Inverse matrix

Given a matrix A of order n, if it exist its inverse, A^{-1}, we have:

$$A \cdot A^{-1} = I_n$$

Inverse matrix

$$A \cdot A^{-1} = I_n$$

$$\boxed{A^{-1} = \frac{1}{|A|}(Adj(A))^t}$$

$Adj(A)$ =Matrix of the Cofactors of A

Inverse matrix

$$\boxed{A^{-1} = \frac{1}{|A|}(Adj(A))^t}$$

If $|A| = 0$ →The inverse of A does not exist

$$A \begin{cases} Regular: If \;\; A^{-1} \; exists \\ Singular: If \; |A| = 0, and \; thus, \;\; A^{-1} \; does \; not \; exist \end{cases}$$

Inverse matrix

$$\boxed{A^{-1} = \frac{1}{|A|}(Adj(A))^t}$$

$$(AB)^{-1} = (B)^{-1}(A)^{-1}$$

$$(\lambda A)^{-1} = \lambda^{-1}(A)^{-1}$$

$$(A^t)^{-1} = (A^{-1})^t$$

Inverse matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightarrow A^{-1}?$$

Inverse matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightarrow |A| = -2$$

$$1)\ Adj(A) = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

Inverse matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightarrow |A| = -2$$

$$1)\ Adj(A) = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$2)\ (Adj(A))^t = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

Inverse matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \to |A| = -2$$

$$1)\ Adj(A) = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$2)\ (Adj(A))^t = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$3)\ A^{-1} = \frac{\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}}{|A|} = \frac{\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}}{-2} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Inverse matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \rightarrow A^{-1}?$$

Inverse matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \rightarrow |A| = 0$$

$$A^{-1}\ does\ not\ exist$$

Rank and systems of equations

Rank of a Matrix

Concept

$$\left.\begin{array}{r} x + y = 2 \\ 2x - 10y = 4 \end{array}\right\} \text{2 different equations}$$

The matrix associated with this system will have rank = 2

Rank of a Matrix

$$\left.\begin{array}{r} x + y = 2 \\ 2x + 2y = 4 \end{array}\right\}$$

2 equations, but they represent the same relationship

The matrix associated with this system will have rank = 1

<u>Rank of a Matrix</u>
Concept

$$The\ rank\ of\ a\ matrix\ tell\ us\ the\ number\ of\ \underline{linearly\ independent\ equations}\ we\ have\ in\ the\ associated\ system$$

Calculation of the rank

1) Elementary row operations
2) Obtain a matrix in row echelon form
3) Rank = Number of rows that are not completely zeroes

Calculation of the rank

1) With elementary row operations we obtain an equivalent matrix, it will have the same rank.

Row operations: Gaussian Elimination

Calculation of the rank

2) Matrix in row echelon form: In each row there is a minimum of one zero more at the beginning of the row than the preceding row

Rank

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow$$

Rank

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

2*(Row 2)- (Row 1)

Rank

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \rightarrow rank = 2$$

2*(Row 2)- (Row 1)

Calculo del rango

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \rightarrow$$

Calculo del rango

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \rightarrow B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

(Row 2) – 2*(Row 1)

Calculo del rango

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \rightarrow B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow rank = 1$$

(Row 2) – 2*(Row 1)

Propiedades del rango

$$rk(A) = rk(A^t)$$

$$\mathrm{A} \in M_n \begin{cases} |A| = 0 \leftrightarrow rk(A) < n \\ |A| \neq 0 \leftrightarrow rk(A) = n \end{cases}$$

Exercise: 7) B

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 4 & 1 & 0 \\ 10 & 1 & 5 \end{pmatrix}$$

$$rk(A) = rk\begin{pmatrix} 2 & 0 & 3 \\ 4 & 1 & 0 \\ 10 & 1 & 5 \end{pmatrix} = rk\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 1 & -10 \end{pmatrix}$$

$$r_2 - 2r_1$$
$$r_3 - 5r_1$$

$$rk(A) = rk\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 1 & -10 \end{pmatrix} = rk\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & -4 \end{pmatrix}$$

$$r_3 - r_2$$

$$rk(A) = rk\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & -4 \end{pmatrix} = 3$$

$$|A| = 2 * 2 * (-4) = -8$$

$$|A| \neq 0 \ \ \rightarrow \ \ rk(A) = n = 3$$

System of linear equations

$$AX = B$$

A = coefficient matrix

X = matrix of unkowns

B = matrix of independent terms

n = number of variables

We will use: $\begin{cases} A & (Coefficient\ matrix) \\ A|B & (Augmented\ matrix) \end{cases}$

System of linear equations

$$AX = B$$

Equivalent matrices will have the same solution

1) We obtain the equivalent matrix of (A|B) in echelon form
2) We find the solution of the system

System of linear equations

$$x - y = 0$$
$$x + y = 2$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

System of linear equations

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 2 \end{array}\right) \rightarrow$$

System of linear equations

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 2 \end{array}\right)$$

System of linear equations

$$(A|B) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 2 \end{array}\right)$$

We transform the matrix into a system again:

$$\left.\begin{array}{r} x - y = 0 \\ 2y = 2 \end{array}\right\} \begin{cases} y = 1 \\ x = 1 \end{cases}$$

System of linear equations

$$Clasification:$$

1)

$$rk(A) = rk\{A|B\} = n$$

$$Determined, consistent\ system\ (DCS)$$

(One unique solution)

System of linear equations

$$Clasification:$$

2)

$$rk(A) = rk\{A|B\} < n$$

$$Indetermined, consistent\ system\ (ICS)$$

(infinitely many solution)

Example

$$\begin{aligned} 2x - y &= 0 \\ -4x + 2y &= 0 \end{aligned}$$

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{cc|c} 2 & -1 & 0 \\ -4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$(A|B) = \begin{pmatrix} 2 & -1 & | & 0 \\ -4 & 2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$$

$$rk(A) = rk(A|B) = 1 < 2 = \mathrm{n}$$

ICS

$$(A|B) = \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$2x - y = 0 \quad \}\{ \quad \boxed{y = 2x}$$

SCI $\rightarrow$ We have infinitely many solutions

$$rg(A) = 1$$

$$\underline{d = \big(n - rg(A)\big) = 2 - 1 = 1 \;\; degrees\; of\;\; freedom}$$

The solution can be written in terms of d variables

$$2x - y = 0 \quad \Big\} \begin{cases} y = 2x \\ x = x \end{cases}$$

System of linear equations

Clasification:

3)

$$rk(A) < rk\{A|B\}$$

$$Inconsistent\ system\ (IS)$$

(There is no solution)

Example

$$3x - y = 0$$
$$6x - 2y = 3$$

$$A = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 6 & -2 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 3 \end{array}\right)$$

$$(A|B) = \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 6 & -2 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 3 \end{array}\right)$$

$$1 = rk(A) < rk(A|B) = 2$$

Inconsistent System (IS)