Multipurpose Multiphysics Package (MP2P)
FEM 1D Steady-State Diffusion Module
Updated: 26 October 2024

Case 1

Consider a $L = 1.00\,\mathrm{m}$ long rod with a thermal conductivity of $k = 1.0\,\mathrm{W\,m^{-1}\,K^{-1}}$. The left end of the rod is maintained at a constant temperature of $T_\mathrm{A} = 50.0\,°\mathrm{C}$, while the right end is heated at a constant inward flux of $q_\mathrm{B} = 2.0\,\mathrm{W\,m^{-2}}$. Heat is generated within the rod at a rate of $\dot{Q} = 100.0\,\mathrm{W\,m^{-3}}$. It is desired to determine the steady-state temperature profile in the rod.

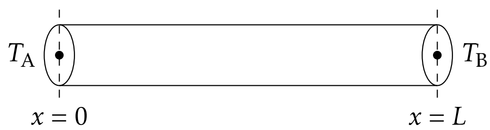
The above problem may be described by the 1D heat equation with the following boundary conditions: In these equations, $\mathbf{n}$ is the outward normal vector (so $-\mathbf{n}$ points inwards), and $\mathbf{q}$ is the diffusive heat flux (i.e., $\mathbf{q} = -k\nabla T$).

$$0 = -\nabla \cdot (-k\nabla T) + \dot{Q} \tag{1}$$
$$T|_{x=0} = T_\mathrm{A} \tag{2}$$
$$-\mathbf{n} \cdot \mathbf{q}|_{x=L} = +q_\mathrm{B} \tag{3}$$

Note that the sign of q_B in Equation 3 is positive because the heat flux points inwards. Because flux-type boundary conditions are defined with respect to the normal vector, the sign of flux-type boundary conditions dictates whether the flux enters or exits the domain, and not whether the flux points towards the positive or negative coordinates.

Case 2

Consider a $L = 1.00\,\mathrm{m}$ long rod with a thermal conductivity of $k = 1.0\,\mathrm{W\,m^{-1}\,K^{-1}}$. The left end of the rod is maintained at a constant temperature of $T_\mathrm{A} = 50.0\,°\mathrm{C}$, while the right end is at a constant temperature of $T_\mathrm{B} = 60.0\,°\mathrm{C}$. Heat is generated within the rod at a rate of $\dot{Q} = 100.0\,\mathrm{W\,m^{-3}}$ It is desired to determine the steady-state temperature profile in the rod.



The above problem may be described by the 1D heat equation with the following boundary conditions:

$$0 = -\nabla \cdot (-k\nabla T) + \dot{Q} \tag{4}$$
$$T|_{x=0} = T_\mathrm{A} \tag{5}$$
$$T|_{x=L} = T_\mathrm{B} \tag{6}$$

Case 3

Consider a $L = 1.00\,\mathrm{m}$ long rod. The left end of the rod loses heat at a constant flux of $q_\mathrm{A} = 2.0\,\mathrm{W\,m^{-2}}$, while the right end is held at a constant temperature of $T_\mathrm{B} = 50.0\,°\mathrm{C}$.

The thermal conductivity k of the rod is non-constant and depends on the temperature as given by Equation 7. Moreover, the linear heat generation $\dot{Q}$ varies with the position in the rod as given by Equation 8. It is desired to determine the steady-state temperature profile in the rod.

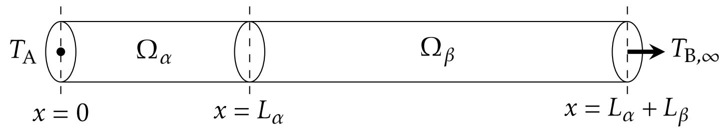
$$k = 1.0 + 0.01T + 500T^{-1} \quad k\,[=]\,\mathrm{W\,m^{-1}\,K^{-1}};\ T\,[=]\,\mathrm{K} \tag{7}$$
$$\dot{Q} = 10.0 + 10.0x^{0.5} - 2.0x^{1.5} \quad \dot{Q}\,[=]\,\mathrm{W\,m^{-3}};\ x\,[=]\,\mathrm{m} \tag{8}$$

The above problem may be described by the 1D heat equation with the following boundary conditions:

$$0 = -\nabla \cdot (-k\nabla T) + \dot{Q} \tag{9}$$
$$-\mathbf{n} \cdot \mathbf{q}|_{x=0} = -q_\mathrm{A} \tag{10}$$
$$T|_{x=L} = T_\mathrm{B} \tag{11}$$

Case 4
Two rods, the first with a length of $L_\alpha = 0.5\,\mathrm{m}$ and the second with a length of $L_\beta = 1.0\,\mathrm{m}$, are joined end-to-end to form a longer rod. The first rod has a thermal conductivity of $k_\alpha = 1.0\,\mathrm{W\,m^{-1}\,K^{-1}}$ and a linear heat generation of $\dot{Q}_\alpha = 500.0\,\mathrm{W\,m^{-3}}$. The second rod has a thermal conductivity of $k_\beta = 5.0\,\mathrm{W\,m^{-1}\,K^{-1}}$ and does not generate heat (i.e., $\dot{Q}_\beta = 0$).

The left end of the combined rod is held at a constant temperature of $T_\mathrm{A} = 50\,°\mathrm{C}$, while the right end is exposed to a cooling fluid at $T_{\mathrm{B},\infty} = 10\,°\mathrm{C}$. The heat flux on the right end follows Newton's law of cooling with a heat transfer coefficient of $h_\mathrm{B} = 5.0\,\mathrm{W\,m^{-2}\,K^{-1}}$. It is desired to determine the steady-state temperature profile in the rod.

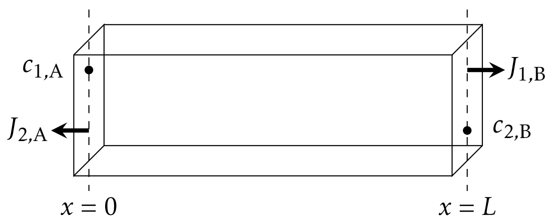


The above problem may be described by the 1D heat equation with the following boundary conditions:

$$0 = -\nabla \cdot (-k_\alpha \nabla T) + \dot{Q}_\alpha \quad \text{in } \Omega_\alpha \tag{12}$$
$$0 = -\nabla \cdot (-k_\beta \nabla T) + \dot{Q}_\beta \quad \text{in } \Omega_\beta \tag{13}$$
$$T|_{x=0} = T_\mathrm{A} \tag{14}$$
$$-\mathbf{n} \cdot \mathbf{q}|_{x=L_\alpha+L_\beta} = -h_\mathrm{B}(T|_{x=L_\alpha+L_\beta} - T_{\mathrm{B},\infty}) \tag{15}$$

Observe the formulation of Newton's law of cooling as a boundary condition. If heat exits the tip of the rod, then the tip must be warmer than the surroundings (i.e., $T|_{x=L_\alpha+L_\beta} > T_{\mathrm{B},\infty}$). The quantity $(T|_{x=L_\alpha+L_\beta} - T_{\mathrm{B},\infty})$ is therefore positive. However, the right-hand side of Equation 15 must be negative since heat is leaving the boundary, thus a negative sign is included.

Case 5
Two chemical species, referred to as species 1 and 2, diffuse in a tank of length $L = 1.00\,\mathrm{m}$. At the left end of the tank, the concentration of species 1 is held at $c_{1,\mathrm{A}} = 0.10\,\mathrm{mol\,m^{-3}}$, while the outward flux of species 2 is $J_{2,\mathrm{A}} = 0.002\,\mathrm{mol\,m^{-2}\,s^{-1}}$. At the right end of the tank, the outward flux of species 1 is $J_{1,\mathrm{B}} = 0.003\,\mathrm{mol\,m^{-2}\,s^{-1}}$, while the concentration of species 2 is held at $c_{2,\mathrm{B}} = 0.40\,\mathrm{mol\,m^{-3}}$.



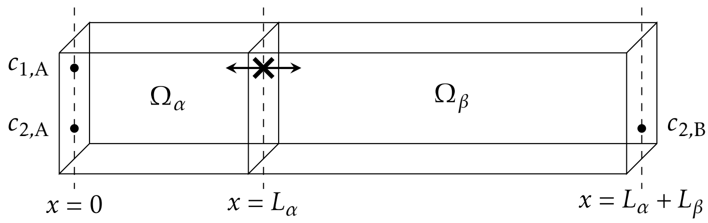
Species 1 has a diffusion coefficient of $D_1 = 0.050\,\mathrm{m^2\,s^{-1}}$, while species 2 has a much lower diffusion coefficient of $D_2 = 0.006\,\mathrm{m^2\,s^{-1}}$. A chemical reaction exists wherein species 1 is converted to species 2 via the following second-order reaction. The rate constant is $0.07\,\mathrm{m^3\,mol^{-1}\,s^{-1}}$. It is desired to determine the steady-state concentration profiles of both species.

$$\dot{R}_{1\to 2} = k_{1\to 2} c_1 c_2 \tag{16}$$

The above problem may be described by the 1D diffusion equation with the following boundary conditions:

$$0 = -\nabla \cdot (-D_1 \nabla c_1) - \dot{R}_{1\to 2} \tag{17}$$
$$0 = -\nabla \cdot (-D_2 \nabla c_2) + \dot{R}_{1\to 2} \tag{18}$$
$$c_1|_{x=0} = c_{1,\mathrm{A}} \tag{19}$$
$$-\mathbf{n} \cdot \mathbf{J}_2|_{x=0} = J_{2,\mathrm{A}} \tag{20}$$
$$-\mathbf{n} \cdot \mathbf{J}_1|_{x=L} = J_{1,\mathrm{B}} \tag{21}$$
$$c_2|_{x=L} = c_{2,\mathrm{B}} \tag{22}$$

Case 6
A tank is divided into two sections by a membrane. The section on the left is $L_\alpha = 0.50\,\mathrm{m}$ long, while the section on the right is $L_\beta = 1.00\,\mathrm{m}$ long. Two chemical species are injected on the left side of the tank. The concentration of species 1 is held at $c_{1,\mathrm{A}} = 0.30\,\mathrm{mol\,m^{-3}}$, while that of species 2 is $c_{2,\mathrm{A}} = 0.20\,\mathrm{mol\,m^{-3}}$. The membrane, however, prevents species 1 from entering the right section of the tank, so species 1 is found only on the left section. The concentration of species 2 on the right side of the tank is $c_{2,\mathrm{B}} = 0.10\,\mathrm{mol\,m^{-3}}$.



Species 1 has a diffusion coefficient of $D_1 = 0.050\,\mathrm{m^2\,s^{-1}}$, while species 2 has a much lower diffusion coefficient of $D_2 = 0.006\,\mathrm{m^2\,s^{-1}}$. A chemical reaction exists wherein species 1 is converted to species 2 via the following second-order reaction. The rate constant is $0.40\,\mathrm{m^3\,mol^{-1}\,s^{-1}}$. It is desired to determine the steady-state concentration profiles of both species.

$$\dot{R}_{1\to 2} = k_{1\to 2} c_1 c_2 \tag{23}$$

The above problem may be described by the 1D diffusion equation with the following boundary conditions:

$$0 = -\nabla \cdot (-D_1 \nabla c_1) - \dot{R}_{1\to 2} \quad \text{in } \Omega_\alpha \tag{24}$$
$$0 = -\nabla \cdot (-D_2 \nabla c_2) + \dot{R}_{1\to 2} \quad \text{in } \Omega_\alpha, \Omega_\beta \tag{25}$$
$$c_1|_{x=0} = c_{1,\mathrm{A}} \tag{26}$$
$$c_2|_{x=0} = c_{2,\mathrm{A}} \tag{27}$$
$$-\mathbf{n} \cdot \mathbf{J}_1|_{x=L_\alpha} = 0 \tag{28}$$
$$c_2|_{x=L_\alpha+L_\beta} = c_{2,\mathrm{B}} \tag{29}$$