Technische Universität Berlin
Institut für Mathematik
Dr. R. Kruse, M. Eisenmann, R. Weiske

WiSe 2017/18

Numerical Mathematics for Engineering II

Assignment 10

Theoretical exercises: submit to exercise class on January 24th or 25th, 2017.
Programming exercises: upload to ISIS until Friday 18:00, January 26th, 2017.

1. Exercise: Quadrature for 1D finite elements **4 points**

Let $\Omega = (0,1)$ and consider the boundary value problem

$$\begin{cases} -u'' = f, & \text{in } \Omega, \\ u(0) = u(1) = 0. \end{cases}$$

Let $(\varphi_i)_{i=1}^n$ be the standard piecewise linear finite element basis with equidistant nodes $z_i = ih$, $i = 1, \dots, n$, where $h = \frac{1}{n+1}$, $n \in \mathbb{N}$. In order to assemble the right hand side of the resulting linear system $A_n u_n = f_n$, quadrature is in general required, i.e., $F(\varphi_i) = \langle \varphi_i, f \rangle_{L^2(\Omega)} = \int_0^1 \varphi_i(x) f(x) \, \mathrm{d}x$ needs to be approximated.

a) Derive formulas for the approximation of $\langle \varphi_i, f \rangle_{L^2(\Omega)}$ with the left point rule, the midpoint rule and the trapezoidal rule, using the nodes $(z_i)_{i=1}^n$ as mesh points.

b) Compare the resulting linear system $A_n u_n = f_n$ of the Galerkin finite element method and its solution to the 3-point stencil finite difference method, if the left point rule is used on the right hand side.

2. Programming exercise: Finite elements for a Dirichlet BVP in 1D **8+6 points**

Let $\Omega = (a,b) \subset \mathbb{R}$, $f \colon \Omega \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ and consider the boundary value problem

$$\begin{cases} -u'' + u = f, & \text{in } \Omega, \\ u(a) = \alpha,\ u(b) = \beta. \end{cases}$$

a) Recall from Section III.1 how to transform an inhomogeneous boundary value problem to one with zero boundary conditions. Transform the equation, state its variational formulation and formulate a Galerkin finite element method for the problem in the space of piecewise linear functions.

Please turn the page!

b) Write a function `[x,U]=a10e03getPDE(a,b,alpha,beta,f,N)` which takes the data `a,b,alpha,beta` being real numbers, a function handle `f`, a positive integer N and returns a vector `x` containing the grid $(a, a+h, \dots, b)$ of $N+2$ uniformly distributed points of $[a,b]$ and a vector `U` containing the finite element approximation on the N interior nodes with the boundary conditions `x(0)=alpha` and `x(N+1)=beta` included. To assemble the system matrix use Assignment 9 Exercise 1 and for the right hand side use quadrature with the trapezoidal rule.

c) Explore the convergence of your code for the equation with $a = \alpha = -1$, $b = \beta = 1$, $f(x) = x^3 - 6x + (\frac{\pi^2}{4} + 1)\cos(\frac{\pi}{2}x)$ by computing the EOC or making an error plot.

d) **BONUS:** Consider the boundary value problem $-u'' + cu = f$ on the same domain and the same boundary conditions as above, where $c\colon \Omega \to \mathbb{R}$. Write a function `[x,U]=a10e03getPDEBonus(a,b,c,alpha,beta,f,N)` which solves the more general boundary value problem. Here the data `c` should be a function handle. Quadrature should be used to compute the part of the system matrix corresponding to the term cu.

e) **BONUS:** Explore the convergence of your code for the problem

$$\begin{cases} -u''(x) + e^{x/4}u(x) = \log(x+4), & x \in (-\pi, \pi), \\ \quad u(-\pi) = -1, \quad u(\pi) = 1, \end{cases}$$

by computing the EOC or making an error plot. Here you need to compare with a *benchmark solution.* This is a numerical approximation which is approximated with a much finer precision than the others, and which works as a substitute for the exact solution when it is not available.

3. Exercise: Preparations for 2D finite elements **8 points**

Consider the reference triangle $\widehat{\Omega}$ with nodes $z_1 = (0,0)$, $z_2 = (1,0)$, and $z_3 = (0,1)$.

a) Determine explicit expressions for the (piecewise) linear basis functionsφ_i on $\widehat{\Omega}$ such that $\varphi_i(z_j) = \delta_{i,j}$, $i,j \in \{1,2,3\}$.

b) Determine the weak gradients $\nabla\varphi_i$, $i \in \{1,2,3\}$.

c) Determine $\langle \varphi_i, \varphi_j \rangle_{L^2(\widehat{\Omega})}$ and $a(\varphi_i, \varphi_j) = \int_{\widehat{\Omega}} \nabla\varphi_i(x) \cdot \nabla\varphi_j(x)\,\mathrm{d}x$ for $i,j \in \{1,\dots,3\}$ (Try to make use of symmetries to avoid unnecessary computations).

d) Let Ω be the triangle with vertices $x_1, x_2, x_3 \in \mathbb{R}^2$. Use the affine linear transformation T determined by $T(z_i) = x_i$ for all $i \in \{1,2,3\}$ to map $\widehat{\Omega}$ onto Ω. Let ψ_i be the (piecewise) linear basis functions on Ω such that $\psi_i(x_j) = \delta_{i,j}$, $i,j \in \{1,2,3\}$. Show that $\psi_i(T(y_1,y_2)) = \varphi_i(y_1,y_2)$ for all $(y_1,y_2) \in \hat{\Omega}$. Use change of variables and the value for $\langle \varphi_i, \varphi_j \rangle_{L^2(\widehat{\Omega})}$ in order to compute $\langle \psi_i, \psi_j \rangle_{L^2(\Omega)}$.

total sum: 20+6 points