Sign of the Main Minors for a matrix of size 3

	Order 1	Order 2	Order 3
Positive Definite	$+$	$+$	$+$
Negative Definite	$-$	$+$	$-$
Positive Semidefinite	$+/0$	$+/0$	0
Negative Semidefinite	$-/0$	$+/0$	0
Indefinite	All other cases		

2º Method: Eigenvalues:

- Calculate the eigenvalues of the matrix.

(PD): The eigenvalues are positive

(PSD): The eigenvalues are positive, one or more are 0

(ND): The eigenvalues are negative

(NSD): The eigenvalues are negative, one or more are 0

Indefinite (I): There are positive and negative eigenvalues

Example:

$$Q(X) = (x - y)^2$$

$$Q(X) = (x - y)^2 = x^2 + y^2 - 2xy$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

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Eigenvalues, λ_i:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 1 =$$

$$= \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$$

$$(\lambda)(\lambda - 2) = 0 \begin{cases} \lambda = 0 \\ \\ \lambda = 2 \end{cases}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Positive Semidefinite

$$Q(X) = (x-y)^2 = x^2 + y^2 - 2xy$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{cases} Order\ 1 \to 1 \quad (+) \\ Order\ 2 \to 0 \end{cases}$$

Positive Semidefinite

Sign of an equation with a restriction

We have an equation with *n* variables.
We incorporate *r* restrictions.

1) Study the sign without the restriction (matrix of size n)

2) Incorporate the restriction into the equation and,

3) Study the sign again (matrix of size n-r)

Example 2: Classify the expression

$$x^2 + 2y^2 + 6xy$$

Restricted to:

$$x - y = 0$$

First, we classify the equation without the restriction:

$$x^2 + 2y^2 + 6xy \rightarrow A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

Order 1 → $|1| = 1 > 0$

Order 2 → $\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7 < 0$

Indefinite.

With the restriction: $x - y = 0 \rightarrow y = x$

We will have a matrix of size n – r = 2 – 1 = 1

$$x^2 + 2y^2 + 6xy = 9x^2 \rightarrow A = (9)$$

Order 1 $\rightarrow |9| = 9 > 0$

Positive Definite

Example 3: Classify the expression

$$x^2 + y^2 - 9z^2$$

Restricted to:

$$x + 3z = 0$$

Without the restriction:

$$x^2 + y^2 - 9z^2 \to A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

Order 1 $\to |1| = 1 > 0$

Order 2 $\to \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0$

Order 3 $\to |A| = -9 < 0$

Indefinite.

With the restriction: $x + 3z = 0 \rightarrow x = -3z$

We will have a matrix of size n – r = 3 – 1 = 2

$$x^2 + y^2 - 9z^2 = y^2 \rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

De orden 1 $\rightarrow |1| = 1 > 0$

De orden 2 $\rightarrow \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

Positive Semidefinite

Exercise 6) Prove that:

$$x^2 + y^2 + z^2 \geq xy + xz + yz$$

$$x^2 + y^2 + z^2 - xy - xz - yz \rightarrow A = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \rightarrow |1| = 1 > 0 \\ \\ \text{Order 2} \rightarrow \begin{vmatrix} 1 & -1/2 \\ -1/2 & 1 \end{vmatrix} = \dfrac{3}{4} > 0 \\ \\ \text{Order 3} \rightarrow |A| = 0 \end{array}\right\} \{Positive\ Semidefinite$$

Exercise 11: As a function of parameter a

$$x^2 + 2y^2 + az^2 - 2xz$$

$$x^2 + 2y^2 + az^2 - 2xz \rightarrow A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & a \end{pmatrix}$$

Order 1 $\rightarrow |1| = 3 > 0$

Order 2 $\rightarrow \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 > 0$

Order 3 $\rightarrow |A| = 2(a-1) \begin{cases} a = 1 \rightarrow |A| = 0 \rightarrow PSD \\ a > 1 \rightarrow |A| > 0 \rightarrow PD \\ a < 1 \rightarrow |A| < 0 \rightarrow I \end{cases}$

12.- Classify the quadratic form $Q(x, y, z) = x^2 + y^2 - 2z^2$ restricted to:

a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + z = 0\}$.

b) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = -z\}$.

$$x^2 + y^2 - 2z^2 \to A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \to |1| = 1 > 0 \\ \\ \text{Order 2} \to \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0 \\ \\ \text{Order 3} \to |A| = -2 < 0 \end{array}\right\} \{Indefinite$$

Also: The eigenvalues have different signs so: Indefinite a)

With the restriction: $x + z = 0 \rightarrow z = -x$

We will have a matrix of size n – r = 3 – 1 = 2

$$x^2 + y^2 - 2z^2 = -x^2 + y^2$$

$$\rightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \rightarrow |-1| = -1 < 0 \\ \\ \text{Order 2} \rightarrow \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 < 0 \end{array}\right\} \{Indefinite$$

b)

With the restriction: $x = y = -z$

We will have a matrix of size n – r = 3 – 2 = 1

$$x^2 + y^2 - 2z^2 = 0z^2 = 0$$

$$\to A = (0)$$

$$No\ sign$$