Technische Universität Berlin
Institut für Mathematik
Dr. R. Kruse, M. Eisenmann, R. Weiske

WiSe 2017/18

Numerical Mathematics for Engineering II
Assignment 4

Theoretical exercises: submit to exercise class on November 22th or 23th, 2017.
Programming exercises: upload to ISIS until Friday 18:00, November 24th, 2017.

1. Excercise: On matrix norms **5 points**

Let $\|\cdot\|\colon \mathbb{R}^N \to \mathbb{R}$ be a norm on the $\mathbb{R}$-vector space $\mathbb{R}^N$. We consider the matrix norm $|||\cdot|||\colon \mathbb{R}^{N\times N} \to \mathbb{R}$ induced by $\|\cdot\|$ which is given by

$$|||A||| := \sup_{v\in\mathbb{R}^N \colon \|v\|=1} \|Av\|$$

for every $A \in \mathbb{R}^{N\times N}$.

a) Show that $|||\cdot|||$ is a norm on the vector space $\mathbb{R}^{N\times N}$ of all $N \times N$ matrices.

b) Show that $0 \le |||A||| < \infty$ for every $A \in \mathbb{R}^{N\times N}$.

c) Show that for all $A, B \in \mathbb{R}^{N\times N}$ and $v \in \mathbb{R}^N$ it holds

$$\begin{aligned} \|Av\| &\le |||A|||\,\|v\| && \text{(compatibility with } \|\cdot\|\text{)},\\ |||A\cdot B||| &\le |||A|||\,|||B||| && \text{(sub-multiplicativity)}. \end{aligned}$$

d) Show that the induced matrix norm of the identity matrix I_N is equal to one, that is $|||I_N||| = 1$.

e) Consider the *Frobenius norm* $\|\cdot\|_F\colon \mathbb{R}^{N\times N} \to \mathbb{R}$, which is given by

$$\|A\|_F = \Big(\sum_{i,j=1}^{N} |A_{i,j}|^2\Big)^{\frac{1}{2}},$$

for every $A = [A_{i,j}]_{i,j=1}^N \in \mathbb{R}^{N\times N}$. Compute $\|I_N\|_F$. Does there exists a vector norm $\|\cdot\|\colon \mathbb{R}^N \to \mathbb{R}$ such that $|||A||| = \|A\|_F$ for every $A \in \mathbb{R}^{N\times N}$?

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2. Exercise: The five point difference stencil **6 points**

Consider the Laplace equation

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega = (0,1) \times (0,1), \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $g(x_1, x_2) = 16x_1$ for all $(x_1, x_2) \in \partial\Omega$.

a) Explicitly determine and write out in full detail the matrix $L_h \in \mathbb{R}^{N^2 \times N^2}$ and the vector $f_h \in \mathbb{R}^{N^2}$ for the linear system $L_h u_h = f_h$ of the five point difference stencil approximation of (1) with $N = 3$ and $h = \frac{1}{N+1}$. Use the lexicographical ordering from the lecture.

b) Compute the discrete solution $u_h \in \mathbb{R}^9$ for the system in part a). Hint: You may use the solver in Python, for example.

c) What is the exact solution u? Compare u_h with u at $x = (0.5, 0.5)$.

3. Programming exercise: Two-point boundary value problem (BVP) **6 points**

Implement a finite difference scheme for the problem

$$\begin{aligned} &-au''(x) + bu'(x) + cu(x) = f(x), \quad x \in (0,1) \\ &u(0) = \alpha, \quad u(1) = \beta, \end{aligned} \tag{2}$$

for a given (probably non-uniform) mesh `xh` $= \bar{\Omega}_h$, with constants a, b, c, α, β.

a) Consider the reduced linear system `Lh*uh=fh` with $L_h \in \mathbb{R}^{N \times N}$ and $N = |\bar{\Omega}_h| - 2$ to problem (2). The matrix L_h is written in compact form as

$$L_h = -a\Big(\tfrac{2}{h_i(h_i+h_{i+1})}, -\tfrac{2}{h_i h_{i+1}}, \tfrac{2}{h_{i+1}(h_i+h_{i+1})}\Big) + b\Big(\tfrac{-1}{h_i+h_{i+1}}, 0, \tfrac{1}{h_i+h_{i+1}}\Big) + c(0,1,0),$$

for $i \in \{1, \ldots, N\}$, where the same notation as in Assignment 3, Exercise 3 is used. Determine the right hand side $f_h \in \mathbb{R}^N$. How does L_h look like if we use the forward or backward differences D^+, D^- instead of D^0 for the approximation of the first order derivative?

b) Write a function `[Lh,fh] = a04e03getPDE(xh,f,consts,flag)` which sets up the matrix `Lh` and the right hand side `fh` for the reduced linear system. The model parameters are contained in `consts = [a, b, c, alpha, beta]` and the character `flag` selects the approximation for u' with `'-'`, `'+'`, `'0'` for D^-, D^+, D^0 respectively.

See next page!

4. Programming excercise: 2D Poisson FDM **6 points**

Consider for $i \in \{1, 2\}$ the boundary value problems:

$$\begin{cases} -\Delta u_i = f_i, & \text{in} \quad \Omega = (0,1)^2 \subset \mathbb{R}^2, \\ u_i = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

having the solutions

$$\begin{aligned} u_1(x,y) &= (xy + x^3y^3 - x^3y - xy^3), \\ u_2(x,y) &= \sin(3\pi x)\sin(\pi y). \end{aligned}$$

a) Determine some f_1, f_2 such that u_1, u_2 are the solutions to the BVP.

b) Write a function `[Lh,fh] = a04e04getPDE(p,i)` that sets up the sparse matrix `Lh` and the right hand side `fh` of the linear system `Lh*uh=fh` for the refinement level `p` on the domain $(0,1) \times (0,1)$. Here we mean that $N = 2^p - 1$ and $h = \frac{1}{N+1}$. Use the standard five point stencil on a uniform mesh with lexicographical order.

c) Write a function `errors = a04e04solve(i)` that solves the discretized problem for f_1 if $i = 1$ and for f_2 if $i = 2$, where f_1, f_2 were found in (a), for $p \in \{1, \dots, 9\}$. Determine for each `p` the error between the computed approximation and the restricted exact solution in the maximum norm and store it in `errors(p)`. For each value of i plot the errors in a `loglog` plot with the corresponding step size h on the x-axis. Determine also the experimental order of convergence (EOC), which is given by

$$\mathrm{EOC}_p = \frac{\log(\mathrm{error}_p) - \log(\mathrm{error}_{p-1})}{\log(h_p) - \log(h_{p-1})}, \quad p \in \{2, \dots, 9\},$$

and present the values in a nicely formatted table.

total sum: 23 points