FUNCTIONS FROM $\mathbb{R}^n$ TO $\mathbb{R}^m$

Note: it will be understood that $\log x = \log_{10} x$ and $\ln x = \log_e x$

1.- Determine and represent graphically the domain of the following functions:

a) $f(x) = \sqrt{x^2 - 16}$

b) $f(x) = \dfrac{1}{x^2 - 1}$

c) $f(x) = \dfrac{1}{\sqrt{x-3}}$

d) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

e) $f(x) = \dfrac{\sqrt[3]{x+1}}{x^2 + x + 1}$

f) $f(x) = e^{\sqrt{\frac{x+1}{x-3}}}$

g) $f(x) = \sqrt{-x} + \dfrac{1}{\sqrt{2+x}}$

h) $f(x) = \ln(x^2 - 4)$

i) $f(x) = \ln\dfrac{x^2 - 3x + 2}{x+1}$

j) $f(x) = \ln(x+2) + \ln(x-2)$

k) $f(x,y) = 3x + y$

l) $f(x,y) = \dfrac{3x+y}{x^2 + 2y^2}$

m) $f(x,y) = \dfrac{1}{\sqrt{x-y}}$

n) $f(x,y) = \dfrac{y}{x^2}$

o) $f(x,y) = \ln(2x - y + 1)$

p) $f(x,y) = x + \sqrt{y}$

q) $f(x,y) = \sqrt{1-x^2} + \sqrt{y^2 - 1}$

r) $f(x,y) = \sqrt{1 - x^2 - y^2}$

s) $f(x,y) = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$

t) $f(x,y) = \ln(3x^2 + y^2 + 4)$

u) $f(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$

v) $f(x,y) = \left(\ln(x+y),\ \sqrt{4 - x^2 - y^2}\right)$

2.- Represent graphically the level (contour) curves of the following functions:

a) $f(x,y) = x + y$

b) $f(x,y) = (x+y)^2$

c) $f(x,y) = x^2 + y^2$

d) $f(x,y) = \dfrac{x}{y}$

e) $f(x,y) = \sqrt{xy}$

f) $f(x,y) = x^y,\ x > 0$

g) $f(x,y) = \dfrac{y}{x^2}$

h) $f(x,y) = \ln(2x - y + 1)$

i) $f(x,y) = \dfrac{x+y}{x}$

3.- Compute the following limits if possible:

a) $\lim_{x\to 1} \dfrac{x^2-1}{2x^2-x-1}$

b) $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x^3+3}-2}$

c) $\lim_{x\to 0} \left(\dfrac{1+x}{1-x}\right)^x$

d) $\lim_{x\to 0} \dfrac{x^2}{1+2^{\frac{1}{x}}}$

e) $\lim_{x\to 0} \dfrac{2}{3+4^{\frac{1}{x}}}$

f) $\lim_{x\to +\infty} \dfrac{5x^2-6x}{3x^2-8}$

g) $\lim_{x\to +\infty} \left(1-\dfrac{2}{3x}\right)^x$

h) $\lim_{x\to +\infty} \left(\dfrac{2x+1}{2x+4}\right)^{\frac{x^2}{x+1}}$

i) $\lim_{x\to +\infty} \dfrac{x^5}{e^{-x}}$

j) $\lim_{x\to +\infty} \left(\dfrac{2x+1}{x}\right)^x$

k) $\lim_{x\to +\infty} \left(\dfrac{5x+1}{8x-1}\right)^{-x}$

l) $\lim_{(x,y)\to(1,2)} \dfrac{3x^2y}{4x-y+1}$

m) $\lim_{(x,y)\to(0,0)} \dfrac{3x^2}{5x^2+y^2}$

n) $\lim_{(x,y)\to(0,1)} \left(x^2+5xy-8,\ e^{x^2y},\ \ln(2x+y)\right)$

4.- Study the continuity of the following functions:

a) $f(x)=\begin{cases} \dfrac{e^x}{x-1} & \text{if } \ x\le 0 \\ x^3+1 & \text{if } \ x>0 \end{cases}$

b) $f(x,y)=\begin{cases} \dfrac{2x^2-3y^2}{7x^2+y^2} & \text{if } \ (x,y)\ne(0,0) \\ 0 & \text{if } \ (x,y)=(0,0) \end{cases}$

5.- Compute the first derivative of the following functions:

a) $f(x)=e^5$

b) $f(x)=\dfrac{e^x}{x}$

c) $f(x)=\dfrac{1}{x}$

d) $f(x)=e^{-\frac{x}{2}}$

e) $f(x)=\dfrac{1+x}{1-x}$

f) $f(x)=\sqrt{2x-1}$

g) $f(x)=\ln x+\sqrt{x^2+1}$

h) $f(x)=\dfrac{\pi}{x}+\ln 2$

i) $f(x)=\dfrac{2}{2x-1}-\dfrac{1}{x}$

j) $f(x)=\dfrac{1+\sqrt{x}}{1-\sqrt{x}}$

k) $f(x)=\ln x\ \log x-\ln a\ \log_a x$

l) $f(x)=\dfrac{3}{56(2x-1)^7}-\dfrac{1}{24(2x-1)^6}-\dfrac{1}{40(2x-1)^5}$

m) $f(x)=\sqrt[3]{2e^x-2^x+1}+\ln^5 x$

n) $f(x)=x^2\,10^{2x}$

o) $f(x)=\ln^2 x-\ln(\ln x)$

p) $f(x)=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})$

q) $f(x)=x^4(a-2x^3)^2$

r) $f(x)=\sqrt{(x+a)(x+b)(x+c)}$

s) $f(x)=\sqrt[3]{x+\sqrt{x}}$

t) $f(x)=(2x+1)(3x+2)\sqrt[3]{3x+2}$

u) $f(x)=\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}$

v) $f(x)=\ln(\sin x)$

w) $f(x)=\sin x^4$

x) $f(x)=\cos^4 x$

y) $f(x)=\sin^2 x\ \cos x^6$

6.- Compute the partial derivatives of the following functions:

a) $f(x,y,z)=e^{\frac{x}{y}}+e^{\frac{z}{y}}$

b) $f(x,y)=\ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$

c) $f(x,y,z)=e^{x^2+y^2+z^2}$

d) $f(x,y)=\sqrt{xy+\frac{x}{y}}$

e) $f(x,y)=x^3+y^3-3axy$

f) $f(x,y)=\frac{x-y}{x+y}$

g) $f(x,y)=\ln(x+\sqrt{x^2+y^2})$

h) $f(x,y)=x^y$

i) $f(x,y,z)=z^{xy}$

j) $f(x,y)=\frac{x+y}{\sqrt[3]{x^2+y^2}}$

k) $f(x,y)=x^2\sin^2 y$

l) $f(x,y)=\frac{e^{ax}(\sin x+a\cos y)}{a^2+b^2}$

7.- Check that $y=xe^{-x}$ verifies the equation $x\frac{dy}{dx}=(1-x)y$.

8.- Compute y’ starting from the following expressions:

a) $2x^2+5xy+y^2=19$

b) $x^2+y^2=25$

c) $y=1+xe^y$

d) $\ln y+e^{-\frac{y}{x}}=8$

e) $\ln y+\frac{x}{y}=7$

f) $x^y=y^x$

9.- Find the equation of the tangent straight line to the graph of the function $f(x)=\frac{1}{x}$ at $x=1$. Find an approximate value of $f(1.1)$.

10.- Compute the following limits:

a) $\lim\limits_{x\to 0}\frac{\sin x}{x}$

b) $\lim\limits_{x\to 0}\frac{2(1-\cos x)}{x^2}$

c) $\lim\limits_{x\to 0}\frac{e^x-1}{x}$

d) $\lim\limits_{x\to 0}\frac{\sin 3x^2}{3x^2}$

e) $\lim\limits_{x\to 0}\frac{1-\cos\sqrt{x}}{\frac{1}{2}x}$

f) $\lim\limits_{x\to 0}\frac{1-\cos(\sin x)}{x^2}$

g) $\lim\limits_{x\to 0}\frac{\ln(1+x^2)}{x^2}$

h) $\lim\limits_{x\to 0}\frac{e^{x^2}-1}{x^2}$

i) $\lim\limits_{x\to 0}\frac{\ln(1+\sin x)}{\sin x}$

j) $\lim\limits_{x\to 0}\frac{e^{2x}-1}{x^3}$

k) $\lim\limits_{x\to 0}\frac{\sin x \, tg x}{1-\cos x}$

l) $\lim\limits_{x\to 0}\frac{1-\cos 4x}{5x^2}$

m) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{1-e^x}$

n) $\lim\limits_{x\to 0}\frac{\sin^2 x}{1-\cos 2x}$

o) $\lim\limits_{x\to 0}\frac{4(x-\ln(1+x))}{x\ln(1+x)}$

p) $\lim\limits_{x\to 0^+} x^{\frac{1}{x}}$

q) $\lim\limits_{x\to 0^+} (\cos x)^{\frac{1}{x}}$

r) $\lim\limits_{x\to +\infty} x\left[\ln(x+1)-\ln x\right]$

s) $\lim\limits_{x\to +\infty}\frac{1}{x}\ln x$

t) $\lim\limits_{x\to +\infty}\frac{\ln x}{x^2}$

u) $\lim\limits_{x\to +\infty}\frac{\ln(1+x^6)}{x^2}$

v) $\lim\limits_{x\to +\infty}(\ln x^3)e^{-x}$

11.- Compute $\nabla f(x,y)$ when $f(x,y)=x^2y+y^3$.

12.- Find the Jacobian matrix of the following functions:

a) $f(x)=\ln^2 x$

b) $f(x_1,x_2)=(x_1-x_2)^2$

c) $f(x_1,x_2)=(x_1+x_2,x_1-x_2)$

d) $f(t)=(t,\ e^t,\ e^{-t})$

13.- Compute $\frac{dz}{dt}$ in the functions:

a) $z=3x+y$, with $x=t^2+1$; $y=e^t$.

b) $z=xy+yu+xu$, with $x=t$; $y=e^{-t}$; $u=\ln(t)$.

c) $z=e^{xy}$, with $x=t\cos t$; $y=t\sin t$.

14.- Compute $\frac{dz}{du}(1)$ in the function $z=3x^2+2xy-y^2$, with $x=u^2+3u$; $y=2u^2-u$.

15.- Compute $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$ in the function $u = z\,sin\dfrac{y}{x}$, with $x = 3r^2 + 2s$; $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$.

16.- Compute $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ and evaluate them at the point $(x, y) = (0,1)$, in the following cases:

a) $z = u + v$, with $u = x + e^y$; $v = \ln(y) + e^{-x}$.

b) $z = \dfrac{sin\ u}{v}$, with $u = s - t$; $v = s + x$; $s = y^2 - x$; $t = e^y$.

17.- Prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2$, if $z = ln(x^2 + xy + y^2)$.

18.- Prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = xy + z$, if $z = xy + x\,e^{y/x}$.

19.- Prove that it holds

$$(x^2 - y^2)\frac{\partial z}{\partial x}(x, y) + xy\frac{\partial z}{\partial y}(x, y) = xyz\ ,$$

where $z = e^y F\left(y\,e^{x^2/2y^2}\right)$ and F being a differentiable real function of a real variable.

20.- Let $f(x, y) = g(x^2 + y^2)$, being $g : \mathbb{R} \to \mathbb{R}$ a differentiable function at any point of the real line. If $g'(5) = 3$, compute $\nabla f(1,2)$.

21.- Determine the differential of the following functions at the indicated points:

a) $y = \ln x$, at 2

b) $y = \dfrac{x^2}{e^x}$, at 0

c) $z = \sqrt{x^2 + y^2}$, at $(1,1)$

d) $u = \left(xy + \dfrac{x}{y}\right)^z$, at $(1,1,1)$.

e) $z = \ln\dfrac{2x}{y^2}$, at $(1, e)$

j) $f(x, y) = x^2 e^{\frac{y}{x}}$, at (x, y) with $x \neq 0$

k) $z = (e^{x+y} + y,\ xy^2)$, at $(0,0)$

22.- Let $f(x)=\sqrt[3]{x}$, using the concept of differential, study the approximate variation of the function when we change $x=27$ into $x=26.9$.

23.- Compute the directional derivative of the following functions in the indicated direction and at the indicated point:

a) $f(x,y)=3-2x^2+y^3$, in the direction of $v=\left(\frac{1}{2},\frac{-\sqrt{3}}{2}\right)$ at the point $P=(1,2)$

b) $f(x,y)=x^2+y+1$, in the direction $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ at the point (0,0)

c) $f(x,y)=\log(x^2+y^3)$, in the direction of $v=\left(\frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}}\right)$ at the point $P=(1,3)$

d) $f(x,y,z)=x\,sin(yz)$, in the direction $\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$ at the point (1,3,0)

e) $f(x,y,z)=x\,siny+y\cos z+z\,sinx$, in the direction $\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)$ at the point (0,0,0)

f) $f(x,y,z)=(x^2+yz^2,\ sin(x^2+y^2))$, in the direction $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ at the point $(\sqrt{\pi},\sqrt{\pi},1)$

24.- Let $f(x,y)$ be a differentiable function at the point (1,2). Knowing that $f_v(1,2)=5$ and $f_w(1,2)=6$ with $v=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $w=\left(\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\right)$, compute $\nabla f(1,2)$.

25.- Given the vector field $f(x,y)=(xy,\ x^2,\ y^2)$,

a) Study its differentiability at the point $(-1,2)$.

b) In the case of being differentiable, compute its differential at that point.

c) Find $f_v{}'$ at the point $(-1,2)$ being $v=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

26.- Compute y'' in $y=\ln(x+\sqrt{4+x^2})$.

27.- Given $f(x,y)=x\,e^{2y-x}$, compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$.

28.- Compute the Hessian matrix of the following functions:

a) $f(x,y) = x^4 + y^4 + 4xy - 2x^2 - 2y^2$.

b) $f(x,y) = x^2 + y^2 + xy - 4\ln x - 10\ln y,\ x,y > 0$.

c) $f(x,y) = 9x^2 + y^2 + 6xy + 12x + 4y$.

d) $f(x,y) = x^3 + y^3 - 9xy + 27$.

e) $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$.

f) $f(x,y,z) = xyz$.

29.- If F and G are real functions of a real variable with continuous second order derivatives, prove that the function $z = x\,F\left(\frac{y}{x}\right) + G\left(\frac{y}{x}\right)$ satisfies the following equation:

$$x^2 \frac{\partial^2 z}{\partial x^2}(x,y) + 2xy \frac{\partial^2 z}{\partial x \partial y}(x,y) + y^2 \frac{\partial^2 z}{\partial y^2}(x,y) = 0 .$$

30.- Knowing that f and g are real functions of a real variable with derivatives of the second order and $z = f(x^2+y^2) + g(x^2+y^2)$, compute:

$$\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}(x,y) - \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}(x,y) - \frac{1}{x^3}\frac{\partial z}{\partial x}(x,y) + \frac{1}{y^3}\frac{\partial z}{\partial y}(x,y) .$$

31.- Check that $z = \log(x^2+y^2)$ satisfies the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

32.- If f is a real function of a real variable with second derivative, check that the function $z = x\,f(x+y) + y\,f(x+y)$ satisfies the equation:

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 .$$

33.- Let $f, g : \mathbb{R} \to \mathbb{R}$ be real functions of a real variable and $h(x,y) = f(y \cdot g(x))$. Assuming that there exist f'' and g'' in all of $\mathbb{R}$, compute $\frac{\partial^2 h}{\partial x^2}(x,y),\ \frac{\partial^2 h}{\partial y^2}(x,y)$.

34.- Compute the Taylor expansion of order 2 of the following functions in a neighbourhood of the indicated points:

a) $f(x) = \ln(1+x),\quad x = 0$

b) $f(x) = e^x,\quad x = 0$

c) $f(x)=\dfrac{1}{1+x}, \quad x=0$

d) $f(x)=\dfrac{1}{(1+x)^2}, \quad x=0$

e) $f(x)=\sqrt{1+x}, \quad x=0$

f) $f(x)=sin\,x, \quad x=0$

g) $f(x)=cos\,x, \quad x=0$

h) $f(x)=\dfrac{2x^2-3}{(x-1)^2}, \quad x=0$

i) $f(x)=\ln(x-2x^2), \quad x=\dfrac{1}{3}$

j) $f(x)=\ln\left(\dfrac{1+x}{1-x}\right), \quad x=0$

k) $f(x)=\ln\left(x+\sqrt{1+x^2}\right), \quad x=1$

l) $f(x)=\sqrt[3]{6+x}, \quad x=2$

m) $f(x)=(1+x)e^{-x}, \quad x=0$

n) $f(x)=(1+e^x)^3, \quad x=\dfrac{1}{2}$

o) $f(x)=sin^2 2x, \quad x=0$

p) $f(x)=\ln(2x)-\dfrac{1}{x-1}, \quad x=2$

q) $f(x)=e^x\ln(1-x), \quad x=0$

r) $f(x)=\ln(9-x^2), \quad x=0$

s) $f(x,y)=e^{2x-3y}, \quad (x,y)=(0,0)$

t) $f(x,y)=\dfrac{x^2y}{x^2+y^2}, \quad (x,y)=(1,1)$

u) $f(x,y,z)=x+yz+e^y, \quad (x,y,z)=(1,0,1)$

35.- Given the function $f(x,y)=\sqrt{x+y}\,\ln y$, we consider the equation $f(x,y)=2e$:

a) Prove that this equation defines y as an implicit function of x in a neighbourhood of the point $(0,e^2)$.

b) Compute $y'(0)$.

36.- Given the equation $e^z sin(x+y)+e^y sin(x+z)+e^x sin(y+z)=0$:

a) Prove that it defines z as an implicit function of x and y in a neighbourhood of the point $(\pi,0,\pi)$.

b) Compute $\dfrac{\partial z}{\partial x}(\pi,0)$ and $\dfrac{\partial z}{\partial y}(\pi,0)$.

37.- Prove that the equation $x^2y+xy^2=16$ defines y as an implicit function of x in a neighbourhood of the point (2,2). Is $y(x)$ increasing or decreasing at $x=2$?

38.- Consider the equation $3\alpha x^2 - \ln(yz) - \dfrac{3\alpha x}{yz} = 0$. For which values of the parameter α we can assure that the previous equation defines implicitly $x = x(y,z)$ in a neighbourhood of the point (1,1,1)?

39.- Given the equation $z\, sin\, x - y\ sin\, z = 0$:

a) Prove that it defines z as an implicit function of (x,y) in a neighbourhood of $\left(\dfrac{\pi}{2},\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

b) Find the Taylor polynomial of degree 1 in a neighbourhood of the point $\left(\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ of the function defined in a).

40.- Let $h:\mathbb{R}^2 \to \mathbb{R}$ be given by $h(x,y) = sin(x^2+y) + x\,y + a^2\,y$.

a) For which values of a the equation $h(x,\ y) = 0$ defines y as an implicit function of $x,\ y = \varphi(x)$, in a neighbourhood of (0, 0)?

b) Compute $\varphi'(0)$ when possible.

c) Does it define the same equation x as an implicit function y in a neighbourhood of (0, 0) for some value of a?

d) Let $F(x,\ t) = (e^{x+t} + x^2 - 1,\ e^{\varphi(x)} + t\cos x - 1)$, with $\varphi(x)$ the implicit function of a). Prove that $JF(0,0)$ is a regular matrix.

41.- Given the equation $e^{x^2-y^2} + \alpha\,(x^2+y^2) = 1+\alpha$:

a) For which values of $\alpha \in \mathbb{R}$ does it define $y = y(x)$ as an implicit function in a neighbourhood of the point $\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$?

b) For the previous values of $\alpha \in \mathbb{R}$, compute $\dfrac{dy}{dx}\left(\dfrac{1}{\sqrt{2}}\right)$.

42.- Let $f:\mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = \alpha\,y\,z - y\ln\left(1+z^2\right) + z\,cos(x+2y) + 1,\ \ \alpha \in \mathbb{R}$.

a) Prove that $f(x,\ y,\ z) = 0$ defines z as an implicit function of x and y in a neighbourhood of the point $(\pi,\ 0,\ 1)$ for any value of α.

b) Compute α such that $\dfrac{\partial z}{\partial x}(\pi,0) = 0 = \dfrac{\partial z}{\partial y}(\pi,0)$.

43.- Given the equation $xy^2 - yx^2 + z^2 cos(xz) = 1$:

a) Prove that it defines $z(x,y)$ as an implicit function in a neighbourhood of the point $(0,\sqrt{2},1)$.

b) Find the tangent plane of $z(x,y)$ at the point $(0,\sqrt{2})$.

c) Compute the directional derivative of $z(x,y)$ at $(0,\sqrt{2})$ respect to the direction $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

44.- Let $F : \mathbb{R}^4 \to \mathbb{R}$ be given by $F(x,y,z,t) = x^3 z + y^3 t^2 - 1$:

a) Prove that the equation $F(x,y,z,t) = 0$ defines t as an implicit function of the variables (x, y, z) in a neighbourhood of the point (0, 1, 0, 1).

b) If $t = \varphi(x,y,z)$ is the implicit function of a), compute $\nabla\varphi(0,1,0)$.

45.- Study whether the following functions are homogeneous, indicating the degree of homogeneity if in the affirmative:

a) $f(x,y) = \dfrac{\sqrt{x^4 + y^4}}{x}$

b) $f(x,y) = 3x^4 + 4x^2y^2 + 5y^4$

c) $f(x,y) = \dfrac{x^2y^2}{x^3 + y^3}$

d) $f(x,y) = \dfrac{2xy}{x^2 + y^2}$

e) $f(x,y) = (x^2 + y^2)^{-1/3}$

f) $f(x,y) = x^2 + 3xy^2 - 15x - 12y$

g) $f(x,y) = 90x^{1/3}y^{1/3}$

h) $f(x,y) = x^a y^b$

i) $f(x,y,z) = \left(\dfrac{x^3y + x^2yz - 4xz^3}{x - 2y}\right)^5$

j) $f(x,y,z) = \dfrac{1}{x^2} ln \dfrac{y}{z}$

k) $f(x,y,z) = \sqrt[3]{x^2 - y^2} + \sqrt{x + z}$

l) $f(x,y,z) = \sqrt[5]{\dfrac{x^6 + y^4x^2 + yz^5}{2z^3}}$

m) $f(x,y,z) = \ln \dfrac{x - 2y}{y + 3z}$

n) $f(x,y,z) = e^{3x+y} + \sqrt[3]{xz}$

o) $f(x,y,z) = e^{\sqrt{\frac{x^2}{yz}}}$

p) $f(x,y,z) = x \ln \dfrac{y}{z} + \sqrt{2yz} + 7$

46.- For the functions of the previous exercise, compute the partial derivatives. Check that Euler's Theorem holds in the appropriate cases.

47.- Given $f(x,y)=x^4y^2e^{y/x}$, check that $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=6f$. What can be deduced from the previous equality?

48.- Let f be an homogeneous function of degree m, such that $f(-1,1)=1$ and $f(-2,2)=1$. Compute m.

49.- Let f be a differentiable, homogeneous function of degree 2 with $\dfrac{\partial f}{\partial x}(3,2)=3$ and $\dfrac{\partial f}{\partial y}(3,2)=4$. Compute $f(3,2)$.

50.- Let $f(x,y,z)$ be a differentiable, homogeneous function of degree 3 such that the components of its gradient vector at $(1,2,3)$ are $(5,2,2)$. What is the value of the function at the point $(1,2,3)$?

51.- Let $z=z(x,y)$ be such that it verifies the equation $F\left(\dfrac{y}{x},\dfrac{z}{x}\right)=0$, F being a function with derivatives of first order, the second of them being not zero.

a) Prove that $z=z(x,y)$ is homogeneous of degree 1.

b) It is homogeneous $\dfrac{\partial z}{\partial x}$? If in the affirmative, of what degree? Reason out the answer.

52.- Let $f(x,y)$ be homogenous of degree 1 with derivativeds of second order. Prove that:

$$\frac{1}{y^2}\frac{\partial^2 f}{\partial x^2}=-\frac{1}{xy}\frac{\partial^2 f}{\partial x\partial y}=\frac{1}{x^2}\frac{\partial^2 f}{\partial y^2}.$$

53.- Let $f(x,y)=\dfrac{x^2y^2}{x^2+y^2}$. Prove that it holds:

$$x^2\frac{\partial^2 f}{\partial x^2}+y^2\frac{\partial^2 f}{\partial y^2}=2\frac{x^2y^2}{x^2+y^2}-2xy\frac{\partial^2 f}{\partial x\partial y}.$$

54.- Given the function $f(x,y,z)=e^{tg\frac{x^2+y^2+z^2}{xy}}$:

a) Check that it is homogeneous of degree 0.

b) Compute $x\dfrac{\partial f}{\partial x}(x,y,z)+y\dfrac{\partial f}{\partial y}(x,y,z)+z\dfrac{\partial f}{\partial z}(x,y,z)$.

55.- Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be differentiable functions and homogeneous of degrees 4 and 1, respectively. Prove that if $h(x,y,z)=f(x,y,z).g(x,y,z)$, it is verified:

$$x\frac{\partial h}{\partial x}(x,y,z) + y\frac{\partial h}{\partial y}(x,y,z) + z\frac{\partial h}{\partial z}(x,y,z) = 5\,h(x,y,z)\,.$$

56.- Let $f(x,y)$ be differentiable such that $\frac{\partial f}{\partial x}(1,1) = 1$, $\frac{\partial f}{\partial y}(1,1) = 1$, $f(1,1) = 2$, $\frac{\partial f}{\partial x}(1,2) = 1$, $\frac{\partial f}{\partial y}(1,2) = 1$, $f(1,2) = 5$. Say whether f is homogeneous and if in the affirmative, of what degree.

57.- Let $f(x,y) = e^{x^2+y^2}$.

a) Compute the contour curves of $f(x, y)$ and represent them graphically.

b) Compute $\nabla f(2,1)$.

c) Compute the directional derivative of f at the point (2,1) following the direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

d) Taking into account $\begin{cases} x = 2 + \ln t^2 \\ y = e^{t^3-1} \end{cases}$, compute $\frac{df}{dt}(1)$.

58.- Let the function $f(x,y) = ln(1 + x^2 + 2y^2)$.

a) Compute the domain of f.

b) Compute the gradient vector of f at (x,y).

c) Is the function f diferentiable?

d) Compute the directional derivative of f respect to the direction of $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ at the point (1,1).

e) Does the function f verify the conditions of Schwarz's Theorem?

f) Using the result of e), compute the Hessian matrix of f at (x, y).

59.- Let the function $f(x,y)=\ln\left(\frac{y}{x^2}+1\right)-1$.

a) Determine the contour curves of the function and represent them graphically.

b) Compute $\nabla f(1,2e)$.

c) Given the vector $v=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, compute the directional derivative $f_v(1,2e)$.

d) Prove that the contour curve of level zero of *f* defines *y* as an implicit function of *x*. Taking the implicit differentiation, compute $y'(1)$.

60.- Consider the function of two variables $f(x,y)=\frac{y}{x+2y}$.

a) Determine its domain and represent it graphically.

b) Determine and represent the contour curves of *f*.

c) Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

d) Determine the pairs $(a,b)\in\mathbb{R}^2$ such that the equation $f(x,y)=\frac{b}{a+2b}$ defines *y* as an implicit function of *x* in a neighbourhood of the point (a,b). For each of the previous pairs, compute $y'(a)$ by implicit differentiation.

61.- Let $F(x,y,z)=x^2zy+e^{xz}-z^2y+4y$:

a) Without computing the partial derivatives, reason out whether the following relation is true:

$$x\frac{\partial F}{\partial x}(x,y,z)+y\frac{\partial F}{\partial y}(x,y,z)+z\frac{\partial F}{\partial z}(x,y,z)=4F(x,y,z).$$

b) Compute $\nabla F(0,2,1)$.

c) Compute the directional derivative of $F(x,y,z)$ with respect to the direction $\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$ at the point (0,2,1).

d) Prove that the equation $F(x,y,z)=7$ defines *x* as an implicit function of *y*, *z* (*x* = *f*(*y*, *z*)) in a neighbourhood of the point (0,2,1).

e) Compute $\nabla f(2,1)$.

f) Compute the directional derivative of *f*(*y*,*z*) with respect to the direction $\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ at the point (2,1).

62.- It is known that the supply of a good as a function of his price p is given by:

$$S(p) = \begin{cases} \dfrac{p^2}{20} & \text{if } 0 \le p < 10 \\ 2p - 15 & \text{if } 10 \le p \le 30 \end{cases}$$

a) Determine the domain of the supply function.

b) Which is the supply if the price is 5 monetary units?

63.- The demand of a good as a function of his price is given by $D(p) = ap^2 - b$. Determine the values of a and b knowing that $D(8)$ = 1 and that $\lim_{p \to 0} D(p) = 5$.

64.- The total cost to produce q units of an article is $C(q) = 4q^3 - 6q^2 + 18q + 12$.

a) Calculate the cost of producing 1 unit.

b) If they are produced 20 units, which is the mean cost?

c) Write the function of mean cost.

d) Write the function of marginal cost.

65.- The profit in terms of the sale price of a product is $B(p)$, defined for $p \in [2,6]$. This function $B(p)$ consist of two straight segments whose slopes are 0.2. Moreover it is known that $B(3)=50$ and that when the price is increased beyond the threshold of 4 monetary units a subsidy is lost so that the profit abruptly decreases by 10 units. Write and sketch graphically this profit function in terms of the price.

66.- The cost function in terms of the number of hours worked, x, is of the form $C(x) = \begin{cases} ax^2 + b & \text{if } x \in [0,100] \\ c\sqrt{x} & \text{if } x \in (100,200] \end{cases}$. Determine a, b, c knowing that $C(x)$ is continuous, that the slope of the straight tangent at x = 50 is 8 and that 121 hours worked imply a cost equal to 990.

67.- The production function of a good is $Q(K,L) = 8\sqrt[3]{K^2 L}$, where Q is the quantity of production, K and L are the quantity of inputs capital and labor, respectively.

Show that the productivity of the work is a function of the ratio capital-labor (only depends of the proportion between the capital and the labor).

68.- Let the production function of a firm be $Q(K,L)=2K^{\frac{1}{2}}L^{\frac{2}{5}}$, where K is the capital, L the labour and Q the obtained production.

a) Assuming that they are used 9 units of capital and 32 of work, what input increase implies greater production increase, keeping constant the other input?

b) Assuming that they are used 9 units of capital and 32 of work, what would be the approximate variation in the production when the capital is increased 2 units and the labour 1 unit?

c) Without computing the derivatives, calculate $K\dfrac{\partial Q}{\partial K}(K,L)+L\dfrac{\partial Q}{\partial L}(K,L)$.

69.- A firm produces two products with quantities q_1 y q_2 respectively, and the income is $I(q_1,q_2)=q_1q_2$. Each one of the products has a production function that depends on the capital K and on the labor L as follows:

$$q_1(K,L)=3K+2L, \qquad q_2(K,L)=6\sqrt{K^2L}$$

a) Calculate $\dfrac{\partial I}{\partial K}(K,L),\dfrac{\partial I}{\partial L}(K,L)$ when they are used 4 units of capital and 9 of labor.

b) Are the production functions homogeneous? of which degree?

c) Without calculating the derivatives, compute $K\dfrac{\partial I}{\partial K}(K,L)+L\dfrac{\partial I}{\partial L}(K,L)$.

INTEGRALS OF FUNCTIONS OF ONE VARIABLE

1.- Compute the following indefinite integrals:

1) $\int \frac{3}{1+2y}\, dy$ 2) $\int \frac{6z}{(z^2-5)^5}\, dz$ 3) $\int \frac{1}{t^2}\sqrt{-1+\frac{1}{t}}\, dt$

4) $\int (5-2x+\sqrt[4]{x^3}+8e^x)\, dx$ 5) $\int \frac{t}{1+t^2}\, dt$ 6) $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\, dy$

7) $\int \left(\frac{1}{x}+\frac{1}{x^3}+8\frac{1}{\sqrt{x^3}}\right) dx$ 8) $\int \left(\frac{1}{3}x+5\right)^9 dx$ 9) $\int \frac{\ln t}{t}\, dt$

10) $\int \frac{5^t-3^t}{7^t}\, dt$ 11) $\int \frac{(\sqrt{x}-2x)^2}{\sqrt{x}}\, dx$ 12) $\int \frac{x-2}{x^2-4x+13}\, dx$

13) $\int \frac{(x+1)(x-1)}{x^2}\, dx$ 14) $\int tg(x)\, dx$ 15) $\int \sqrt[3]{cos^2 x}\; sin\, x\; dx$

16) $\int \frac{x}{1+x^2}\, dx$ 17) $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$ 18) $\int \frac{e^{2x}-1}{e^x-1}\, dx$

19) $\int \frac{2x+1}{x^2+x+1}\, dx$ 20) $\int \frac{4-2x}{x^2-4x+3}\, dx$ 21) $\int \frac{x-3}{x^2-6x+3}\, dx$

22) $\int x\sqrt{1-x^2}\, dx$ 23) $\int \frac{x^2 dx}{(x^3-1)^2}$ 24) $\int \frac{dx}{\sqrt{x}e^{\sqrt{x}}}$

2.- Compute the following definite integrals:

1) $\int_1^2 \left(x^3-\frac{1}{x^2}+2\right) dx$ 2) $\int_0^1 e^{-t+1}\, dt$ 3) $\int_{-1}^1 \frac{x}{x^2+1}\, dx$

4) $\int_1^4 \left(\frac{1}{\sqrt{x^3}}-\sqrt{x}\right) dx$ 5) $\int_{-2}^0 \left(\frac{x+4}{3}\right)^2 dx$ 6) $\int_e^{e^2} \frac{1}{x\,\ln x}\, dx$

3.- Solve the following problems of application of integral calculus to the determination of areas:

1) Calculate the area limited by the graphics of the functions $3y=x^2$ *and* $y=-x^2+4x$.

2) Calculate the area of the figure limited by the parabolas $y=x^2-2x$ *and* $y=-x^2+4x$.

3) Calculate the area of the figure limited by the curves $y = x^2$, $y = 2x$ and $y = \frac{x^2}{2}$.

4) Calculate the area of the figure limited by the curves $x = 0$, $x = 2$, $y = 2^x$ and $y = 2x - x^2$.

5) Calculate the area limited by the curve $y = x^2 - 5x + 6$ and the line $y = 2x$.

6) Calculate the area limited by the parabola $y^2 = 4x$ and the straight line $y = x$.

7) Calculate the area limited by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

8) Calculate the area limited by the parabolas $x = -2y^2$ and $x = 1 - 3y^2$.

4.- Solve the following problems of application of integral calculus:

1) For a certain country, the marginal propensity to the consumption is given by $\frac{dC}{dI} = \frac{3}{4} - \frac{1}{2\sqrt{3I}}$, where the consumption C is a function of the national income I (in million euros). Determine the consumption function for that country if it is known that the consumption is 1 million euros when I=12.

2) The marginal cost function for a certain product is given by: $\frac{dc}{dq} = 10 - \frac{100}{q+10}$ where c is the total cost in euros when they are produced q units. When they are produced 100 units the mean cost is 50 euros each unit. Determine the fixed cost of the manufactured rounding off to the nearest integer of euro.

3) The marginal cost of the manufacture of a certain product is given by the function $CMg(x) = 6 - \frac{2}{\sqrt{x}}$, where x are the units produced. Knowing that the cost of operation is 84000 euros, find the total cost function of the manufacture of the cited product.

4) In the process of recovery of a patient, it is observed that the rhythm by which eliminates a toxic substance is given by the function $f(t) = -1.19\,e^{-0.22t}$, where t is the elapsed time in hours from the ingestion of the cited substance. Find the function that expresses the concentration of the substance in the blood, knowing that an hour after his ingestion the concentration is 1 gram by liter.

5) It is estimated that by t months the population of some city will change at a rate of $4 + 5t^{2/3}$ people by month. If the current population is 10.000 people, which is the population after 8 months have passed?

6) The promoters of a fair estimate that t hours after opening the doors, from 9:00 a.m., the visitors will enter to the fair at a rate of $N'(t)$ people by hour. Find an expression for the number of people that will go into the fair between the 11:00 a.m. and 1:00 p.m.

7) It is estimated that after t years have elapsed, the population of some community living on a lake shore will change at a rate of $0.6t^2 + 0.2t + 0.5$ thousands of people per year. The environmentalists have found that the level of pollution of the lake increases at a rate of, roughly, 5 units by 1.000 people. Find how much it will increase the pollution of the lake in the next 2 years.

8) A manufacturer claims that his marginal cost is $6q + 1$ euros by unit when they are produced q units. The total cost (included the indirect expenses) of producing the first unit is 130 euros. Which is the total cost of producing the first 10 units?