14, a)

$$mx + y - z = 1$$
$$x + 2y + z = 2$$
$$x + 2y - z = 0$$

$$A = \begin{pmatrix} m & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$|A| = -2m + 2 - 2 + 1 - 2m + 1 = 2 - 4m$$

$$|A| = 2 - 4m \begin{cases} |A| = 0 \rightarrow m = \dfrac{1}{2} \rightarrow rk(A) < 3 \\ |A| \neq 0 \rightarrow m \neq \dfrac{1}{2} \rightarrow rk(A) = 3, \qquad DCS \end{cases}$$

For the DCS:

$$x = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 0 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} m & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{-2 - 4 - 2 + 2}{2 - 4m} = \frac{-6}{2 - 4m}$$

For the DCS:

$$y = \frac{\begin{vmatrix} m & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} m & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{-2m + 1 + 2 + 1}{2 - 4m} = \frac{4 - 2m}{2 - 4m} =$$

$$= \frac{2 - m}{1 - 2m}$$

For the DCS:

$$z = \frac{\begin{vmatrix} m & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{vmatrix}}{\begin{vmatrix} m & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{2 + 2 - 2 - 4m}{2 - 4m} = \frac{2 - 4m}{2 - 4m} = 1$$

For the DCS:

$$S.C.D \rightarrow m \neq \frac{1}{2} \rightarrow \begin{cases} x = \dfrac{-6}{2-4m} \\ y = \dfrac{4-2m}{2-4m} \\ z = 1 \end{cases}$$

$$If\ m = \frac{1}{2}$$

$$A = \begin{pmatrix} 1/2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{ccc|c} 1/2 & 1 & -1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1/2 & 1 & -1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$$r_3 - r_2$$
$$r_2 - 2r_1$$

$$(A|B)=\left(\begin{array}{ccc|c} 1/2 & 1 & -1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1/2 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

$$r_3+\frac{2}{3}r_2$$

$$Si\ m = \frac{1}{2} \rightarrow rk(A) < 3$$

$$(A|B) = \left(\begin{array}{ccc|c} 1/2 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

$$rk(A) < rk(A|B) \ \rightarrow Inconsistent\ System$$

15)

$$A = \begin{pmatrix} 1 & 1 & m \\ 3 & 2 & 4m \\ 2 & 1 & 3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|A| = 6 + 3m + 8m - 4m - 4m - 9 = 3m - 3$$

$$|A| = 3m - 3 \begin{cases} |A| = 0 \rightarrow m = 1 \rightarrow rk(A) < 3 \\ |A| \neq 0 \rightarrow m \neq 1 \rightarrow rk(A) = 3, \qquad DCS \end{cases}$$

Cramer's Rule

$$x = \frac{\begin{vmatrix} 0 & 1 & m \\ 0 & 2 & 4m \\ 0 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & m \\ 3 & 2 & 4m \\ 2 & 1 & 3 \end{vmatrix}} = \frac{0}{3m-3} = 0$$

$$y = \frac{\begin{vmatrix} 1 & 0 & m \\ 3 & 0 & 4m \\ 2 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & m \\ 3 & 2 & 4m \\ 2 & 1 & 3 \end{vmatrix}} = \frac{0}{3m-3} = 0$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & m \\ 3 & 2 & 4m \\ 2 & 1 & 3 \end{vmatrix}} = \frac{0}{3m-3} = 0$$

$$\text{DCS} \rightarrow m \neq 1 \rightarrow B = 0 \rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

$$If\ m = 1$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & 4 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$r_2 - 3r_1$$
$$r_3 - 2r_1$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$r_3 - r_2$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$rk(A) = rk(A|B) = 2 < 3 = n \rightarrow S.C.I$$

$$\left.\begin{array}{r} x + y + z = 0 \\ -y + z = 0 \end{array}\right\} \begin{cases} x = -2y \\ y = y \\ z = y \end{cases}$$

Degrees of Freedom= n-rk(A) = 1

32.- Let $A = \begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix}$ with $a, b \in \mathbb{R}$. Calculate the values of the parameters a and b so that the vector (2,-1) is eigenvector of A associated to the eigenvalue 2.

$$\begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$2 - a = 4$$
$$4 - b = -2$$

$$\begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\left.\begin{array}{r} 2-a=4 \\ 4-b=-2 \end{array}\right\}\left\{\begin{array}{r} a=-2 \\ b=6 \end{array}\right.$$

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}$$

31.- A matrix $A \in M_2$ verifies the following conditions: $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and (2,-1) is an eigenvector of A associated to the eigenvalue $\lambda = -2$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$a - b = 3$$

$$c - d = 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

$$2a - b = -4$$

$$2c - d = 2$$

$$\left.\begin{array}{c} a-b=3\\ c-d=1\\ 2a-b=-4\\ 2c-d=2 \end{array}\right\}\left\{\begin{array}{c} a=-7\\ b=-10\\ c=1\\ d=0 \end{array}\right.$$

$$A=\begin{pmatrix} -7 & -10\\ 1 & 0 \end{pmatrix}$$

Exercise 5:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Analyze if any of these three vectors is an eigenvector of the matrix A: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

b) Is $\begin{pmatrix}1\\1\\1\end{pmatrix}$ an eigenvector?

$$AX = \lambda X$$

$$\begin{pmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\3\\5\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$AX = \lambda X$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix}$$

$$\left.\begin{matrix} 1 = \lambda \\ 3 = \lambda \\ 5 = \lambda \end{matrix}\right\} \to \; Imposible. \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ is not an eigenvector}$$

b) Is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ an eigenvector?

$$AX = \lambda X$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$AX = \lambda X$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$$

$$\left.\begin{matrix} 0 = 0 \\ 0 = 0 \\ 2 = \lambda \end{matrix}\right\} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ is an eigenvector, with an eigenvalue of $\lambda = 2$

b) Is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ an eigenvector?

$$AX = \lambda X$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$AX = \lambda X$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} 0 = 0 \\ 1 = \lambda \\ 0 = 0 \end{matrix}\right\} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ is an eigenvector, with an eigenvalue of $\lambda = 1$