PUBLICACIONES DE PRIMER CURSO

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MATRICES

1.- Calculate the following determinants:

a) $\begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{vmatrix}$ b) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 5 \\ 6 & 0 & -2 \end{vmatrix}$ c) $\begin{vmatrix} 3 & 1 & 5 & 0 \\ 5 & 4 & 6 & 3 \\ 1 & 3 & 2 & 1 \\ 6 & 7 & 5 & 4 \end{vmatrix}$

d) $\begin{vmatrix} 7 & 6 & 8 & 5 \\ 6 & 7 & 10 & 6 \\ 7 & 8 & 8 & 9 \\ 8 & 7 & 9 & 6 \end{vmatrix}$ e) $\begin{vmatrix} 1 & 3 & 2 & 1 \\ 3 & 5 & 3 & 2 \\ 3 & 6 & 2 & 2 \\ 6 & 4 & 5 & 3 \end{vmatrix}$ f) $\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$

g) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$ h) $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$ i) $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$

j) $\begin{vmatrix} 3 & 4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ k) $\begin{vmatrix} 3 & 4 & 1 & 2 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ l) $\begin{vmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 & 1 \end{vmatrix}$

m) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 3 & 4 & 5 \\ -1 & -2 & 0 & 4 & 5 \\ -1 & -2 & -3 & 0 & 5 \\ -1 & -2 & -3 & -4 & 0 \end{vmatrix}$ n) $\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{vmatrix}$ ñ) $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix}$

o) $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$ p) $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$ q) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$

r) $\begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & 2 \\ 1 & 2 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix}$ s) $\begin{vmatrix} 1 & 1 & 1 & 1 & . & . & 1 \\ -1 & a & 1 & 1 & . & . & 1 \\ -1 & -1 & a & 1 & . & . & 1 \\ . & . & . & . & . & . & . \\ -1 & -1 & -1 & -1 & . & . & a \end{vmatrix}$

t) $\begin{vmatrix} 1 & 2 & 3 & . & . & n-1 \\ 2 & 3 & 4 & . & . & n \\ 3 & 4 & 5 & . & . & n+1 \\ . & . & . & . & . & . \\ n-2 & n-1 & n & . & . & 2n-2 \\ n-1 & n & n+1 & . & . & 2n-1 \end{vmatrix}$

2.- Solve the equation $\begin{vmatrix} 4 & x & 6 \\ 5 & 7 & 12 \\ 3 & -1 & x \end{vmatrix} = 0$.

3.- Show that:

a) $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = -2(x^3+y^3)$ b) $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix} = x^2z^2$

4.- Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 6 & -3 & 2 \\ 4 & 1 & 3 \end{pmatrix}$. Calculate the determinant $\left|A - \lambda I_3\right|$, with $\lambda \in \mathbb{R}$.

5.- Knowing that $\begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1$, obtain the value of $\begin{vmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+4 & y+4 & z+4 \end{vmatrix}$.

6.- Let $A, B \in M_4$ with $|A|$=3, $|B|$=-2. Calculate:

a) $|2A|$.

b) $|\frac{1}{2}B|$.

c) $|BA^t|$.

d) $|(BA)^t|$.

e) $|(B^tA^tB)^t|$.

7.- Calculate the rank of the following matrices:

$A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 0 & 3 \\ 4 & 1 & 0 \\ 10 & 1 & 5 \end{pmatrix}$ $C = \begin{pmatrix} 2 & -1 & 0 & 1 & 3 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & -1 & 3 & 6 \\ 5 & -2 & -1 & 4 & 9 \end{pmatrix}$

$D = \begin{pmatrix} -2 & -4 & 3 & 4 & 1 & 8 \\ 1 & 2 & 0 & -2 & 1 & -1 \\ 3 & 6 & 3 & -6 & 6 & 3 \\ 2 & 4 & 2 & -4 & 4 & 2 \end{pmatrix}$ $E = \begin{pmatrix} 1 & -2 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 1 \\ 1 & 2 & 0 & -2 & 1 \\ 2 & 1 & 3 & -3 & 4 \\ 3 & -2 & 2 & -2 & 5 \end{pmatrix}$ $F = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 3 & 3 & 5 \\ 0 & 3 & 4 \end{pmatrix}$

$$G=\begin{pmatrix} 1 & 1 & -2 & 1 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 2 & 1 \\ 3 & 2 & 1 & 2 \end{pmatrix} \qquad H=\begin{pmatrix} 0 & 2 & 1 & -1 & 2 \\ 1 & 0 & -1 & 3 & 2 \\ 0 & 4 & 2 & -2 & 4 \end{pmatrix} \qquad I=\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

8.- Calculate the rank of the following matrices depending on the values of the real parameter a:

$$A=\begin{pmatrix} 1 & 1 & 0 & 1 \\ 3 & 2 & -1 & 3 \\ a & 3 & -2 & 0 \\ -1 & 0 & -4 & 3 \end{pmatrix} \qquad B=\begin{pmatrix} 1 & 1 & -1 & 1 \\ a & 1 & 1 & 1 \\ 1 & -1 & 3 & 0 \\ 4 & 2 & 0 & a \end{pmatrix} \qquad C=\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 5 & 1 & a \end{pmatrix}$$

9.- Study if the following matrices are invertible. If affirmative, calculate the inverse matrix.

$$A=\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \\ 4 & 5 & 7 \end{pmatrix} \qquad B=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad C=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad D=\begin{pmatrix} 0 & 3 & 1 \\ -3 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$$

$$E=\begin{pmatrix} 1 & -1 & 3 \\ 0 & 4 & -2 \\ -2 & 6 & -8 \end{pmatrix} \quad F=\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad G=\begin{pmatrix} 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

10.- Study the existence of the inverse matrix depending on the values of $m \in \mathbb{R}$. Calculate the inverse matrix in the invertible cases.

$$A=\begin{pmatrix} m & -1 & 0 \\ 1 & m-1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad B=\begin{pmatrix} m & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{pmatrix} \qquad C=\begin{pmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{pmatrix}$$

11 .- Study for what values of the parameter $m \in \mathbb{R}$, the following matrices do not have inverse.

$$A=\begin{pmatrix} 1-m & 2 \\ 3 & 2-m \end{pmatrix} \qquad B=\begin{pmatrix} -m & 5 \\ 2 & 3-m \end{pmatrix} \qquad C=\begin{pmatrix} 2-m & 3 & 1 \\ 1 & 1-m & 4 \\ 0 & 1 & 1-m \end{pmatrix}$$

$$D=\begin{pmatrix} m & 9 & 4 \\ 4 & m & -1 \\ 7 & 7 & 7 \end{pmatrix} \qquad E=\begin{pmatrix} 1 & 1 & 1 \\ m & 1 & 0 \\ -1 & 3 & m-1 \end{pmatrix}$$

12.- Given the matrices $P=\begin{pmatrix}1&1&1\\1&0&1\\0&1&1\end{pmatrix}$ and $B=\begin{pmatrix}1&2&-1\\0&-3&1\\2&0&1\end{pmatrix}$, calculate the matrix A that verifies $P^{-1}AP=B$.

13.- Solve, using Cramer's Rule:

a) $\left.\begin{array}{r}x+y-z=3\\5x-y+2z=5\\-3x+3y-4z=1\end{array}\right|$ b) $\left.\begin{array}{r}x+y-z=1\\x+2y+z=2\\x+3y-z=0\end{array}\right|$ c) $\left.\begin{array}{r}x+y-2z+t+3s=1\\2x-y+2z+2t+6s=2\\3x+2y-4z-3t-9s=3\end{array}\right|$

14 .- Discuss the following systems of linear equations as the real parameters (*m, a and b)* varies, and solve them using Cramer's Rule:

a) $\left.\begin{array}{r}mx+y-z=1\\x+2y+z=2\\x+3y-z=0\end{array}\right|$ b) $\left.\begin{array}{r}mx+y+z=m\\x+my+z=1\\x+y+mz=1\end{array}\right|$ c) $\left.\begin{array}{r}x+y+z=m\\x+(1+m)y+z=2m\\x+y+z=4\end{array}\right|$

d) $\left.\begin{array}{r}x+my-z=0\\2x-3y-2z=0\\x+2y+z=0\end{array}\right|$ e) $\left.\begin{array}{r}mx+y+3z=3\\x-y-z=0\\5x-3y-2z=6\end{array}\right|$ f) $\left.\begin{array}{r}x-2y+z=-1\\x+y+3z=4\\5x-y+mz=10\end{array}\right|$

g) $\left.\begin{array}{r}-x+2y-2z=0\\2x-y+az=b\\2x-2y+3z=1+b\end{array}\right|$ h) $\left.\begin{array}{r}2x+ay+z=7\\x+ay+z+t=b\\x+2ay+t=-1\\bx+ay=b\end{array}\right|$

15.- Given the matrix $A=\begin{pmatrix}1&1&m\\3&2&4m\\2&1&3\end{pmatrix}$ with $m\in\mathbb{R}$:

a) For what values of the parameter *m*, is the matrix *A* invertible?

b) Solve the linear system $AX=0_3$ using the results of point a).

16.- Consider the matrices $A=\begin{pmatrix}1&a&0&0\\a&1&0&1-a\\0&0&1&0\end{pmatrix}$, $B=\begin{pmatrix}a\\a^2\\a^3\end{pmatrix}$ and $X=\begin{pmatrix}x\\y\\z\\t\end{pmatrix}$, with *a* being a real parameter.

a) Discuss the system $AX=B$ as *a* varies.

b) In the compatible cases, calculate the solutions of $AX=B$.

17 .- In a market with perfect competition the functions of supply and demand of the goods are given by:

$$Q_{1d} = 10 - 2P_1 + 4P_2 \qquad\qquad Q_{2d} = 10 + 5P_1 - 3P_2$$

$$Q_{1s} = 20 + 3P_1 - 2P_2 \qquad\qquad Q_{2s} = 30 - 7P_1 + 5P_2$$

Where Q_{id} is the demanded quantity of good *i*, Q_{is} is the supplied quantity of good *i* and P_i is the price of the good *i*, for *i*=1,2. Calculate the prices for which the market is in equilibrium and the demanded and supplied quantities of each good in this situation.

18 .- The condition of equilibrium for the price of three goods in a given market is determined by the following equations:

$$\begin{aligned} 11P_1 - P_2 - P_3 &= 31 \\ -P_1 + 6P_2 - 2P_3 &= 26 \\ -P_1 - 2P_2 + 7P_3 &= 24 \end{aligned}$$

where P_1, P_2 and P_3 are the prices of these three goods. Calculate the price of equilibrium for each good.

19.- Consider the vectors *u* = (1, -3, 2) and *v* = (2, -1, 1) of $\mathbb{R}^3$:

a) Write, if it is possible, the vectors (1, 7, -4) and (2, -5, 4) as a linear combination of *u* and *v*.

b) For which values of *x* is the vector (1, *x*, 5) a linear combination of *u* and *v*?

20.- The vectors $v_1 = (1, 1, -1)$, $v_2 = (2,1,3)$ and $v_3 = (5, 2, 10)$ of $\mathbb{R}^3$, are linearly independent? If not, find the relation of dependency.

21.- Given the vectors $u_1 = (2,-1,0)$, $u_2 = (0,1,-1)$ and $u_3 = (8,3,1)$ of $\mathbb{R}^3$, study if they are linearly dependent or independent.

22.- Given the vectors $u_1 = (1,0,-1,0)$, $u_2 = (2,0,3,-1)$, $u_3 = (1,1,-1,1)$ and $u_4 = (2,1,-2,1)$ of $\mathbb{R}^4$, study if they are linearly independent. If not, find the relation of dependency.

23.- Given the vectors of $\mathbb{R}^4$ $v_1 = (1,1,0,m)$, $v_2 = (3,-1,n,-1)$ and $v_3 = (-3,5,m,-4)$, determine the values of the parameters *m* and *n* so that the three vectors be linearly dependent.

24.- Sean $u = (-1, 0, 0)$, $v = (1, 1, 0)$ and $w = (-1, 1, -1)$ vectors of $\mathbb{R}^3$:

a) Show that $\{u, v, w\}$ is a basis of $\mathbb{R}^3$.

b) Find the coordinates with respect to this basis of the vector whose coordinates with respect to the canonical basis are 1, 0, 2.

c) Find the coordinates concerning the canonical basis of the vector $a = 3u - v + 5w$

25.- Let $A \in M_n$, $\lambda \in \mathbb{R}$ an eigenvalue of A and $\mathbf{x} \in \mathbb{R}^n$ an eigenvector of A associated to λ

Prove that:

a) $\alpha\lambda$ is an eigenvalue of the matrix αA for any $\alpha \in \mathbb{R}$ and $\mathbf{x}$ is an eigenvector of αA associated to $\alpha\lambda$.

b) λ^p is an eigenvalue of A^p and $\mathbf{x}$ is an eigenvector of A^p associated to λ^p, with $p \in N$.

c) $|A| = 0 \Leftrightarrow \lambda = 0$ is an eigenvalue of A.

d) If A invertible then $\lambda \neq 0$. Moreover, λ^{-1} is an eigenvalue of A^{-1} and $\mathbf{x}$ is an eigenvector of A^{-1} associated to λ^{-1}.

26.- Let $A, B \in M_n$ be similar matrices. Prove that:

a) $|A| = |B|$.

b) A^p is similar to B^p for any $p \in N$.

c) If A is invertible then B is invertible and A^{-1} is similar to B^{-1}.

27.- Let $A \in M_n$. Prove that A and A^t have the same characteristic polynomial.

28.- Given the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$ answer the following questions:

a) Study if 3 is eigenvalue of A or not.

b) Are the vectors (1,1,1) and (0,0,1) eigenvectors of A? If in the affirmative, find the associated eigenvalues.

29.- Given the matrix $A=\begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$ answer the following questions:

a) Study if the vector $(-1,\frac{1}{3},\frac{1}{2})$ is an eigenvector or not of the matrix *A*. If in the affirmative, determine the associated eigenvalue.

b) The same question for the vector (-1,0,1).

30.- For each one of the following matrices reason out if it is diagonalizable or not. Besides:

a) If in the affirmative, give a diagonal matrix *D* and an invertible matrix *P* such that $D=P^{-1}AP$.

b) If in the negative, calculate the eigenvalues and the eigenvectors associated to each eigenvalue.

$$A_1=\begin{pmatrix} 0 & 3 & 0 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{pmatrix} \qquad A_2=\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \qquad A_3=\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

$$A_4=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad A_5=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \qquad A_6=\begin{pmatrix} 4 & -4 & 6 \\ 3 & -4 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$

$$A_7=\begin{pmatrix} 2 & 0 & -2 \\ -3 & -1 & 2 \\ 2 & 0 & -2 \end{pmatrix} \qquad A_8=\begin{pmatrix} 2 & 4 & 3 \\ -2 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} \qquad A_9=\begin{pmatrix} -1 & 2 & -2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

31.- A matrix $A\in M_2$ verifies the following conditions: $A\begin{pmatrix} 1 \\ -1 \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and (2,-1) is an eigenvector of *A* associated to the eigenvalue $\lambda=-2$. Find the matrix *A* indicating if it is diagonalizable or not. If affirmative, give a diagonal matrix *D* and an invertible matrix *P* such that $D=P^{-1}AP$.

32.- Let $A=\begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix}$ with $a,b\in\mathbb{R}$. Calculate the values of the parameters *a* and *b* so that the vector (2,-1) is eigenvector of *A* associated to the eigenvalue 2.

33.- Find a matrix $A\in M_3$ of eigenvalues -1, 1, 2 with associated eigenvectors (1,0,-1), (1,1,0), (3,-3,1), respectively.

34.- Find a matrix $A \in M_3$ with eigenvalue $\lambda_1 = 1$ (simple) and eigenvalue $\lambda_2 = 2$ (double) with associated eigenvectors (1,1,0), (1,0,0), respectively.

35.- Find a matrix $A \in M_3$ with eigenvalue $\lambda_1 = -1$ (double) and eigenvalue $\lambda_2 = 3$ (simple) with associated eigenvectors (1,0,2), (-1,0,0) and (0,1,1), respectively.

36.- Let $A = \begin{pmatrix} 3 & 2 \\ a & b \end{pmatrix}$ with $a, b \in \mathbb{R}$. Calculate the values of the parameters *a* and *b* so that *A* has eigenvalues 1 and -1. Is *A* a diagonalizable matrix?

37.- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & a \\ 3 & 0 & 1 \end{pmatrix}$ with $a \in \mathbb{R}$.

a) For which values of the parameter *a is* $\lambda = -2$ an eigenvalue of *A*?

b) For which values of the parameter *a* is the matrix *A* diagonalizable?

38.- Determine a matrix $A \in M_3$ such that $A\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ and that his eigenvectors be the non-null vectors of $\mathbb{R}^3$ belonging to the sets $\{(x, y, z) \in \mathbb{R}^3 \mid x = z\}$, $\{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0, z = 0\}$.

39.- The matrix $A = \begin{pmatrix} a & 1 & p \\ b & 2 & q \\ c & -1 & r \end{pmatrix}$ admits as eigenvectors (-1,-1,0), (1,0,-2) and $(0,-1,1)$ associated to the eigenvalues 3, 0 and 3/2 respectively. Answer the following items:

a) Find the unknown elements of *A*.

b) Is *A* diagonalizable? If affirmative, perform the diagonalization.

40.- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}$

a) Find its eigenvalues and eigenvectors. It is *A* diagonalizable?

b) Check that that the determinant of the matrix *A* is the product of its eigenvalues.

41.- Check that the matrices $A=\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{pmatrix}$ have the same eigenvalues but, however, are not similar.

42.- Calculate A^{100} and, in general, A^k, with $k \in N$, for the matrix $A=\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.

43.- Calculate A^k with $k \in N$ odd, for the matrix $A=\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

44.- Calculate A^n $\forall n \in N$ in each one of the following cases:

a) $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ b) $A=\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$

45.- Calculate a symmetric matrix $A \in M_3$ that verify: v=(1,-1,0) is eigenvector of A, $A\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $|A|$ =0. Calculate A^{50}.

46.- Determine for which values of the parameters $b, c \in \mathbb{R}$ the following matrices are diagonalizable. If affirmative, find a diagonal matrix similar to the given one.

$A_1=\begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & b \\ 3 & 0 & c \end{pmatrix}$ $A_2=\begin{pmatrix} b & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $A_3=\begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

47.- For each one of the following matrices A_i, i=1,2,3, find, if it is possible, an invertible matrix P and a diagonal matrix D so that $D=P^{-1}A_iP$.

$A_1=\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$ $A_2=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $A_3=\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

48.- The characteristic polynomial of a matrix A of order 3 is $P(\lambda)=-\lambda^3+21\lambda-20$. With this information, can you justify whether A is invertible and/or diagonalizable?

49.- The eigenvalues of a certain diagonalizable matrix A $\in$M(n $\times$ n) are the roots of the characteristic polynomial $P(\lambda) = \lambda^5 + \lambda^4 - 5\lambda^3 - 5\lambda^2 + 4\lambda + 4$.

a) Determine the eigenvalues and their multiplicity.

b) Determine the dimension of the matrix A and Rk(A) .

c) Calculate, if possible, $\left|A^{-1}\right|$ and $\left|-\frac{1}{2}A^{-1}\right|$.

d) Calculate the eigenvalues of A^2 and their multiplicity.

50.- Consider the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

a) Calculate the eigenvalues and eigenvectors of A.

b) Is it diagonalizable? If it is, calculate a similar diagonal matrix D and a matrix P such that $D = P^{-1}AP$.

c) Is there any value of the parameter *a* so that (3, -6, a) is an eigenvector of A?

51.- Let A be a matrix of order 3 whose characteristic polynomial is

$$P(\lambda) = -\lambda^3 - 2\lambda^2 + 5\lambda + 6 :$$

a) Calculate the eigenvalues of A and the determinant of A.

b) Explain if the following statements are true or false:

i) The homogeneous linear system, AX = 0, is consistent and determined.

ii) There is a diagonal D matrix and a regular P matrix such that $D = P^{-1}AP$.

52.- Given a symmetric matrix A of order 4 whose eigenvalues are 1, 3, -5 and -5:

i) Reason if A^{-1} exists and, if so, calculate its determinant.

ii) Obtain the rank of the matrix $A + 5I_4$?

53.- Let $A=\begin{pmatrix} m & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ with m being a real number:

a) Study the existence of the inverse matrix of A according to the values of the parameter m. Calculate, when possible, A^{-1}.

b) Determine the values of the parameter m so that A is diagonalizable.

c) Calculate the characteristic polynomial and the eigenvalues of A.

d) For m = 0, calculate a diagonal matrix D and a matrix P such that $D=P^{-1}AP$.

54.- Given the matrix $A=\begin{pmatrix} -1 & 0 & -3 \\ 3 & 3 & 3 \\ -3 & 0 & -1 \end{pmatrix}$,

a) Calculate the eigenvalues of A.

b) Calculate the eigenvectors of A.

c) Is it diagonalizable? If it is, calculate a regular matrix P and a diagonal matrix D such that $D=P^{-1}AP$.

d) Calculate the eigenvalues of A^3.

QUADRATIC FORMS

1.- Consider the quadratic form $Q(\mathbf{x})$ of associated matrix (respectively)

$$A_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

Answer to the following points:

a) Express $Q(\mathbf{x})$ in polynomial form.

b) Find, by the method of eigenvalues, a diagonal expression for Q ($\mathbf{x}$).

c) Classify $Q(\mathbf{x})$.

2.- Consider the quadratic form Q($\mathbf{x}$) of associated matrix (respectively)

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Answer to the following points:

a) Express $Q(\mathbf{x})$ in polynomial form.

b) Find, by completing squares, a diagonal expression for Q ($\mathbf{x}$).

c) Classify $Q(\mathbf{x})$.

3.- Consider the quadratic form $Q(x, y, z) = 2x^2 + 5y^2 + 5z^2 + 4xy - 4xz - 8yz$ and answer:

a) Find a diagonal expression for Q.

b) Classify Q.

4.- For each one of the following matrices, answer:

a) Find a congruent diagonal matrix to the matrix given.

b) Classify the quadratic form that it represents.

$$A_1 = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad A_3 = \begin{pmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 5 & -2 & 0 & -1 \\ -2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -1 & 1 & -2 & 2 \end{pmatrix} \quad A_5 = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad A_7 = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 3 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

5.- Classify by their sign the following quadratic forms:

a) $Q(x, y) = 3x^2 - 4xy + 7y^2$.

b) $Q(x, y) = x^2 + y^2 + 2xy$.

c) $Q(x, y) = 6xy - 2x^2 - 5y^2$.

d) $Q(x, y) = 4y^2 + 8xy$.

e) $Q(x, y, z) = y^2 + 2xy + 2yz$.

f) $Q(x, y, z) = x^2 + 4y^2 + z^2 + 2xy + 2xz + 4yz$

g) $Q(x, y, z) = x^2 + 2y^2 + z^2 + xz + 2xy + 2yz$.

h) $Q(x, y, z) = 4x^2 + 4y^2 + z^2 - 4xy$.

i) $Q(x, y, z) = 2xy + 2xz + 2yz$.

j) $Q(x, y, z) = x^2 + y^2 + z^2$.

k) $Q(x, y, z) = -4x^2 + y^2 + 3z^2 + 3xz + yz$.

l) $Q(x, y, z) = 2x^2 + 2xz + 3y^2 + 2z^2$.

m) $Q(x, y, z) = 2x^2 - y^2 + 3z^2 - 3xy + 4yz - 2xz$.

n) $Q(x, y, z) = x^2 + 10y^2 + 6xy$.

o) $Q(x, y, z) = x^2 + 4y^2 + 3z^2 + 4xy + 2xz + 4yz$.

p) $Q(x, y, z, t) = 2xz - 3yt + 2xt - t^2$.

q) $Q(x, y, z) = x^2 - y^2 - 2z^2 + 2xy + 4yz$.

r) $Q(x, y, z) = 2xy + 4yz - 4xz - x^2 - y^2 + 4z^2$.

6.- Show that for all $\forall x, y, z \in \mathbb{R}$ it holds $x^2 + y^2 + z^2 \geq xy + xz + yz$.

7.- Determine for what value of $\alpha \in \mathbb{R}$ the following quadratic forms are semidefinite indicating if it is positive or negative.

a) $Q(x, y, z) = x^2 + 2y^2 + \alpha z^2 - 2xz$.

b) $Q(x, y, z) = x^2 + \alpha y^2 + \alpha z^2 + 2yz$.

8.- Classify, depending on the values of $\beta \in \mathbb{R}$, the quadratic form of expression $Q(x, y, z, t) = x^2 + y^2 + z^2 + t^2 + 2\beta yt + 2\beta xz$.

9.- Study, depending on the values of the real parameter a, the sign of the quadratic form of associated matrix $A = \begin{pmatrix} a & 2 & 0 \\ 2 & a & 3 \\ 0 & 3 & 3 \end{pmatrix}$.

10.- Consider the quadratic form of associated matrix $A = \begin{pmatrix} -1 & a & 2 & 1 \\ a & 0 & 1 & 0 \\ 2 & 1 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$. Is it negative definite for any value of the real parameter a?.

11.- Classify the following restricted quadratic forms:

a) $Q(x, y) = 2x^2 + y^2 + 2\sqrt{2}xy$ to $S = \{(x, y) \in \mathbb{R}^2 \mid x - \sqrt{2}y = 0\}$.

b) $Q(x, y, z) = x^2 + 4y^2 + 5z^2 + 2xy - 2xz + 4yz$ to the vectors (x, y, z) such that $x + 2y - z = 0$ and $2x - 3y + z = 0$.

c) $Q(x, y, z) = 2x^2 + y^2 - 4xy + 2yz$ to the vectors (x, y, z) such that $x - y + z = 0$.

d) $Q(x, y, z, t) = x^2 - z^2 + 2xz + xt + 2yz$ to the vectors (x, y, z, t) such that $x + y - z = 0,\ y - t = 0$.

12.- Classify the quadratic form $Q(x, y, z) = x^2 + y^2 - 2z^2$ restricted to:

a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + z = 0\}$.

b) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = -z\}$.

13.- Classify $Q(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$ restricted to:

a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2z = 0\}$.

b) $S = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

14.- Classify the quadratic form $Q(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ restricted to the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid x - y - 2z = 0\}$.

15.- Consider the quadratic form $Q(x,\ y,\ z) = 4x^2 + 5y^2 + z^2 - 4xz$.

a) Find a diagonal expression for Q.

b) Classify the sign of Q.

c) Classify Q restricted to $S = \{(x,\ y,\ z) \in \mathbb{R}^3 \mid x - z = 0\}$.

16.- Consider the quadratic form of associated matrix $A = \begin{pmatrix} 1 & 0 & b \\ 0 & 2 & 0 \\ b & 0 & 1 \end{pmatrix}$, where b is a real parameter.

a) Determine its sign as a function of the values of the parameter b.

b) Determine its sign restricted to the vectors (x, y, z) satisfying $y = 2z$, for any value of b.

17.- Let $Q(x,\ y,\ z) = y^2 - xy - xz - yz$.

a) Find a diagonal expression for Q.

b) Classify Q.

c) Classify, depending on the values of the real parameter α, Q restricted to the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = \alpha z\}$.

18.- Let $Q(x,\ y,\ z) = 2ax^2 + y^2 + z^2 + 4axz$, with $a \in \mathbb{R}$.

a) Classify Q depending on the values of the real parameter a.

b) If a = -1, find subsets S_1 and S_2 of $\mathbb{R}^3$ such that Q restricted to S_1 be positive definite and Q restricted to S_2 be negative definite.

19.- Considering the quadratic form Q (x, y, z) represented by the matrix $A = \begin{pmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ answer reasonably the following questions:

a) Does the quadratic form Q admit the expression $Q(\tilde{x}, \tilde{y}, \tilde{z}) = -2\tilde{x}^2 - 2\tilde{y}^2 - 5\tilde{z}^2$?

b) Justify if there is any $(x_0, y_0, z_0) \in \mathbb{R}^3$ for which it is verified that $Q(x_0, y_0, z_0) > 0$.

c) Define, if possible, a subset $\mathbb{R}^3$ where the quadratic form Q is definite negative.

20.- Given the quadratic form $Q(x,y,z) = 2x^2 - y^2 - 4z^2 + 4zy$. It is requested:

a) Calculate a diagonal expression for Q.

b) Justify if there it exist some $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that $Q(x_0, y_0, z_0) < 0$.

c) Study the sign of Q constrained to $S = \left\{(x,y,z) \in \mathbb{R}^3 \mid y - 2z = 0\right\}$.

21.- Given the quadratic form $Q(x,y,z) = x^2 + 3y^2 + z^2 + 2xz$:

a) Classify Q (x, y, z).

b) Can $Q(\bar{x}, \bar{y}, \bar{z}) = \bar{x}^2 + 3\bar{y}^2 + 2\bar{z}^2$ be a diagonal expression of Q (x, y, z)?

22.- Given the high value of the public deficit of an imaginary country, the government decides to create a new tax *T* whose amount is a function of the payments (or reimbursements if applicable) of the income tax, *R*, and of the payments of the heritage tax, *P*, in such way that $T = 2R^2 + 4P^2 - 4RP$. The government, before setting up the new tax, wants to make sure that no taxpayer will be reimbursed by this new tax. Check that the new tax will fulfil its purpose with all and each one of the taxpayers.

23.- The economists of a company claim that the production function is of the type: $P = L^2 + K^2 - 2LK$, being *L* and *K* the number of workers and of machines, respectively. Besides that, it is known that each machine needs two workers for it to work. Check that indeed, *P* is a production function with the condition given.

24.- An investor estimates that by investing in three securities, A, B and C, the resulting portfolio will have a profitability of $U(x,y,z) = 2x^2 - 2y^2 - 7z^2 + 2xy + 6xz$, being x, y and z, the returns at the time of purchase of each value A, B and C respectively.

a) Analyze if said portfolio can generate losses.

b) Analyze how the previous response affects an economic situation in which the three products have the same profitability.

c) If the investor knows that the profitability of the product A is double of that of B, Will there be gains?