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WiSe 2017/18

Numerical Mathematics for Engineering II

Assignment 8

Theoretical exercises: submit to exercise class on December 20th or 21st, 2017.
Programming exercises: upload to ISIS until Friday 18:00, December 22nd, 2017.

1. Exercise: Transport equation with varying coefficients **6 points**

Consider the transport equation

$$\begin{cases} \text{Find } u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R} \text{ such that} \\ u_t = a(t)u_x, & \text{for all } t\in(0,\infty), x\in\mathbb{R}, \\ u(0,x) = v(x), & \text{for all } x\in\mathbb{R}, \end{cases} \tag{1}$$

where $v\colon \mathbb{R}\to\mathbb{R}$ is a smooth function denoting the initial condition. The parameter $a\colon [0,\infty)\to\mathbb{R}$ is an integrable mapping and there exists $a_0\in[0,\infty)$ such that $a(t)\geq a_0>0$ for all $t\in[0,\infty)$.

a) Verify that the general solution to (1) is given by

$$u(t,x) := v\Big(x+\int_0^t a(s)\,\mathrm{d}s\Big)$$

for all $t\in[0,\infty)$, $x\in\mathbb{R}$. Given an arbitrary point $(\tau,\xi)\in[0,\infty)\times\mathbb{R}$, what is the characteristic associated to (1) through (τ,ξ)?

b) Derive the upwind scheme for (1).

c) Assume we want to approximate the exact solution at time $t=10$ and position $x=0$ by the upwind scheme. Assuming the spatial step size equals $h=0.01$, explain how to choose the temporal step size k such that the upwind scheme is stable. If the parameter function a is given by

1. $a(t)=\exp(t)$, $t\in[0,\infty)$, and
2. $a(t)=2-\exp(-t)$, $t\in[0,\infty)$,

provide an estimate of the total computational effort in time if a single application of the upwind scheme costs approximatively 1 ms $=10^{-3}$ s.

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2. Exercise: Transformation into a variational problem **4 points**

For $\Omega = (0,1) \times (0,1)$ consider the Dirichlet boundary value problem

$$\begin{cases} \text{Find } u\colon \Omega \to \mathbb{R} \text{ such that} \\ -((2x+3)u_{xx} + (4x-1)u_{xy} + 3u_{yx} + (2x+3)u_{yy}) - 2(u_x + u_y) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \overline{\Omega}. \end{cases} \tag{2}$$

a) Show that (2) is an elliptic equation in Ω.

b) Rewrite the problem (2) in *divergence form*. A general differential operator L is in divergence form if it is written as

$$\begin{aligned} Lu &= -\nabla \cdot \big(A(x)\nabla u\big) + b(x) \cdot \nabla u + c(x)u \\ &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\big(a_{ij}(x)u_{x_i}\big) + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u(x), \end{aligned}$$

where $x = (x_1, \dots, x_n)^T \in \Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $A(x) = [a_{ij}(x)]$ denotes the diffusion matrix of L at $x \in \Omega$, and $b(x) = (b_1(x), \dots, b_n(x))$.

c) Derive the variational formulation of problem (2).

3. Programming exercise: Lax-Wendroff method for the wave equation **8 points**

Consider the following linear problem: Find $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } (0,T] \times \mathbb{R}, \\ u(0,x) = f(x) & \text{for } x \in \mathbb{R}, \\ u_t(0,x) = g(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{3}$$

Here $f, g \in C^1(\mathbb{R})$ are given continuously differentiable mappings. Our goal is to use the Lax-Wendroff method in order to approximate the solution u.

a) Denoting $v(t,x) = [cu_x(t,x), u_t(t,x)]^T$, rewrite (3) as a system of first order equation

$$v_t + Mv_x = 0 \quad \text{in } (0,T] \times \mathbb{R}, \tag{4}$$

where M is a 2×2 matrix, and give the initial value vector $v(0,x)$. Equation (4) will be implemented using the Lax-Wendroff method

$$\begin{aligned} V_j^{\ell+1} &= V_j^\ell - \frac{\lambda}{2} M\big(V_{j+1}^\ell - V_{j-1}^\ell\big) + \frac{\lambda^2}{2} M^2\big(V_{j+1}^\ell - 2V_j^\ell + V_{j-1}^\ell\big), \quad \ell, j \in \mathbb{Z}, \ell \geq 0 \\ V_j^0 &= v(x_j), \quad j \in \mathbb{Z} \end{aligned}$$

How can we obtain a discrete approximation U_j^ℓ for (3) from the knowledge of V_j^ℓ?

b) Write a function `U = a08e03LaxWend(t,xmin,xmax,c,f,g,h,k)` which returns a vector `U` consisting of the values U_j^ℓ with $\ell = \lfloor \frac{t}{k} \rfloor$ and for all j with $\texttt{xmin} \leq x_j \leq \texttt{xmax}$ for the given velocity c, initial values f, g, and step sizes $h, k > 0$.

c) Write a script `a08e03errors.py` that uses your solution to compute the approximation of $u(1,2)$ in the case $c = 1$, $f(x) = \frac{1}{(x-4)^2+1} + \frac{1}{(x+4)^2+1}$, $g(x) = x\exp(-\frac{1}{2}x^2)$. Determine the errors for the step sizes $k = \frac{1}{2}h$, $h = \frac{1}{N}$, $N = 2^p$, $p = 3, \ldots, 10$.

Hint: the exact solution was already studied in `a03e03`.

d) Write a script `a08e03movie.py` that creates a movie displaying the time evolution of the Lax-Wendroff approximation in the case $c = 1$, $f(x) = \frac{1}{(x-4)^2+1} + \frac{1}{(x+4)^2+1}$, $g(x) = x\exp(-\frac{1}{2}x^2)$. Your animation should cover the region $[-15, 15]$ in space and $[0, 10]$ in time and be saved to an MP4 file. Use $k = \frac{1}{2}h$, $h = \frac{1}{N}$, $N = 2^p$ with $p = 6$.

Hint : use the module FuncAnimation to create and save your animation.

total sum: 18 points