) = f(x^* + \lambda y - \lambda x^*) =$$
$$= f(\lambda y + (1-\lambda)x^*) \leq \lambda f(y) + (1-\lambda) f(x^*)$$

combining the two:

$$f(x^*) \leq \lambda f(y) + (1-\lambda) f(x^*)$$
$$\cancel{f(x^*)} \leq \lambda f(y) + \cancel{f(x^*)} - \lambda f(x^*)$$
$$0 \leq \lambda f(y) - \lambda f(x^*) \qquad /\lambda$$
$$\Leftrightarrow \quad f(x^*) \leq f(y)$$

this means that $f(x^*)$ is smaller or equal than all the other values of $f(y)$ where y is an arbitrary point

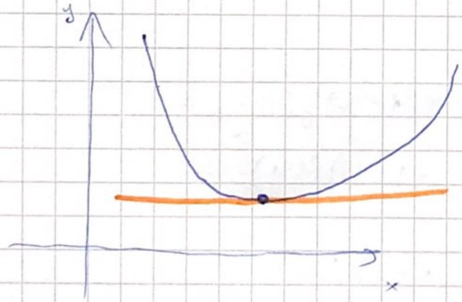
$\Leftrightarrow$ x^* is global minimum

$\square$

④ If $\bar{x}$ is a stationary point of the convex function f, then $\bar{x}$ is a global minimizer of f.

geometric intuition:

If we have a convex function ~~x~~



In a stationary point $\bar{x}$, $\nabla f(\bar{x}) = 0$.
f is convex, so no line between two points lies below it.
if you fit a tangent to a convex function it is always below it.
$\Rightarrow$ so in $\bar{x}$, no other function value is lower.

$\Rightarrow$ $\bar{x}$ is a global minimizer.

⑤ I have them installed.
(Not sure how to prove this, but mine github might be the proof.)