Exercise 1: Calculate the rank of A, depending on the values of the real parameter a

$$A = \begin{pmatrix} 3 & 1 \\ 6 & a \end{pmatrix}$$

$$rk(A) = rk\begin{pmatrix} 3 & 1 \\ 6 & a \end{pmatrix} = rk\begin{pmatrix} 3 & 1 \\ 0 & a-2 \end{pmatrix}$$

$$r_2 - 2r_1$$

$$rk(A) = rk\begin{pmatrix} 3 & 1 \\ 0 & a-2 \end{pmatrix} \begin{cases} a = 2 \rightarrow rk(A) = 1 \\ a \neq 2 \rightarrow rk(A) = 2 \end{cases}$$

$$rg(A) = rk\begin{pmatrix} 3 & 1 \\ 0 & a-2 \end{pmatrix}\begin{cases} a = 2 \rightarrow rk(A) = 1 \\ a \neq 2 \rightarrow rk(A) = 2 \end{cases}$$

Another option:

$$|A| = 3a - 6\begin{cases} a = 2 \rightarrow |A| = 0 \rightarrow rk(A) = 1 \\ a \neq 2 \rightarrow |A| \neq 0 \rightarrow rk(A) = 2 \end{cases}$$

Exercise 2: Obtain the solution of the following system

$$
\begin{aligned}
x + 2y &= 5 \\
x - 3y &= -5 \\
3x + y &= 5
\end{aligned}
$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -3 \\ 3 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & -3 & -5 \\ 3 & 1 & 5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{array}\right)$$

$$r_2 - r_1$$
$$r_3 - 3r_1$$

$$(A|B) = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{array}\right)$$

$$r_3 - r_2$$

$$(A|B) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{array}\right)$$

$$rk(A) = rk(A|B) = 2 = \text{nº of variables}$$

Determined, Consistent System (DCS)

$$\left.\begin{aligned} x + 2y &= 5 \\ -5y &= -10 \end{aligned}\right\} \begin{cases} y = 2 \\ x = 1 \end{cases}$$

Exercise 3: Obtain the solution of the following system

$$x - y = 0$$
$$y + z = 1$$
$$x + z = 0$$
$$y - z = 1$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$r_3 - r_1$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

$$r_3 - r_2$$
$$r_4 - r_2$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$$r_3 \leftrightarrow r_4$$

$$(A|B) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$$rk(A) = 3 < 4 = rk(A|B)$$

Inconsistent System: There is no solution

Exercise 4: Obtain the solution of the following system

$$x + y + z + t = 0$$
$$x + 3y + 2z + 4t = 0$$
$$2x + z - t = 0$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 0 \\ 2 & 0 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & -2 & -1 & -3 & 0 \end{array}\right)$$

$$r_2 - r_1$$
$$r_3 - 2r_1$$

$$(A|B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & -2 & -1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$r_3 + r_2$$

$$(A|B) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$rk(A) = rk(A|B) = 2 < 4 = n$$

Indetermined, consistent system

Degrees of freedom:
n - rk(A)= 4 – 2 = 2

We will put the answer with only two variables at the right

$$\left.\begin{matrix} x + y + z + t = 0 \\ 2y + z + 3t = 0 \end{matrix}\right\} \left\{\begin{matrix} x = y + 2t \\ y = y \\ z = -2y - 3t \\ t = t \end{matrix}\right.$$

Exercise 5: Analyze the solution of the following system, depending on the value of m

$$x - 2y + z = -1$$
$$x + y + 3z = 4$$
$$5x - y + mz = 10$$

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 3 \\ 5 & -1 & m \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} -1 \\ 4 \\ 10 \end{pmatrix}$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 1 & 3 & 4 \\ 5 & -1 & m & 10 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 9 & m-5 & 15 \end{array}\right)$$

$$r_2 - r_1$$
$$r_3 - 5r_1$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 9 & m-5 & 15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & m-11 & 0 \end{array}\right)$$

$$r_3 - 3r_2$$

$$(A|B) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & m-11 & 0 \end{array}\right)$$

If $m \neq 11 \rightarrow 3 = rk(A) = rk(A|B) = n = 3 \rightarrow DCS$

$$x - 2y + z = -1$$
$$3y + 2z = 5$$
$$(m - 11)z = 0$$

$$(A|B) \to \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & m-11 & 0 \end{array}\right)$$

$$\left.\begin{array}{c} x-2y+z=-1 \\ 3y+2z=5 \\ (m-11)z=0 \end{array}\right\} \begin{cases} x = \dfrac{7}{3} \\ y = \dfrac{5}{3} \\ z = 0 \end{cases}$$

If $m = 11 \rightarrow 2 = rk(A) = rk(A|B) < n = 3$

$$(A|B) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{ICS}$$

$$\left.\begin{array}{c} x - 2y + z = -1 \\ 3y + 2z = 5 \end{array}\right\} \begin{cases} x = \dfrac{-7 + 7y}{2} \\ y = y \\ z = \dfrac{5 - 3y}{2} \end{cases}$$