Exercise 1

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Is A Diagonalizable?

Eigenvalues:

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda)(2-\lambda) = 0 \begin{cases} \lambda = 1 \ (2\ times) \\ \\ \lambda = 2 \end{cases}$$

$$\lambda = 2$$

$$(A - (2)I_n)X = 0$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 2$

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\lambda = 2$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} -x = 0 \\ -y = 0 \end{matrix}\right\} \begin{cases} x = 0 \\ y = 0 \\ z = z \end{cases} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$D_2 = n - rk(A) = 3 - 2 = 1$$

$$\lambda = 1$$

$$(A - (1)I_n)X = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 1$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} 2x = 0 \\ 3x + z = 0 \end{array}\right\}\left\{\begin{array}{l} x = 0 \\ y = y \\ z = 0 \end{array}\right. \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$D_1 = n - rk(A) = 3 - 2 = 1$$

$The\ matrix\ is\ not\ diagonalizable$

Multiplicity of the first eigenvalue = 1
Degrees of freedom of first eigenvector = 1

But:
Multiplicity of the second eigenvalue = 2
Degrees of freedom of the second eigenvectgor = 1

30) A3

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & -3 & 0 \\ -2-\lambda & -5-\lambda & -2-\lambda \\ 0 & -6 & -2-\lambda \end{vmatrix}$$

$$C_1 + C_2$$

$$C_3 + C_2$$

$$\begin{vmatrix} -2-\lambda & -3 & 0 \\ -2-\lambda & -5-\lambda & -2-\lambda \\ 0 & -6 & -2-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & -3 & 0 \\ 0 & -2-\lambda & -2-\lambda \\ 0 & -6 & -2-\lambda \end{vmatrix}$$

$$r_2 - r_1$$

$$\begin{vmatrix} -2-\lambda & -3 & 0 \\ 0 & -2-\lambda & -2-\lambda \\ 0 & -6 & -2-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & -3 & 0 \\ 0 & -2-\lambda & -2-\lambda \\ 0 & -4+\lambda & 0 \end{vmatrix}$$

$$r_3 - r_2$$

$$= (-4+\lambda)(-2-\lambda)(-2-\lambda) = 0$$

$$|A - \lambda I_n| = 0$$

$$(-4 + \lambda)(-2 - \lambda)(-2 - \lambda) = 0 \begin{cases} \lambda = -2 \ (2\ times) \\ \\ \lambda = 4 \end{cases}$$

Sarrus is not recommended, but this is the solution with that method:

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-5-\lambda)(4-\lambda) + (3)(3)(-6) + (-3)(3)(6) -$$
$$-(3)(6)(-5-\lambda) - (3)(-6)(1-\lambda) - (-3)(3)(4-\lambda) =$$

$$= [\lambda^2 + 4\lambda - 5](4-\lambda) - 108 -$$
$$-18(-5-\lambda) + 18(1-\lambda) + 9(4-\lambda) =$$

$$= -\lambda^3 - 4\lambda^2 + 5\lambda + 4\lambda^2 + 16\lambda - 20 - 108 +$$
$$+90 + 18\lambda + 18 - 18\lambda + 36 - 9\lambda =$$

$$= -\lambda^3 + 12\lambda + 16 \rightarrow Ruffini \rightarrow \begin{cases} \lambda = -2 \; (2\ times) \\ \\ \lambda = 4 \end{cases}$$

$$\lambda = 4$$

$$(A - (4)I_n)X = 0$$

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 4$

$$\begin{pmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} -3x - 3y + 3z = 0 \\ -12y + 6z \end{matrix}\right\} \begin{cases} x = y \\ y = y \\ z = 2y \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 2y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$D_4 = n - rk(A - \lambda I) = 3 - 2 = 1$$

$$\lambda = -2$$

$$(A - (-2)I_n)X = 0$$

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = -2$

$$\begin{pmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x - 3y + 3z = 0\} \begin{cases} x = y - z \\ y = y \\ z = z \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$: D_{-2} = n - rk(A - \lambda I) = 3 - 1 = 2$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

46) A1. Determine for which values of b and c, the matrix is diagonalizable

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & b \\ 3 & 0 & c \end{pmatrix}$$

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & b \\ 3 & 0 & c-\lambda \end{vmatrix} = 0$$

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & b \\ 3 & 0 & c-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-1-\lambda)(c-\lambda) = 0 \begin{cases} \lambda = 5 \\ \lambda = -1 \\ \lambda = c \end{cases}$$

$$|A - \lambda I_n| = 0$$

$$(5-\lambda)(-1-\lambda)(c-\lambda) = 0 \begin{cases} \lambda = 5 \\ \lambda = -1 \\ \lambda = c \end{cases}$$

If the eigenvalues are different, A will be diagonalizable:

$$c \neq 5 \neq -1$$

If an eigenvalue appears two times, we have to study the matrix more carefully