Problem Sheet 2 (Solutions)

MATH1710 Probability and Statistics I

University of Leeds, 2022-23

A: Short questions

A1. Suppose you toss a coin 4 times.

(a) What would you suggest for a sample space Ω **(i)** if you only care about the total number of heads; **(ii)** if you care about the result of each coin toss?

(b) For each of the cases in part (a), what is $|\Omega|$?

> *Solution.*
>
> **(i)** We can take $\Omega = \{0, 1, 2, 3, 4\}$, with $|\Omega| = 5$.
>
> **(ii)** Here, $\Omega = \{\mathrm{HHHH}, \mathrm{HHHT}, \mathrm{HHTH}, \dots, \mathrm{TTTT}\}$ should be the set of all sequences of four "H"s or "T"s. So here, $|\Omega| = 2^4 = 16$.

A2. Let A, B and C be events in a sample space Ω. Write the following events using only A, B, C and the complement, intersection, and union operations.

(a) C happens but A doesn't.

> *Solution.* This is "C and not A": $C \cap A^{\mathsf{c}}$.

(b) At least one of A, B and C happens.

> *Solution.* This is simply the union $A \cup B \cup C$.

(c) Exactly one of B or C happens.

> *Solution.* One way to write this is to split it up as " 'B but not C' or 'C but not B' ", which is $(B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C)$.
>
> An alternative is to split it up as " 'B or C' but not 'both B and C' ", which is $(B \cup C) \cap (B \cap C)^{\mathsf{c}}$.
>
> You can check these are equal by (for example) using De Morgan's law and the distributive law to expand out the second version.

(d) Exactly two of A, B and C happens.

> *Solution.* I would split this up into "A and B but not C", "A and C but not B", and "B and C but not A" and take the union. This gives
>
> $$(A \cap B \cap C^{\mathsf{c}}) \cup (A \cap B^{\mathsf{c}} \cap C) \cup (A^{\mathsf{c}} \cap B \cap C).$$
>
> There are other equivalent formulations.

A3. What is the value of the following expressions?

(a) $6!$

Solution.

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

(b) 8^4

Solution.

$$8^4 = 8 \times 8 \times 8 \times 8 = 4096$$

(c) $8^{\underline{4}}$

Solution.

$$8^{\underline{4}} = 8 \times 7 \times 6 \times 5 = 1680$$

(d) $\binom{10}{4}$

Solution.

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

A4. An urn contains 4 red balls and 6 blue balls. Two balls are drawn from the urn. What is the probability that both balls are red, if the balls are drawn **(a)** with replacement; **(b)** without replacement?

Solution.

(a) There are $|\Omega| = 10^2 = 100$ ways to draw two balls with replacement. There are $|A| = 4^2 = 16$ ways to draw two blue balls. So $\mathbb{P}(A) = \frac{16}{100} = 0.16$.

(b) There are $|\Omega| = 10^{\underline{2}} = 10 \times 9 = 90$ ways to draw two balls without replacement. There are $|A| = 4^{\underline{2}} = 4 \times 3 = 12$ to draw two blue balls. So $\mathbb{P}(A) = \frac{12}{90} = \frac{2}{15} = 0.133$.

B: Long questions

B1. Starting from just the three probability axioms, prove the following statements:

(a) $\mathbb{P}(\varnothing) = 0$.

Solution. Let A be any event (such as $A = \varnothing$ or $A = \Omega$, for example). Then $A \cup \varnothing = A$, and the union is disjoint – since $\varnothing$ contains no sample points, it certainly can't contain any sample points that are also in A. Then applying Axiom 3, we get $\mathbb{P}(A) + \mathbb{P}(\varnothing) = \mathbb{P}(A)$. Subtracting $\mathbb{P}(A)$ from both sides gives the result.

Alternatively, if you prove part (b) first, you can apply that with $A = \varnothing$. Since $\varnothing^{\mathsf{c}} = \Omega$ and Axiom 2 tells us that $\mathbb{P}(\Omega) = 1$, the result follows.

Group feedback: With this, and most "prove from the axioms" questions, the key is to find a relevant disjoint union, which then allows us to use Axiom 3. So if we can find $C = A \cup B$ as a disjoint union (hopefully containing some events relevant to the question at hand), Axiom 3 allows us to write $\mathbb{P}(C) = \mathbb{P}(A) + \mathbb{P}(B)$.

(b) $\mathbb{P}(A^{\mathsf{c}}) = 1 - \mathbb{P}(A)$.

Solution. A very useful and relevant disjoint union is $A \cup A^{\mathsf{c}} = \Omega$. Applying Axiom 3 gives us $\mathbb{P}(A) + \mathbb{P}(A^{\mathsf{c}}) = \mathbb{P}(\Omega)$. But Axiom 2 tells us that $\mathbb{P}(\Omega) = 1$, so $\mathbb{P}(A) + \mathbb{P}(A^{\mathsf{c}}) = 1$. Rearranging gives the result.

B2. In this question, you will have to use the standard two-event form of the addition rule for unions

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(a) Using the two-event addition rule, show that

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D \cup E) - \mathbb{P}\big(C \cap (D \cup E)\big).$$

Solution. As with the Cauchy–Schwarz question from Problem Sheet 1, the key is to make a good choice for what A and B should be. This time, $A = C$ and $B = D \cup E$ will work well, since $C \cup (D \cup E) = C \cup D \cup E$. (You can call this "associativity", if you like.) Making that substitution immediately gives us

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D \cup E) - \mathbb{P}\big(C \cap (D \cup E)\big),$$

as required.

(b) Using your result from part (a), the two-event addition rule, the distributive law, and the two-event addition rule again, prove the three-event form of the addition rule for unions:

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(C \cap D) - \mathbb{P}(C \cap E) - \mathbb{P}(D \cap E) + \mathbb{P}(C \cap D \cap E).$$

Solution. Let's take the three terms on the right of the equation from part (a) separately.

The first term is $\mathbb{P}(C)$, which is fine as it is.

The second term is $\mathbb{P}(D \cup E)$. This is the probability of the union of two events, so we can use addition rule for the union of two events to get

$$\mathbb{P}(D \cup E) = \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(D \cap E).$$

The third term is $\mathbb{P}\big(C \cap (D \cup E)\big)$. If we use the distributive law, as suggested in the question, we get $C \cap (D \cup E) = (C \cap D) \cup (C \cap E)$, so we want to find $\mathbb{P}\big((C \cap D) \cup (C \cap E)\big)$. But this is another union of two events again, this time with $A = C \cap D$ and $B = C \cap E$. So the two-event addition rule gives

$$\mathbb{P}\big((C \cap D) \cup (C \cap E)\big) = \mathbb{P}(C \cap D) + \mathbb{P}(C \cap E) - \mathbb{P}(C \cap D \cap E),$$

since $(C \cap D) \cap (C \cap E) = C \cap D \cap E$.

Finally, we put this all together, and get

$$\begin{aligned}
&\mathbb{P}(C \cup D \cup E) \\
&\quad = \mathbb{P}(C) + \big(\mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(D \cap E)\big) - \big(\mathbb{P}(C \cap D) + \mathbb{P}(C \cap E) - \mathbb{P}(C \cap D \cap E)\big) \\
&\quad = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(C \cap D) - \mathbb{P}(C \cap E) - \mathbb{P}(D \cap E) + \mathbb{P}(C \cap D \cap E),
\end{aligned}$$

which is what we wanted.

B3. Suppose we pick a number at random from the set $\{1, 2, \dots, 2023\}$.

(a) What is the probability that the number is divisible by 5?

Solution. The sample space is $\Omega = \{1, 2, \dots, 2023\}$. Clearly $|\Omega| = 2023$. The event in question is $A = \{5, 10, \dots, 2020\}$ of numbers up to 2023 that are divisible by 5. Thus $|A|$ is the largest integer no bigger than $\frac{2023}{5} = 404.6$, which is 404, as this is how many times 5 "goes into" 2023. Hence

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{404}{2023} = 0.1997,$$

just a tiny bit smaller than $\frac{1}{5}$.

Group feedback: With these "classical probability" questions, the steps should always be:

1. State clearly what the sample space Ω is.
2. Count how many outcomes $|\Omega|$ are in the sample space.
3. State clearly what the event A is.
4. Count how many outcomes $|A|$ are in the event.
5. The desired probability is then $\mathbb{P}(A) = |A|/|\Omega|$.

(b) What is the probability the number is divisible by 5 or by 7?

Solution. With the same Ω and A, now let B be the numbers up to 2023 divisible by 7; so we're looking for $\mathbb{P}(A \cup B)$. By the addition rule for unions, this is

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

We already know $\mathbb{P}(A) = \frac{404}{2022}$, so need to find out $\mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$.

This time, 7 goes into 2023 exactly, so $|B|$ is $\frac{2023}{7} = 289$. So

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{289}{2023} = \frac{1}{7}.$$

Now, $A \cap B$ is the set of numbers divisible by both 5 and 7, which is precisely the numbers divisible by their least common multiple $5 \times 7 = 35$. Then $|A \cap B|$ is $\frac{2023}{35} = 57.8$ rounded down, so $\mathbb{P}(A \cap B) = \frac{57}{2023}$.

So finally, we have

$$\mathbb{P}(A \cup B) = \frac{404}{2023} + \frac{289}{2023} - \frac{57}{2023} = \frac{636}{2023} = 0.314,$$

just a tiny bit larger than $\frac{11}{35}$

B4. Eight friends are about to sit down at random at a round table. Find the probability that

(a) Ashley and Brook sit next to each other, with Chris directly opposite Brook;

Solution. Let Ω be the sample space of ways the friends can sit around the table. This is an ordering problem, so $|\Omega| = 8!$.

Let A be the event in the question. What is $|A|$? Well,

- Ashley can sit anywhere, so has 8 choices of seat.
- Brook can sit either directly to Ashley's left or directly to Ashley's right, so has 2 choices of seat.
- Chris must sit directly opposite Brook, so only has 1 choice of seat.
- The remaining five friends can fill up the remaining seats however they like, so have 5, 4, 3, 2, and 1 choices respectively.

Hence $|A| = 8 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1$. Thus we get

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{8 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{2 \times 1}{7 \times 6} = \frac{1}{21}.$$

Group feedback: As we have discussed more recently, after this Problem Sheet was assigned, often "classical probability" problems also can be equivalently solved by the step-by-step "chain rule" method. Can you use a chain rule argument to find the same answer as

$$\mathbb{P}(A) = 1 \times \frac{2}{7} \times \frac{1}{6} \times 1 \times 1 \times 1 \times 1 \times 1 = \frac{1}{21}?$$

(b) neither Ashley, Brook nor Chris sit next to each other.

Solution. The sample space Ω is as before. Let's count the outcomes in B, the event in the question.

- Ashley can sit anywhere, so has 8 choices of seat.
- Chris's number of choices will depend on where Brook sits, so we'll have to count Brook's and Chris's choices together:
  - Brook cannot sit next to Ashley.

- If Brook sits next-but-one to Ashley – of which there are 2 choices – then Chris has 3 choices: Chris cannot sit on the seat directly between Ashley and Brook, nor directly next to Ashley on the other side, nor directly next to Brook on the other side, leaving $6-3=3$ choices.
- If Brook sits neither next nor next-but-one to Ashley – of which there are 3 choices – then Chris has 2 choices: he cannot sit to the right or left of Ashley, nor to the right or left of Brook, leaving $6-4=2$ choices.

- The remaining friends have 5, 4, 3, 2, and 1 choices again.

Hence, $|B| = 8 \times (2\times 3 + 3 \times 2) \times 5 \times 4 \times 3 \times 2 \times 1$. So

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{8 \times (2\times 3 + 3 \times 2) \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{2\times 3 + 3 \times 2}{7 \times 6} = \frac{12}{42} = \frac{2}{7}.$$

Alternatively, in a previous year's tutorial, a MATH1710 student suggested to me the following rather elegant solution. Suppose the five other friends are already sat at a round table with five chairs. Ashley, then Brook, then Chris will each bring along their own chair, and push into one of the gaps between the friends.

Ashley has 5 gaps to choose from, then Brook will have 6 gaps (Ashley joining the table will have increased the number of gaps by 1), then Chris will have 7, so the total number of ways they can push in is $|\Omega| = 5 \times 6 \times 7$.

To not sit next to each other, Ashley can push in any of the 5 gaps, Brook only has $6-2=4$ choices (not in the gap directly to the left or right of Ashley), and Chris only has $7-4=3$ choices (not in the gaps directly to the left or right of Ashley nor the gaps directly to the left or right of Brook – these four gaps are distinct assuming Brook was not next to Ashley). Hence $|B| = 5 \times 4 \times 3$, and we have

$$\mathbb{P}(B) = \frac{5 \times 4 \times 3}{5 \times 6 \times 7} = \frac{4 \times 3}{6 \times 7} = \frac{12}{42} = \frac{2}{7}.$$

Group feedback: Again, an equivalent answer can be derived using the step-by-step "chain rule" method.

B5. A "random digit" is a number chosen at random from $\{0, 1, \dots, 9\}$, each with equal probability. A statistician chooses n random digits (with replacement).

(a) For $k = 0, 1, \dots, 9$, let A_k be the event that all the digits are k or smaller. What is the probability of A_k, as a function of k and n?

Solution. The sample space is $\Omega = \{0, 1, \dots, 9\}^n$, the set of length-$n$ sequences of digits between 0 and 9. The number of these is $|\Omega| = 10^n$, as there are 10 choices for each of the n digits.

The event A_k is $\{0, 1, \dots, k\}^n$, the set of length-n sequences of digits that are between 0 and k. The number of these is $|A_k| = (k+1)^n$. (Note that it's $k+1$ because we're allowing 0 as well.)

Hence, the probability is

$$\mathbb{P}(A_k) = \frac{|A_k|}{|\Omega|} = \frac{(k+1)^n}{10^n}.$$

(b) Let B_k be the event that the largest digit chosen is equal to k. By finding a relationship between B_k,

A_{k-1} and A_k, or otherwise, show that

$$\mathbb{P}(B_k) = \frac{(k+1)^n - k^n}{10^n}.$$

Solution. Consider the event A_k that all the digits are at most k. Within A_k, *either* one or more of the digits equal k, in which case that k is the largest digit and we are in B_k; *or* none of the digits equal k, in which case they are all at most $k-1$, and we are in A_{k-1}. Only one of these two possibilitie can occur, so we have a disjoint union

$$A_k = B_k \cup A_{k-1}.$$

Applying Axiom 3 to the disjoint union gives

$$\mathbb{P}(A_k) = \mathbb{P}(B_k) + \mathbb{P}(A_{k-1}).$$

Rearranging this gives

$$\mathbb{P}(B_k) = \mathbb{P}(A_k) - \mathbb{P}(A_{k-1}).$$

Substituting in the answer from part (a) gives

$$\mathbb{P}(B_k) = \frac{(k+1)^n}{10^n} - \frac{(k-1+1)^n}{10^n} = \frac{(k+1)^n - k^n}{10^n}.$$

C: Assessed questions

C1. Let Ω be a sample space with a probability measure $\mathbb{P}$, and let $A, B \subset \Omega$ be events. For each of the following statements, state whether the statement is true or false (that is, always true or sometimes false). If it is true, briefly justify the statement; if it is false, give a counterexample.

(a) If $\mathbb{P}(A) \leq \mathbb{P}(B)$, then $A \subset B$.

> *Hint.* Try to find a counterexample. Make sure you're paying attention to the direction of implication (the other direction is true).

(b) $\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^{\mathsf{c}}) = \mathbb{P}(A)$.

> *Hint.* Is there a relevant disjoint union here?

(c) $\mathbb{P}(A \cup B) \leq \mathbb{P}(A)$

> *Hint.* You'd expect the inequality to be the other way round – so it should be possible to find a counterexample.

(d) If A and B are disjoint, then $\mathbb{P}((A \cup B)^{\mathsf{c}}) = 1 - \mathbb{P}(A) - \mathbb{P}(B)$.

> *Hint.* Can you use the complement rule to start off with?

C2. An urn contains 15 balls: 4 red balls, 5 blue balls, and 6 green balls.

(a) If three balls are drawn *with* replacement, what is the probability that all three balls are the *same* colour?

> *Hint.* If A is the event all three balls are the same colour, then we have a disjoint union $A = A_{\text{red}} \cup A_{\text{blue}} \cup A_{\text{green}}$, where A_{red} is the event all three balls are red, and so on.

(b) If three balls are drawn *without* replacement, what is the probability that all three balls are *different* colours?

> *Hint.* One way to do this is to look at the ordered collection of balls, and look at all 3! possible orderings red-blue-green, red-green-blue, etc.
>
> Another way is to look at the unordered collection, so the denominator is $\binom{15}{3}$.