Mirror Descent

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Recap on (sub)-gradient descent

When we used a norm $\|\cdot\|$ we meant the 2-*norm*, i.e.

$$\|x\| = \Big(\sum_{i=1}^{d} x_i^2\Big)^{1/2}.$$

In **gradient descent** we used Lipschitz continuity:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$$

(Lead to a complexity of $\mathcal{O}(\frac{L}{k})$)
For **sub-gradient descent** we used $\|g\| \leq G$ which lead to a complexity of $\mathcal{O}(\frac{G}{\sqrt{k}})$.
But there are other norms

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\|x\|_\infty$$

It can happen that $\|g\|_\infty \leq G$ but $\|g\|_2 \approx \sqrt{d}G$

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Different norms?

But where did we use the norm in the ***method****?*

Gradient Descent

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

equivalently

$$x_{k+1} = \underset{x \in \mathbb{R}^d}{\arg\min} \left\{ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2\alpha_k} \|x - x_k\|_2^2 \right\}$$

We can replace the 2-norm with a more general **distance**.

Bregman distance

$h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex

(i) h is differentiable of the interior of dom h

(ii) h is 1-strongly convex w.r.t. $\|\cdot\|_2$

Then

$$\mathcal{D}_h(x, y) := h(x) - h(y) - \langle \nabla h(y), x - y \rangle.$$

Properties

- $\mathcal{D}_h(x, y) \geq 0$
- $\mathcal{D}_h(x, y) \neq \mathcal{D}_h(y, x)$
- $\mathcal{D}_h(\cdot, y)$ is convex for all y

$$\mathcal{D}_h(x, y) \approx \frac{1}{2}\langle \nabla^2 h(y)(x - y), x - y \rangle = \frac{1}{2}\|x - y\|^2_{\nabla^2 h(y)}$$

- $\mathcal{D}_h(x, y) \geq \frac{1}{2}\|x - y\|^2$ (1-strong convexity)

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Examples

(i) $h(x) = \frac{1}{2}\|x\|_2^2$ gives $\mathcal{D}_h(x, y) = \|x - y\|^2$

(ii) $h(x) = \frac{1}{2(p-1)}\|x\|_p^2$ with $p \in [1, 2]$

(iii) $\Delta^d = \{x \in \mathbb{R}_+^d : \sum_{i=1}^d x_i = 1\}$ the *unit simplex* and

$$h(x) = \begin{cases} \sum_{i=1}^d x_i \log(x_i) & x_i > 0 \\ +\infty & \text{otherwise} \end{cases}$$

the **Negative entropy**.

Negative entropy

- Negative entropy: $h(x) = \sum_{i=1}^{d} x_i \log(x_i)$ for $x_i > 0$.
- Then $\nabla h(x) = \log(x) + 1$ (coordinatewise) and

$$
\begin{aligned}
\mathcal{D}_h(x, y) &= \sum_{i=1}^{d} x_i \log(x_i) - y_i \log(y_i) - \langle \log(y) + 1, x - y \rangle \\
&= \sum_{i=1}^{d} x_i \log(x_i) - \sum_{i=1}^{d} x_i \log(y_i) \\
&= \sum_{i=1}^{d} x_i \log\left(\frac{x_i}{y_i}\right)
\end{aligned}
$$

  Known as Kullback-Leibler divergence $K(X \| Y)$.
- Is strongly convex over Δ

$$\mathcal{D}(x, y) \geq \frac{1}{2}\|x - y\|_1^2 \quad \text{Pinsker's ineq.}$$

Mirror descent

Idea: replace squared Euclidian norm with more general object:

$$
\begin{aligned}
x_{k+1} &= \underset{x\in\mathbb{R}^d}{\arg\min}\{f(x_k) + \langle \nabla f(x_k), x - x_k\rangle + \frac{1}{\alpha_k}\mathcal{D}_h(x, x_k)\} \\
&= \underset{x\in\mathbb{R}^d}{\arg\min}\{\langle \nabla f(x_k), x\rangle + \frac{1}{\alpha_k}\mathcal{D}_h(x, x_k)\} \\
&= \underset{x\in\mathbb{R}^d}{\arg\min}\{\langle \nabla f(x_k), x\rangle + \frac{1}{\alpha_k}(h(x) - h(x_k) - \langle \nabla h(x_k), x - x_k\rangle)\} \\
&= \underset{x\in\mathbb{R}^d}{\arg\min}\{\langle \nabla f(x_k), x\rangle + \frac{1}{\alpha_k}(h(x) - \langle \nabla h(x_k), x\rangle)\} \\
&= \underset{x\in\mathbb{R}^d}{\arg\min}\{\langle \alpha_k \nabla f(x_k) - \nabla h(x_k), x\rangle + h(x)\}
\end{aligned}
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Question: But why *mirror* descent?

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The Mirror part

$$x_{k+1} = \arg\min_{x \in \mathbb{R}^d} \{ \langle \alpha_k \nabla f(x_k) - \nabla h(x_k), x \rangle + h(x) \}$$

By optimality condition:

$$0 = \alpha_k \nabla f(x_k) - \nabla h(x_k) + \nabla h(x_{k+1})$$

Therefore

$$\nabla h(x_{k+1}) = \nabla h(x_k) - \alpha_k \nabla f(x_k)$$

Why it's called mirror descent

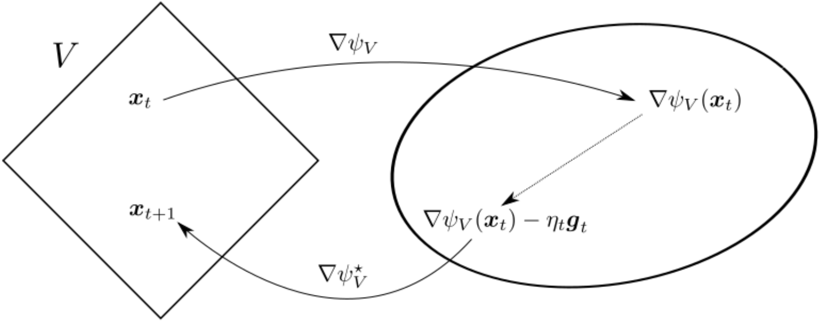

Figure: $\psi = h$

Mirror Descent on the unit simplex

Negative entropy: $h(x) = \sum_{i=1}^{d} x_i \log(x_i)$ for $x_i > 0$.
We define $a := \alpha_k \nabla f(x_k) - \nabla h(x_k)$. Then

$$x_{k+1} = \arg\min_{x \in \Delta} \{\langle a, x \rangle + h(x)\}$$

with $x_i \geq 0$ and $\sum x_i = 1$.

How to solve this?

Via **Lagrange**

$$L(x, \mu) = \langle a, x \rangle + h(x) - \mu(x_1 + \cdots + x_d - 1)$$

Mirror Descent on the unit simplex [contd]

Then,

$$\begin{aligned}\partial_{x_i} L(x,\mu) &= a_i + \log(x_i) + 1 - \mu \stackrel{!}{=} 0\\ \log(x_i) &= \mu - 1 - a_i\\ x_i &= e^{\mu-1-a_i} = \beta e^{-a_i}\end{aligned}$$

with $\beta = e^{\mu-1}$.

Second constraint:

$$\sum_{i=1}^{d} x_i \stackrel{!}{=} 1 \Rightarrow \sum_{i=1}^{d} \beta e^{-a_i} = 1 \Rightarrow \beta = \frac{1}{\sum_{i=1}^{d} e^{-a_i}} \Rightarrow x_i = \frac{e^{-a_i}}{\sum_{j=1}^{d} e^{-a_j}}$$

Final mirror descent update:

$$x_{k+1}(i) = \frac{x_k(i) e^{\alpha_k [\nabla f(x_k)]_i}}{\sum_{j=1}^{d} e^{\alpha_k [\nabla f(x_k)]_j}}$$

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(General) mirror descent convergence statement

$$\|y\|_* := \max_{\|x\|=1} \{\langle y, x\rangle\}$$

Theorem

In $(\mathbb{R}^d, \|\cdot\|)$ *and subgradients bounded in dual norm* $\|g_k\|_* \leq G$*, then*

$$f(\hat{x}_k) - f^* \leq \frac{\mathcal{D}(x^*, x_1) + \frac{1}{2}\sum_i \alpha_i^2 G^2}{\sum_i \alpha_i}$$

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Convergence on the unit simplex

What about $\mathcal{D}(x^*, x_1)$**?** Let $x_1 = (\frac{1}{n}, \cdots, \frac{1}{n})$, then

$$\mathcal{D}(x, x_1) = \sum x_i \log\left(\frac{x_i}{\frac{1}{n}}\right) = \sum x_i \log(x_i) + \log(n) \leq log(n)$$

while $\|x_0 - x^*\|^2 \leq 2$.
But if

$$\|g\|_\infty = \|g\|_1^* \leq G$$

we can still have

$$\|g\|_2 \approx \sqrt{d}G.$$