Technische Universität Berlin
Institut für Mathematik
Dr. R. Kruse, M. Eisenmann, R. Weiske

WiSe 2017/18

Numerical Mathematics for Engineering II

Assignment 9

Theoretical exercises: submit to exercise class on January 17th or 18th, 2017.
Programming exercises: upload to ISIS until Friday 18:00, January 19th, 2017.

1. Exercise: Inner products and hat-functions **6 points**

Let $\Omega = (-10, 10) \subset \mathbb{R}$. Recall that $L^2(\Omega)$ is the space of all functions $u\colon \Omega \to \mathbb{R}$ which satisfy

$$\|u\|_{L^2(\Omega)} = \Big(\int_\Omega |u(x)|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} < \infty.$$

It is equipped with the inner product

$$\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x) \,\mathrm{d}x, \quad u, v \in L^2(\Omega),$$

and norm $\|u\|_{L^2(\Omega)} = \sqrt{\langle u, u \rangle_{L^2(\Omega)}}$. Furthermore, let $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form, defined for weakly differentiable mappings u, v by

$$a(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x.$$

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be the function $\varphi(x) = 0$ for $|x| \geq 1$, $\varphi(x) = x + 1$ for $x \in [-1, 0]$ and $\varphi(x) = 1 - x$ for $x \in [0, 1]$. For $h > 0$ and $n \in \mathbb{Z}$, let $\varphi_n^h(x) := \varphi(h^{-1}x - n)$.

a) Draw by hand the functions $\varphi_n^{0.25}$, $n = 1, \dots, 4$, in the same graph.

b) Determine the weak derivative of $\varphi_1^{0.5}$.

c) Compute the terms $\langle \varphi_m^h, \varphi_n^h \rangle_{L^2(\Omega)}$ and $a(\varphi_m^h, \varphi_n^h)$ as functions of h, m, n. Formulate the $\mathbb{R}^{N\times N}$ matrices $M_{ij} = \langle \varphi_i^h, \varphi_j^h \rangle_{L^2(\Omega)}$ and $S_{ij} = a(\varphi_i^h, \varphi_j^h)$, $i, j = 1, \dots, N$.

d) Prove that $a(\cdot, \cdot)$ is an inner product on the space $C_0^1(\bar{\Omega})$ consisting of $C^1(\bar{\Omega})$-functions u with $u(x) = 0$ on $\partial\Omega$. Show that it is not an inner product on $C^1(\bar{\Omega})$.

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2. Programming exercise: Polynomial Galerkin approximation **10 points**

For $\Omega = (0,1)$ consider the boundary value problem

$$\begin{cases} -u''(x) = -e^x, & x \in \Omega, \\ u(0) = 0,\ u'(1) = -1. \end{cases} \tag{1}$$

Let $P^N(\bar{\Omega})$ be the space of all N-th order polynomials on $[0,1]$. The monomials $\psi_n(x) = x^n$, $n \in \{0,\dots,N\}$, form a basis of this space. We want to approximate the solution u in the space $P^N(\bar{\Omega})$. For this we take $U_N = \sum_{n=1}^N \xi_n \psi_n$, where $\xi_1,\dots,\xi_N \in \mathbb{R}$. Due to the boundary condition $u(0) = 0$ we do not have the constant basis function ψ_0 in the sum.

a) Find the exact solution to (1).

b) Use integration by parts to derive the variational formulation of (1) containing only the first derivative of u. Take care of the inhomogeneous Neumann boundary condition.

c) Formulate the Galerkin equation for the approximation of u in $P^N(\bar{\Omega})$ using the formulation obtained in **(b)**. From this derive a linear system of equations in order to determine the coefficients $\xi_1,\dots,\xi_N \in \mathbb{R}$.

d) Write a function `coeff = a09e02getpoly(N)` that returns the vector `coeff` containing the coefficients $\xi_1,\dots,\xi_N$.

e) Write a script `a09e02error.py` that plots the error in a `loglog`-plot for some different values of N.

f) Explain from a numerical linear algebra perspective why the polynomial Galerkin approximation is not favorable (one sentence).

3. Exercise: Equivalence to a minimization problem **3 points**

Let $A \in \mathbb{R}^{N\times N}$ be a symmetric and positive definite matrix and $b \in \mathbb{R}^N$ be arbitrary. Show that finding the solution $x \in \mathbb{R}^N$ to $Ax = b$ is equivalent to determining the minimizer $x \in \mathbb{R}^N$ of the expression $J(x) = \frac{1}{2}x^T Ax - b^T x$.

total sum: 19 points