A is diagonalizable if we can write the following expression

$$A = PDP^{-1}$$

In this sense, we can say that A and D are similar. They will have the same rank and determinant.

$$A = PDP^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

$$P = \begin{pmatrix} X_1 & X_2 & \dots & X_n \end{pmatrix}$$

A is diagonalizable if we can write the following expression

$$A = PDP^{-1}$$

**First Theorem:**

If $A \in M_n$ has n linearly independant eigenvectors, A is diagonalizable

A is diagonalizable if we can write the following expression

$$A = PDP^{-1}$$

**<u>Second Theorem:</u>**

If $A \in M_n$ has n different eigenvalues, A is diagonalizable

Example 1)

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

# 1) To obtain the eigenvalues

Characteristic Polynomial:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(1-\lambda) = 0 \begin{cases} \lambda = 1 \\ \lambda = 5 \end{cases}$$

# 2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = 5 \rightarrow \begin{pmatrix} 3-(5) & 2 \\ 2 & 3-(5) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

# 2) To obtain the eigenvectors:

For $\lambda = 5$:

$$(A - (5)I_n)X = 0$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$r_2 + r_1$$

$$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

## 2) To obtain the eigenvectors:

For $\lambda = 5$:

$$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{-2x + 2y = 0\} \begin{cases} x = x \\ y = x \end{cases}$$

Eigenvector: $\begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

# 2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = 1 \rightarrow \begin{pmatrix} 3-(1) & 2 \\ 2 & 3-(1) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

# 2) To obtain the eigenvectors:

For $\lambda = 1$:

$$(A - (1)I_n)X = 0$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$r_2 - r_1$$

$$\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

# 2) To obtain the eigenvectors:

For $\lambda = 1$:

$$\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{2x + 2y = 0\} \begin{cases} x = x \\ y = -x \end{cases}$$

Eigenvector: $\begin{pmatrix} x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

# Summary:

$$1)\ \ |A-\lambda I_n| = 0 \ \begin{cases} \lambda = 5 \\ \lambda = 1 \end{cases}$$

$$2)\ \ (A-\lambda I_n)X = 0 \begin{cases} x\begin{pmatrix}1\\1\end{pmatrix}, for\ \ \lambda = 5 \\ x\begin{pmatrix}1\\-1\end{pmatrix}, \quad for\ \ \lambda = 1 \end{cases}$$

A is diagonalizable, we can write the following expression

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = P \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

A is diagonalizable if we can write the following expression

$$A = PDP^{-1}$$

**<u>Third Theorem:</u>**

If $A \in M_n$ has, for each eigenvalue, that:
Mutiplicity of $\lambda_i$ = Degrees of freedom of $X_i$,
A is diagonalizable

# Example 2)

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

# 1) To obtain the eigenvalues

Characteristic Polynomial:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(3-\lambda) = 0 \begin{cases} \lambda = 3 \\ \lambda = 3 \end{cases}$$

# 2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = 3 \rightarrow \begin{pmatrix} 3-(3) & 0 \\ 0 & 3-(3) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = 3$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{0 = 0\} \begin{cases} x = x \\ y = y \end{cases}$$

Multiplicity of $\lambda = 3 \rightarrow 2$

Degrees of freedom = 2

## 2) To obtain the eigenvectors:

For $\lambda = 3$:

$$\{0 = 0\} \begin{cases} x = x \\ y = y \end{cases}$$

Eigenvectors:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

# Summary:

$$1)\ \ |A - \lambda I_n| = 0 \ \begin{cases} \lambda = 3 \\ \lambda = 3 \end{cases}$$

$$2)\ \ (A - \lambda I_n)X = 0 \begin{cases} x\begin{pmatrix} 1 \\ 0 \end{pmatrix}, for\ \ \lambda = 3 \\ y\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad for\ \ \lambda = 3 \end{cases}$$

A is diagonalizable, we can write the following expression

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = P \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} P^{-1}$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Obviously…

# Example 3)

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

# 1) To obtain the eigenvalues

Characteristic Polynomial:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(3-\lambda) = 0 \begin{cases} \lambda = 3 \\ \lambda = 3 \end{cases}$$

# 2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = 3 \rightarrow \begin{pmatrix} 2-(3) & 1 \\ -1 & 4-(3) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

## 2) To obtain the eigenvectors:

For $\lambda = 3$:

$$(A - (3)I_n)X = 0$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$r_2 - r_1$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

## 2) To obtain the eigenvectors:

For $\lambda = 3$:

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{-x + y = 0\} \begin{cases} x = x \\ y = x \end{cases}$$

$$\left.\begin{array}{r} \text{Multiplicity of } \lambda = 3 \rightarrow 2 \\ \text{Degrees of freedom} = 1 \end{array}\right\} \text{Not diagonalizable}$$

2) To obtain the eigenvectors:

For $\lambda = 3$:

$$\{-x + y = 0\} \begin{cases} x = x \\ y = x \end{cases}$$

Eigenvectors:

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

# Summary:

$$1)\ \ |A-\lambda I_n| = 0\ \ \begin{cases}\lambda = 3\\ \lambda = 3\end{cases}$$

$$2)\ \ (A-\lambda I_n)X = 0\left\{x\begin{pmatrix}1\\1\end{pmatrix}, for\ \ \lambda = 3\right.$$

$$A = PDP^{-1}$$

There is only 1 eigenvector, if we write:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow |P| = 0 \rightarrow P^{-1} \text{ does not exist}$$

**<u>A is not diagonalizable</u>**

A is diagonalizable if we can write the following expression

$$A = PDP^{-1}$$

**Fourth Theorem:**

If $A \in M_n$ is symmetric, A is diagonalizable

$$A = PDP^{-1}$$

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

$$\dots$$

$$A^k = PD^kP^{-1}$$

# Example 4)

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$$

$$A^5?$$

## 1) To obtain the eigenvalues

Characteristic Polynomial:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) = 0 \begin{cases} \lambda = 4 \\ \lambda = 2 \end{cases}$$

2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = 4 \rightarrow \begin{pmatrix} 1-(4) & 3 \\ -1 & 5-(4) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = 4$:

$$(A - (4)I_n)X = 0$$

$$\begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3r_2 - r_1$$

$$\begin{pmatrix} -3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = 4$:

$$\begin{pmatrix} -3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{-3x + 3y = 0\} \begin{cases} x = x \\ y = x \end{cases}$$

Eigenvector: $\begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

# 2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = 2 \rightarrow \begin{pmatrix} 1-(2) & 3 \\ -1 & 5-(2) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

# 2) To obtain the eigenvectors:

For $\lambda = 2$:

$$(A - (2)I_n)X = 0$$

$$\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$r_2 - r_1$$

$$\begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = 2$:

$$\begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{-x + 3y = 0\}\begin{cases} x = 3y \\ y = y \end{cases}$$

Eigenvector: $\begin{pmatrix} 3y \\ y \end{pmatrix} = y\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

# Summary:

$$1)\ \ |A-\lambda I_n|=0\ \begin{cases}\lambda=4\\ \lambda=2\end{cases}$$

$$2)\ \ (A-\lambda I_n)X=0\begin{cases}x\begin{pmatrix}1\\1\end{pmatrix}, for\ \ \lambda=4\\ y\begin{pmatrix}3\\1\end{pmatrix},\quad for\ \ \lambda=2\end{cases}$$

$$\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} = P \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$$

$$P = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}^5 = P \begin{pmatrix} 4^5 & 0 \\ 0 & 2^5 \end{pmatrix} P^{-1}$$