Eigenvalues & Eigenvectors:

From a matrix A, square of size n,
we are goinf to find the eigenvalues and eigenvectors associated with that matrix.

We will use as an example:

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Basic definition:

$$AX = \lambda X \quad \begin{cases} \lambda: Eigenvalue \\ X: Eigenvector \end{cases}$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} X = \lambda X$$

$$AX = \lambda I_n X$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} X = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X$$

$$AX - \lambda I_n X = 0_n$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} X - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I_n)X = 0$$

$$\begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I_n)X = 0$$

$$\begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Homogeneous system:

$$\begin{cases} rk(A - \lambda I_n) = n \leftrightarrow |A - \lambda I_n| \neq 0 \leftrightarrow X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ rk(A - \lambda I_n) < n \leftrightarrow |A - \lambda I_n| = 0 \leftrightarrow ICS \end{cases}$$

1) To obtain the eigenvalues

Characteristic Polynomial:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\,(1-\lambda) - 9 =$$
$$= (-2-\lambda)(4-\lambda) = 0$$

1) To obtain the eigenvalues

Characteristic Polynomial:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(4-\lambda) = 0 \begin{cases} \lambda = -2 \\ \lambda = 4 \end{cases}$$

2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = -2 \rightarrow \begin{pmatrix} 1-(-2) & 3 \\ 3 & 1-(-2) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = -2$:

$$(A - (-2)I_n)X = 0$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$r_2 - r_1$$

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = -2$:

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{3x + 3y = 0\} \begin{cases} x = x \\ y = -x \end{cases}$$

Eigenvector: $\begin{pmatrix} x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2) To obtain the eigenvectors:

For each eigenvalue we solve:

$$(A - \lambda I_n)X = 0$$

$$\lambda = 4 \to \begin{pmatrix} 1-(4) & 3 \\ 3 & 1-(4) \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = 4$:

$$(A - (4)I_n)X = 0$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$r_2 + r_1$$

$$\begin{pmatrix} -3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2) To obtain the eigenvectors:

For $\lambda = 4$:

$$\begin{pmatrix} -3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{-3x + 3y = 0\} \begin{cases} x = x \\ y = x \end{cases}$$

Eigenvector: $\begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Summary:

$$1)\ \ |A-\lambda I_n| = 0\ \ \begin{cases}\lambda = -2\\ \lambda = 4\end{cases}$$

$$2)\ \ (A-\lambda I_n)X = 0\begin{cases}x\begin{pmatrix}1\\-1\end{pmatrix}, for\ \ \lambda = -2\\ x\begin{pmatrix}1\\1\end{pmatrix},\quad for\ \ \lambda = 4\end{cases}$$

Important concepts:

A matrix of size *n* will have *n* eigenvalues.

Multiplicity of λ: Number of times that an Specific value appear as an eigenvalue.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \rightarrow \begin{cases} \lambda = 3 \\ \lambda = 3 \end{cases}$$

The multiplicity of λ=3 is 2.

$$(A - \lambda I_n)X = 0$$

With

$$|A - \lambda I_n| = 0$$

Indetermined Consistent System

Each eigenvector will have some degrees of freedom

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$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & 1-\lambda & 1-\lambda \end{vmatrix}$$

$$r_3 + r_2$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$c_2 + c_3$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda)(1-\lambda) = 0 \begin{cases} \lambda = 1 \\ \lambda = 2 \\ \lambda = 3 \end{cases}$$

Those are the eigenvalues

$$(A - \lambda I_n)X = 0$$

$$\lambda = 1 \to \begin{pmatrix} 2-1 & 0 & 0 \\ 0 & 2-1 & -1 \\ 0 & -1 & 2-1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r_3 + r_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x = 0 \\ y - z = 0 \end{cases} \begin{cases} x = 0 \\ y = z \\ z = z \end{cases}$$

Eigenvector: $\begin{pmatrix} 0 \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$(A - \lambda I_n)X = 0$$

$$\lambda = 2 \rightarrow \begin{pmatrix} 2-2 & 0 & 0 \\ 0 & 2-2 & -1 \\ 0 & -1 & 2-2 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r_1 \leftrightarrow r_3$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{Bmatrix} -y = 0 \\ -z = 0 \end{Bmatrix} \begin{cases} x = x \\ y = 0 \\ z = 0 \end{cases}$$

Eigenvector: $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$(A - \lambda I_n)X = 0$$

$$\lambda = 3 \to \begin{pmatrix} 2-3 & 0 & 0 \\ 0 & 2-3 & -1 \\ 0 & -1 & 2-3 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r_3 - r_2$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{matrix} -x = 0 \\ -y - z = 0 \end{matrix} \right\} \begin{cases} x = 0 \\ y = y \\ z = -y \end{cases}$$

Eigenvector: $\begin{pmatrix} 0 \\ y \\ -y \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Summary:

$$1)\quad |A-\lambda I_n| = 0 \begin{cases} \lambda = 1 \\ \lambda = 2 \\ \lambda = 3 \end{cases}$$

$$2)\quad (A-\lambda I_n)X = 0 \begin{cases} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, for\ \ \lambda = 1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad for\ \ \lambda = 2 \\ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad for\ \ \lambda = 3 \end{cases}$$