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WiSe 2017/18

Numerical Mathematics for Engineering II

Assignment 11

Theoretical exercises: submit to exercise class on January 31th or February 1st, 2018.
Programming exercises: upload to ISIS until Friday 18:00, February 2nd, 2018.

1. Exercise: Properties of pyramid functions **4 points**

Consider a triangle $\mathcal{T} \subset \mathbb{R}^2$ with vertices $z_1, z_2, z_3 \in \mathbb{R}^2$. We denote by $\varphi_i \colon \mathcal{T} \to \mathbb{R}$, $i \in \{1,2,3\}$, linear mappings satisfying $\varphi_i(z_j) = \delta_{i,j}$ for all $i, j \in \{1,2,3\}$. Thus φ_i is the restriction of a pyramid function to the triangle $\mathcal{T}$. Prove the following assertions:

a) For all $z \in \mathcal{T}$ it holds true that

$$\sum_{i=1}^{3} \varphi_i(z) = 1.$$

b) For all $z \in \mathcal{T}$ it holds true that

$$\sum_{i=1}^{3} \varphi_i(z)^2 \leq 1.$$

c) For all $z \in \mathcal{T}$ it holds true that

$$\sum_{i=1}^{3} z_i \varphi_i(z) = z.$$

2. Programming exercise: Preparation for 2D finite element **6 points**

Consider the triangulation of $\Omega = (0,1)^2$ in Figure 3.4 in the lecture notes by J. Liesen (JL) and let $x_{ij} = (ih, jh)$, $i, j \in \mathbb{N}$, denote the associated nodes. Let φ_{ij}^h be the *pyramid function*, which is affine linear on each triangle and satisfies $\varphi_{ij}^h(x_{ij}) = 1$ and $\varphi_{ij}^h(x_{kl}) = 0$ if $(i,j) \neq (k,l)$, see Figure 3.5 in JL.

Please turn the page!

a) Write a function `phi = a11e02pyramid(i,j,h,order)` which returns the pyramid function φ_{ij}^h as a function handle if `order = 0` and its weak gradient $\nabla\varphi_{ij}^h$ if `order = 1`. Notice that it is no problem to define a vector valued function handle using the `lambda` function.

b) Create a surface plot of $\varphi_{11}^{0.25}$.

c) Use **(a)**, to write a function `c = a11e02matrixelement(i,j,k,l,h,order)` which returns an approximation of $\langle\varphi_{ij}^h, \varphi_{k\ell}^h\rangle_{L^2(\Omega)}$ if `order=0` and of $a(\varphi_{ij}^h, \varphi_{kl}^h)$ if `order=1`. Here $a(u,v) = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x$. Use an appropriate function from the package `scipy.integrate` for the quadrature.

3. Programming exercise: Selected applications **10 points**

The objective is to solve an applied problem of interest using FEniCS and Python, to evaluate the results obtained and draw some conclusions. Use your creativity and focus on features of interest. Note that the problems are not precisely formulated. You thus have to think of:

1. An interesting real world problem.
2. Mathematical modeling including for instance the choice of boundary conditions and truncation of the computational domain in case of unbounded domains.
3. Computational aspects.

Write a report in Latex (preferred) or Word which describes your model and presents your results. A good report is interesting, concise and contains at most 3 pages of text and plots. On the ISIS webpage you can find a code example using FEniCS as well as an introduction to FEniCS.

Convection-diffusion-absorbtion/reaction: Consider a 2D convection- diffusion-absorbtion/reaction problem of the type

$$-\nabla\cdot(\varepsilon\nabla u) + b\cdot\nabla u + cu = f$$

on a domain Ω together with suitable boundary conditions. Here $u\colon\Omega\to\mathbb{R}$ is the unknown concentration, $\varepsilon\colon\Omega\to(0,\infty)$ is a given (small) diffusion coefficient, $b\colon\Omega\to\mathbb{R}$ is a given velocity field, $c\colon\Omega\to\mathbb{R}$ is a given absorbtion/reaction coeficient and $f\colon\Omega\to\mathbb{R}$ is a given production term. Solve a convection dominated problem of this form for instance related to pollution control, where f is a delta function at some point $p\in\Omega$. Determine for instance the width of the smoke plume. In case you have knowledge of this problem compare with theory.

Electrostatics: Consider the basic problem of 2D electrostatics

$$\nabla\cdot(\varepsilon E) = \varrho, \quad E = -\nabla\varphi,$$

See next page!

together with suitable boundary conditions corresponding to a part of the boundary of Ω being a perfect conductor and the remaining part being insulated. Here $E\colon \Omega \to \mathbb{R}^2$ is the electric field, $\varphi\colon \Omega \to \mathbb{R}$ the electric potential, $\varepsilon\colon \Omega \to \mathbb{R}$ the dielectric coefficient and $\varrho\colon \Omega \to \mathbb{R}$ the charge density. Solve a problem of this form in a configuration of interest for instance on a domain whose boundary has sharp non-convex corners. Study the behavior of the electric field in the vicinity of the corner.

2D fluid flow The velocity $u = (u_1, u_2)$ of an incompressible irrotational 2D fluid may be expressed through a potential $\varphi\colon \Omega \to \mathbb{R}$ by $u = \nabla\varphi$. Coupled with the incompressibility condition $\nabla \cdot u = 0$ this gives the Laplace equation for φ

$$\Delta\varphi = 0,$$

together with suitable boundary conditions expressing, for instance, that $u \cdot \nu = 0$ on solid boundaries. Note that it is not possible to use Neumann boundary conditions on the entire boundary. Solve a problem of the following type:

- Flow through a 2D nozzle.
- Flow around a disc or wing profile.

Use the gradient plot to visualize the flow.

Heat conduction Consider the 2D stationary heat equation

$$\nabla \cdot q = f, \quad q = -\kappa\nabla u,$$

together with suitable boundary conditions. Here $u\colon \Omega \to \mathbb{R}$ is the temperature, $q\colon \Omega \to \mathbb{R}^2$ the heat flow, $\kappa\colon \Omega \to \mathbb{R}$ the heat conduction coefficient and $f\colon \Omega \to \mathbb{R}$ a given production term. Solve for instance a problem of this form modeling a hot water pipe buried in a half space and determine the temperature above the pipe using a Robin boundary condition on the surface.

Quantum physics Consider the 2D stationary Schrödinger eigenvalue problem

$$-\frac{\hbar}{2m}\Delta u + V(x)u = \lambda u,$$

where V is a given potential, $\hbar$ is Planck's constant divided by 2π and m is the particle mass. Give a quantum physical interpretation of the eigenvalues and corresponding eigenfunctions determined by this equation. Normalize the constants and solve the problem for some suitable domain and potential. Discuss your computational results from a quantum physical viewpoint.

total sum: 20 points