46) A1. Determine for which values of b and c, the matrix is diagonalizable

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & b \\ 3 & 0 & c \end{pmatrix}$$

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & b \\ 3 & 0 & c-\lambda \end{vmatrix} = 0$$

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$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & b \\ 3 & 0 & c-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-1-\lambda)(c-\lambda) = 0 \begin{cases} \lambda = 5 \\ \lambda = -1 \\ \lambda = c \end{cases}$$

$$|A - \lambda I_n| = 0$$

$$(5-\lambda)(-1-\lambda)(c-\lambda) = 0 \begin{cases} \lambda = 5 \\ \lambda = -1 \\ \lambda = c \end{cases}$$

If the eigenvalues are different, A will be diagonalizable:

$$c \neq 5 \neq -1$$

If an eigenvalue appears two times, we have to study the matrix more carefully

$$\lambda = 5 \ \ (with\, c = 5)$$

$$(A - (5)I_n)X = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & b \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 5 \ (c = 5)$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -6 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D_5 = n - rk(A - \lambda I) = 3 - 2 = 1$$

If c = 5:

$$\lambda = 5 \rightarrow \begin{cases} Multiplicity = 2 \\ Degrees\ of\ freedom = 1 \end{cases}$$

Not Diagonalizable

$$\lambda = -1 \ \ (c = -1)$$

$$(A - (-1)I_n)X = 0$$

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & b \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = -1 \; (c = -1)$

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If c = -1:

$$\lambda = -1 \rightarrow \begin{cases} Multiplicity = 2 \\ Degrees\ of\ freedom: \end{cases}$$

$$\begin{cases} b = 0 \rightarrow D_{-1} = n - rk(A - \lambda I) = 3 - 1 = 2 \\ b \neq 0 \rightarrow D_{-1} = n - rk(A - \lambda I) = 3 - 2 = 1 \end{cases}$$

$It\ is\ Diagonalizable\ if$:

$$-1 \neq c \neq 5$$

Or

$$c = -1\ \&\ b = 0$$

Exercise 1

$$3x^2 + 3y^2 + 3z^2 + 2xy + 2xz + 2yz$$

$$3x^2 + 3y^2 + 3z^2 + 2xy + 2xz + 2yz$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1 } \rightarrow |3| = 3 > 0 \\ \\ \text{Order 2 } \rightarrow \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 > 0 \\ \\ \text{Order 3 } \rightarrow |A| = 20 > 0 \end{array}\right\} \text{ Positive Definite}$$

Exercise 2

$$-x^2 - 2y^2 - 3z^2 + 2xy + 2yz$$

$$-x^2 - 2y^2 - 3z^2 + 2xy + 2yz$$

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \rightarrow |-1| = -1 < 0 \\[2ex] \text{Order 2} \rightarrow \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1 > 0 \\[2ex] \text{Order 3} \rightarrow |A| = -2 < 0 \end{array}\right\} \text{Negative Definite}$$

Exercise 3

$$x^2 + y^2 + 5z^2 - 2xy + 4xz$$

$$x^2 + y^2 + 5z^2 - 2xy + 4xz$$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 5 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \ \rightarrow |1| = 1 > 0 \\[2em] \text{Order 2} \ \rightarrow \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0 \quad \xrightarrow{Main\,minors} \quad \begin{cases} 1 \\ 5 \end{cases} \\[2em] \text{Order 3} \ \rightarrow |A| = -4 < 0 \end{array}\right\} \text{Indefinite}$$

Exercise 4: As a function of a

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$ax^2 + y^2 + z^2 + 2yz \rightarrow A = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \rightarrow |a| = a \\ \\ \text{Order 2} \rightarrow \begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} = a \\ \\ \text{Order 3} \rightarrow |A| = 0 \end{array}\right\} \begin{cases} a < 0 \rightarrow Indefinite \\ a = 0 \rightarrow Positive\ Semidefinite \\ a > 0 \rightarrow Positive\ Semidefinite \end{cases}$$

Exercise 5: As a function of a

$$ax^2 - y^2 - z^2 + 4xz$$

$$ax^2 - y^2 - z^2 + 4xz \rightarrow A = \begin{pmatrix} a & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \rightarrow |a| = a \\ \text{Order 2} \rightarrow \begin{vmatrix} a & 0 \\ 0 & -1 \end{vmatrix} = -a \\ \text{Order 3} \rightarrow |A| = a + 4 \end{array}\right\} \begin{cases} a > -4 \rightarrow Indefinite \\ a = -4 \rightarrow Negative\ Semidefinite \\ a < -4 \rightarrow Negative\ Definite \end{cases}$$