Exercise 4

① If the subdifferential of $f: \mathbb{R}^d \to \mathbb{R}$ is nonempty at every point, then f is a convex function.

The subdifferential of f, $\partial f(x)$, is the set of all sugradients of f at x.

$$\partial f(x) := \{ g \text{ s.t. } f(y) \geq f(x) + g^T(y-x) \ \forall y \in X \}$$

We know that $\partial f(x)$ is nonempty at every point
$\Rightarrow \forall x, \exists g$ s.t. $f(y) \geq f(x) + g^T(y-x)$

Let $z = \lambda x + (1-\lambda)y$, where $\lambda \in [0,1]$, $g \in \partial f(x)$.

$$\begin{aligned} f(x) &\geq f(z) + g^T(x-z) = \\ &= f(z) + g^T(x - \lambda x - (1-\lambda)y) = \\ &= f(z) + g^T(x(1-\lambda) - (1-\lambda)y) = \\ &= f(z) + g^T((1-\lambda)(x-y)) = \\ &= f(z) + (1-\lambda)g^T(x-y) \end{aligned}$$

$$\begin{aligned} f(y) &\geq f(z) + g^T(y-z) = \\ &= f(z) + g^T(\cancel{y} - \lambda x - \cancel{y} + \lambda y) = \\ &= f(z) + g^T(-\lambda(x-y)) = \\ &= f(z) - \lambda g^T(x-y) \end{aligned}$$

$$\begin{aligned} \lambda f(x) + (1-\lambda)f(y) &\geq \lambda f(z) + \cancel{\lambda(1-\lambda)g^T(x-y)} + (1-\lambda)f(z) - \cancel{(1-\lambda)\lambda g^T(x-y)} = \\ &= \lambda f(z) + (1-\lambda)f(z) \\ &= f(z) \end{aligned}$$

We have that:

$$\begin{aligned} \lambda f(x) + (1-\lambda)f(y) &\geq f(z) = \\ &= f(\lambda x + (1-\lambda)y) \end{aligned}$$

$\Rightarrow f$ is convex

$\square$