4.1

(a) The 2nd order Taylor approximation of f(t) centered at t_0 is given by.

$$T_2(t) = f(t_0) + f'(t_0)(t-t_0) + \frac{f''(t_0)}{2}(t-t_0)^2$$

$$f(t_0) = E^T(w_t) \qquad f(t) = E^T(w_{t+1})$$

$$\underset{\substack{\uparrow \\ R^{n,1}}}{w_{t+1}} = w_t - \eta_t d_t$$

$$T_2 : E^T(w_{t+1}) = E^T(w_t) + (\nabla E^T(w_t))^T(w_{t+1} - w_t) + \frac{1}{2}(w_{t+1} - w_t)^T H_{E^T(w_t)}(w_{t+1} - w_t)$$

(b) $$E^T(w_{t+1}) = E^T(w_t) + (\nabla E^T(w_t))^T(w_{t+1} - w_t) + \frac{1}{2}(w_{t+1} - w_t)^T H_{E^T(w_t)}(w_{t+1} - w_t) \overset{!}{\leq} E^T(w_t)$$

$$(\nabla E^T(w_t))^T(w_{t+1} - w_t) + \frac{1}{2}(w_{t+1} - w_t)^T H_{E^T(w_t)}(w_{t+1} - w_t) \overset{!}{\leq} 0$$

$$\xrightarrow{w_{t+1} = w_t - \eta_t d_t}$$

$$(\nabla E^T(w_t))^T(w_t - \eta_t d_t - w_t) + \frac{1}{2}(w_t - \eta_t d_t - w_t)^T H_{E^T(w_t)}(w_t - \eta_t d_t - w_t) \overset{!}{\leq} 0$$

$$(\nabla E^T(w_t))^T(-\eta_t d_t) + \frac{1}{2}(-\eta_t d_t)^T H_{E^T(w_t)}(-\eta_t d_t) \overset{!}{\leq} 0$$

$$\frac{1}{2}\eta_t^2 d_t^T H_{E^T(w_t)} d_t - \eta_t(\nabla E^T(w_t))^T d_t \overset{!}{\leq} 0$$

(c) $$E^T(w) = \frac{1}{2}(w - w^*)^T H(w - w^*)$$

from (b) $$\underbrace{\frac{1}{2}\eta_t^2 d_t^T H_{E^T(w)} d_t}_{a} - \underbrace{\eta_t(\nabla E^T(w_t))^T d_t}_{b} \leq 0$$

$$\text{minimum } \eta_t^* = -\frac{b}{2a} = -\frac{-(\nabla E^T(w_t))^T d_t}{2 \times \frac{1}{2} d_t^T H_{E^T(w_t)} d_t} = \frac{(\nabla E^T(w_t))^T d_t}{d_t^T H_{E^T(w_t)} d_t}$$

$$E^T(w_t) = \frac{1}{2}(w_t - w^*)^T H(w_t - w^*)$$

$$\nabla E^T(w_t) = H(w_t - w^*)$$

$$H_{E^T(w_t)} = H$$

$$\eta_t^* = \frac{(H(w_t - w^*))^T d_t}{d_t^T H d_t}$$

(d) orthogonal $\longleftrightarrow \nabla E^T(w_{t+1}) \cdot d_t = 0 \rightarrow d_t^T \cdot \nabla E^T(w_{t+1}) = 0.$

$$\nabla E^T(w_{t+1}) = H(w_{t+1} - w^*) = H(w_t - \eta_t d_t - w^*)$$

$$= H\left(w_t - \frac{(H(w_t - w^*))^T d_t}{d_t^T H d_t} - w^*\right)$$

$$H\left((w_t - w^*) - d_t \frac{(w_t - w^*)^T H d_t}{d_t^T H d_t}\right)$$

$$\nabla E^T(w_{t+1}) \cdot d_t = d_t^T \nabla E^T(w_{t+1})$$

$$= d_t^T H\left((w_t - w^*) - d_t \frac{(w_t - w^*)^T H d_t}{d_t^T H d_t}\right)$$

$$= d_t^T H(w_t - w^*) - d_t^T H d_t \frac{(w_t - w^*) H d_t}{d_t^T H d_t}$$

$$d_t^T H(w_t - w^*) d_t^T H d_t - d_t^T H d_t (w_t - w^*)^T H d_t = 0.$$

$$\nabla E^T_{w_{t+1}} \cdot d_t = 0$$

So they are orthogonal.