Technische Universität Berlin
Institut für Mathematik
Dr. R. Kruse, M. Eisenmann, R. Weiske WiSe 2017/18

Numerical Mathematics for Engineering II
Assignment 12

Notice: This is the last assignment for this course and the ISIS submission deadline is updated.

Theoretical exercises: submit to exercise class on February 7th or February 8th, 2018.
Programming exercises: upload to ISIS until Friday 14:00, February 9th, 2018.

1. Exercise: The Ritz projector **8 points**

Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex and polygonal domain and $a\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the usual energy scalar product

$$a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x, \quad u,v \in H_0^1(\Omega).$$

Let $V_h \subset H_0^1(\Omega)$, $h \in (0,1)$, be a family of finite element spaces consisting of piecewise linear functions. Let $R_h\colon H_0^1(\Omega) \to V_h$ be the Ritz projector, i.e., the unique operator which satisfies

$$a(w - R_h w, v_h) = 0, \quad \text{for all } w \in H_0^1(\Omega),\ v_h \in V_h. \tag{1}$$

a) Use the Lax-Milgram Theorem to show that (1) uniquely determines R_h.

b) Prove that R_h is linear, idempotent and symmetric, i.e., that $R_h(u+\beta v) = R_h u + \beta R_h v$, $R_h^2 = R_h$ and that $a(R_h u, v) = a(u, R_h v)$ for all $u,v \in H_0^1(\Omega)$, $\beta \in \mathbb{R}$.

c) Prove that $\|R_h w\|_{H_0^1(\Omega)} \leq \|w\|_{H_0^1}(\Omega)$ for all $w \in H_0^1(\Omega)$.

d) Let $u \in H_0^1(\Omega)$ be the weak solution to a variational problem of the form $a(u,v) = F(v)$ for all $v \in H_0^1(\Omega)$. Compare $R_h u$ to the finite element approximation $u_h \in V_h$ of the weak solution u.

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2. Programming exercise: Finite elements for the 2D Poisson equation **6 points**

Consider on $\Omega = (0,1) \times (0,1)$ the Poisson equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where f is given by

$$f(x_1, x_2) = \begin{cases} 1, & \text{if } 0 \leq x_1, x_2 \leq \frac{1}{2}, \\ 0, & \text{else.} \end{cases}$$

Write a function `a12e02Poisson2D(N)` that creates a surface plot of the corresponding finite element approximation u_h on the triangulation described on Assignment 11 Exercise 2 with $h = \frac{1}{N+1}$, $N \in \mathbb{N}$. Hereby, the entries in the load vector should be approximated by $(I_h f, \varphi_{i,j}^h)$, where $I_h f$ denotes the piecewise linear interpolation of f.

Hint : Use a lexicographical ordering of the nodes in order to assemble the stiffness- and mass-matrix.

3. Programming exercise The 1D heat equation **6 points**

Consider on $\Omega = (0,1)$ the heat equation

$$\begin{cases} u_t = \frac{1}{100} u_{xx}, & (t,x) \in (0,1] \times \Omega, \\ u(t,0) = 0, \ u(t,1) = 0, & t \in (0,1], \\ u(0,x) = e^{-9(x-1/2)^2}, & x \in \Omega. \end{cases}$$

Formulate a combined Galerkin finite element and backward Euler method for this equation. Write a function `[x,xi] = a12e03Heat(M,N,mov)` which computes the finite element/backward Euler approximation with M interior nodes and N time steps and `mov` being either 0 or 1. The function should return the space grid `x` and a vector `xi` with the approximate solution at the node points including the boundary points. If `mov=1` the function should create an animation and save it to a file. Verify your code by some numerical convergence analysis in space and time. Plot also the computational time of your algorithm when $M = N = 2^k$, for $k \in \{3, \ldots, K\}$ with K of your choice.

total sum: 20 points