Last week: Analyze the solution of the following system, depending on the value of m

$$x - 2y + z = -1$$
$$x + y + 3z = 4$$
$$5x - y + mz = 10$$

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 3 \\ 5 & -1 & m \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} -1 \\ 4 \\ 10 \end{pmatrix}$$

$$(A|B) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & m-11 & 0 \end{array}\right)$$

If $m \neq 11$

$$\left.\begin{array}{c} x - 2y + z = -1 \\ 3y + 2z = 5 \\ (m-11)z = 0 \end{array}\right\} \begin{cases} x = \dfrac{7}{3} \\ y = \dfrac{5}{3} \\ z = 0 \end{cases}$$

If $m = 11 \to 2 = rk(A) = rk(A|B) < n = 3$

$$(A|B) \to \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{ICS}$$

$$\left.\begin{array}{c} x - 2y + z = -1 \\ 3y + 2z = 5 \end{array}\right\} \begin{cases} x = \dfrac{-7 + 7y}{2} \\ y = y \\ z = \dfrac{5 - 3y}{2} \end{cases}$$

Teorema de Rouché-Frobenius

$$Consistent\ System \leftrightarrow rk(A) = rk(A|B) = r$$

$$r = n \rightarrow DCS$$
$$r < n \rightarrow ICS$$

Complete System:

$$AX = B, \qquad B \neq 0$$

Homogeneous System:

$$AX = B, \qquad B = 0$$

Homogeneous System:

$$AX = B, \qquad B = 0$$

It is always consistent: $rk(A) = rk(A|B) = r$

$$\begin{cases} r = n \rightarrow DCS \rightarrow solution = X = 0 \\ r < n \rightarrow ICS \end{cases}$$

Example

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$$

$$(A|B) \rightarrow \left(\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right) \rightarrow \left(\begin{matrix} 2 & 1 \\ 0 & 3 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right)$$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right)$$

$$rk(A) = rk(A|B) = 3 = n$$

$$DCS\begin{cases} x = 0 \\ y = 0 \end{cases}$$

Cramer's Rule

$$It\ is\ only\ applicable\ if\ we\ want\ to\ find$$
$$the\ solution\ of\ a\ DCS$$

$$rk(A) = n$$
$$\downarrow$$
$$|A| \neq 0$$

Cramer’s Rule

How do we do it?

$$x_i = \frac{\begin{vmatrix} a_{11} & \dots & b_{1i} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & b_{ni} & \dots & a_{nn} \end{vmatrix}}{|A|}$$

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$

DCS

$$We\ can\ apply\ Cramer's\ Rule$$

$$x = \frac{\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{3}{3} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{3}{3} = 1$$

Is very useful in exercises with parameters:

1) Obtain the values of the parameter that makes $|A| = 0$
2) Normal case (matrix in row echelon form) for those values $\begin{cases} S.C.I \\ S.I \end{cases}$

3) Cramer's Rule otherwise

Example:

$$\begin{pmatrix} 2 & a \\ 1 & b \end{pmatrix} X = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$|A| = 2b - a$$

$$|A| = 2b - a \begin{cases} |A| = 0 \rightarrow a = 2b \rightarrow rk(A) < 2 & ICS \\ |A| \neq 0 \rightarrow a \neq 2b \rightarrow rk(A) = 2 = n & DCS \end{cases}$$

1) a = 2b:

$$\begin{pmatrix} 2 & 2b \\ 1 & b \end{pmatrix} X = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \rightarrow$$

$$(A|B) \rightarrow \left(\begin{matrix} 2 & 2b \\ 1 & b \end{matrix}\middle|\begin{matrix} 6 \\ 3 \end{matrix}\right) \rightarrow \left(\begin{matrix} 2 & 2b \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 6 \\ 0 \end{matrix}\right)$$

$$2r_2 - r_1$$

$$\left(\begin{matrix} 2 & 2b \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 6 \\ 0 \end{matrix}\right)$$

$$rk(A) = rk(A|B) = 1 < n = 2$$

$$2x + 2by = 6$$

$$S.C.I \begin{cases} x = 3 - by \\ y = y \end{cases}$$

2) $a \neq 2b$, we can use Cramer's Rule

$$x = \frac{\begin{vmatrix} 6 & a \\ 3 & b \end{vmatrix}}{\begin{vmatrix} 2 & a \\ 1 & b \end{vmatrix}} = \frac{6b - 3a}{2b - a} = \frac{3(2b - a)}{2b - a} = 3$$

$$y = \frac{\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & a \\ 1 & b \end{vmatrix}} = \frac{6 - 6}{2b - a} = 0$$

Diagonalization:

Eigenvalues & Eigenvectors

Definition of the relationship between A and B $\in M_{nxm}$:

Equivalents: $B = NAM \ \rightarrow rk(A) = rk(B)$

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Similars: $B = P^{-1}AP \rightarrow \begin{cases} rk(A) = rk\,(B) \\ Det(A) = Det(B) \end{cases}$

Eigenvalues & Eigenvectors

$$\boxed{AX = \lambda X}$$

A: Matrix

X: Eigenvector

λ: Eigenvalue

Eigenvalues & Eigenvectors

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{cases} 3 \text{ is an eigenvalue of } A \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector of } A \end{cases}$$

Eigenvalues & Eigenvectors

$$\boxed{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{cases} 1\ is\ an\ eigenvalue\ of\ A \\ \begin{pmatrix} -1 \\ 1 \end{pmatrix}\ is\ an\ eigenvector\ of\ A \end{cases}$$

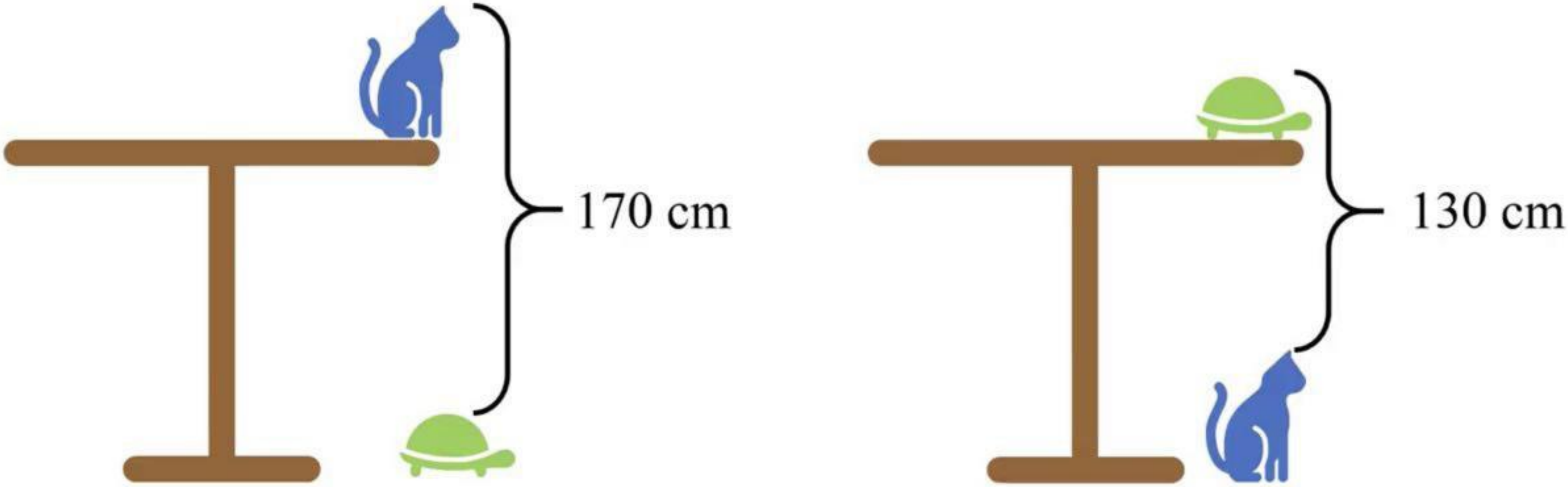
$$A\ matrix\ of\ size\ n\ will\ have\ n\ eigenvalues.$$

$$|A| = \lambda_1 \lambda_2 \dots \lambda_n$$

$$|A| = 4 - 1 = 3 = \lambda_1 \lambda_2 = 3 \cdot 1$$

Homework in China for elementary school students: how tall is the table?

What is the minimum height of the cat?



Adapted from tweet by China Focus
https://twitter.com/China__Focus/status/1039338689187332096

$$x + y - z = 170$$
$$x - y + z = 130$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 170 \\ 1 & -1 & 1 & 130 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 170 \\ 0 & -2 & 2 & -40 \end{array}\right)$$

$$r_2 - 2_1$$

$$\left(\begin{matrix} 1 & 1 & -1 \\ 0 & -2 & 2 \end{matrix}\middle| \begin{matrix} 170 \\ -40 \end{matrix}\right)$$

$$ICS$$

$$\left.\begin{matrix} x + y - z = 170 \\ -2y + 2z = -40 \end{matrix}\right\} \left\{\begin{matrix} x = 150 \\ y = z + 20 \\ z = z \end{matrix}\right.$$

Table = x = 150 cm

Cat = y = A mínimum of 20 cm