Sign of a Quadratic Form: Third method.

Sum of squares: We transform the equation so that there is only squares.

$$x^2 + 2y^2 + 4xy$$

The first coefficient has to be 1

$$x^2+2y^2+4xy$$

We want to eliminate 4xy. We add and subtract the expression:

Half

$$(x+2y)^2=(x^2+4y^2+4xy)$$

$$x^2+2y^2+4xy+(x+2y)^2-(x^2+4y^2+4xy)=$$
$$=(x+2y)^2-2y^2$$

$$x^2+2y^2+4xy = 1(x+2y)^2 - 2y^2$$

We now have only terms with squares. We can construct a matrix
With the coefficients in the diagonal and see the sign

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

Indefinite

$$-2x^2 - 8y^2 + 8xy$$

$$-2(x^2+4y^2 - 4xy)$$

We want to eliminate $-4xy$, we add and subtract

$$(x-2y)^2 = (x^2+4y^2 - 4xy)$$

$$-2[\,(x^2+4y^2-4xy) + (x-2y)^2 - (x^2+4y^2-4xy)] =$$

$$= -2[(x-2y)^2 + 0y^2]$$

$$-2x^2 - 8y^2 + 8xy = -2(x - 2y)^2 + 0y^2$$

Matrix:

$$D = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

Negative Semidefinite

$$x^2 - y^2 + 2xy + 4xz + 4yz$$

$$x^2 - y^2 + 2xy + 4xz + 4yz$$

We begin eliminating the terms with x: $2xy + 4xz$,

We add and subtract:

$$(x + y + 2z)^2 = (\, x^2 + y^2 + 4z^2 + 2xy + 4xz + 4yz)$$

$$(x^2 - y^2 + 2xy + 4xz + 4yz) +$$
$$+(x + y + 2z)^2 - (\, x^2 + y^2 + 4z^2 + 2xy + 4xz + 4yz)$$

$$= (x + y + 2z)^2 - 2y^2 - 4z^2$$

$$(x + y + 2z)^2 - 2y^2 - 4z^2$$

Matrix:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

Indefinite

Linear Independence of vectors

Being $v_1, v_2\dots, v_n$ some vectors

$$If\ \ rk(v_1 \quad v_2 \quad \dots \quad v_n) < n \rightarrow Linearly\ dependent$$

$$If\ \ rk(v_1 \quad v_2 \quad \dots \quad v_n) = n \rightarrow Linearly\ independent$$

If the n vectors are linearly independent, they form a <u>**Base**</u>.

Exercise 1: Classify the expression

$$y^2 - 2xy + 4xz$$

Restricted to:

$$x - az = 0$$
$$y - az = 0$$

Without the restriction

$$y^2 - 2xy + 4xz \rightarrow A = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Order 1 $\rightarrow |0| = 0 \quad \overbrace{\rightarrow}^{Main\ Minors} \begin{cases} 0 \\ 1 \\ 0 \end{cases}$

Order 2 $\rightarrow \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1 < 0$

Order 3 $\rightarrow |A| = 0 - 4 = -4 < 0$

Indefinite

With the restrictions: $\left.\begin{matrix} x - az = 0 \\ y - az = 0 \end{matrix}\right\} \begin{cases} x = az \\ y = az \end{cases}$

We will have a matrix of size n – r = 3 – 2 = 1

$$y^2 - 2xy + 4xz = a^2z^2 - 2a^2z^2 + 4az^2 =$$
$$= a(4-a)z^2 \rightarrow A = (a(4-a))$$

$$0 < a < 4 \rightarrow Positive\ Definite$$
$$a < 0 \rightarrow Negative\ Definite$$
$$a > 4 \rightarrow Negative\ Definite$$

Exercise 2: Profit function:

$$x^2 + y^2 + 10z^2 - 2yz - 6xz$$

Are we sure that the firm will only have positive profits?

$$x^2 + y^2 + 10z^2 - 2yz - 6xz \rightarrow A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ -3 & -1 & 10 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1 } \rightarrow |1| = 1 > 0 \\ \\ \text{Order 2 } \rightarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0 \\ \\ \text{Order 3 } \rightarrow |A| = 0 \end{array}\right\} \{Positive\ Semidefinite$$

12.- Classify the quadratic form $Q(x, y, z) = x^2 + y^2 - 2z^2$ restricted to:

a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + z = 0\}$.

b) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = -z\}$.

b)

With the restriction: $x = y = -z$

We will have a matrix of size n – r = 3 – 2 = 1

$$x^2 + y^2 - 2z^2 = 0z^2 = 0$$

$$\rightarrow A = (0)$$

$$No\ sign$$

15.- Clasificar $Q(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$ restringida a:

a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2z = 0\}$.

b) $S = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

Ejercicio 14

$$2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz \to A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \to |2| = 2 > 0 \\[2em] \text{Order 2} \to \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0 \\[2em] \text{Order 3} \to |A| = 4 > 0 \end{array}\right\} \{Positive\ Definite$$

a) With the restriction: $x - 2z = 0 \rightarrow x = 2z$

$$2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$$
$$= 2y^2 + 14z^2 + 6yz$$

$$\rightarrow A = \begin{pmatrix} 2 & 3 \\ 3 & 14 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Order 1} \rightarrow |2| = 2 > 0 \\ \\ \text{Order 2} \rightarrow \begin{vmatrix} 2 & 3 \\ 3 & 14 \end{vmatrix} = 19 > 0 \end{array}\right\} Positive\ Definite$$

b) With the restriction: $x = y = 0$

$$2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz = 2z^2$$
$$\rightarrow A = (2)$$

Positive Definite