A is diagonalizable if we can write the following expression

$$A = PDP^{-1}$$

Fourth Theorem:

If $A \in M_n$ is symmetric, A is diagonalizable

$$A = PDP^{-1}$$

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

$$\dots$$

$$A^k = PD^kP^{-1}$$

Example 4)

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$$

$$A^5?$$

Summary:

$$1)\ \ |A - \lambda I_n| = 0 \ \begin{cases} \lambda = 4 \\ \lambda = 2 \end{cases}$$

$$2)\ \ (A - \lambda I_n)X = 0 \begin{cases} x\begin{pmatrix} 1 \\ 1 \end{pmatrix}, for\ \ \lambda = 4 \\ y\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad for\ \ \lambda = 2 \end{cases}$$

$$\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} = P \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$$

$$P = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}^5 = P \begin{pmatrix} 4^5 & 0 \\ 0 & 2^5 \end{pmatrix} P^{-1}$$

Real Quadratic Forms

$$Q(X) = 2x^2 + 3y^2 + 4xy$$

Is it always positive? Negative?
Can it be 0?

Sign, without considering the case of X=0:

Positive Definite (PD): $Q(X) > 0$

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Positive semidefinite (PSD): $\begin{cases} Q(X) \geq 0 \\ Q(X) = 0,\ for\ some\ X \end{cases}$

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<u>Positive Definite (PD)</u>: $Q(X) > 0$

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<u>Indefinite (I)</u>: Can be positive or negative

$$Q(X) = 2x^2 + 3y^2 + 4xy$$

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$

$$Q(X) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$Q(X) = x^2 + 5y^2 + 6xy + 2xz + 8yz$$

$$Q(X) = x^2 + 5y^2 + 6xy + 2xz + 8yz$$

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 5 & 4 \\ 1 & 4 & 0 \end{pmatrix}$$

How to know the sign? Three methods:

1º: Main Minors $\rightarrow$ This week

$$\left.\begin{array}{l}\text{2º: Eigenvalues}\\[2em]\text{3º: Sum of squares}\end{array}\right\} Next\ Week$$

2º Method: Main Minors

Main Minor of order p:

Determinant with the same p rows and columns

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Minors\ of\ order\ 1$$
$$row\ 1, column\ 1$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Minors\ of\ order\ 1$$
$$row\ 2, column\ 2$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Minors\ of\ order\ 1$$
$$row\ 3, column\ 3$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Minors\ of\ order\ 2$$
$$rows\ 1\ \&\ 2, columns\ 1\ \&\ 2$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{Minors of order } 2$$
$$\text{rows } 1 \,\&\, 3, \text{columns } 1 \,\&\, 3$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Minors\ of\ order\ 2$$
$$rows\ 2\ \&\ 3, columns\ 2\ \&\ 3$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Minors\ of\ order\ 3$$
$$rows\ 1, 2\ \&\ 3, columns\ 1, 2\ \&\ 3$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Sign of the Main Minors

	Odd order	Even order	Last order
Positive Definite	$+$	$+$	$+$
Negative Definite	$-$	$+$	Odd - Even +
Positive Semidefinite	$+/0$	$+/0$	0
Negative Semidefinite	$-/0$	$+/0$	0
Indefinite	All other cases		

Sign of the Main Minors for a matrix of size 3

	Order 1	Order 2	Order 3
Positive Definite	$+$	$+$	$+$
Negative Definite	$-$	$+$	$-$
Positive Semidefinite	$+/0$	$+/0$	0
Negative Semidefinite	$-/0$	$+/0$	0
Indefinite	All other cases		

$$Q(X) = 2x^2 + 3y^2 + 4xy$$

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{cases} Order\ 1 \rightarrow 2, 3\ \ (+) \\ Order\ 2\ \rightarrow 2\ \ (+) \end{cases}$$

Positive Definite

$$Q(X) = x^2 + 5y^2 + 6xy + 2xz + 8yz$$

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 5 & 4 \\ 1 & 4 & 0 \end{pmatrix} \begin{cases} Order\ 1 \rightarrow 1, 5, 0 \quad (+) \\ Order\ 2 \rightarrow -4, -1, -16 \quad (-) \\ Order\ 3 \rightarrow 3 \quad (+) \end{cases}$$

Indefinite

You can check the value of the first Main Minor of each order and use that sign, <u>only if it is different than 0</u>.

$$Minors\ of\ order\ 1$$
$$row\ 1, column\ 1$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{Minors of order } 2$$
$$\text{rows } 1 \ \& \ 2, \text{columns } 1 \ \& \ 2$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$Minors\ of\ order\ 3$$
$$rows\ 1, 2\ \&\ 3, columns\ 1, 2\ \&\ 3$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 1

$$3x^2 + 7y^2 - 4xy$$

Example 1

$$3x^2 + 7y^2 - 4xy \to A = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}$$

Order 1 $\to |3| = 3 > 0$

Order 2 $\to \begin{vmatrix} 3 & -2 \\ -2 & 7 \end{vmatrix} = 17 > 0$

Positive Definite

Example 2

$$-2x^2 - 5y^2 + 6xy$$

Example 2

$$-2x^2 - 5y^2 + 6xy \to A = \begin{pmatrix} -2 & 3 \\ 3 & -5 \end{pmatrix}$$

Order 1 $\to |-2| = -2 < 0$

Order 2 $\to \begin{vmatrix} -2 & 3 \\ 3 & -5 \end{vmatrix} = 1 > 0$

Negative Definite

Example 3

$$4xy$$

Example 3

$$4xy \to A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Order 1 $\to |0| = 0$ $\overbrace{\xrightarrow{}}^{We\ look\ all\ Main\ Minors\ of\ order\ 1}$ $\begin{cases} 0 \\ 0 \end{cases}$

Order 2 $\to \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0$

Indefinite