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WiSe 2017/18

Numerical Mathematics for Engineering II

Assignment 5

Theoretical exercises: submit to exercise class on November 29th or 30th, 2017.
Programming exercises: upload to ISIS until Friday 18:00, December 1st, 2017.

1. Exercise: On M-matrices **4 points**

Let $A \in \mathbb{R}^{N\times N}$ be an M-matrix. Show the following statements:

a) If $v \in \mathbb{R}^N$ is a vector with $Av \geq 0$, then $v \geq 0$.

b) If $v_1, v_2 \in \mathbb{R}^N$ are vectors such that $|Av_1| \leq Av_2$, then $|v_1| \leq v_2$.

(Note that "$\geq$", "$\leq$", and "$|\cdot|$" are understood componentwise as in the lecture.)

2. Exercise: Stability of 1D FDM **4 points**

Consider the 1-dimensional boundary value problem

$$\begin{cases} -u''(x) + bu'(x) + cu(x) = f(x), & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

with $b, c > 0$. Let $L_h u_h = f_h$ be the finite difference approximation with the central difference for u' on a uniform grid. For grid sizes satisfying $h \leq \frac{2}{b}$ the resulting matrix L_h is an M-matrix (this does not have to be shown). Determine a vector $v \in \mathbb{R}^N$ with $L_h v \geq (1, \dots, 1)^T$. What can be said about $\left|\left|\left| L_h^{-1} \right|\right|\right|_\infty$ for $h \to 0$?

Hint: Follow the same steps as in the proof of Theorem II.17 with $w(x) = -\frac{1}{2}\alpha x^2 + \beta x$ for suitable $\alpha > 0$, $\beta \in \mathbb{R}$.

3. Exercise: Kronecker product **4 points**

Let $A, B, C, D \in \mathbb{R}^{N\times N}$. Show the following properties of the Kronecker product:

a) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \in \mathbb{R}^{N^2\times N^2}$,

Please turn the page!

b) $(A \otimes B)^T = A^T \otimes B^T$,

c) **(Bonus 2 pts:)** Let $x, y \in \mathbb{R}^{N^2}$ satisfy $(A \otimes B)x = y \in \mathbb{R}^{N^2}$. Define the matrices $X, Y \in \mathbb{R}^{N \times N}$ as

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \\ x_{N+1} & x_{N+2} & \cdots & x_{2N} \\ \vdots & & & \vdots \\ x_{N(N-1)+1} & \cdots & & x_{N^2} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \\ y_{N+1} & y_{N+2} & \cdots & y_{2N} \\ \vdots & & & \vdots \\ y_{N(N-1)+1} & \cdots & & y_{N^2} \end{bmatrix}$$

and show that $Y = AXB^T$. Hint: Use that $Y_{i,j} = y_{(i-1)N+j}$ for $i, j = 1, \ldots, N$.

4. Programming exercise: Singularly perturbed BVP **8 points**

Consider the two-point boundary value problem

$$\begin{cases} -\varepsilon u''(x) + u'(x) = 1, & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

with small but positive values for ε. Simply setting $\varepsilon = 0$ does not help, because the solution (if it exists) is not close to the one for small values for $\varepsilon > 0$. It also changes the order of the BVP. This behaviour is typical for *singularly perturbed problems.*

The exact solution to (1) is given by

$$u_\varepsilon(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}, \quad x \in (0,1)$$

and shown in the figure below. As we see $u_0(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = x$ for $0 \le x < 1$, but $u_0(x)$ does not satisfy the boundary conditions at $x = 1$ in a smooth way. Instead, the solution exhibits a thin region near $x = 1$ (also known as boundary layer) where u_ε changes rapidly. The width of the region depends on ε and thereby derivatives of u_ε become large as $x \to 1$ and $\varepsilon \to 0$. That means that the constant $u^{(k)}(\xi)h^{k-2}$ $(k > 2)$ in the remainder of the difference quotient for u'' is large. In order to make the remainder small, we need to make h very small.

a) Write a function `uh = a05ex04solve(eps,xh,flag)` which returns the FDM solution of the singularly perturbed problem (1) for given `eps`$=\epsilon$, grid `xh`, and approximation for the first derivative selected by `flag`. Hint: You can use the solution to Assignment 4 Exercise 3.

b) Write a function `[err,uex] = a05ex04error(eps,xh,uh)` that returns the error `err` between `uh` and the restricted exact solution, which is also to be returned as `uex`.

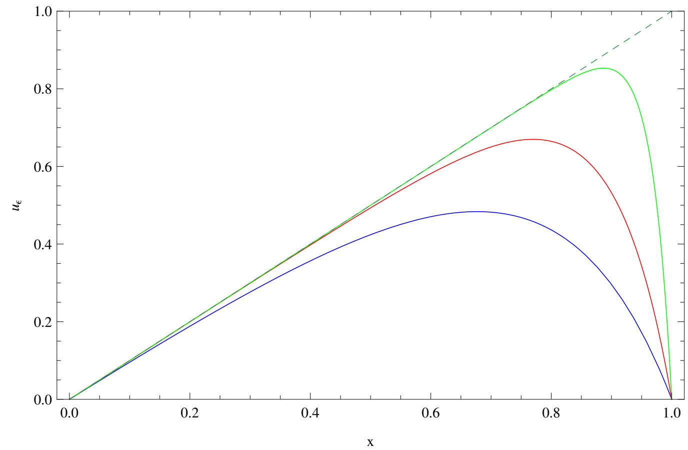

Fig.: u_ε for $\varepsilon \in \{1/5, 1/10, 1/30\}$ (blue,red,green) and the function $u_0(x) = x$ (dashed)

c) Write a function `xh = a05ex04shishkin(N,sigma)` that generates a column vector of size 2N+1 describing a "Shishkin" grid `xh` that is defined by

$$\texttt{xh(i)} = \begin{cases} (\texttt{i} - 1)H, & \text{for} \quad \texttt{i} = 1, \dots, \texttt{N}, \\ (1 - \texttt{sigma}) + (\texttt{i} - \texttt{N} - 1)h, & \text{for} \quad \texttt{i} = \texttt{N} + 1, \dots, 2\texttt{N} + 1, \end{cases}$$

where $H = (1 - \texttt{sigma})/\texttt{N}$ and $h = \texttt{sigma}/\texttt{N}$.

d) Write a script `a05ex04experiment.py` that plots the exact and approximated solution for `eps`=0.001 and all $N \in \{5, 50, 500, 5000\}$ on a

i) uniform grid with $h = \frac{1}{2N}$ and forward difference operator D^+ (`flag='+'`),
ii) uniform grid with $h = \frac{1}{2N}$ and central difference operator D^0 (`flag='0'`).
iii) uniform grid with $h = \frac{1}{2N}$ and backward difference operator D^- (`flag='-'`).
iv) non-uniform Shishkin grid with `N` and `sigma=4*eps*log(2*N)` and central difference operator D^0 (`flag='0'`).

Use the commands `plt.figure(1)`, ..., `plt.figure(4)` to group plots corresponding to i,ii,iii,iv in plot 1,2,3,4. Results with different N but same operator should be plotted into the same figure. Use the command `plt.title` to add information about the used operator, the grid and the error for different N.

total sum: 20 points