Matrices: Basic concepts

Matrices

$$\begin{pmatrix} 1 & 5 & 1 \\ 6 & 3 & 2 \end{pmatrix} \in M_{2x3}$$

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$$\begin{pmatrix} 1 & 5 & 1 \\ 6 & 3 & 2 \end{pmatrix} \in M_{2x3}$$

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 4 \\ 6 & 3 & 2 \end{pmatrix} \in M_{3x3}, \text{ square}$$

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 4 \\ 6 & 3 & 2 \end{pmatrix} \in M_3 \text{ (order 3, size 3)}$$

Basic definitions

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 6 & 7 & 4 \end{pmatrix}$$

Transposed Matrix: interchange rows by columns

$$A^t = \begin{pmatrix} 1 & 6 \\ 3 & 7 \\ 2 & 4 \end{pmatrix}$$

Simmetric matrix

If, and only if, $A = A^t$

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} = A^t$$

Diagonal matrix

When there are only numbers in the diagonal

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Identity matrix

There are only ``1´´ in the diagonal

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Basic operations

Sum of matrices

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 6 & 7 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1+3 & 3+4 & 2+2 \\ 6+1 & 7+2 & 4+1 \end{pmatrix} = \begin{pmatrix} 4 & 7 & 4 \\ 7 & 9 & 5 \end{pmatrix}$$

Product by an scalar

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 6 & 7 & 4 \end{pmatrix}$$

$$3A = 3\begin{pmatrix} 1 & 3 & 2 \\ 6 & 7 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 9 & 6 \\ 18 & 21 & 12 \end{pmatrix}$$

Matrix product

If we want to multiply A and B:
The number of columns of A = The number of rows in B

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 5 & 1 \\ 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ 4 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{2} & 3 \\ \mathbf{5} & 1 \\ \mathbf{4} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{25} & \\ & \end{pmatrix}$$

$$1 \cdot 2 + 3 \cdot 5 + 2 \cdot 4 = 2 + 15 + 8 = 25$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ 4 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & \mathbf{3} \\ 5 & \mathbf{1} \\ 4 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 25 & \mathbf{8} \\ & \end{pmatrix}$$

$$1 \cdot 3 + 3 \cdot 1 + 2 \cdot 1 = 3 + 3 + 2 = 8$$

$$\begin{pmatrix} 1 & 3 & 2 \\ \mathbf{4} & \mathbf{6} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{2} & 3 \\ \mathbf{5} & 1 \\ \mathbf{4} & 1 \end{pmatrix} = \begin{pmatrix} 25 & 8 \\ \mathbf{42} & \end{pmatrix}$$

$$4 \cdot 2 + 6 \cdot 5 + 1 \cdot 4 = 8 + 30 + 4 = 42$$

$$\begin{pmatrix} 1 & 3 & 2 \\ \mathbf{4} & \mathbf{6} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} 2 & \mathbf{3} \\ 5 & \mathbf{1} \\ 4 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 25 & 8 \\ 42 & \mathbf{19} \end{pmatrix}$$

$$4 \cdot 3 + 6 \cdot 1 + 1 \cdot 1 = 12 + 6 + 1 = 19$$

1) Exercise

$$A = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

$$AB?$$
$$BA?$$

1) Exercise

$$A = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} =$$
$$= 1 \cdot 5 + 2 \cdot 3 + 4 \cdot 0 = 11$$

1) Exercise

$$A = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 5 & 10 & 20 \\ 3 & 6 & 12 \\ 0 & 0 & 0 \end{pmatrix}$$

2) Exercise

$$\mathrm{X} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Verify:

$$X^2 - 7X + 10I_2 = 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2) Exercise

$$X = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Verify:

$$X^2 - 7X + 10I_2 =$$

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - 7\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} + 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2) Exercise

$$\mathrm{X} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Verify:

$$X^2 - 7X + 10I_2 =$$

$$\begin{pmatrix} 18 & 14 \\ 7 & 11 \end{pmatrix} - \begin{pmatrix} 28 & 14 \\ 7 & 21 \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

2) Exercise

$$X = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Verify:

$$X^2 - 7X + 10I_2 =$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_2$$

Determinants

Order 1

$$(2) \rightarrow |2| = 2$$

$$(-5) \rightarrow |-5| = -5$$

Determinants

Order 2

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \rightarrow \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} =$$

Determinants

Order 2

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \rightarrow \begin{vmatrix} \mathbf{1} & 2 \\ 4 & \mathbf{3} \end{vmatrix} = 1 \cdot 3$$

Determinants

Order 2

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \rightarrow \begin{vmatrix} 1 & \mathbf{2} \\ \mathbf{4} & 3 \end{vmatrix} = 1 \cdot 3 - 2 \cdot 4 = -5$$

Determinants

Order 3: Sarrus

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 5 & 3 \end{pmatrix} \to \begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 5 & 3 \end{vmatrix} =$$

Determinants

Order 3: Sarrus

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 5 & 3 \end{pmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 5 & 3 \end{vmatrix} =$$

$$= 1 \cdot 1 \cdot 3 + 2 \cdot 3 \cdot 5 + 4 \cdot 2 \cdot 1$$

Determinants

Order 3: Sarrus

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 5 & 3 \end{pmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 5 & 3 \end{vmatrix} =$$

$$= 1 \cdot 1 \cdot 3 + 2 \cdot 3 \cdot 5 + 4 \cdot 2 \cdot 1$$
$$-2 \cdot 1 \cdot 4 - 1 \cdot 1 \cdot 5 - 3 \cdot 3 \cdot 2 = 41 - 31 = 10$$

3) Exercise

Order 3

$$\begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{vmatrix} =$$

3) Exercise

Order 3

$$\begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{vmatrix} =$$

$$= 3 \cdot 1 \cdot 6 + 5 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 4$$
$$-5 \cdot 1 \cdot 3 - 3 \cdot 1 \cdot 4 - 2 \cdot 2 \cdot 6 =$$
$$= 52 - 51 = 1$$

4) Exercise

Order 3

$$\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} =$$

4) Exercise

Order 3

$$\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} =$$

$$= 0 + abc + abc - 0 - 0 - 0 =$$

$$= 2abc$$

Determinantes

$$\begin{vmatrix} 0 & 1 & 2 & 2 \\ 2 & 8 & 3 & 6 \\ 0 & 3 & 1 & 1 \\ 0 & 4 & 5 & 3 \end{vmatrix}$$

Determinants

$$a_{ij}.$$

i = row

j = column

$$A = \begin{pmatrix} 1 & 3 & 6 \\ 5 & 2 & 4 \\ 7 & 1 & 8 \end{pmatrix} \rightarrow \begin{cases} a_{12} = 3 \\ a_{23} = 4 \\ a_{11} = 1 \\ a_{31} = 7 \end{cases}$$

Complementary minor

Complementary minor of a_{ij}: The determinant of what remains after having eliminating the row i and the column j

$$\text{Complementary minor of } a_{11} = \begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix} = 12$$

$$\text{Complementary minor of } a_{23} = \begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix} = -20$$

Cofactor

Cofactor of a_{ij}: $(-1)^{i+j} \cdot complementary\ minor\ of\ a_{ij} =$
$= A_{ij}$

Cofactor of $a_{11} = (-1)^2 \begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix} = 12 = A_{11}$

Cofactor of $a_{23} = (-1)^5 \begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix} = 20 = A_{23}$

Determinants

For a matrix A of order n: $A \in M_n$

Calculation by the expansion of rows (select row i):

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

Calculation by the expansion of columns (select column j):

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

Determinants

$$\begin{vmatrix} 0 & 1 & 2 & 2 \\ 2 & 8 & 3 & 6 \\ 0 & 3 & 1 & 1 \\ 0 & 4 & 5 & 3 \end{vmatrix}$$

Determinants

$$\begin{vmatrix} 0 & 1 & 2 & 2 \\ 2 & 8 & 3 & 6 \\ 0 & 3 & 1 & 1 \\ 0 & 4 & 5 & 3 \end{vmatrix} = 0 \cdot A_{11} + 2 \cdot A_{21} + 0 \cdot A_{31} + 0 \cdot A_{41}$$

Determinants

$$\begin{vmatrix} 0 & 1 & 2 & 2 \\ 2 & 8 & 3 & 6 \\ 0 & 3 & 1 & 1 \\ 0 & 4 & 5 & 3 \end{vmatrix} = 0 \cdot A_{11} + 2 \cdot A_{21} + 0 \cdot A_{31} + 0 \cdot A_{41}$$

$$= 2A_{21} = 2(-1)^3 \begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 5 & 3 \end{vmatrix} = -2 \cdot 10 = -20$$

Properties

- We can interchange two rows or two columns without changing the determinant

- We can add or substract two rows or two columns without changing the determinant

- If a row or a column is multiplied by a number, the determinant will also be multiplied by that number

Properties

- If we have a row or a column full of zeros: Determinant = 0

- If two rows or two columns are equal: Determinant = 0

- If two rows or two columns are proportional: Determinant = 0

Properties

$$|AB| = |A||B|$$

Being A a square matrix:

$$|A^t| = |A|$$

Being A a square matrix of order n:

$$|\lambda A| = \lambda^n |A|$$

Properties

$$\begin{vmatrix} a_{11} & \dots & a_{1j}+b_{1j} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j}+b_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj}+b_{nj} & \dots & a_{nn} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & b_{1j} & \dots & a_{1n} \\ a_{21} & \dots & b_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & b_{nj} & \dots & a_{nn} \end{vmatrix}$$