Maximum Likelihood Estimate, Poisson Distribution and Monte Carlo Simulations

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***This document contains both the mathematical part as well as the simulation results.**

Problem Analysis

We need to calculate the MLE for λ, in the Poisson Distribution. Post that, we have also been given another setup with a different value of λ, particularly because of another consideration in the experiment. For both these cases we need to determine the MLE for λ mathematically. Post that, we need to perform a Monte Carlo simulation to estimate λ using the same setup and compare our results.

Solution

We have,

$$P(r|\lambda) = \mathrm{e}^{(-\lambda)} \frac{\lambda^r}{r!}$$

Finding the likelihood function for the given distribution, we have

$$L(\lambda) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{r_i}}{r_i!}$$

Here, we need to find for r = 9. So, we have,

$$L(\lambda) = \mathrm{e}^{(-\lambda)} \frac{\lambda^9}{9!}$$

Differentiating with respect to λ, and equating it to zero to find the critical point we have,

$$\frac{dP}{d\lambda} = (-1)e^{-\lambda}\frac{\lambda^9}{9!} + e^{-\lambda}\frac{9\lambda^8}{9!} = 0$$

$$\frac{\lambda^8 e^{-\lambda}}{8!}\left[\frac{-\lambda}{9} + 1\right] = 0$$

$$\lambda = 9$$

Finding the Second – order derivative to determine whether the obtained value is a maxima, minima or a saddle point.

$$\frac{d^2P}{d\lambda^2} = \frac{\lambda^8 e^{-\lambda}}{8!}\left[\frac{-1}{9}\right] + \left[\frac{-\lambda}{9} + 1\right]\{\frac{8\lambda^7 e^{-\lambda}}{8!} - \frac{\lambda^8 e^{-\lambda}}{8!}\}$$

$$= \frac{\lambda^7 e^{-\lambda}}{8!}\left[\frac{-\lambda}{9} + \left(\frac{-\lambda}{9} + 1\right)[8 - \lambda]\right]$$

Putting the value λ = 9, we get

$$= \frac{9^7 e^{-9}}{8!}(-1) < 0$$

Since, the second derivative is less than zero, we can conclude that λ = 9 represents a maxima.

Ans 3 (b):

Background rate of photons = 13 photons/minute

New mean for the Poisson distribution (λ_{new}) $= \lambda + b = \lambda + 13$

Now, we have:

$$P(r|\lambda_{new}) = \mathrm{e}^{(-\lambda_{new})} \frac{{\lambda_{new}}^r}{r!}$$

$$P(r|\lambda_{new}) = \mathrm{e}^{(-\lambda-13)} \frac{(\lambda+13)^r}{r!}$$

Again, we need to find for r = 9. So, we have,

$$L(\lambda_{new}) = \mathrm{e}^{(-\lambda-13)} \frac{(\lambda+13)^9}{9!}$$

Differentiating with respect to λ_{new}, and equating it to zero to find the critical point we have,

$$\frac{dP}{d\lambda_{new}} = (-1)e^{(-\lambda-13)} \frac{(\lambda+13)^9}{9!} + e^{-\lambda-13} \frac{9(\lambda+13)^8}{9!} = 0$$

$$\frac{(\lambda+13)^8 e^{-\lambda-13}}{8!} \left[\frac{-(\lambda+13)}{9} + 1\right] = 0$$

On solving the above question, we get

$$\lambda = -13, \qquad \lambda = -4$$

Since $\lambda \in (0, \infty)$, no appropriate maximum likelihood estimate can be made for λ given r = 9. Hence, no solution. We may not find the second derivative.

Simulation Methodology, Results and Analysis

In order to verify and compare our results obtained using mathematics, a Monte Carlo simulation experiment was designed. The experiment was driven by 3 distributions that gave the choice for λ.

The 1st distribution was a sequential one with numbers being generated with an increment of 0.5.

The other two distributions were uniform distributions. 1000 values for λ were distributed using each distribution. Thus, in total there were 3000 choices for λ (All 3000 may not be unique).

After this, the likelihood was calculated with r = 9 and using different values for λ and plotted.

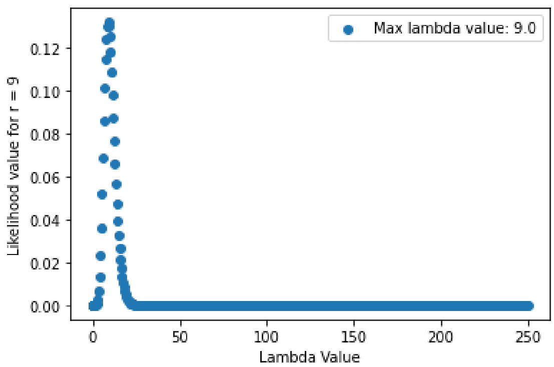

Figure 1: Likelihood values for corresponding λ values and r = 9

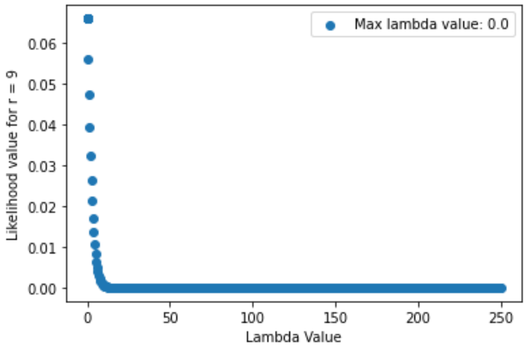

Figure 2: Likelihood values for corresponding λ values taking into consideration b = 13 and r = 9.

For the second part, where another background photon was taken into consideration, the effective value for λ increased to λ + 13. This, however, did not give a maximum value for λ. This result matches from the one calculated mathematically.

These results (both mathematical and simulated) can also be interpreted as finding the value for which $\lambda^r > e^\lambda$. These are the two deciding factors that are determining the MLE for a fixed r.

Takeaways

The results obtained through the simulation are similar to those obtained on solving the question mathematically. This, thus, verifies our calculations for the MLE for λ with r = 9.

There is also an important point that we must note. The second part of the question where another factor b was added in the λ, making the effective λ = λ+13, we had already got the maximum value at λ = 9. With no other changes in r or the probability distribution, it was intuitive that no positive value for λ will be possible to be able to maximize the function.

Additional information

(This section discusses about the way for MLE for more than one random variable in a distribution.)

For more than one random variable,

Taking log on both sides and simplifying, we have

$$\log\big(L(\lambda)\big) = \sum_{i=1}^{n} \log\,(e^{-\lambda}\frac{\lambda^{r_i}}{r_i!})$$

$$= -n\lambda + \sum_{i=1}^{n} \mathrm{r}_i \log(\lambda) - \log\,(r_i!)$$

Now, for maximizing, we will consider the first derivative. We can directly take the derivative of the logarithmic function as well.

$$\frac{d\log\big(L(\lambda)\big)}{d\lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} \mathrm{r}_i = 0$$

This gives us: $\lambda = \frac{1}{n}\sum_{i=1}^{n} \mathrm{r}_i$

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