1) Level sets

Level set of value c: Set of points in which the function is equal to c

n = 2 (we can draw the level sets):

Given a function and a level set of value K:

$$f(x,y) = K \begin{cases} 1)\ We\ obtain\ the\ equation\quad y = g(x) \\ 2)\ We\ draw\ the\ equation\quad y = g(x) \end{cases}$$

2) Partial Derivative

Partial Derivative: Gives information about the increase of the function when one of the variables increases

- A function of n variables will have n partial derivatives

$$f_i = \frac{\partial f(\bar{x})}{\partial i}$$

Gradient / Jacobian Matrix: Vector with the partial derivatives

$$\nabla f(\bar{x}) = \left(\frac{\partial f(\bar{x})}{\partial x_1}, \frac{\partial f(\bar{x})}{\partial x_2}, \dots, \frac{\partial f(\bar{x})}{\partial x_n} \right)$$

Partial Derivatives:

$$f(x, y) = x^2 y$$

Partial Derivatives:

$$f(x,y) = x^2 y \quad \begin{cases} \dfrac{\partial f}{\partial x} = 2xy \\ \dfrac{\partial f}{\partial y} = x^2 \end{cases}$$

Gradient:

$$\nabla f(\bar{x}) = (2xy, x^2)$$

Partial Derivatives:

$$f(x, y, z) = xyz + x$$

Partial Derivatives:

$$f(x, y, z) = xyz + x \begin{cases} \dfrac{\partial f}{\partial x} = yz + 1 \\ \dfrac{\partial f}{\partial y} = xz \\ \dfrac{\partial f}{\partial z} = xy \end{cases}$$

Gradient:

$$\nabla f(\bar{x}) = (yz + 1, xz, xy)$$

3) Differential

Differential:

$$df(\bar{x}) = \frac{\partial f(\bar{x})}{\partial x}dx + \frac{\partial f(\bar{x})}{\partial y}dy + \cdots + \frac{\partial f(\bar{x})}{\partial z}dz$$

Differential:

$$f(x, y) = x^2 y$$

Differential:

$$f(x,y) = x^2y \quad \begin{cases} \dfrac{\partial f}{\partial x} = 2xy \\ \dfrac{\partial f}{\partial y} = x^2 \end{cases}$$

$$df(\bar{x}) = \frac{\partial f(\bar{x})}{\partial x}dx + \frac{\partial f(\bar{x})}{\partial y}dy = (2xy)dx + (x^2)dy$$

In the point (3,1) $\rightarrow df(3,1) = (6)dx + (9)dy$

Differential:

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$$f(x,y,z) = xyz + x \begin{cases} \dfrac{\partial f}{\partial x} = yz + 1 \\ \dfrac{\partial f}{\partial y} = xz \\ \dfrac{\partial f}{\partial z} = xy \end{cases}$$

$$df(\bar{x}) = \frac{\partial f(\bar{x})}{\partial x}dx + \frac{\partial f(\bar{x})}{\partial y}dy + \frac{\partial f(\bar{x})}{\partial z}dz = (yz+1)dx + (xz)dy + (xy)dz$$

In the point (1,2,1) $\rightarrow df(1,2,1) = (3)dx + (1)dy + (2)dz$

4) Directional Derivative

Directional Derivative, with respect to $\vec{v}$ (unitary vector)

$$D_{\vec{v}}f(\bar{x}) = \left(\nabla f(\bar{x})\right)' \, \vec{v} = \frac{\partial f(\bar{x})}{\partial x} v_1 + \frac{\partial f(\bar{x})}{\partial y} v_2 + \cdots + \frac{\partial f(\bar{x})}{\partial z} v_n$$

$$D_{\vec{v}}f(\bar{x}) = \text{gradient} \cdot vector$$

Directional Derivative:

$$f(x, y) = x^2 y$$

In the direction: $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

Directional Derivative:

$$f(x,y) = x^2 y \quad \begin{cases} \dfrac{\partial f}{\partial x} = 2xy \\ \dfrac{\partial f}{\partial y} = x^2 \end{cases}$$

In the direction: $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$$D_{\vec{v}} f(\bar{x}) = \frac{\partial f(\bar{x})}{\partial x} v_1 + \frac{\partial f(\bar{x})}{\partial y} v_2 = 2xy\left(\frac{2}{\sqrt{5}}\right) + x^2\left(\frac{1}{\sqrt{5}}\right)$$

In the point (1,1) $\rightarrow D_{\vec{v}} f(1,1) = 2\left(\frac{2}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}}\right) = \sqrt{5}$

Directional Derivative:

$$f(x, y, z) = xyz + x$$

In the direction: $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

Directional Derivative:

$$f(x,y,z) = xyz + x \begin{cases} \dfrac{\partial f}{\partial x} = yz + 1 \\ \dfrac{\partial f}{\partial y} = xz \\ \dfrac{\partial f}{\partial z} = xy \end{cases}$$

In the direction: $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$$D_{\vec{v}} f(\bar{x}) = (yz+1)\left(\frac{1}{\sqrt{6}}\right) + (xz)\left(\frac{2}{\sqrt{6}}\right) + (xy)\left(\frac{1}{\sqrt{6}}\right)$$

In the point (2,1,2) $\rightarrow D_{\vec{v}} f(2,1,2) = (3)\left(\frac{1}{\sqrt{6}}\right) + (4)\left(\frac{2}{\sqrt{6}}\right) + (2)\left(\frac{1}{\sqrt{6}}\right) = \frac{13}{\sqrt{6}}$

5) Tangent plane of $f(x, y)$ in the point (x_0, y_0):

$$z = f(x_0, y_0) + \left(\frac{\partial f(x_0, y_0)}{\partial x}\right)(x - x_0) + \left(\frac{\partial f(x_0, y_0)}{\partial y}\right)(y - y_0)$$

Tangent plane of $f(x, y) = x^2 y$:

$$z(\bar{x}, \bar{y}) = x^2 y + (2xy)(\bar{x} - x) + (x^2)(\bar{y} - y)$$

Those are not substituted

In the point $(1, 3)$:

$$z = 3 + 6(x - 1) + 1(y - 3)$$

Tangent plane of f(x,y,z)=$xyz + x$:

$$t = xyz + x + (yz + 1)(\bar{x} - x) + xz(\bar{y} - y) + xy(\bar{z} - z)$$

Tangent plane in the point $(1,1,1)$:

$$t = 2 + 2(x - 1) + 1(y - 1) + 1(z - 1)$$

Second-order partial derivatives: The derivatives of the derivatives.

$$f(x,y)\begin{cases}\dfrac{\partial f}{\partial x} & \begin{cases}\dfrac{\partial^2 f}{\partial x^2}\\[2ex]\dfrac{\partial^2 f}{\partial x \partial y}\end{cases}\\[6ex]\dfrac{\partial f}{\partial y} & \begin{cases}\dfrac{\partial^2 f}{\partial y \partial x}\\[2ex]\dfrac{\partial^2 f}{\partial y^2}\end{cases}\end{cases}$$

$$
x^2y \begin{cases} \dfrac{\partial f}{\partial x} = 2xy & \begin{cases} \dfrac{\partial^2 f}{\partial x^2} = 2y \\ \dfrac{\partial^2 f}{\partial x \partial y} = 2x \end{cases} \\ \dfrac{\partial f}{\partial y} = x^2 & \begin{cases} \dfrac{\partial^2 f}{\partial y \partial x} = 2x \\ \dfrac{\partial^2 f}{\partial y^2} = 0 \end{cases} \end{cases}
$$

$$
xyz + x \begin{cases}
\dfrac{\partial f}{\partial x} = yz + 1 & \left\{ \dfrac{\partial^2 f}{\partial x^2} = 0 \right., \quad \dfrac{\partial^2 f}{\partial x \partial y} = z, \quad \dfrac{\partial^2 f}{\partial x \partial z} = y \\[2ex]
\dfrac{\partial f}{\partial y} = xz & \left\{ \dfrac{\partial^2 f}{\partial y \partial x} = z \right., \quad \dfrac{\partial^2 f}{\partial y^2} = 0, \quad \dfrac{\partial^2 f}{\partial y \partial z} = x \\[2ex]
\dfrac{\partial f}{\partial z} = xy & \left\{ \dfrac{\partial^2 f}{\partial z \partial x} = y \right., \quad \dfrac{\partial^2 f}{\partial z \partial y} = x, \quad \dfrac{\partial^2 f}{\partial z^2} = 0
\end{cases}
$$

6) Hessian matrix:

$$Hf(\bar{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Hessian matrix of x^2y

$$Hf(\bar{x}) = \begin{bmatrix} 2y & 2x \\ 2x & 0 \end{bmatrix}$$

Hessian matrix of $xyz + x$

$$Hf(\bar{x}) = \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$$

Taylor Formula (only with 1 variable). Of order N, in point x=a

$$f(x) = f(a) + \sum_{n=1}^{N} \frac{f^{(n}}{!n}(x-a)^n + R_N$$

First order:

$$f(x) = f(a) + f'(a)(x-a) + R_1(a)$$

Second order:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + R_2(a)$$

$f(x) = x^3$, in x=2

First order:

$$f(x) = f(2) + f'(2)(x-2) = 8 + 3 \cdot 2^2\,(x-2) + R_1 = 8 + 12(x-2) + R_1$$

Second order:

$$f(x) = f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + R_2 =$$
$$= 8 + 12(x-2) + 6(x-2)^2 + R_2$$