MATH1710 Probability and Statistics I

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Contents

Schedule

Week 6 (6–10 November):

- **Lecture 11:** Binomial and geometric distributions (Monday 6 November)
- **Lecture 12:** Poisson distribution (Wednesday 8 November)
- **Tutorial** on Problem Sheet 3
- **Problem Sheet 3:** Work through in preparation for your tutorial. Deadline for assessed questions: Monday 13 November.
- **R Worksheet 5**
- **Office hours:** Friday 10 November, 11–12 and 1–2, Physics Research Deck 9.320
- Mid-semester check-in

Week 5 (30 October–3 November):

- **Lecture 9:** Discrete random variables (Monday 30 October)
- **Lecture 10:** Expectation and variance (Wednesday 1 November)
- **Problem Sheet 3:** Work through in preparation for your tutorial in Week 6. Deadline for assessed questions: Monday 13 November.
- **R Worksheet 4** – deadline for assessed questions: Monday 6 November
- **Office hours:** Friday 3 November, 11–1, Physics Research Deck 9.320 (*note change of time*)

Week 4 (23–27 October):

- **Lecture 7:** Independence and conditional probabilty (Monday 23 October)
- **Lecture 8:** Two theorems on conditional probability (Wednesday 25 October)
- **Tutorial** on Problem Sheet 2
- **Problem Sheet 2:** Work through in preparation for your tutorial. Deadline for assessed questions: Monday 30 October.
- **R Worksheet 3**
- **Office hours:** Friday 27 October, 11–12 and 1–2, Physics Research Deck 9.320 (*note change of venue*)

Week 3 (16–20 October):

- **Lecture 5:** Classical probability I (Monday 16 October)
- **Lecture 6:** Classical probability II (Wednesday 18 October)
- **R Practical**
- **Problem Sheet 2:** Work through in preparation for your tutorial in Week 4. Deadline for assessed questions: Monday 30 October.
- **R Worksheet 2:** Deadline for assessed exercises: Monday 23 October
- **Office hours:** Friday 20 October, 11–12 and 1–2, Maths Boardroom

Week 2 (9–13 October):

- **Lecture 3:** Sample spaces and events (Monday 9 October)
- **Lecture 4:** Probability (Wednesday 11 October)
- **Tutorial** on Problem Sheet 1
- **R Practical**
- **Problem Sheet 1:** Work through in preparation for your tutorial. Deadline for assessed questions: Monday 16 October.
- **R Worksheet 1**
- **Office hours:** Friday 13 October, 11–12 and 1–2, Maths Boardroom

Week 1 (2–6 October):

- **Lecture 1:** Summary statistics (Monday 2 October)
- **Lecture 2:** Data visualisation (Wednesday 4 October)
- **Problem Sheet 1:** Work through in preparation for your tutorial in Week 2. Deadline for assessed questions: Monday 16 October.
- **Office hours:** Friday 6 October, 11–12 and 1–2, Maths Boardroom

About MATH1710

Organisation of MATH1710

This module is **MATH1710 Probability and Statistics I**. (A small number of second-year scientists are taking this module as the first half of **MATH2700 Probability and Statistics for Scientists**.)

This module lasts for 11 weeks from 2 October to 15 December 2023. The exam will take place between 15 and 26 January 2024.

The module leader, the lecturer, and the main author of these notes is Dr Matthew Aldridge (you can call me "Matt", "Matthew", or "Dr Aldridge", pronounced "*old*-ridge").

Lectures

The main way you will learn new material for this module is by attending lectures. There are two lectures per week. Because this is a very large class each lecture will be delivered twice:

- **Mondays** at 1200 or at 1400, in Chemistry West LT F
- **Wednesdays** at 1500 in Chemistry West LT F or at 1600 in Roger Stevens LT 20

Check your timetable to see which lecture you are assigned to each day (or click this link for a map of how to get to Chemistry West LT F.

I recommend taking your own notes during the lecture. I will put brief summary notes from the lectures on this website, but will not reflect all the details I say and write during the lectures. Lectures will go through material quite quickly and the material may be quite difficult, so it's likely you'll want to spend time reading through your notes after the lecture.

You are probably reading the web version of the notes. If you want a PDF copy (to read offline or to print out), it can be downloaded via the top ribbon of the page. (Warning: I have not made as much effort to make the PDF as neat and tidy as I have the web version, and there may be formatting errors.) I am very keen to hear about errors in the notes, mathematical, typographical or otherwise. Please email me if think you may have found any.

Attendance at lectures is compulsory.

Problem Sheets

There will be 5 problem sheets. Each problem sheet has a number of short and long questions, for you to work on in your own time to help you learn the material, and two assessed questions, which you should submit for marking. The assessed questions on each problem sheet make up 3% of your mark on this module, for a total of 15%. Deadlines are 2pm on Mondays, although I'd recommend completing and submitting the work in the previous week.

Problem Sheet	Lectures covered	Deadline for assessed work
1	1 and 2	Monday 16 October (Week 3)
2	3–6	Monday 30 October (Week 5)
3	7–10	Monday 13 November (Week 7)
4	11–14	Monday 27 November (Week 9)
5	15–18	Monday 11 December (Week 11)

An informal Problem Sheet 6 covering material from Lectures 19 and 20 will be available. Lectures 21 and 22 are revision lectures with no new material.

Assessed questions should be submitted in online through the Gradescope platform. Most students choose to hand-write their solutions on paper and then scan and submit on their phone using the Gradescope app. Further Gradescope details to follow nearer the first deadline.

Tutorials

Tutorials are small groups of about a dozen students. You have been assigned to one of 34 tutorial groups, each with a member of staff as the tutor. Your tutorial group will meet five times, in Weeks 2, 4, 6, 8, and 10; you should check your timetable to see when and where your tutorial group meets.

The tutorials are an interactive session, where the main goal will be to go over your answers to the non-assessed questions on the problems sheets, which you will have worked on in advance of the tutorial. In this smaller group, you will be able to ask detailed questions of your tutor, and have the chance to discuss your answers to the problem sheet. Your tutor may ask you to present some of your work to your fellow students, or may give you the opportunity to work together with others during the tutorial. Your tutor may be willing to give you a hint on the assessed questions if you've made a first attempt but have got stuck. Because of the much smaller groups, the tutorials are the most valuable type of teaching on the module; you should make sure you attend, and you should be well prepared to ensure you make the most of the opportunity.

My recommended approach to problem sheets and tutorials is the following:

- Work through the problem sheet before the tutorial, spending plenty of time on it, and making multiple efforts at questions you get stuck on. I recommend spending *at least 4 hours per problem sheet*. This is a long time, but you shouldn't expect to be able to answer the hardest questions on a problem sheet without making multiple attempts. You don't have to wait until all lectures in a section are complete until starting to work on some of the questions. Collaboration is encouraged when working through the non-assessed problems, but I recommend writing up your work on your own; answers to assessed questions must be solely your own work.
- Take advantage of the small group setting of the tutorial to ask for help or clarification on questions you weren't able to complete.
- After the tutorial, attempt again the questions you were previously stuck on.
- If you're still unable to complete a question after this second round of attempts, *then* consult the solutions.

Your tutor will also be the marker of your answers to the assessed questions on the problem sheets.

Attendance at tutorials is compulsory.

R Worksheets and Practicals

R is a programming language that is particularly good at working with probability and statistics. Learning to use R is an important part of this module, and is used in many other modules in the University, including MATH1712 Probability and Statistics II. R is used by statisticians throughout academia and increasingly in industry too. Learning to program is a valuable skill for all students, and learning to use R is particularly valuable for students interested in statistics and related topics like actuarial science.

You will learn R by working through one R worksheet each week in your own time, starting from Week 2. Even-numbered worksheets will also contain a few questions for assessment, which will be due by 2pm Monday the following week (except the last one). Each of these is worth 3% of your mark for a total of 15%. You will submit your answers through a Microsoft Form (details to follow later). I recommend spending one hour per week on the week's R worksheet, plus one extra hour if there are assessed questions that week.

Week	Worksheet	Deadline for assessed work
2	1: R basics	—
3	2: Vectors	Monday 23 October (Week 4)
4	3: Data in R	—
5	4: Plots I – Making plots	Monday 6 November (Week 6)
6	5: Plots II – Making plots better	—
7	6: Discrete distributions	Monday 20 November (Week 8)
8	7: Discrete random variables	—
9	8: Normal distribution	Monday 4 December (Week 10)
10	9: Law of large numbers	—

Week	Worksheet	Deadline for assessed work
11	10: Recap	Thursday 14 December (Week 11)

R Practical sessions: You will be introduced you to R in your first Practical session, in Week 2. You will first see how to use R on University computers (these sessions will take place in computer "clusters"). There will then be an opportunity to install R on your own device – if you have a laptop on which you want to install R, bring it along to the practical session. A second practical, in Week 3, will allow you to get help on the R Worksheet 2, which is the first worksheet with assessed questions.

There are 11 R practical session groups – check your timetable for Weeks 2 and 3 to see when and where your group meets.

Attendance at the first R practical session (Week 2) is compulsory.

"Office hours" drop-in sessions

If you there is something in the module you wish to discuss one-on-one with the module leader, the place for the is the optional weekly "office hours", which will operate as drop-in sessions. These sessions are an optional opportunity for you to ask questions you have to me; these are particularly useful if there's something on the module that you are stuck on or confused about, but I'm happy to discuss any statistics-related issues or questions you have.

I currently plan two "office hours" drop-in sessions per week:

- Fridays 1100–1200 and 1300–1400 in the Mathematics Boardroom (map).

I may change arrangements as term continues – if attendance levels are low, I will move office hours to be actual office.

If neither time is possible, you may email me to arrange an alternative time to talk to me.

Attendance at "office hours" sessions is optional.

Time management

It is, of course, up to you how you choose to spend your time on this module. But my recommendations for your work would be something like this:

- **Lectures:** 2 hours per week, plus 1 hour per week reading through notes.
- **Problem sheets:** 4 hours per problem sheet, plus 1 extra hour for writing up and submitting answers to assessed questions.

- **R worksheets:** 1 hour per week, plus 1 extra hour if there are assessed questions.
- **Tutorials:** 1 hour every other week.
- **Revision:** 16 hours total at the end of the module.
- **Exam:** 2 hours.

That makes about 100 hours in total. (MATH1710 is a 10-credit module, so is supposed to represent 100 hours work. MATH2700 students are expected to be able to use their greater experience to get through the material in just 75 hours, so should scale these recommendations accordingly.)

Exam

There will be an exam in January, which makes up the remaining 70% of your mark. The exam will consist of 20 short and 2 long questions, and will be time-limited to 2 hours. We'll talk more about the exam format near the end of the module.

Who should I ask about...?

There are over 440 students registered for this module. If each student emails me once a week, and if each email takes me 10 minutes to read and respond, that will take more than 15 hours of my time every day! Generally, it's much better to come to speak to me at the "office hours" drop-in session or, if it will be very quick, before or after a lecture.

- *I don't understand something in the notes or on a problem sheet*: Come to office hours, or ask your tutor in your next tutorial.
- *I'm having difficulties with R:* In Weeks 2 or 3, you should ask at your R practical session; at other times, come to office hours.
- *I have an admin question about arrangements for the module:* Come to office hours or talk to me before/after lectures.
- *I have an admin question about arrangements for my tutorial:* Contact your tutor.
- *I have an admin question about general arrangements for my programme as a whole:* Contact the Student Information Service or speak to your personal tutor.
- *I have a question about the marking of my assessed work on the Problem Sheets:* First, check your feedback on Gradescope; if you still have questions, contact your tutor.
- *I have a question about the marking of my assessed work on the R Worksheets:* You can email me about this.
- *Due to truly exceptional and unforeseeable personal circumstances I require an extension on or exemption from assessed work:* You can apply by filling in the mitigating circumstances form at this link. Neither I nor your tutor can unilaterally offer an extension or exemption, so please don't ask. (Extensions of up to 4 days are available for Problem Sheets. Only exemptions are available for R Worksheets.)

Content of MATH1710

Prerequisites

The formal prerequisite for MATH1710 is "Grade B in A-level Mathematics or equivalent". I'll assume you have some basic school-level maths knowledge, but I won't assume you've studied probability or statistics in detail before (although I recognise that many of you will have). If you have studied probability and/or statistics at A-level (or post-16 equivalent) level, you'll recognise some of the material in this module; however you should find that we go deeper in many areas, and that we treat the material through with a greater deal of mathematical formality and rigour. "Rigour" here means precisely stating our assumptions, and carefully *proving* how other statements follow from those assumptions.

Syllabus

The module has three parts: a short first part on "exploratory data analysis", a long middle part on probability theory, and a short final part on a statistical framework called "Bayesian statistics". There's also the weekly R worksheets, which you could count as a fourth part running in parallel, but which will connect with the other parts too.

An outline plan of the topics covered is the following.

- **Exploratory data analysis** [2 lectures]: Summary statistics, data visualisation
- **Probability** [16 lectures]:
  - Probability with events: Probability spaces, probability axioms, examples and properties of probability, "classical probability" of equally likely events, independence, conditional probability, Bayes' theorem [6 lectures]
  - Probability with random variables: Discrete random variables, expectation and variance, binomial distribution, geometric distribution, Poisson distribution, multiple random variables, law of large numbers, continuous random variables, exponential distribution, normal distribution, central limit theorem [10 lectures]
- **Bayesian statistics** [2 lectures]: Bayesian framework, Beta prior, normal–normal model
- Summary and revision [2 lectures]

You'll notice that this module is heavier on the "Probability" than the "Statistics" of its title. MATH1712 Probability and Statistics II, on the other hand, which many students on this module will take next semester, is almost entirely "Statistics", but uses probabilistic techniques developed here.

Books

You can do well on this module by attending the lectures and tutorials, and working on the problem sheets and R worksheets, without needing to do any further reading beyond this. However, students can benefit from optional pre-reading in advance, extra background reading, or an alternative view on the material, especially in the parts of the module on probability. These books are also a good place to look if you want extra exercises and problems for revision.

For exploratory data analysis, you can stick to Wikipedia, but if you really want a book, I'd recommend:

- GM Clarke and D Cooke, *A Basic Course in Statistics*, 5th edition, Edward Arnold, 2004.

For the probability section, any book with a title like "Introduction to Probability" would do. Some of my favourites are:

- JK Blitzstein and J Hwang, *Introduction to Probability*, 2nd edition, CRC Press, 2019.
- G Grimmett and D Welsh, *Probability: An Introduction*, 2nd edition, Oxford University Press, 2014. (The library has online access.)
- SM Ross, *A First Course in Probability*, 10th edition, Pearson, 2020.
- RL Scheaffer and LJ Young, *Introduction to Probability and Its Applications*, 3rd edition, Cengage, 2010.
- D Stirzaker, *Elementary Probability*, 2nd edition, Cambridge University Press, 2003. (The library has online access.)

I also found lecture notes by Prof Oliver Johnson (University of Bristol) and Prof Richard Weber (University of Cambridge) to be useful.

On Bayesian statistics, we will only taste a brief introduction, but if you want a book, I recommend:

- JV Stone, *Bayes' Rule: A Tutorial Introduction to Bayesian Analysis*, Sebtel Press, 2013.

For R, there are many excellent resources online.

(For all these books I've listed the newest editions, but older editions are usually fine too.)

About these notes

These notes were written by Matthew Aldridge in 2021, and were edited and updated a lot in 2022 and a little bit in 2023. They are based in part on previous notes by Dr Robert G Aykroyd and Prof Wally Gilks. Dr Jason Susanna

Anquandah and Dr Aykroyd advised on the R worksheets. Dr Aykroyd's help and advice on many aspects of the module was particularly valuable.

These notes (in the web format) should be accessible by screenreaders. If you have accessibility difficulties with these notes, contact me.

Part I: Exploratory data analysis

Chapter 1

Summary statistics

1.1 What is EDA?

Statistics is the study of data. **Exploratory data analysis** (or **EDA**, for short) is the part of statistics concerned with taking a "first look" at some data. Later, toward the end of this module, we will see more detailed and complex ways of building models for data, and in MATH1712 Probability and Statistics II (for those who take it) you will see many other statistical techniques – in particular, ways of testing formal hypotheses for data. But here we're just interested in first impressions and brief summaries.

In this section, we will concentrate on two aspects of EDA:

- **Summary statistics:** That is, calculating numbers that briefly summarise the data. A summary statistic might tell us what "central" or "typical" values of the data are, how spread out the data is, or about the relationship between two different variables.
- **Data visualisation:** Drawing a picture based on the data is an another way to show the shape (centrality and spread) of data, or the relationship between different variables.

Even before calculating summary statistics or drawing a plot, however, there are other questions it is important to ask about the data:

- *What is the data?* What variables have been measured? How were they measured? How many datapoints are there? What is the possible range of responses?
- *How was the data collected?* Was data collected on the whole population or just a smaller sample? If a sample: How was that sample chosen? Is that sample representative of the population?
- *Are there any outliers?* "Outliers" are datapoints that seem to be very different from the other datapoints – for example, are much larger or much smaller than the others. Each outlier should be investigated to seek

the reason for it. Perhaps it is a genuine-but-unusual datapoint (which is useful for understanding the extremes of the data), or perhaps there is an extraordinary explanation (a measurement or recording error, for example) meaning the data is not relevant. Once the reason for an outlier is understood, it then *might* be appropriate to exclude it from analysis (for example, the incorrectly recorded measurement). It's usually bad practice to exclude an outlier merely for being an outlier before understanding what caused it.

- *Ethical questions:* Was the data collected ethically and, where necessary, with the informed consent of the subjects? Has it been stored properly? Are their privacy issues with the collection and storage of the data? What ethical issues should be considered before publishing (or not publishing) results of the analysis? Should the data be kept confidential, or should it be openly shared with other researchers for the betterment of science?

1.2 What is R?

R is a programming language that is particularly good at working with probability and statistics. A convenient way to use the language R is through the program **RStudio**. An important part of this module is learning to use R, by completing weekly worksheets – you can read more in the R section of these notes.

R can easily and quickly perform all the calculations and draw all the plots in this section of notes on exploratory data analysis. In this text, we'll show the relevant R code. Code will appear like this:

```
data <- c(4, 7, 6, 7, 4, 5, 5)
mean(data)
```

```
[1] 5.428571
```

Here, the code in the first shaded box is the R commands that are typed into RStudio, which you can type in next to the `>` arrow in the RStudio "console". The numerical answers that R returns are shown here in the second unshaded box. The `[1]` can be ignored (this is just R's way of saying that this is the first part of the answer – but the answer here only has one part anyway). Plots produced by R are displayed in these notes as pictures.

Most importantly for now, *you are not expected to understand the R code in this section yet.* The code is included so that, in the future, as you work through the R worksheets week by week, you can look back at the code in the section, and it will start to make sense. By the time you have finished R Worksheet 5 in Week 6, you should be able understand most of the R code in this section.

1.3 Statistics of centrality

Suppose we have collected some data on a certain variable. We will assume here that we have n datapoints, each of which is a single real number. We can write this data as a vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n).$$

A **statistic** is a calculation from the data $\mathbf{x}$, which is (usually) also a real number. In this section we will look at two types of "summary statistics", which are statistics that we feel will give us useful information about the data.

We'll look here at two types of summary statistic:

- **Statistics of centrality**, which tell us where the "middle" of the data is.
- **Statistics of spread**, which tell us how far the data typically spreads out from that middle.

Some measures of centrality are the following.

Definition 1.1. Consider some real-valued data $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

- The **mode** is the most common value of x_i. (If there are multiple joint-most common values, they are all modes.)
- Suppose the data is ordered as $x_1 \leq x_2 \leq \cdots \leq x_n$. Then the **median** is the central value in the ordered list. If n is odd, this is $x_{(n+1)/2}$; if n is even, we normally take halfway between the two central points, $\frac{1}{2}(x_{n/2} + x_{n/2+1})$.
- The **mean** $\bar{x}$ is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

In that last expression, we've made use of Sigma notation to write down the sum. (If Sigma notation is new to you, I recommend this PDF from MathCentre, or Section 2.4 of Clarke and Cooke, *A Basic Course in Statistics*.)

Example 1.1. Some packets of Skittles (a small fruit-flavoured sweet) were opened, and the number of Skittles in each packet counted. There were 13 packets, and the number of sweets (sorted from smallest to largest) were:

$$59,\ 59,\ 59,\ 59,\ 60,\ 60,\ 60,\ 61,\ 62,\ 62,\ 62,\ 63,\ 63.$$

The mode is 59, because there were 4 packets containing 59 sweets; more than any other number.

Since there are $n = 13$ packets, the middle packet is number $i = 7$, so the median is $x_7 = 60$.

The mean is

$$\bar{x} = \frac{1}{13}(59 + 59 + \cdots + 63) = \frac{789}{13} = 60.7.$$

The median is one example of a "quantile" of the data. Suppose our data is increasing order again. For $0 \leq \alpha \leq 1$, the **α-quantile** $q(\alpha)$ of the data is the datapoint α of the way along the list. Generally, $q(\alpha)$ is equal to $x_{1+\alpha(n-1)}$ when $1 + \alpha(n-1)$ is an integer. (If $1 + \alpha(n-1)$ isn't an integer, there are various conventions of how to choose that we won't go into here. R has *nine* different settings for choosing quantiles! – we will always just use R's default choice.)

- The **median** is the $\frac{1}{2}$-quantile $q(\frac{1}{2})$, which is $q(\frac{1}{2}) = x_7 = 60$ for this data.
- The **minimum** is the 0-quantile $q(0)$, which is $q(0) = x_1 = 59$ for this data.
- The **maximum** is the 1-quantile $q(1)$, which is $q(1) = x_{13} = 63$ for this data
- The **lower quartile** (that's "quartile", as in "quarter" – not "quantile") is the $\frac{1}{4}$-quantile $q(\frac{1}{4})$, which is $q(\frac{1}{4}) = x_4 = 59$ for this data.
- The **upper quartile** is the $\frac{3}{4}$-quantile $q(\frac{3}{4})$, which is $q(\frac{3}{4}) = x_{10} = 62$ for this data.

The following R code reads in some data which has the daily average temperature in Leeds in 2020, divided into months. We can find, for example, the mean October temperature or the lower quartile of the July temperature.

```
temperature <- read.csv("https://mpaldridge.github.io/math1710/data/temperature.csv")
jul <- temperature[temperature$month == "jul", ]
oct <- temperature[temperature$month == "oct", ]

mean(oct$temp)
```

```
[1] 11.93548
```

```
quantile(jul$temp, probs = 1 / 4)
```

```
25%
 15
```

1.4 Statistics of spread

Some measures of spread are:

Definition 1.2. The **number of distinct observations** is precisely that: the number of different datapoints we have after removing any repeats.

The **interquartile range** is the difference between the upper and lower quartiles $\text{IQR} = q(\frac{3}{4}) - q(\frac{1}{4})$.

The **sample variance** is

$$s_x^2 = \frac{1}{n-1}\left((x_1 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2\right) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2,$$

where $\bar{x}$ is the sample mean from before. The **standard deviation** $s_x = \sqrt{s_x^2}$ is the square-root of the sample variance.

Example 1.2. We continue with the Skittles data.

The number of distinct observation is 5. (These are 59, 60, 61, 62, and 63.)

The interquartile range is $x_{10} - x_4 = 62 - 59 = 3$.

You will calculate the sample variance on Problem Sheet 1.

The formula we've given for sample variance is sometimes called the "definitional formula", as it's the formula used to *define* the sample variance. We can rearrange that formula as follows:

$$\begin{aligned}
s_x^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}2x_i\bar{x} + \sum_{i=1}^{n}\bar{x}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - 2\bar{x}\sum_{i=1}^{n}x_i + \bar{x}^2\sum_{i=1}^{n}1\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - 2n\bar{x}^2 + n\bar{x}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right).
\end{aligned}$$

Here, the first line is the definitional formula; the second line is from expanding out the bracket; the third line is taking the sum term-by-term; the fourth line takes any constants (things not involving i) outside the sums; the fifth line uses $\sum_{i=1}^{n} x_i = n\bar{x}$, from the definition of the mean, and $\sum_{i=1}^{n} 1 = 1 + 1 + \cdots 1 = n$; and the sixth line simplifies the final two terms.

This has left us with

$$s_x^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right).$$

This is sometimes called the "computational formula"; this is because it usually takes fewer presses of calculator buttons to compute the sample variance with this formula rather than the definitional formula. (But make sure you keep enough decimal points in $\bar{x}^2$.)

Going back to our weather data in R, we can find the sample variance of the October weather or the interquartile range of the July weather.

```
var(oct$temp)
```

```
[1] 2.862366
```

```
IQR(jul$temp)
```

```
[1] 3
```

Summary

- Exploratory data analysis is about taking a first look at data.
- Summary statistics are numbers calculated from data that give us useful information about the data.
- Summary statistics that measure the centre of the data include the mode, median, and mean.
- Summary statistics that measure the spread of the data include the number of distinct outcomes, the interquartile range, and the sample variance.

Recommended reading:

- Wikipedia: Exploratory data analysis, Mode (statistics), Median, Arithmetic mean, Quantile, Interquartile range.
- Clarke and Cooke, *A Basic Course in Statistics*, Sections 2.1–2.4, 2.7, 4.1–4.4, 4.6, 4.7.

On Problem Sheet 1, you should now be able to complete Questions A1, B1, B2, B4, C1.

Chapter 2

Data visualisations

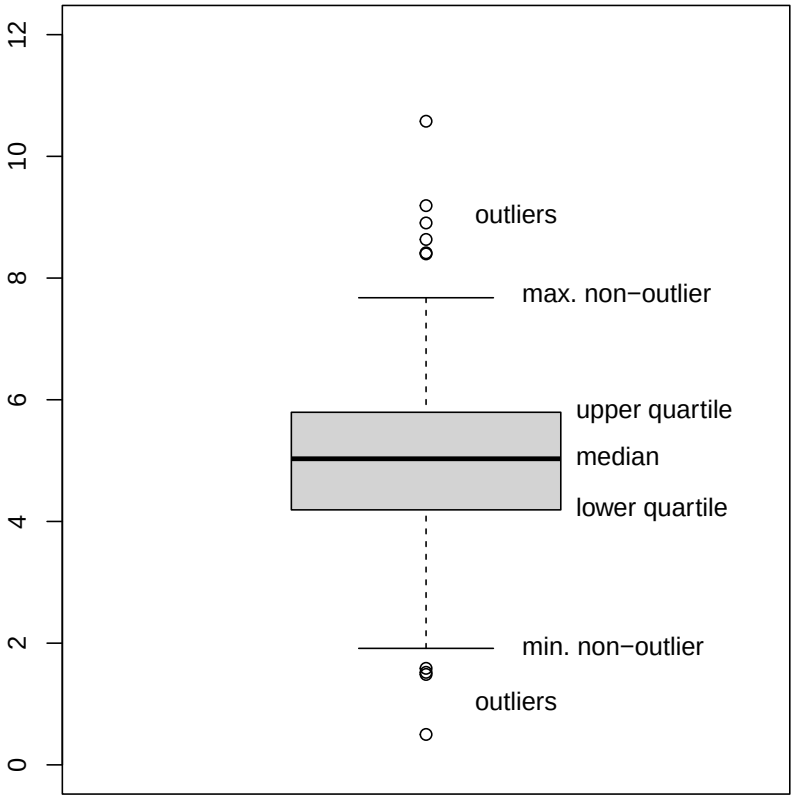
Data visualisations – drawings or graphs based on data – can help us to understand the "shape" of a dataset as part of exploratory data anlaysis. In this lecture, we'll look at three types of data visualisation.

2.1 Boxplots

A **boxplot** is a useful way to illustrate numerical data. It can be easier to tell the difference between different data sets "by eye" when looking at a boxplot, rather than examining raw summary statistics.

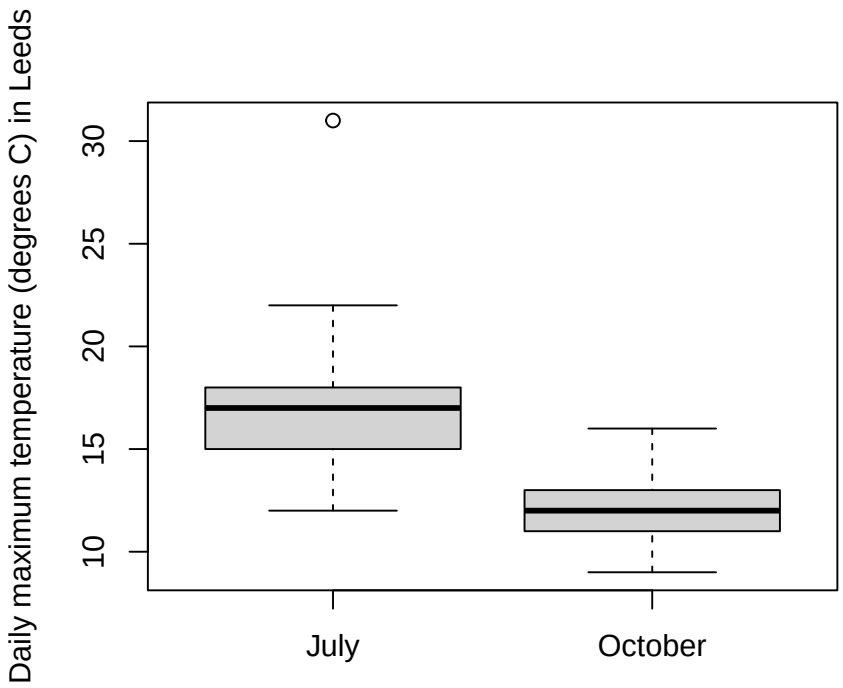
A boxplot is drawn as follows:

- The vertical axis represents the data values.
- Draw a box from the lower quartile $q(\frac{1}{4})$ to the median $q(\frac{1}{2})$.
- Draw another box on top of this from the median $q(\frac{1}{2})$ to the upper quartile $q(\frac{3}{4})$. Note that size of these two boxes put together is the interquartile range.
- Decide which datapoints are outliers, and plot these with circles. (The R default is that any data point less than $q(\frac{1}{4}) - 1.5 \times \text{IQR}$ or greater than $q(\frac{3}{4}) + 1.5 \times \text{IQR}$ is an outlier.)
- Out from the two previous boxes, draw "whiskers" to the minimum and maximum non-outlier datapoints.



When we have multiple datasets, drawing boxplots next to each other can help us to compare the datasets. Here are two boxplots from the July and October temperature data we used in the last lecture. What do you conclude about the data from these boxplots?

```
boxplot(jul$temp, oct$temp,
        names = c("July", "October"),
        ylab = "Daily maximum temperature (degrees C) in Leeds"
)
```



(And yes, I did check the outlier to make sure it was a genuine datapoint.)

2.2 Histograms

Often when collecting data, we don't collect exact data, but rather collect data clumped into "bins". For example, suppose a student wished to use a questionnaire to collect data on how long it takes people to reach campus from home; they might not ask "Exactly how long does it take?", but rather give a choice of tick boxes: "0–5 minutes", "5–10 minutes", and so on.

Consider the following binned data, from $n = 100$ students:

Time	Frequency	Relative frequency
0–5 minutes	4	0.04
5–10 minutes	8	0.08
10–15 minutes	21	0.21
15–30 minutes	42	0.42
30–45 minutes	15	0.15
45–60 minutes	8	0.08
60–120 minutes	2	0.02

Time	Frequency	Relative frequency
Total	100	1

Here the **frequency** f_j of bin j is simply the number of observations in that bin; so, for example, 42 students had journey lengths of between 15 and 30 minutes. The **relative frequency** of bin j is f_j/n; that is, the proportion of the observations in that bin.

Which bin would you say is the most popular – that is, the "modal" bin? The bin with the most observations in it is the "15–30 minute" bin. But this bin covers 15 minutes, while some of the other bins only cover 5 minutes. It would be a fairer comparison to look at the **frequency density**: the relative frequency divided by the size of the bin.

Time	Frequency	Relative frequency	Frequency density
0–5 minutes	4	0.04	0.008
5–10 minutes	8	0.08	0.016
10–15 minutes	21	0.21	0.042
15–30 minutes	42	0.42	0.028
30–45 minutes	15	0.15	0.010
45–60 minutes	8	0.08	0.005
60–120 minutes	2	0.02	0.0003
Total	100	1	

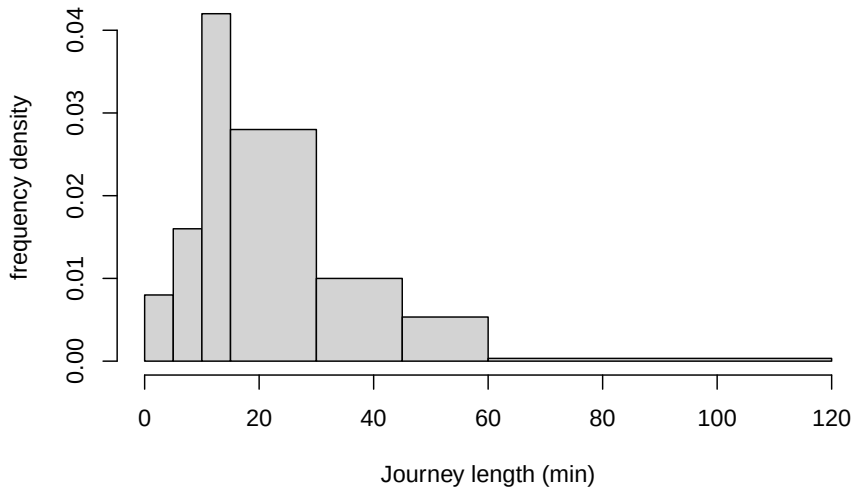
In the first row, for example, the relative frequency is 0.04 and the size of the bin is 5 minutes, so the frequency density is $0.04/5 = 0.008$. We now see that the modal bin – the bin with the highest frequency *density* – is in fact the "10–15 minutes" bin. This bin has somewhat fewer datapoints that the "15–30 minutes" bin, but they're squashed into a much smaller bin.

Data in bins can be illustrated with a **histogram**. A histogram has the measurement on the x-axis, with one bar across the width of each bin, where bars are drawn up to the height of the corresponding frequency density. Note that this means that the area of the bar is exactly the relative frequency of the corresponding bin.

If all the bins are the same width, frequency density is directly proportional to frequency and to relative frequency, so it can be clearer use one of those as the y-axis instead in the equal-width-bins case.

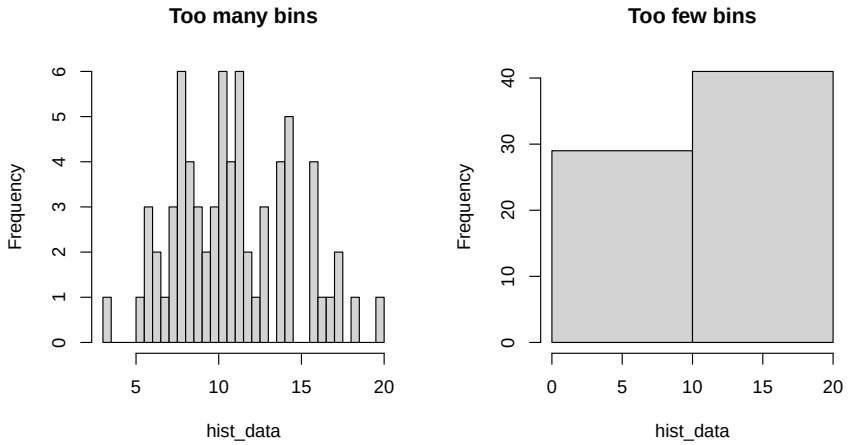
Here is a histogram for our journey-time data:

```
journeys <- read.csv("https://mpaldridge.github.io/math1710/data/journeys.csv")
bins <- c(0, 5, 10, 15, 30, 45, 60, 120)

hist(journeys$midpoint, breaks = bins,
     xlab = "Journey length (min)", ylab = "frequency density", main = ""
)
```



Often we draw histograms because the data was collected in bins in the first place. But even when we have exact data, we might *choose* to divide it into bins for the purposes of drawing a histogram. In this case we have to decide where to put the "breaks" between the bins. Too many breaks too close together, and the small number of observations in each bin will give "noisy" results (see left); too few breaks too far apart, and the wide bins will mean we lose detail (see right).

```
set.seed(2172)
hist_data <- c(rnorm(30, 8, 2), rnorm(40, 12, 3))  # Some fake data

hist(hist_data, breaks = 40, main = "Too many bins")
hist(hist_data, breaks = 2,  main = "Too few bins")
```



We can also calculate some summary statistics even when we have binned data. We mentioned the mode earlier, where the modal bin is the bin of highest frequency density.

What is the median journey length? Well, we don't know exactly, but $0.04 + 0.08 + 0.21$ (the first three bins) is less than 0.5, while $0.04 + 0.08 + 0.21 + 0.42$ (including the fourth bin) is greater than 0.5. So we know that the median student is in the fourth bin, the "15–30 minute" bin, and we can say that the median journey length is between 15 and 30 minutes.

Since we don't have the exact data, it's not possible to exactly calculate the mean and variance. However, we can often get a good estimate by assuming that each observation was in fact right in the centre of its bin. So, for example, we could assume that all 4 observations in the "0–5 minutes" bin were journeys of exactly 2.5 minutes. Of course, this isn't true (or is highly unlikely to be true), but we can often get a good approximation this way.

For our journey-time data, our approximation of the mean would be

$$\bar{x} = \frac{1}{100}(4 \times 2.5 + 8 \times 7.5 + \cdots + 2 \times 90) = 24.4.$$

More generally, if m_j is the midpoint of bin j and f_j its frequency, then we can calculate the binned mean and binned variance by

$$\bar{x} = \frac{1}{n}\sum_j f_j m_j$$
$$s_x^2 = \frac{1}{n-1}\sum_j f_j (m_j - \bar{x})^2$$

2.3 Scatterplots

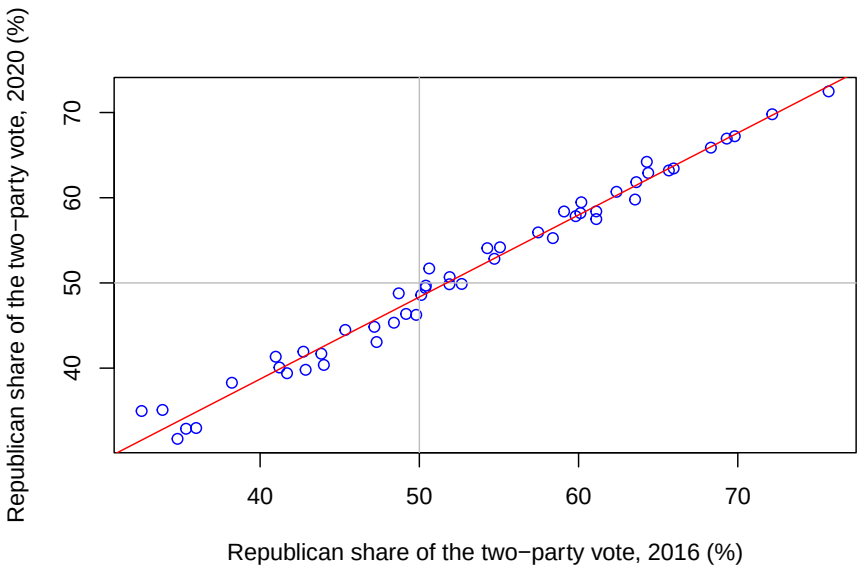
Often, more than one piece of data is collected from each subject, and we wish to compare that data, to see if there is a relationship between the variables.

For example, we could take n second-year maths students, and for each student i, collect their mark x_i in MATH1710 and their mark y_i in MATH1712. This gives is two "paired" datasets, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. We can calculate sample statistics of draw plots for $\mathbf{x}$ and for $\mathbf{y}$ individually. But we might also want to see if there is a relationship *between* $\mathbf{x}$ and $\mathbf{y}$: Do students with high marks in MATH1710 also get high marks in MATH1712?

A good way to visualise the relationship between two variables is to use a **scatterplot**. In a scatterplot, the ith data pair (x_i, y_i) is illustrated with a mark (such as a circle or cross) whose x-coordinate has the value x_i and whose y-coordinate has the value y_i.

In the following scatterplot, we have $n = 50$ datapoints for the 50 US states; for each state i, x_i is the Republican share of the vote in that state in the 2016 Trump–Clinton presidential election, and y_i is the Republican share of the vote in that state in the 2020 Trump–Biden election.

```
elections <- read.csv("https://mpaldridge.github.io/math1710/data/elections.csv")

plot(elections$X2016, elections$X2020,
     col = "blue",
     xlab = "Republican share of the two-party vote, 2016 (%)",
     ylab = "Republican share of the two-party vote, 2020 (%)")

abline(h = 50, col = "grey")
abline(v = 50, col = "grey")
abline(0.195, 0.963, col = "red")
```



We see that there is a strong relationship between $\mathbf{x}$ and $\mathbf{y}$, with high values of x corresponding to high values of y and vice versa. Further, the points on the scatterplot lie very close to a straight line.

A useful summary statistic here is the **correlation**

$$r_{xy} = \frac{s_{xy}}{s_x s_y},$$

where s_{xy} is the **sample covariance**

$$\begin{aligned} s_{xy} &= \frac{1}{n-1}((x_1 - \bar{x})(y_1 - \bar{y}) + \cdots + (x_n - \bar{x})(y_n - \bar{y})) \\ &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}), \end{aligned}$$

and $s_x = \sqrt{s_x^2}$ and $s_y = \sqrt{s_y^2}$ are the standard deviations.

The correlation r_{xy} is always between -1 and $+1$. Values of r_{xy} near $+1$ indicate that the scatterpoints are close to a straight line with an upward slope (big x

= big y); values of r_{xy} near -1 indicate that the scatterpoints are close to a straight line with a downward slope (big x = small y); and values of r_{xy} near 0 indicate that there is a weak linear relationship between x and y.

For the elections data, the correlation is

```
cor(elections$X2016, elections$X2020)
```

```
[1] 0.9919659
```

which, as we expected, is extremely high.

Summary

- Boxplots show the shape of numerical data, and can compare different datasets.
- Histograms show the shape of binned data.
- Scatterplots show the relationship between two datasets.

Recommended reading:

- Wikipedia: Box plot, Histogram, Grouped data, Scatter plot, Pearson correlation coefficient.
- Clarke and Cooke, *A Basic Course in Statistics*, Sections 1.2, 2.5, 4.5, 4.6, 21.2, 21.3.

On Problem Sheet 1, you should now be able to complete all questions.

Problem Sheet 1

Solutions are now available to all non-assessed questions.

You can download this problem sheet as a PDF file

This is Problem Sheet 1, which covers material from Lectures 1 and 2 of the notes. You should work through all the questions on this problem sheet in advance of your tutorial in Week 2. Questions C1 and C2 are assessed questions, and are due in by **2pm on Monday 16 October**. I recommend spending about 4 hours on this problem sheet, plus 1 extra hour to neatly write up and submit your answers to the assessed questions.

A: Short questions

The first three questions are **short questions**, which are intended to be mostly not too difficult. Short questions usually follow directly from the material in the lectures. Here, you should clearly state your final answer, and give enough working-out (or a short written explanation) for it to be clear how you reached that answer. You can check your answers with the solutions-without-working at the bottom of this sheet; solutions-with-working will be available after Friday 13 October. If you get stuck on any of these questions, you might want to ask for guidance in your tutorial.

A1. Consider again the "number of Skittles in each packet" data from Example 1.1.

$$59,\ 59,\ 59,\ 59,\ 60,\ 60,\ 60,\ 61,\ 62,\ 62,\ 62,\ 63,\ 63.$$

(a) Calculate the mean number of Skittles in each packet.

(b) Calculate the sample variance using the definitional formula.

(c) Calculate the sample variance using the computational formula.

(d) Out of (b) and (c), which calculation did you find easier, and why?

A2. Consider the following data sets of the age of elected politicians on a local council. (The "18–30" bin, for example, means from one's 18th birthday to the moment before one's 30th birthday, so lasts 12 years.)

Age (years)	Frequency	Relative frequency	Frequency density
18–30	1		
30–40	2		
40–45	4		
45–50	5		
50–60	6		
60–80	2		
Total	20	1	—

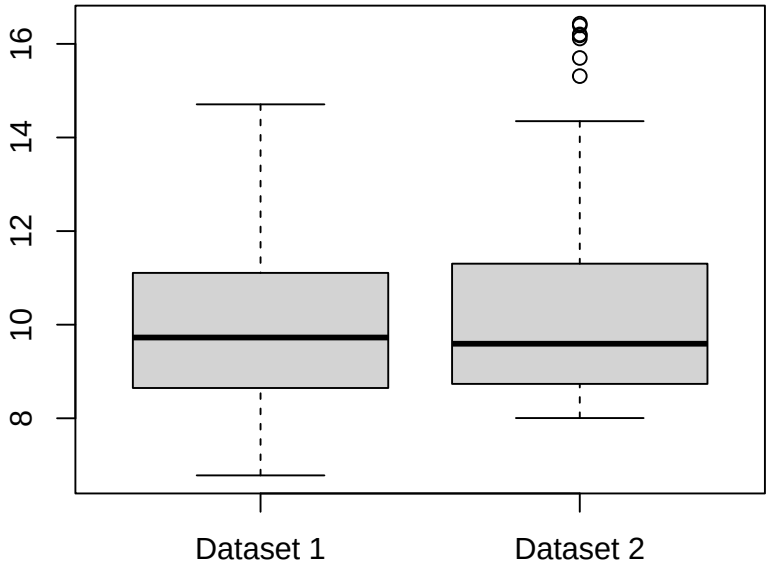
(a) Complete the table by filling in the relative frequency and frequency densities.

(b) What is the median age bin?

(c) What is the modal age bin?

(d) Calculate (the standard approximation of) the mean age of the politicians.

A3. Consider the two datasets illustrated by the boxplots below. Write down some differences between the two datasets.



B: Long questions

The next four questions are **long questions**, which are intended to be harder. Long questions often require you to think originally for yourself, not just directly follow procedures from the notes. You may not be able to solve all of these questions, although you should make multiple attempts to do so. Here, your answers should be written in complete sentences, and you should carefully explain in words each step of your working. Your answers to these questions – not only their mathematical content, but also how to write good, clear solutions – are likely to be the main topic for discussion in your tutorial. Solutions will be available after Friday 13 October.

B1. For each of the two datasets below, calculate the following summary statistics, or explain why it is not possible to do so: mode; median; mean; number of distinct outcomes; inter-quartile range; and sample variance.

(a) Shirt sizes for the $n = 16$ members of a university football squad:

Colour	Xtra Small	Small	Medium	Large	Xtra Large
Number of shirts	0	1	6	4	5

(b) Six packets of Skittles are opened together, a total of $n = 361$ sweets. The colours of these sweets is recorded as follows:

Colour	Red	Orange	Yellow	Green	Purple
Number of Skittles	67	71	87	74	62

B2. A summary statistic is informally said to be "robust" if it typically doesn't change much if a small number of outliers are introduced to a large dataset, or "sensitive" if it often changes a lot when a small number of outliers are introduced. Briefly discuss the robustness or sensitivity of the following summary statistics: **(a)** mode; **(b)** median; **(c)** mean; **(d)** number of distinct outcomes; **(e)** inter-quartile range; and **(f)** sample variance.

B3. Let $\mathbf{a} = (a_1, a_2, \dots a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be two real-valued vectors of the same length. Then the *Cauchy–Schwarz inequality* says that

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

(a) By making a clever choice of (a_i) and (b_i) in the Cauchy–Schwarz inequality, show that $s_{xy}^2 \leq s_x^2 s_y^2$.

(b) Hence, show that the correlation r_{xy} satisfies $-1 \leq r_{xy} \leq 1$.

B4. A researcher wishes to study the effect of mental health on academic achievement. The researcher will collect data on the mental health of a cohort

of students by asking them to fill in a questionnaire, and will measure academic achievement via the students' scores on their university exams. Discuss some of the ethical issues associated with the collection, storage, and analysis of this data, and with the publication of the results of the analysis. Are there ways to mitigate these issues?

(It's not necessary to write an essay for this question – a few short bulletpoints will suffice. There may be an opportunity to discuss these issues in more detail in your tutorial.)

C: Assessed questions

The last two questions are **assessed questions**. This means you will submit your answers, and your answers will be marked by your tutor. These two questions count for 3% of your final mark for this module. If you get stuck, your tutor may be willing to give you a small hint in your tutorial.

The deadline for submitting your solutions is **2pm on Monday 16 October** at the beginning of Week 3. Submission will be via Gradescope, which you can access via Minerva or on the Gradescope mobile app. You should submit your answers as a single PDF file. Most students choose to hand-write their work on paper, then scan-and-submit it to using the Gradescope mobile app. Your work will be marked by your tutor and returned on Monday 23 October, when solutions will also be made available.

Question C1 is a "short question", where brief explanations or working are sufficient; Question C2 is a "long question", where the marks are not only for mathematical accuracy but also for the clarity and completeness of your explanatory writing.

You should not collaborate with others on the assessed questions: your answers must represent solely your own work. The University's rules on academic integrity – and the related punishments for violating them – apply to your work on the assessed questions.

C1. The monthly average exchange rate for US dollars into British pounds over a 12-month period was:

$$1.306,\ 1.301,\ 1.290,\ 1.266,\ 1.268,\ 1.302,$$
$$1.317,\ 1.304,\ 1.284,\ 1.268,\ 1.247,\ 1.215.$$

(a) Calculate the median for this data.

(b) Calculate the mean for this data.

(c) Calculate the sample variance for this data.

(d) Is the mode an appropriate summary statistic for this sort of data? Why/why not?

C2. **(a)** Suppose that a dataset $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ (with $n \geq 2$) has sample variance $s_x^2 = 0$. Show that all the datapoints are in fact equal.

(b) Prove the following computational formula for the sample covariance:

$$s_{xy} = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}\right).$$

Solutions to short questions

A1. (a) 60.7, (b) 2.40, (c) 2.40, (d) —.
A2. (a) —, (b) 45–50, (c) 45–50, (d) 48.6 *(corrected)*.
A3. —

Part II: Probability

Chapter 3

Sample spaces and events

3.1 What is probability?

We now begin the big central block of this module, on probability theory.

Probability theory is the study of randomness. Probability, as an area of mathematics, is a fascinating subject in its own right. However, probability is particularly important due to its usefulness in applications – especially in statistics (the study of data), in finance, and in actuarial science (the study of insurance).

Probability is well suited to modelling situations that involve randomness, uncertainty, or unpredictability. If you want to predict the time of the next solar eclipse, a deterministic (that is, non-random) model based on physical laws will tell you when the sun, the moon, and the earth will be in the correct positions; but if you want to predict the weather tomorrow, or the price of a share of Apple stock next month, or the results of an election next year, you will need a probabilistic model that takes into account the uncertainty in the outcome. A probabilistic model could tell you the most likely outcome, or a range of the most probable outcomes.

So what do we mean when we talk about the "probability" of an event occurring? You might say that the probability of an event is a measure of "how likely" it is to occur, or what the "chance" of it occurring is.

More concretely, here are some interpretations of probability:

- **Subjective** (or **Bayesian**) **probability:** The probability of an event is the way someone expresses their degree of belief that the event will occur, based on their own judgement, and given the evidence they have seen. Their belief is measured on a scale from 0 to 1, from probabilities near 0 meaning they believe the event is very unlikely to occur to probabilities near 1 meaning they believe the event is very likely to occur.
  - This interpretation is philosophically sound, but a bit vague to be the basis for a mathematics module.

- **Classical** (or **enumerative**) **probability:** Suppose there are a finite number of equally likely outcomes. Then the probability of an event is the proportion of those outcomes that correspond to the event occurring. So when we say that a randomly dealt card has a probability $\frac{1}{13}$ of being an ace, this is because there are 52 cards of which 4 are aces, so the proportion of favourable outcomes is $\frac{4}{52} = \frac{1}{13}$.
  - This interpretation is good for simple procedures like flipping a fair coin, rolling a dice, or dealing cards, where the "finite number of equally likely outcomes" assumption holds. But we want to be able to study more complicated situations, where some outcomes are more likely than others, or where infinitely many different outcomes are possible.
- **Frequentist probability:** In a repeated experiment, the probability of an event is its long-run frequency. That is, if we repeat an experiment a very large number of times, the probability of the event is (approximately) the proportion of the experiments in which the event occurs. So when we say a biased coin has probability 0.9 of landing heads, we mean that were we toss it 1000 times, we would expect to see very close to $0.9 \times 1000 = 900$ heads.
  - There are two problems with this. First, this doesn't deal with events that can't be repeated over and over again (like "What's the probability that Labour win the 2024 general election?"). Second, to answer the question, "Yes, but *how* close to the probability should the proportion of occurrences be?", you end up having to answer, "Well, it depends on the probability," and you've got a circular definition.
- **Mathematical probability:** We have a function that assigns to each event a number between 0 and 1, called its probability, and that function has to obey certain mathematical rules, called "axioms".

It will not surprise you to learn that, in this mathematics course, we will take the "mathematical probability" approach. However, we will also learn useful things about the other approaches: we will see that classical probability is one special case of mathematical probability; we will see a result called the "law of large numbers" that says that the long-run frequency does indeed get closer and closer to the mathematical probability; and a result called "Bayes' theorem" will advise a subjectivist on how to update her subjective beliefs when she sees new evidence.

3.2 Sample spaces and events

Taking the "mathematical probability" approach, we will want to give a formal mathematical definition of the *probability* of an event. But even before that, we need to give a formal mathematical definition of an *event* itself. Our setup will be this:

- There is a set called the **sample space**, normally given the letter Ω (upper-case Omega), which is the set of all possible outcomes.

- An element of the sample space Ω is a **sample outcome**, sometimes given the letter ω (lower-case omega), represents one of the possible outcomes.
- An **event** is a set of sample outcomes; that is, a subset of the sample space Ω. Events are often given letters like A, B, C. We write $A \subset \Omega$ to mean that A is an event in (or, equivalently, is a subset of) the sample space Ω.

This will be easier to understand with some concrete examples. We write a set (such as a sample space or an event) by writing all the elements of that set inside curly brackets { }, separated by commas.

Example 3.1. Suppose we toss a (possibly biased) coin, and record whether it lands heads or tails. Then our sample space is $\Omega = \{\mathrm{H}, \mathrm{T}\}$, where the sample outcome H denotes heads and the sample outcome T denotes tails.

The event that the coin lands heads is $\{\mathrm{H}\}$.

Example 3.2. Suppose we roll a dice, and record the number rolled. Then our sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$, where the sample outcome 1 corresponds to rolling a one, and so on.

The event "we roll an even number" is $\{2, 4, 6\}$. The event "we roll at least a five" is $\{5, 6\}$.

Example 3.3. Suppose we wish to count how many claims are made to an insurance company in a year. We could model this by taking the sample space Ω to be $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, the set of all non-negative integers.

The event "the company receives less than 1000 claims" is $\{0, 1, 2, \dots, 998, 999\}$.

Example 3.4. Suppose we want a computer to pick a random number between 0 and 1. We could model this by taking the sample space Ω to be the interval $[0, 1]$ of all real numbers between 0 and 1.

The event "the number is bigger than $\frac{1}{2}$" is the sub-interval $(\frac{1}{2}, 1]$ of all real numbers greater than $\frac{1}{2}$ but no bigger than 1. The event "the first digit is a 7" is the sub-interval $[0.7, 0.8)$. The event "the random number is exactly $1/\sqrt{2}$" is $\{1/\sqrt{2}\}$.

In the first two examples, the sample space Ω was finite. In third example, the sample space was infinite but "countably infinite", in that it could be counted using the discrete values of the positive integers. Both of these were for *counting* discrete observations. In the fourth example, the sample space was infinite but "uncountably infinite", in that it had a sliding scale or "continuum" of gradually varying measurements. This was for *measuring* continuous observations. This distinction will be important later in the course.

For any sample space Ω, there are two special events that always exist. There's Ω itself, the event containing all of the sample outcomes, which represents "something happens". There's also the empty set $\emptyset$, which contains none of the sample outcomes, which represents "nothing happens". Common sense suggests that Ω should have probability 1, because *something* is bound to happen – this will later be one of our probability "axioms". Common sense also suggests that $\emptyset$

should have probability 0, because it can't be that *nothing* happens – this will not be one probability axioms, but we'll show that it follows logically from the axioms we do choose.

3.3 Set theory

Since we've now defined events as being sets – specifically, subsets of the sample space Ω – it will be useful to mention a little set basic theory here.

First, there are ways we can build new sets (or events) out of old. It's fine to just read the words and look at the pictures for these definitions, but those who want to read the equations too will need to know this:

- $\omega \in A$ means "ω is in A" or "ω is an element of A", while $\omega \notin A$ means the opposite, that ω is *not* in A;
- a colon $:$ in the middle of set notation should be read as "such that";
- so $\{\omega \in \Omega : \text{fact about } \omega\}$ should be read as "the set of sample outcomes ω in the sample space Ω such that the fact is true".

Definition 3.1. Consider a sample space Ω, and let A and B be events in that sample space.

- **NOT:** The **complement** of A, written A^{c} (and said "A complement" or "not A"), is the set of sample outcomes not in A; that is

  $$A^{\mathsf{c}} = \{\omega \in \Omega : \omega \notin A\}.$$

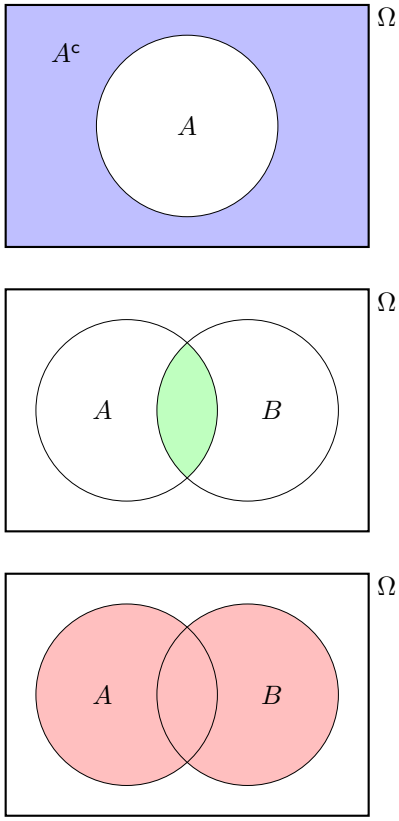
  This represents the event that A does not occur.
- **AND:** The **intersection** of A and B, written $A \cap B$ (and said "A intersect B" or "A and B") is the set of sample outcomess in both A and B; that is,

  $$A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}.$$

  This represents the event that both A and B occur.
- **OR:** The **union** of A and B, written $A \cup B$ (and said "A union B" or "A or B") is the set of sample outcomess in A or in B; that is,

  $$A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}.$$

  This represents the event that A occurs or B occurs. (In mathematics, "or" includes "both", so a sample outcome in both A and B is in $A \cup B$ too.)



Example 3.5. Suppose we are rolling a dice, so our sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{2, 4, 6\}$ be the event that we roll and even number, and let $B = \{5, 6\}$ be the event that we roll at least a 5. Then

$$\begin{aligned} A^c &= \{1, 3, 5\} = \{\text{roll an odd number}\}, \\ A \cap B &= \{6\} = \{\text{roll a 6}\}, \\ A \cup B &= \{2, 4, 5, 6\}. \end{aligned}$$

An important case is when two events A, B cannot happen at the same time; that is, $A \cap B = \emptyset$ ("A intersect B is the empty set"). In this case, we say that A and B are **disjoint** or **mutually exclusive**. For example, when Ω is a deck of cards, then $A = \{\text{the card is a spade}\}$ and $B = \{\text{the card is red}\}$ are disjoint, because a card cannot be both a spade (a black suit) and red.

You might think that if two events are disjoint, then it would be reasonable to find the probability of their union – that is, the probability that one (and, by necessity, only one) of them happens – you can just add the two separate probabilities together. This will be another of our "axioms" of probability.

There are a few rules about ways you can combine the complement, intersection and union operations. These are ways of building new events from old.

- The **double complement law** tells us that not-not-A is the same as A:

$$(A^c)^c = A.$$

This says that if it's not "not-raining", then it's raining!

- The **distributive laws** tells us we can "multiply out of the brackets" with sets:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

  The first says that if you are eating a burger with fries or salad, then you're eating a burger with fries or eating a burger with salad. The second is a bit less intuitive, I find, but it's clear that if A is true then the first of each of the terms on the right is true, while if both B and C are true then the second of each of the terms on the right is true.
- **De Morgan's laws** tell us how complements interact with intersection/unions:

$$(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$$
$$(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$$

  The first of these says that if it's not a Monday in October, then either it's not Monday or it's not October (or both). The second says that if a maths lecture is not "useful or fun", then it's not useful and it's not fun. (Augustus De Morgan was a British mathematician of the 19th century who did important work in logic.)

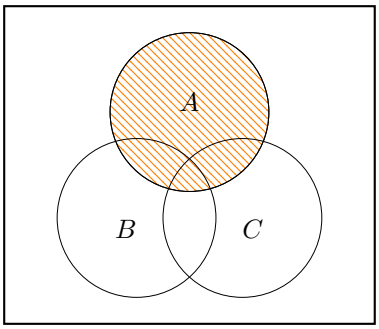
For this module, these mostly count as "common sense" – but if you ever do need to prove one of these statements (or a similar one), one way is to use a Venn diagram.

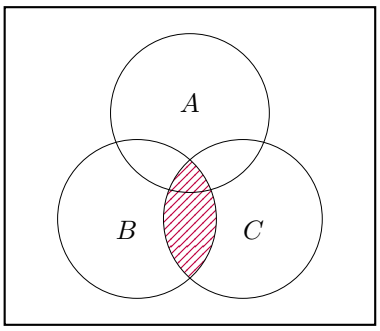
Let's prove the second distributive law,

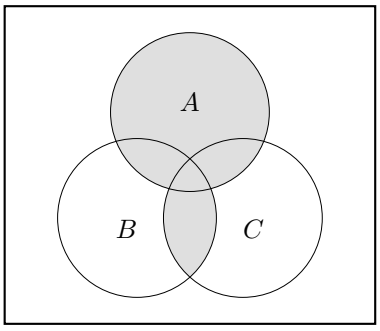
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

with a Venn diagram as an example.

We can build the left-hand side of the law as:

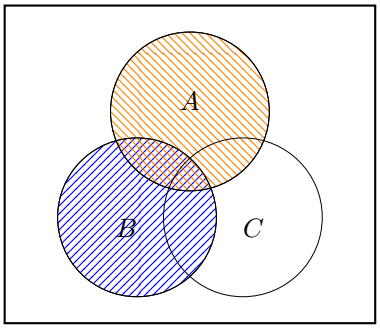


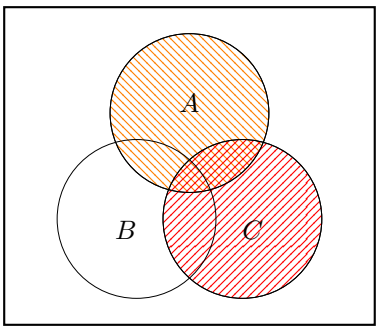


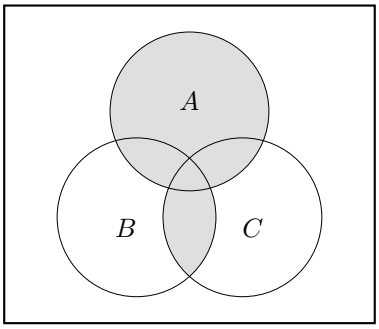


The left-hand figure is A, the middle figure is $B \cap C$, and the right-hand figure is union of these, $A \cup (B \cap C)$.

Then for the right-hand side of the law, we have:







The left-hand figure is $A \cup B$, the middle figure is $A \cup C$, and the right-hand figure is intersection of these, $(A \cup B) \cap (A \cup C)$.

We see that the areas shaded in two right-hand figures are the same, so it is indeed the case that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Summary

- A sample space Ω is a set representing all possible sample outcomes.
- An event is a subset of Ω.
- For events A and B, we also have the complement "not A" A^c, the intersection "A and B" $A \cap B$, and the union "A or B" $A \cup B$.

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 1.1 and 1.2 (plus optionally Chapter 0).
- Grimmett and Welsh, *Probability*, Sections 1.1 and 1.2.

Chapter 4

Probability

4.1 Probability axioms

Recall that, in this mathematics course, the probability of an event will be a real number that satisfies certain properties, which we call **axioms**.

Definition 4.1. Let Ω be a sample space. A **probability measure** on Ω is a function $\mathbb{P}$ that assigns to each event $A \subset \Omega$ a real number $\mathbb{P}(A)$, called the **probability** of A, and that satisfies the following three axioms:

1. $\mathbb{P}(A) \geq 0$ for all events $A \subset \Omega$;
2. $\mathbb{P}(\Omega) = 1$;
3. if $A_1, A_2, ...$ is a finite or infinite sequence of disjoint events, then

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots.$$

The sample space Ω together with the probability measure $\mathbb{P}$ are called a **probability space**.

Axiom 1 says that all probabilities are non-negative numbers. Axiom 2 says the probability that *something* happens is 1. Axiom 3 is about *disjoint* events – recall that these are events where no two can happen at the same time, because the intersection of any pair of them is empty. [***Correction:*** *In the 3pm lecture, I wrongly said that their* union *is empty.*] Axiom 3 says that for disjoint events the probability that one of them happens is the sum of the individual probabilities. (Those who like their mathematical statements very precise should note that an infinite sequence in Axiom 3 must be "countable"; that is, indexed by the natural numbers $1, 2, 3, ...$)

These axioms of probability (and our later results that follow from them) were first written down by the Russian mathematician Andrey Nikolaevich Kolmogorov in 1933. This marked the point from when probability theory could now be considered a proper branch of mathematics – just as legitimate as geometry or number theory – and not just a past-time that can be useful to help

gamblers calculate their odds. I always find it surprising that the axioms of probability are only 90 years old!

There are other properties that it seems natural that a probability measure should have aside from the axioms – for example, that $\mathbb{P}(A) \leq 1$ for all events A. But we will show shortly that other properties can be proven just by starting from the three axioms.

But first, let's see some examples.

Example 4.1. Suppose we wish to model tossing an biased coin the is heads with probability p, where $0 \leq p \leq 1$.

Our probability space is $\Omega = \{\mathrm{H}, \mathrm{T}\}$. The probability measure is given by

$$\mathbb{P}(\emptyset) = 0 \qquad \mathbb{P}(\{\mathrm{H}\}) = p$$
$$\mathbb{P}(\{\mathrm{T}\}) = 1 - p \qquad \mathbb{P}(\{\mathrm{H}, \mathrm{T}\}) = 1.$$

Let's check that the axioms hold:

1. Since $0 \leq p \leq 1$, all the probabilities are greater than or equal to 0.
2. It is indeed the case that $\mathbb{P}(\Omega) = \mathbb{P}(\{\mathrm{H}, \mathrm{T}\}) = 1$.
3. The only nontrivial disjoint union to check is $\{\mathrm{H}\} \cup \{\mathrm{T}\} = \{\mathrm{H}, \mathrm{T}\}$, where we see that

   $$\mathbb{P}(\{\mathrm{H}\}) + \mathbb{P}(\{\mathrm{T}\}) = p + (1 - p) = 1 = \mathbb{P}(\{\mathrm{H}, \mathrm{T}\}),$$

   as required.

Example 4.2. Suppose we wish to model rolling a dice.

Our sample space is $\{1, 2, 3, 4, 5, 6\}$. The probability measure is given by

$$\mathbb{P}(A) = \frac{|A|}{6},$$

where $|A|$ is the number of sample outcomes in A.

So, for example, the probability of rolling an even number is

$$\mathbb{P}(\{2, 4, 6\}) = \frac{3}{6} = \frac{1}{2}.$$

The dice rolling is a particular case of the "classical probability" of equally likely outcomes. We'll look at this more in the next lecture, and prove that the classical probability measure does indeed satisfy the axioms

4.2 Properties of probability

The axioms of Definition 4.1 only gave us some of the properties that we would like a probability measure to have. Our task now (in this subsection and the next) is to carefully prove how these other properties follow from just those axioms. In particular, we're not allowed to make claims that merely "seem likely to be true" or "are common sense" – we can only use the three axioms together with strict logical deductions and nothing else.

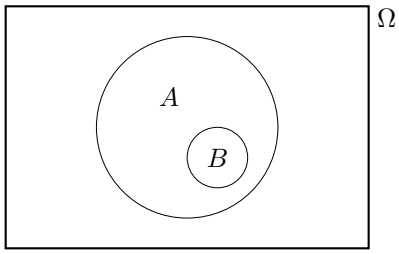
Theorem 4.1. *Let Ω be a sample space with a probability measure $\mathbb{P}$. Then we have the following:*

1. $\mathbb{P}(\emptyset) = 0$.
2. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ *for all events* $A \subset \Omega$.
3. *For events* A *and* B *with* $B \subset A$, *we have* $\mathbb{P}(B) \leq \mathbb{P}(A)$.
4. $0 \leq \mathbb{P}(A) \leq 1$ *for all events* $A \subset \Omega$.

Importantly, the second result here tells us how to deal with complements or "not" events: the probability of A *not* happening is 1 minus the probability it does happen. This is often very useful.

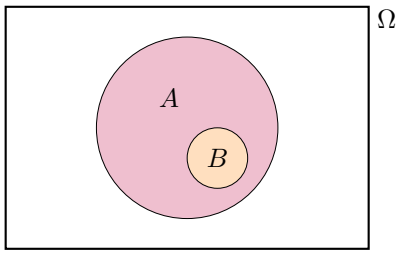
Proof. The key with most of these "prove from the axioms" problems is to think of a way to write the relevant events as part of a *disjoint* union, then use Axiom 3. Statements 1 and 2 are exercises for you on Problem Sheet 2. We'll start with the third statement.

Here, since B is a subset of A, meaning that B is entirely inside A.



It would be useful to write A as a *disjoint* union of B and "the bit of A that isn't in B". That is, we have the disjoint union

$$A = B \cup (A \cap B^c).$$



Applying Axiom 3 to this disjoint union gives

$$\mathbb{P}(A) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c).$$

We're happy to see the term on the left-hand side and the first term on the right-hand side. But what about the awkward $\mathbb{P}(A \cap B^{\mathsf{c}})$? Well, by Axiom 1, we know that the probability of any event is greater than or equal to 0, so in particular. $\mathbb{P}(A \cap B^{\mathsf{c}}) \geq 0$. Hence

$$\mathbb{P}(A) \geq \mathbb{P}(B) + 0 = \mathbb{P}(B),$$

and we are done with the third statement.

For the fourth statement, we have $\mathbb{P}(A) \geq 0$ directly from Axiom 1, so only need to show that $\mathbb{P}(A) \leq 1$. We can do this using the third statement of this theorem. For any event A we have $A \subset \Omega$, so the third statement tells us that $\mathbb{P}(A) \leq \mathbb{P}(\Omega)$. But Axiom 2 tells us that $\mathbb{P}(\Omega) = 1$, so $\mathbb{P}(A) \leq 1$ and we are done. □

4.3 Addition rules for unions

If we have two or more events, we'd like to work out the probability of their union; that is, the probability that at least one of them occurs.

We already have an addition rule for *disjoint* unions.

Theorem 4.2. *Let $A, B \subset \Omega$ be two disjoint events. Then*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

Proof. In Axiom 3, take the finite sequence $A_1 = A$, $A_2 = B$. □

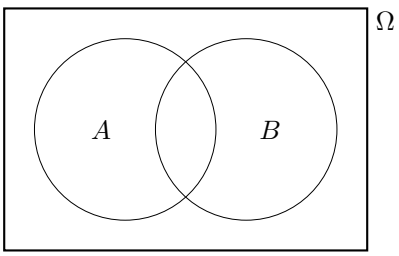
But what about if A and B are not disjoint? Then we have the following.

Theorem 4.3. *Let $A, B \subset \Omega$ be two events. Then*

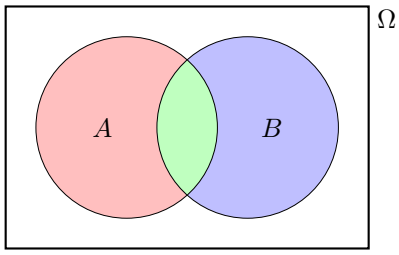
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

You may have seen this result before. You've perhaps justified it by saying something like this: "We can add the two probabilities together, except now we've double-counted the overlap, so we have to take the probability of that away." Maybe you drew a Venn diagram. That's OK as a way to remember the result – but this is a proper university mathematics course, so we have to carefully *prove* it starting from just the axioms and nothing else.

Proof. The problem here is that A and B are not (in general) disjoint, so we can't apply Axiom 3.



Instead, let's split this up into the three disjoint bits: "A but not B" $A \cap B^c$, "B but not A" $B \cap A^c$, and "both" $A \cap B$.



Now we can write A, B and $A \cup B$ in terms of these disjoint bits.

$$A = (A \cap B^c) \cup (A \cap B) \tag{4.1}$$
$$B = (B \cap A^c) \cup (A \cap B) \tag{4.2}$$
$$A \cup B = (A \cap B^c) \cup (B \cap A^c) \cup (A \cap B), \tag{4.3}$$

with all the unions on the right-hand side being disjoint. Applying Axiom 3 to them all gives

$$\mathbb{P}(A) = \mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B) \tag{4.4}$$
$$\mathbb{P}(B) = \mathbb{P}(B \cap A^c) + \mathbb{P}(A \cap B) \tag{4.5}$$
$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cap B^c) + \mathbb{P}(B \cap A^c) + \mathbb{P}(A \cap B). \tag{4.6}$$

Here, (4.6) is looking good, but we need to get rid of the awkward $\mathbb{P}(A \cap B^c)$ and $\mathbb{P}(B \cap A^c)$ terms. We can do that be rearranging (4.4) and (4.5) to get

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) \tag{4.7}$$
$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A \cap B). \tag{4.8}$$

Substituting these into (4.6) gives

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B) \tag{4.9}$$
$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B), \tag{4.10}$$

as required. □

Example 4.3. *Consider picking a card from a standard 52-card deck at random, with $\mathbb{P}(A) = |A|/52$. What's the probability the card is a spade or an ace?*

It is possible to just to work this out directly. But let's use our addition law for unions.

We have $\mathbb{P}(\text{spade}) = \frac{13}{52}$ and $\mathbb{P}(\text{ace}) = \frac{4}{52}$. So we have

$$\mathbb{P}(\text{spade or ace}) = \tfrac{13}{52} + \tfrac{4}{52} - \mathbb{P}(\text{spade and ace}).$$

But $\mathbb{P}(\text{spade and ace})$ is the probability of picking the ace of spades, which is $\frac{1}{52}$. Therefore

$$\mathbb{P}(\text{spade or ace}) = \tfrac{13}{52} + \tfrac{4}{52} - \tfrac{1}{52} = \tfrac{16}{52} = \tfrac{4}{13}.$$

Summary

- The axioms of probability are (1) $\mathbb{P}(A) \geq 0$; (2) $\mathbb{P}(\Omega) = 1$; and (3) that for disjoint events $A_1, A_2, ...$, we have $\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$.
- Other properties can be proven from these axioms, like the complement rule $\mathbb{P}(A^{\mathsf{c}}) = 1 - \mathbb{P}(A)$, and the addition rule for unions $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 1.3 and 1.4.
- Grimmett and Welsh, *Probability*, Sections 1.3 and 1.4.
- My blogpost "How to prove the addition rule for unions"

On Problem Sheet 2, you should now be able to complete Questions A1, A2, B1, B2 and perhaps C1.

Chapter 5

Classical probability I

5.1 Probability with equally likely outcomes

Classical probability is the name we give to probability where there are a finite number of equally likely outcomes.

Classical probability was the first type of probability to be formally studied – partly because it is the simplest, and partly because it was useful for working out how to win at gambling. Tossing fair coins, rolling dice, and dealing cards are all common gambling situations that can be studied using classical probability – in a deck of cards, for example, there are 52 cards that are equally likely to be drawn. Among the first works to seriously study classical probability were "Book on Games of Chance" by Girolamo Cardano (written in 1564, but not published until 1663, one hundred years later), and a famous series of letters between Blaise Pascal and Pierre de Fermat in 1654.

Definition 5.1. Let Ω be a finite sample space. Then the **classical probability measure** on Ω is given by

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}.$$

So to work out a classical probability $\mathbb{P}(A)$, crucially we need to be able to count how many outcomes $|A|$ are in the event A and count how many outcomes $|\Omega|$ are in the whole sample space Ω. (This is why classical probability is also called "enumerative probability" – "enumeration" is another word for counting.) In this lecture and the next, we'll look at some different ways in which we can count the number of outcomes in common events and sample spaces.

Example 5.1. *We roll a dice. What is the probability we get at least 5?*

The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$, with $|\Omega| = 6$. The event that we roll at least 5 is $A = \{5, 6\}$, with $|A| = 2$. Hence

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{2}{6} = \frac{1}{3}.$$

There's something we ought to check before going any further!

Theorem 5.1. *Let Ω be a finite nonempty sample space. Then the classical probability measure on Ω,*

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|},$$

is indeed a probability measure, in that it satisfies the three axioms in Definition 4.1.

Proof. We'll take the axioms one by one.

1. Since $|\Omega| \geq 1$ and $|A| \geq 0$, it is indeed the case that $\mathbb{P}(A) = |A|/|\Omega| \geq 0$.
2. We have $\mathbb{P}(\Omega) = \dfrac{|\Omega|}{|\Omega|} = 1$, as required.
3. Since we have a finite sample space, we only need to show Axiom 3 for a sequence of two disjoint events; the argument can be repeated to get any finite number of events. Let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_l\}$ be two disjoint events with $|A| = k$ and $|B| = l$. Note that we can enumerate the elements of the disjoint union $C = A \cup B$ as

   $$c_1 = a_1, c_2 = a_2, \dots, c_k = a_k, c_{k+1} = b_1, c_{k+2} = b_2, \dots, c_{k+l} = b_l.$$

   Since A and B are disjoint, this list has no repeats, and we see that $|C| = |A \cup B| = k + l$. Hence

   $$\mathbb{P}(A \cup B) = \frac{k+l}{|\Omega|} = \frac{k}{|\Omega|} + \frac{l}{|\Omega|} = \mathbb{P}(A) + \mathbb{P}(B),$$

   and Axiom 3 is fulfilled.

□

5.2 Multiplication principle

In classical probability, to find the probability of an event A, we need to count the number of outcomes in A and the total number of possible outcomes in Ω. This can be easy when we're just looking at one choice – like the 2 outcomes from tossing a single coin, the 6 outcomes of rolling a single dice, or the 52 outcomes from dealing a single card. Now we're going to look at what happens if there are a number of choices one after another – like tossing multiple coins, rolling more than one dice, or dealing a hand of cards.

Here, an important principle is the **multiplication principle**. The multiplication principle says that if you have n choices followed by m choices, than all together you have $n \times m$ total choices. You can see this by imagining the choices in a $n \times m$ grid, with the n columns representing the first choice and m rows representing the second choice. For example, suppose you go to a burger restaurant where there are 3 choices of burger (beefburger, chicken burger, veggie burger)

and 2 choices of sides (fries, salad), then altogether there are $3 \times 2 = 6$ choices of meal.

	Beefburger	Chicken burger	Veggie burger
Fries	1: Beefburger with fries	2: Chicken burger with fries	3: Veggie burger with fries
Salad	4: Beefburger with salad	5: Chicken burger with salad	6: Veggie burger with salad

More generally, if you have m stages of choosing, with n_1 choices in the first stage, then n_2 choices in the second stage, all the way to n_m choices in the final stage, you have $n_1 \times n_2 \times \cdots \times n_m$ total choices altogether.

Example 5.2. *Five fair coins are tossed. What is the probability they all show the same face?*

Here, the sample space Ω is the set of all sequences of 5 coin outcomes. How many sample outcomes are in Ω? Well, the first coin can be heads or tails (2 choices); the second coin can be heads or tails (2 choices) and so on, until the fifth and final coin. So, by the multiplication principle, $|\Omega| = 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$.

The event we're interested in is $A = \{\text{HHHHH}, \text{TTTTT}\}$, the event that the faces are all the same – either all heads or all tails. This clearly has $|A| = 2$ outcomes.

So the probability all five coins show the same face is

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{2}{32} = \frac{1}{16} \approx 0.06.$$

Example 5.3. *Four dice are rolled. What is the probability we get at least one 6?*

Here, Ω is the set of all possible sequences of four dice rolls. Clearly $|\Omega| = 6^4 = 1296$.

The event A is the set of all dice roll sequences with at least one 6. Whenever you see a question with the phrase "at least one" in it, it's very often a good idea to look at the complementary event A^{c} instead. We know from the last lecture that $\mathbb{P}(A) = 1 - \mathbb{P}(A^{\mathsf{c}})$, but in "at least one" questions, it's often easier to count $|A^{\mathsf{c}}|$ than to count $|A|$.

Here, since A is the set of all dice roll sequences with at least one 6, then A^{c} is the set of dice roll sequence without any 6s at all. This means all four dice must have rolled a 1, 2, 3, 4 or 5. Since each of the four dice rolls has five possibilities, this means that $|A^{\mathsf{c}}| = 5^4 = 625$.

Putting this together, we see that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^{\mathsf{c}}) = 1 - \frac{|A^{\mathsf{c}}|}{|\Omega|} = 1 - \frac{625}{1296} = \frac{671}{1296} \approx 0.518.$$

So there's about a 52% chance we get at least one 6.

5.3 Sampling with and without replacement

Probabilists love problems where they pick coloured balls out of a bag!

Example 5.4. *A bag contains 15 balls: 10 black balls and 5 white balls. We draw 3 balls out of the bag. What is the probability all 3 balls are black* ***(a)*** *if we put each ball back into the bag after it is chosen;* ***(b)*** *if we do not put each ball back into the bag after it is chosen.*

Let's start with (a). The number of ways to choose a ball out 15 on three occasions is $|\Omega| = 15^3$. The number of ways to choose a black ball out of 10 on three occasions is $|A| = 10^3$. Hence

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{10^3}{15^3} = \frac{1000}{3375} = \frac{8}{27} \approx 0.30.$$

What about (b)? Here we don't put the ball back in the bag once it has been chosen. There are 15 ways to pick the first ball. But then there are only 14 balls left in the bag for the second choice, and only 13 balls for the third choice. So $|\Omega| = 15 \times 14 \times 13$. Similarly, there are 10 ways the first ball can be black. But once that black ball is removed, only 9 choices for the second black ball, and only 8 for the third. So $|A| = 10 \times 9 \times 8$. So this time we have

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{10 \times 9 \times 8}{15 \times 14 \times 13} = \frac{720}{2730} = \frac{24}{91} \approx 0.26,$$

which is slightly smaller than the answer in part (a).

This example illustrated the difference between **sampling with replacement** (when the balls were put back into the bag) and **sampling without replacement** (when the balls were not put back). If we want to sample k items from a set of n items, then:

- the number of ways to sample with replacement is
$$n^k = n \times n \times \cdots \times n;$$
- the number of ways to sample without replacement is
$$n^{\underline{k}} = n \times (n-1) \times \cdots \times (n-k+1).$$

Here, we've defined the notation $n^{\underline{k}}$ for the number of ways to sample without replacement; this is called the **falling factorial** or **permutation number**. This is still k numbers multiplied together, but decreasing by 1 each time down from n. The final number in the product is the number of choices in the kth an final round: this is the original n items minus the $k-1$ items sampled in the previous $k-1$ rounds; so the final number is $n-(k-1) = n-k+1$. A notation point: Notice that the subscript is underlined in the falling factorial; other notation sometimes used includes $(n)_k$, $P(n,k)$, or ${}^nP_k$.

In our balls-in-a-bag problem, the answer for sampling with replacement was $10^3/15^3$, while the answer for sampling without replacement was $10^{\underline{3}}/15^{\underline{3}}$.

Next time, we will look at more classical probability problems. Do two of your friends share a birthday? Can you shuffle a deck of cards in an order that has never happened before in the history of the universe? And can you win the National Lottery?

Summary

- "Classical probability" describes the situation where there are finitely many equally likely outcomes. The classical probability $\mathbb{P}(A) = |A|/|\Omega|$ requires us to count how many outcomes there are in events or sample spaces.
- The multiplication principle says that n choices followed by m choices makes $n \times m$ choices in total.
- Sampling k objects out of n with replacement gives n^k choices.
- Sampling k objects out of n without replacement gives $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$ choices.

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 3.1 and 3.2.

Chapter 6

Classical probability II

We continue looking at the classical probability $\mathbb{P}(A) = |A|/|\Omega|$, by looking at ways to enumerate Ω and A. Last time we saw:

- The multiplication principle: n_1 choices followed by n_2 choices, ..., up to n_k choices gives $n_1 \times n_2 \times \cdots \times n_k$ choices in total.
- Sampling k objects out of n with replacement gives n^k choices.
- Sampling k objects out of n without replacement gives $n^{\underline{k}} = n(n-1)\ldots(n-k+1)$ choices.

6.1 Ordering

Example 6.1. *Suppose a lecturer marks a pile of n exam papers, all of which receive a different mark. What is the probability she ends up marking them in order from lowest scoring first in the pile to highest scoring last in the pile?*

Here, the sample space Ω is the set of all orderings of the n exam papers by mark, and A is the event that the papers are in order from lowest to highest scoring. It's clear that $|A| = 1$: since the exams scored different marks, there's only one way of putting the exams in the correct lowest-to-highest order. But what's $|\Omega|$?

There are n choices for the first exam paper to be marked. Then, for the second exam paper, there are $n-1$ choices left, because the lecturer is not going to mark the same paper twice. There are $n-2$ choices for the third exam paper. And so on, until she has marked $n-1$ papers, and there is only 1 choice left for the final paper. So we have

$$|\Omega| = n^{\underline{n}} = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

ways to order the exam papers.

Hence, the probability the papers are marked in order is

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{1}{n(n-1)\cdots 2 \cdot 1} = \frac{1}{n!}.$$

This number

$$n! = n^{\underline{n}} = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

is called n **factorial** and denoted $n!$. It is the number of ways that n different objects can be ordered.

The factorial $n!$ gets very large very quickly. **Stirling's formula** gives the approximation $n! \approx \sqrt{2\pi n}\, \mathrm{e}^{-n}\, n^n$.

Example 6.2. Suppose you shuffle a pack of cards. The resulting ordering of the deck has 52! possibilities. This is an unimaginably huge number – the exact value to 3 significant figures is

$$52! = 8.07 \times 10^{67},$$

while Stirling's formula gives the approximation

$$52! \approx \sqrt{2\pi \times 52} \times \mathrm{e}^{-52} \times 52^{52} = 8.05 \times 10^{67}.$$

This is an 8 followed by 67 zeroes.

If every person on the planet (very roughly 10^{10}) had shuffled a deck of cards one million (10^6) times a second for the entire lifetime of the universe (roughly 10^{17} seconds), they could only expect to have got through about 10^{33} shuffles. This is only the most tiny, microscopic fraction of 52!. So every time you have ever shuffled a deck of cards, it is essentially certain that you have created an ordering of the deck that has never existed before.

If we take the ratio of a bigger factorial $n!$ over a smaller factorial $j!$, we get lot of cancellation,

$$\begin{aligned}\frac{n!}{j!} &= \frac{n(n-1)\cdots(j+1)j(j-1)\cdots 1}{j(j-1)\cdots 1} \\ &= n(n-1)\cdots(j+1),\end{aligned}$$

because the last part of the product in the numerator cancels with the whole of the denominator. Replacing j with $n-k$, this gives

$$\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) = n^{\underline{k}}.$$

This gives a way of writing the falling factorial as the ratio of two (normal) factorials, which can sometimes be useful.

6.2 Sampling without replacement in any order

Example 6.3. *In the Lotto, the UK national lottery, you can buy a ticket for £2 and choose 6 numbers between 1 and 59. If your 6 numbers match the 6 numbers on the balls chosen by the lottery machine, you win the jackpot (usually between £2 million and £20 million, shared between the tickets that get all 6 numbers). If you buy a ticket, what is the probability you win the jackpot?*

Here, Ω is the set of all possible sets of 6 winning numbers, and A is the set of numbers on your ticket. Clearly $|A| = 1$, but what is $|\Omega|$?

Well, the first ball out of the machine has 59 possibilities, the second ball has 58 possibilities, and so on, making

$$59 \times 58 \times 57 \times 56 \times 55 \times 54 = 59^{\underline{6}}.$$

But this isn't the correct answer, because the same set of balls could be drawn from the machine in any order! The sets of balls $\{1, 2, 3, 4, 5, 6\}$ and $\{6, 5, 4, 3, 2, 1\}$ and $\{1, 3, 5, 6, 4, 2\}$ are all the same set of numbers. How many ways can we see the same list of numbers? This is precisely the number of orderings of 6 balls, which we know is 6!. So the number of possible sets of 6 balls to come out of the machine is actually

$$\binom{59}{6} = \frac{59^{\underline{6}}}{6!} = \frac{59 \times 58 \times 57 \times 56 \times 55 \times 54}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \approx 45 \text{ million}.$$

Thus the probability that your ticket wins the jackpot is

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{1}{\binom{59}{6}} \approx \frac{1}{45 \text{ million}} \approx 0.000\,000\,02.$$

Here, we have introduced the notation

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2 \cdot 1}$$

for the number of ways to choose k objects out of n without replacement *and where the order they were chosen in doesn't matter.* This is called the **binomial coefficient**, although when we say it out loud we normally just say "n choose k". (Another notation for the binomial coefficient is ${}^nC_k$.)

It can sometimes be useful to remember that $n^{\underline{k}} = n!/(n-k)!$ allows us to write the binomial coefficient in terms of the factorial function as

$$\binom{n}{k} = \frac{n^{\underline{k}}}{(n-k)!} = \frac{n!}{k!(n-k)!}.$$

Example 6.4. *You are dealt a "hand" of 13 cards from a deck of 52 cards. What is the probability that you have the Ace, King, Queen, and Jack of Spades?*

Here, Ω is the set of all 13-card hands from the deck, and A is the subset of those that contain the AKQJ of Spades.

Using the binomial coefficient notation, it's clear that

$$|\Omega| = \binom{52}{13} = \frac{52 \times 51 \times \cdots \times 41 \times 40}{13 \times 12 \times \cdots \times 2 \times 1}.$$

What about $|A|$? If we fix the fact that the hand contains the 4 cards AKQJ of Spades, then it also contains $13 - 4 = 9$ cards out of the other $52 - 4 = 48$ remaining cards in the deck. This makes

$$|A| = \binom{48}{9} = \frac{48 \times 47 \times \cdots \times 41 \times 40}{9 \times 8 \times \cdots \times 2 \times 1}$$

hands.

Thus the probability that the hand contains AKQJ of Spades is

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{48}{9}}{\binom{52}{13}}.$$

Conveniently, we can simplify the expression quite a lot, because plenty of cancellation will occur. We have

$$\begin{aligned}\mathbb{P}(A) = \frac{\binom{48}{9}}{\binom{52}{13}} &= \frac{\frac{48\times 47\times\cdots\times 41\times 40}{9\times 8\times\cdots\times 2\times 1}}{\frac{52\times 51\times\cdots\times 41\times 40}{13\times 12\times\cdots\times 2\times 1}} \\ &= \frac{48\times 47\times\cdots\times 41\times 40}{52\times 51\times\cdots\times 41\times 40} \times \frac{13\times 12\times\cdots\times 2\times 1}{9\times 8\times\cdots\times 2\times 1} \\ &= \frac{13\times 12\times 11\times 10}{52\times 51\times 50\times 49} \\ &\approx 0.0026,\end{aligned}$$

or about 1 in every 380 hands.

6.3 Birthday problem

Example 6.5. *There are $k = 23$ students in a class. What is the probability that at least two of the students share a birthday?*

This a famous problem, known as the "birthday problem". You may have seen this problem before – but let's try to solve it using the techniques from this section of notes. If you haven't seen it before, you might like to guess what you think the answer might be. (We'll assume all days are equally likely for birthdays, and ignore the leap day 29 February.)

The sample space Ω is the set of possible birthdays for all k students. Clearly $|\Omega| = 365^k$.

Let A be the even that at least one pair of student share a birthday. Since this is an "at least" event, it seems like it might be a good idea to look instead at the complementary event A^{c}. If A is the event that there's at least one shared birthday, then A^{c} is the event that there are *no* shared birthdays; that is, A^{c} is the event that all k students have *different* birthdays.

So what is $|A^{\mathsf{c}}|$, the number of ways the k students can have different birthdays? Well, the first student can have any of the 365 days for their birthday. For them to have different birthdays, the second student only has 364 days available. Then the third student must avoid the birthday of students 1 and 2, so has 363 available days, and so on. We see that

$$|A^{\mathsf{c}}| = 365\times 364\times\cdots\times(365-k+1) = 365^{\underline{k}}.$$

Hence, the probability at least two students share a birthday is

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^{\mathsf{c}}) = 1 - \frac{365^{\underline{k}}}{365^k} = 1 - \frac{365}{365}\cdot\frac{364}{365}\cdots\frac{365-k+1}{365}.$$

Setting $k = 23$, we can calculate the required answer in R:

```
k <- 23
1 - prod((365:(365 - k + 1)) / 365)
```

```
[1] 0.5072972
```

The probability is 50.7%. So it's more likely than not that at least two students share a birthday.

Some people find it surprising that only 23 students have such a high probability of sharing a birthday, since 23 is so small compared to 365. But remember there are $\binom{23}{2} = 253$ *pairs* of birthdays, and each of those 253 pairs is a potential match.

Summary

- Ordering n objects can be done in $n! = n^{\underline{n}} = n(n-1)\cdots 2 \cdot 1$ ways.
- The number of ways to sample k objects out of n when the order doesn't matter is given by the binomial coefficient $\binom{n}{k} = n^{\underline{k}}/k!$.

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 3.2 and 3.3.

On Problem Sheet 2, you should now be able to complete all questions.

Problem Sheet 2

Solutions are now available to all non-assessed questions.

This is Problem Sheet 2. This problem sheet covers material from Lectures 3 to 6. You should work through all the questions on this problem sheet in preparation for your tutorial in Week 4. The problem sheet contains two assessed questions, which are due in by **2pm on Monday 30 October**.

A: Short questions

A1. Suppose you toss a coin 4 times.

(a) What would you suggest for a sample space Ω **(i)** if you only care about the total number of heads; **(ii)** if you care about the result of each coin toss?

(b) For each of the cases in part (a), what is $|\Omega|$?

A2. Let A, B and C be events in a sample space Ω. Write the following events using only A, B, C and the complement, intersection, and union operations.

(a) C happens but A doesn't.

(b) At least one of A, B and C happens.

(c) Exactly one of B or C happens.

(d) Exactly two of A, B and C happens.

A3. What is the value of the following expressions?

(a) $6!$

(b) 8^4

(c) $8^{\underline{4}}$

(d) $\binom{10}{4}$

A4. An urn contains 4 red balls and 6 blue balls. Two balls are drawn from the urn. What is the probability that both balls are red, if the balls are drawn **(a)** with replacement; **(b)** without replacement?

B: Long questions

B1. Starting from just the three probability axioms, prove the following statements:

(a) $\mathbb{P}(\emptyset) = 0$.

(b) $\mathbb{P}(A^{\mathsf{c}}) = 1 - \mathbb{P}(A)$.

B2. In this question, you will have to use the standard two-event form of the addition rule for unions

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(a) Using the two-event addition rule, show that

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D \cup E) - \mathbb{P}(C \cap (D \cup E)).$$

(b) Using your result from part (a), the two-event addition rule, the distributive law, and the two-event addition rule again, prove the three-event form of the addition rule for unions:

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(C \cap D) - \mathbb{P}(C \cap E) - \mathbb{P}(D \cap E) + \mathbb{P}(C \cap D \cap E).$$

B3. Suppose we pick a number at random from the set $\{1, 2, \ldots, 2023\}$.

(a) What is the probability that the number is divisible by 5?

(b) What is the probability the number is divisible by 5 or by 7?

B4. Eight friends are about to sit down at random at a round table. Find the probability that

(a) Ashley and Brook sit next to each other, with Chris directly opposite Brook;

(b) neither Ashley, Brook nor Chris sit next to each other.

B5. A "random digit" is a number chosen at random from $\{0, 1, \ldots, 9\}$, each with equal probability. A statistician chooses n random digits (with replacement).

(a) For $k = 0, 1, \ldots, 9$, let A_k be the event that all the digits are k or smaller. What is the probability of A_k, as a function of k and n?

(b) Let B_k be the event that the largest digit chosen is equal to k. By finding a relationship between B_k, A_{k-1} and A_k, or otherwise, show that

$$\mathbb{P}(B_k) = \frac{(k+1)^n - k^n}{10^n}.$$

C: Assessed questions

The last two questions are **assessed questions**. These two questions count for 3% of your final mark for this module.

The deadline for submitting your solutions is **2pm on Monday 30 October** at the beginning of Week 5. You should submit a PDF to Gradescope. Your work will be marked by your tutor and returned on Monday 6 November, when solutions will also be made available.

Both questions are "long questions", where the marks are not only for mathematical accuracy but also for the clarity and completeness of your explanatory writing.

You should not collaborate with others on the assessed questions: your answers must represent solely your own work. The University's rules on academic integrity – and the related punishments for violating them – apply to your work on the assessed questions.

C1. Let Ω be a sample space with a probability measure $\mathbb{P}$, and let $A, B \subset \Omega$ be events. For each of the following statements, state whether the statement is true or false (that is, always true or sometimes false). If it is true, briefly justify the statement; if it is false, give a counterexample.

(a) If $\mathbb{P}(A) \leq \mathbb{P}(B)$, then $A \subset B$.

(b) $\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^{\mathsf{c}}) = \mathbb{P}(A)$.

(c) $\mathbb{P}(A \cup B) \leq \mathbb{P}(A)$

(d) If A and B are disjoint, then $\mathbb{P}((A \cup B)^{\mathsf{c}}) = 1 - \mathbb{P}(A) - \mathbb{P}(B)$.

C2. An urn contains 15 balls: 4 red balls, 5 blue balls, and 6 green balls.

(a) If three balls are drawn *with* replacement, what is the probability that all three balls are the *same* colour?

(b) If three balls are drawn *without* replacement, what is the probability that all three balls are *different* colours?

Solutions to short questions

A1. (a) (i) $\{0, 1, \ldots, 4\}$ (ii) $\{\text{HHHH}, \text{HHHT}, \text{HHTH}, \ldots, \text{TTTT}\}$ (b) (i) 5 (ii) 16.
A2. (a) $C \cap A^{\mathsf{c}}$ (b) $A \cup B \cup C$ (c) $(B \cup C) \cap (B \cap C)^{\mathsf{c}}$ or $(B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C)$ (d) $(A \cap B \cap C^{\mathsf{c}}) \cup (A \cap B^{\mathsf{c}} \cap C) \cup (A^{\mathsf{c}} \cap B \cap C)$ or other equivalent
A3. (a) 720 (b) 4092 (c) 1680 (d) 210
A4. (a) $\frac{4}{25} = 0.16$ (b) $\frac{2}{15} = 0.133$

Chapter 7

Independence and conditional probability

7.1 Independent events

Suppose 40% of people have blond hair, and 20% of people have blue eyes. What proportion of people have both blond hair and blue eyes?

The answer to this question is: we don't know. The question doesn't give us enough information to tell. However, *if* it were the case that having blond hair didn't effect your chance of having blue eyes, *then* we could work out the answer. If that were true, we would think that the 20% of people with blue eyes would equally make up both 20% of the blonds and also 20% of the non-blonds. Thus the proportion of people with blond hair and blue eyes would be this 20% of the 40% of people with blond hair; and 20% of 40% is $0.2 \times 0.4 = 0.08$, or 8%.

To put it in probability language, *if* blond hair and blue eyes were unrelated, then we would expect that

$$\mathbb{P}(\text{blond hair and blue eyes}) = \mathbb{P}(\text{blond hair}) \times \mathbb{P}(\text{blue eyes}).$$

This is an important property known as "independence".

Definition 7.1. Two events A and B are said to be **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

There are two ways we can use this definition.

- If we know $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(A \cap B)$, then we can find out whether or not A and B are independent by checking whether or not $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.
- If we know $\mathbb{P}(A)$ and $\mathbb{P}(B)$ and we know that A and B are independent, then we can find $\mathbb{P}(A \cap B)$ by calculating $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.

In this second case, we might know A and B are independent because we are specifically told they are in a question. But alternatively we might reason that A and B must be independent because the related experiments are not physically related. For example if we roll a dice then toss a coin, we might reason that {roll a 5} and {the coin lands Heads} must be independent because the dice roll doesn't effect the coin toss – we could then use the independence assumption in calculations.

Example 7.1. *I roll a single dice. Let $A = \{$even number$\} = \{2, 4, 6\}$ be the event I roll an even number, and let $B = \{$at least 4$\} = \{4, 5, 6\}$ be the event I roll at least 4. Are the events A and B independent?*

Clearly we have $\mathbb{P}(A) = \frac{3}{6} = \frac{1}{2}$ and $\mathbb{P}(B) = \frac{3}{6} = \frac{1}{2}$. The intersection is $A \cap B = \{4, 6\}$, so $\mathbb{P}(A \cap B) = \frac{2}{6} = \frac{1}{3}$. So we see that

$$\mathbb{P}(A \cap B) = \frac{1}{3} \qquad \text{and} \qquad \mathbb{P}(A)\,\mathbb{P}(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

So $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\,\mathbb{P}(B)$, and the two events are not independent.

Example 7.2. *A biased coin has probability p of landing Heads and probability $1 - p$ of landing Tails. You toss the coin 3 times. Assuming tosses of the coin are independent, calculate the probability of getting exactly 2 Heads.*

There are three ways we could get exactly 2 Heads: HHT, HTH, or THH. For the first of these,

$$\mathbb{P}(\text{HHT}) = \mathbb{P}(\text{first coin H} \cap \text{second coin H} \cap \text{third coin T}).$$

Since tosses of the coin are independent, we therefore have

$$\begin{aligned}\mathbb{P}(\text{HHT}) &= \mathbb{P}(\text{first coin H}) \times \mathbb{P}(\text{second coin H}) \times \mathbb{P}(\text{third coin T}) \\ &= p \times p \times (1 - p) \\ &= p^2(1 - p).\end{aligned}$$

Similarly,

$$\mathbb{P}(\text{HTH}) = \mathbb{P}(\text{THH}) = p^2(1 - p)$$

also.

Finally, because the events are disjoint, we have

$$\mathbb{P}(\text{HHT} \cup \text{HTH} \cup \text{THH}) = \mathbb{P}(\text{HHT}) + \mathbb{P}(\text{HTH}) + \mathbb{P}(\text{THH}) = 3p^2(1 - p).$$

7.2 Conditional probability

Let us to return to the example of blond hair and blue eyes. Suppose the population statistics are like this:

	Brown hair	**Blond hair**	**Total**
Brown eyes	50%	30%	**80%**
Blue eyes	10%	10%	**20%**
Total	**60%**	**40%**	**100%**

It turns out that $\mathbb{P}(\text{blond hair and blue eyes}) = 0.1 \neq 0.08$, so having blond hair and having blue eyes are not in fact independent.

We know from this table that 20% of people have blue eyes. But suppose you already know that someone has blond hair: what *then* is the probability they have blue eyes *given* that they have blond hair?

Well, the 40% of blond-haired people is made up of the 10% of people who also have blue eyes along with their blond hair, and the 30% of people who have brown eyes along with their blond hair. So of the 40% of blond-haired people, only one quarter of that 40% – which makes 10% – have blue eyes. If we use a vertical line | in a probability to mean "given" (or "assuming that" or "conditional upon"), then we can write this as

$$\mathbb{P}(\text{blue eyes} \mid \text{blond hair}) = \frac{\mathbb{P}(\text{blue eyes and blond hair})}{\mathbb{P}(\text{blond hair})} = \frac{0.1}{0.4} = \frac{1}{4}.$$

What we've seen here is called a "conditional probability".

Definition 7.2. Let A and B be events, with $\mathbb{P}(A) > 0$. Then the **conditional probability of B given A** is defined to be

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

The condition $\mathbb{P}(A) > 0$ is just to ensure we don't have any "divide by 0" errors. (I normally won't bother saying this explicitly – any statement about conditional probability will implicitly assume that the event being conditioned on has nonzero probability.)

Conditional probability ties in with independence in an important way. Suppose A and B are independent, so $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$. Then the conditional probability becomes

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)\,\mathbb{P}(B)}{\mathbb{P}(A)} = \mathbb{P}(B),$$

so $\mathbb{P}(B \mid A) = \mathbb{P}(B)$. In other words, if A and B are independent, then A happening doesn't affect the probability of B happening (and vice versa).

So when we have independence, we know that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$, and the mathematics is quite easy. But conditional probability tells us how things work when we don't have independence.

As with independence, we can use the definition of conditional probability in two ways. The first way is that if we know $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A)$, then we can calculate the conditional probability of B given A as

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Example 7.3. *With the dice roll again, what's the probability of rolling at least 4 given that you roll an even number?*

We previously calculated

$$\mathbb{P}(\text{even and at least 4}) = \mathbb{P}(A \cap B) = \tfrac{2}{6}$$
$$\mathbb{P}(\text{even number}) = \mathbb{P}(A) = \tfrac{3}{6}.$$

Hence

$$\mathbb{P}(\text{even} \mid \text{at least 4}) = \mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \tfrac{2}{3}.$$

This is intuitively correct: of the three possibilities of rolling at least 4, $\{4, 5, 6\}$, two of those three are even, so the probability is $\frac{2}{3}$.

Note that with classical probability $\mathbb{P}(A) = |A|/|\Omega|$, where we have finitely many equally likely outcomes, we have

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\frac{|A \cap B|}{|\Omega|}}{\frac{|A|}{|\Omega|}} = \frac{|A \cap B|}{|A|}.$$

This is effectively reducing ourselves from the whole sample space Ω and moving to a smaller "restricted sample space" A.

Note that this worked in the previous example, when

$$\mathbb{P}(B \mid A) = \frac{|A \cap B|}{|A|} = \frac{2}{3}.$$

7.3 Chain rule

The second way to use the definition of conditional probability is that if we know $\mathbb{P}(A)$ and $\mathbb{P}(B \mid A)$, then we can calculate the event that both A and B occur as

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B \mid A).$$

This can be a particularly useful tool when A concerns the first stage of an experiment and B the second stage. This says that the probability A happens then B happens is equal to the probability A happens multiplied the conditional probability, given that A has already happened, that B then happens too.

We can extend this to more events. For three events, we have

$$\begin{aligned}\mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A \cap B)\,\mathbb{P}(C \mid A \cap B) \\ &= \mathbb{P}(A)\,\mathbb{P}(B \mid A)\,\mathbb{P}(C \mid A \cap B),\end{aligned}$$

which can be useful when we have three stages of an experiment.

Continuing that process, we get a general rule.

Theorem 7.1 (Chain rule)**.** *For events $A_1, A_2, \dots, A_n$, we have*

$$\begin{aligned}&\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)\\ &\quad = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}).\end{aligned}$$

At each step, we need to calculate the probability of that step given all the previous steps being successful. We then multiply them all together.

Often questions that can be solved using the classical probability counting methods from Section 3 also be solved "step by step" using the chain rule. (It's a matter of personal taste which you prefer, but I find that chain rule methods are often simpler and more intuitive.)

Example 7.4. *Recall the Lotto problem from Example 6.3: What is the probability we match 6 balls from 59?*

Let $A_1, A_2, \dots, A_6$ be the events that the first, second, ..., sixth balls out of the machine are on our ticket. Clearly $\mathbb{P}(A_1) = \frac{6}{59}$, as we have six numbers on our ticket that the first ball could match. The conditional probability that the second ball matches given that the first ball matched is $\mathbb{P}(A_2 \mid A_1) = \frac{5}{58}$, because there are 58 balls left in the machine and, given that we got the first number right, there are 5 numbers left on our ticket. Similarly, $\mathbb{P}(A_3 \mid A_1 \cap A_2) = \frac{4}{57}$, and so on, until $\mathbb{P}(A_6 \mid A_1 \cap \cdots \cap A_5) = \frac{1}{54}$.

So, using the chain rule, we get

$$\begin{aligned}&\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_6)\\ &\qquad = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_6 \mid A_1 \cap \cdots \cap A_5)\\ &\qquad = \frac{6}{59} \times \frac{5}{58} \times \frac{4}{57} \times \frac{3}{56} \times \frac{2}{55} \times \frac{1}{54}.\end{aligned}$$

The answer we got before was

$$\frac{1}{\binom{59}{6}} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{59 \times 58 \times 57 \times 56 \times 55 \times 54} = \frac{1}{45 \text{ million}}.$$

It's easy to see that this is the same answer. The structure of the answers shows how our previous classical probability method got the answer "all at once", while this new chain rule method gets the answer "one step at a time".

Summary

- Two events are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.
- The conditional probability of B given A is $\mathbb{P}(B \mid A) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$.
- The chain rule is

$$\begin{aligned}&\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)\\ &\qquad = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap \cdots \cap A_{n-1}).\end{aligned}$$

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 2.1 and 2.2.
- Grimmett and Welsh, *Probability*, Sections 1.6 and 1.7.

Chapter 8

Two theorems on conditional probability

Last time we met the conditional probability $\mathbb{P}(B \mid A)$ of one event B given another event A. In this lecture we will be looking two very useful theorems about conditional probability, called the **law of total probability** and **Bayes' theorem** – and they're particularly powerful when used together.

8.1 Law of total probability

Example 8.1. *My friend has three dice: a 4-sided dice, a 6-sided dice, and a 10-sided dice. He picks one of them at random, with each dice equally likely. What is the probability my friend rolls a 5?*

If my friend were to tell which dice he picked, then this question would be very easy! If we write D_4, D_6 and D_{10} to be the events that he picks the 4-sided, 6-sided, or 10-sided dice, then we know immediately that

$$\mathbb{P}(\text{roll } 5 \mid D_4) = 0 \qquad \mathbb{P}(\text{roll } 5 \mid D_6) = \tfrac{1}{6} \qquad \mathbb{P}(\text{roll } 5 \mid D_{10}) = \tfrac{1}{10}.$$

What we need is a way to combine the results for different "sub-cases" into an over-all answer.

Luckily, there exists just such a tool for this job! It's called the "law of total probability" (also known as the "partition theorem"). The important point is to make sure that the different sub-cases cover all possibilities, but that only one of them happens at a time.

Definition 8.1. A set of events $A_1, A_2, \ldots, A_n$ are said to be a **partition** of the sample space Ω if

1. they are disjoint, in that $A_i \cap A_j = \emptyset$ for all $i \neq j$;
2. they cover space, in that $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$.

Theorem 8.1 (Law of total probability)**.** *Let $A_1, A_2, \ldots, A_n$ be a partition, and B another event. Then*

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i).$$

So the law of total probability tells us we can add up the probabilities $\mathbb{P}(B \mid A_i)$ for each of the sub-cases provided we weight them by how likely $\mathbb{P}(A_i)$ by how likely each sub-case is.

Proof. Since the partition of A_is cover space, we can split up B depending on which part of the partition it is in:

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n).$$

[I meant to draw a picture here, but didn't get round to it – perhaps you'd like to draw your own?]

Since the A_i are disjoint, the union on the right is disjoint also. Therefore we can use Axiom 3 to get

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \cdots + \mathbb{P}(B \cap A_n) - \sum_{i=1}^{n} \mathbb{P}(B \cap A_i).$$

But using the definition of conditional probability, each "summand" (term inside the sum) is

$$\mathbb{P}(B \cap A_i) = \mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i).$$

The result follows. □

Example 8.1 continued. Returning to our dice example, we see that $\{D_4, D_6, D_{10}\}$ is indeed a partition, since my friend must choose exactly one of the three dice. So the law of total probability tells us that

$$\mathbb{P}(\text{roll 5}) = \mathbb{P}(D_4)\,\mathbb{P}(\text{roll 5} \mid D_4) + \mathbb{P}(D_6)\,\mathbb{P}(\text{roll 5} \mid D_6) + \mathbb{P}(D_{10})\,\mathbb{P}(\text{roll 5} \mid D_{10}).$$

We were told that all the dice were picked with equal probability, so $\mathbb{P}(D_4) = \mathbb{P}(D_6) = \mathbb{P}(D_{10}) = \frac{1}{3}$, and we've already calculated the individual conditional probabilities as

$$\mathbb{P}(\text{roll 4} \mid D_4) = 0 \qquad \mathbb{P}(\text{roll 4} \mid D_6) = \tfrac{1}{6} \qquad \mathbb{P}(\text{roll 4} \mid D_{10}) = \tfrac{1}{10}.$$

Therefore, we have

$$\mathbb{P}(\text{roll 5}) = \tfrac{1}{3} \times 0 + \tfrac{1}{3} \times \tfrac{1}{6} + \tfrac{1}{3} \times \tfrac{1}{10} = \tfrac{8}{90} = 0.089.$$

8.2 Bayes' theorem

In this section, we will discuss an important result called **Bayes' theorem**. Let's first state and prove this result, and do an example, and then afterwards we'll talk about two reasons why Bayes' theorem is so important.

Theorem 8.2 (Bayes' theorem)**.** *For events A and B with $\mathbb{P}(A), \mathbb{P}(B) > 0$, we have*

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A)\,\mathbb{P}(B \mid A)}{\mathbb{P}(B)}.$$

Bayes' theorem is thought to have first appeared in the writings of Rev. Thomas Bayes, a British church minister and mathematician, shortly after his death, in the 1760s. (Bayes' work was significantly edited by Richard Price, another minister–mathematician, and many historians of mathematics think that Price deserves a good share of the credit.)

Proof. From the definition of conditional probability, we can write $\mathbb{P}(A \cap B)$ in two different ways: we can write it as

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B \mid A),$$

but we can also write it as

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\,\mathbb{P}(A \mid B).$$

Since these are two different ways of writing the same thing, we can equate them, to get

$$\mathbb{P}(A)\,\mathbb{P}(B \mid A) = \mathbb{P}(B)\,\mathbb{P}(A \mid B).$$

Dividing both sides by $\mathbb{P}(B)$ gives the result. □

Example 8.2. *My friend again secretly picks the 4-sided, 6-sided, or 10-sided dice, each with probability $\frac{1}{3}$. He rolls that secret dice, and tells me he rolled a 5. What is the probability he picked the 6-sided dice?*

This is asking us to calculate $\mathbb{P}(D_6 \mid \text{roll } 5)$. Bayes' theorem tells us that

$$\mathbb{P}(D_6 \mid \text{roll } 5) = \frac{\mathbb{P}(D_6)\,\mathbb{P}(\text{roll } 5 \mid D_6)}{\mathbb{P}(\text{roll } 5)} = \frac{\frac{1}{3} \times \frac{1}{6}}{\frac{8}{90}} = \frac{5}{8},$$

since we had calculated $\mathbb{P}(\text{roll } 5) = \frac{8}{90}$ in the previous subsection.

The first way to think about Bayes' theorem is that it tells us how to relate $\mathbb{P}(A \mid B)$ and $\mathbb{P}(B \mid A)$. Remember that $\mathbb{P}(A \mid B)$ and $\mathbb{P}(B \mid A)$ are not the same thing! The conditional probability someone is under 40 given they are a Premiership footballer is very high, but the conditional probability someone is a Premiership footballer given they are under 40 is very low.

Bayes' theorem, in this first view, is a useful technical result that helps us switch the order of a conditional probability from B given A to A given B: we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \times \mathbb{P}(B \mid A).$$

In the dice example, the probability $\mathbb{P}(\text{roll } 5 \mid D_6) = \frac{1}{6}$ was very obvious, but Bayes' theorem allowed us to reverse the conditioning, to find $\mathbb{P}(D_6 \mid \text{roll } 5) = \frac{5}{8}$ instead.

The second way to think about Bayes' theorem is that it tells us how to update our beliefs as we acquire more evidence. That is, we might start by believing that the probability some event A will occur is $\mathbb{P}(A)$. But then we find out that B has occurred, so we want to incorporate this knowledge and update our belief of the probability A will occur to $\mathbb{P}(A \mid B)$, the conditional probability A will occur given this new evidence B.

Bayes theorem, in this second view, tells us how to update from $\mathbb{P}(A)$ to $\mathbb{P}(A \mid B)$: we have

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \times \frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)}.$$

In the dice example, we initially believed there was a $\mathbb{P}(D_6) = \frac{1}{3} = 0.333$ chance our friend had chosen the six-sided dice. But when we heard that our friend had rolled a 5, we updated our belief to now thinking there was now a $\mathbb{P}(D_6 \mid \text{roll } 5) = \frac{5}{8} = 0.625$ chance it was the 6-sided dice.

This second way of thinking about Bayes' theorem is at the heart of **Bayesian statistics**. In Bayesian statistics, we start with a "prior" belief about a model, then, after collecting some data, we update, using Bayes' theorem, to a "posterior" belief about the model. We will discuss Bayesian statistics much more in Week 10.

Quite often we use Bayes' theorem and the law of total probability together. If we have a partition $A_1, A_2, \ldots, A_n$, perhaps representing some possible hypotheses, and we observe some evidence B, then Bayes' theorem tells us how likely each hypothesis is given the evidence:

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i)}{\mathbb{P}(B)}.$$

But this shared denominator $\mathbb{P}(B)$ can be expanded using the law of total probability

$$\mathbb{P}(B) = \sum_{j=1}^{n} \mathbb{P}(A_j)\,\mathbb{P}(B \mid A_j).$$

Putting these together, we get the following.

Theorem 8.3. *Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of a sample space and let B be another event. Then, for all $i = 1, 2, \ldots, n$, we have*

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i)}{\sum_{j=1}^{n} \mathbb{P}(A_j)\,\mathbb{P}(B \mid A_j)}.$$

This is essentially what we did with the dice example – although we split up the calculation into two separate parts rather than using this formula directly.

8.3 Diagnostic testing

Example 8.3. *Members of the public are tested for a certain rare disease. About 2% of the population have the disease. The test is 95% accurate, in the following sense: if you have the disease, there's a 95% chance you correctly get a positive test result; while if you don't have the disease, there's a 95% chance you correctly get a negative test result. Suppose you get a positive test result. What is the probability you have the disease?*

The first thing we have to do is translate the words in the question into probability statements. Let D be the event you have the disease, so D^{c} is the event you don't have the disease, and let $+$ be the event you get a positive result. Then the question tells us that

- $\mathbb{P}(D) = 0.02$ and $\mathbb{P}(D^{\mathsf{c}}) = 0.98$;
- $\mathbb{P}(+ \mid D) = 0.95$;
- $\mathbb{P}(+ \mid D^{\mathsf{c}}) = 0.05$;
- we want to find $\mathbb{P}(D \mid +)$.

So from Bayes' theorem, we have

$$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(D)\,\mathbb{P}(+ \mid D)}{\mathbb{P}(+)} = \frac{0.02 \times 0.95}{\mathbb{P}(+)}.$$

How can we find $\mathbb{P}(+)$? Well, importantly, D (you have the disease) and D^{c} (you don't) make up a partition – so we can use the law of total probability. We have

$$\begin{aligned}\mathbb{P}(+) &= \mathbb{P}(D)\,\mathbb{P}(+ \mid D) + \mathbb{P}(D^{\mathsf{c}})\,\mathbb{P}(+ \mid D^{\mathsf{c}}) \\ &= 0.02 \times 0.95 + 0.98 \times 0.05.\end{aligned}$$

Putting the Bayes' theorem result and the law of total probability result together, we get

$$\mathbb{P}(D \mid +) = \frac{0.02 \times 0.95}{0.02 \times 0.95 + 0.98 \times 0.05} = 0.28.$$

So if you get a positive result on this 95%-accurate test, there's still only about a 1 in 4 chance you actually have the disease.

Many people find this result surprising. It sometimes helps to put more concrete numbers on things. Suppose 1000 people get tested. On average, we expect about 20 of them to have the disease, and 980 of to not have the disease. Of the 20 with the disease, on average 19 will correctly test positive, while 1 will test negative. Of the 980 without the disease, an average 931 will correctly test negative, but 49 will wrongly test positive. So of the $19 + 49 = 68$ people with positive tests, only 19 of them actually have the disease, which is 28%.

The key point is that the disease is rare – only 2% of people have it. So even though positive test increases the likelihood you have the disease a lot (it's about 14 times more likely), it's not enough to make it a very large probability.

Summary

- The law of total probability says that if $A_1, A_2, \ldots A_n$ is a partition of the sample space (that is, exactly one of them occurs), then

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i).$$

- Bayes' theorem says that $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A)\,\mathbb{P}(B \mid A)}{\mathbb{P}(B)}$.

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 2.1 and 2.2.
- Grimmett and Welsh, *Probability*, Section 1.8.

On Problem Sheet 3, you should now be able to complete Questions A1, A2, B1–3 and C1.

Chapter 9

Discrete random variables

9.1 What is a random variable?

Let's consider again the case of rolling two dice. We know that the sample space is the set of pairs of numbers between 1 and 6. But if we are rolling the two dice as part of a board game, we might only care about the *total* score on the two dice, and the actual the two individual dice scores might be irrelevant. Let us wrote write X for the total score. This X is a sort of "numerical summary" of the experiment. Probabilists call such a numerical summary a **random variable**.

One we've defined this random variable X, it can be easier to work with X than with sample spaces and events. For example, if we want to know the probability that our two dice rolls add up to 5, it's more convenient to write

$$\mathbb{P}(X = 5)$$

rather than

$$\mathbb{P}(A) \quad \text{where} \quad A = \big\{(1,4),(2,3),(3,2),(4,1)\big\}.$$

The probability that $X = 5$ is $\mathbb{P}(X = 5) = \frac{4}{36} = \frac{1}{9}$. We might also be interested in other things about the total score X, like what the average total score over many pairs if dice rolls is.

The good news is that, once we have properly set up our random variable X, we can often then choose to ignore things like the sample space, the probability measure, and individual events.

Random variables are typically given capital letters from late in the alphabet, like X, Y, Z. Values that those random variables take are often given the lower-case equivalent, like x, y, z.

This idea of a random variable as a numerical summary of an experiment is how we *think* about random variables when solving problems. On the other hand, as mathematicians, we also want to define carefully what a random variable is as a mathematical object. We'll discuss that now (but if you find the next couple of paragraphs difficult to follow, you won't miss much if you skip them).

In this formal mathematical view, our experiment of rolling two dice is represented by a sample space

$$\Omega = \{\omega = (\omega_1, \omega_2) : \omega_1, \omega_2 \in \{1, 2, 3, 4, 5, 6\}\}$$

of pairs $\omega = (\omega_1, \omega_2)$ of numbers from 1 to 6, where ω_1 represents the first dice roll and ω_2 the second dice roll. The random variable X is the score on the first dice plus the score on the second dice – that is,

$$X(\omega) = \omega_1 + \omega_2.$$

In other words, X is a *function* which takes in a sample outcome $\omega \in \Omega$ and outputs a real number $X = X(\omega) = \omega_1 + \omega_2$.

Definition 9.1. Let Ω be a sample space. Then a **random variable** is a function X from Ω to the real numbers $\mathbb{R}$; that is, to each sample outcome ω it assigns a real number $X(\omega)$.

Expressions like $\mathbb{P}(X = x)$ or $\mathbb{P}(X \in A)$ should be understood as representing more formal probabilities

$$\mathbb{P}(\{\omega : X(\omega) = x\}) \quad \text{or} \quad \mathbb{P}(\{\omega : X(\omega) \in A\}).$$

It's useful to have a notation for the values a random variable can take.

Definition 9.2. The set of values a random variable X can take is called its **range**, $\text{Range}(X) = \{X(\omega) : \omega \in \Omega\}$.

So, for example, the range of the dice total X is $\text{Range}(X) = \{2, 3, \ldots, 12\}$, because those are the possible outcome from the sum of two dice rolls.

Random variables that we will consider in this module will be one of two types:

- **Discrete random variables** have a range that is a collection of discrete separate counts, so $\text{Range}(X)$ is finite (like the dice total being an integer between 2 and 12) or countably infinite (like the positive integers). Discrete random variables can be used as models for "count data".

- **Continuous random variables** have a range that is a continuum of slowly varying measurements, so $\text{Range}(X)$ is uncountably infinite (like the real numbers, the positive real numbers, or the interval $[0, 1]$). Continuous random variables can be used as models for "measurement data".

The techniques we will use to study discrete random variable and continuous random variables are quite different, although have some similarities. For this week and the two weeks after, we will just look at discrete random variables; later in Lectures 15 to 17 we will look at continuous random variables.

9.2 Probability mass function

We now consider only discrete random variables X, where the range $\text{Range}(X)$ is finite or countably infinite.

To fully understand a discrete random variable X, we need only understand the probabilities $p(x) = \mathbb{P}(X = x)$. These are captured by the probability mass function.

Definition 9.3. Let X be a discrete random variable. Theb the **probability mass function** (or **PMF**) p_X of X is given by

$$p_X(x) = \mathbb{P}(X = x) \qquad \text{for } x \in \text{Range}(X).$$

(When the random variable X is obvious from context, we'll just write $p(x)$ without the subscript.)

Once we have the PMF, then Axiom 3 tells us that for any set A, we have

$$\mathbb{P}(X \in A) = \sum_{x \in A} \mathbb{P}(X = x) = \sum_{x \in A} p(x).$$

(Recall that the symbol $\in$ means "is an element of", or just "is in" for short.) So the probability that X is in some set A can be found by simply adding up $p(x)$ for all the values x in A. Thus the PMF $p(x)$ is the only thing we need to know.

Example 9.1. Consider tossing a biased coin, that is Heads with probability p and Tails with probability $1-p$. Let $X = 1$ if the coin lands Heads, and $X = 0$ if the coin lands Tails. The PMF p_X of this random variable is given by

$$p_X(0) = 1 - p \qquad p_X(1) = p.$$

We could alternatively think of the same random variable X as representing the result of an experiment, where $X = 1$ represents a success, with probability $p_X(1) = p$, and $X = 0$ represents a failure, with probability $p_X(0) = 1 - p$.

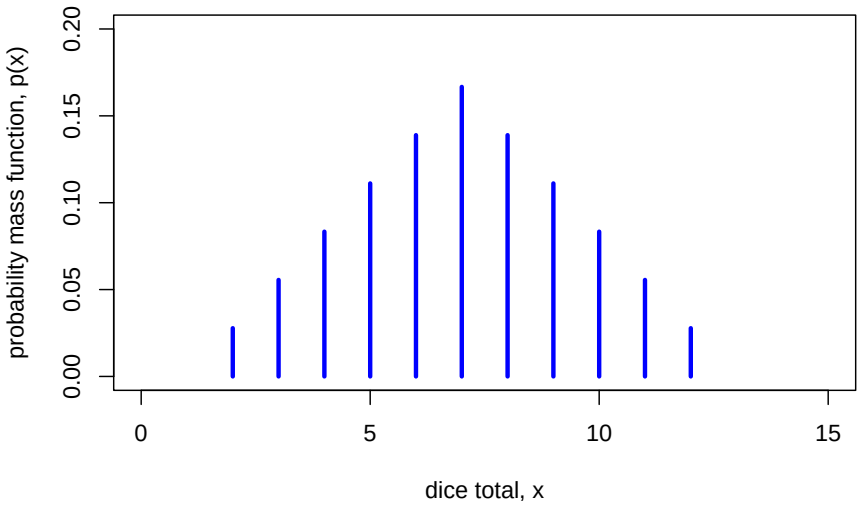
A random variable X with this PMF is called a **Bernoulli trial** (or a "Bernoulli random variable", or is said to "follow the Bernoulli distribution" – after the seventeenth-century Swiss mathematician Jacob Bernoulli). We use the notation $X \sim \text{Bern}(p)$ for short.

Example 9.2. Let X being the sum of two dice rolls. As this is a classical probability problem, the probability $p(x)$ of rolling a total of x is $n(x)/36$, where $n(x)$ is the number of ways of rolling a total of x. So, for example, there is only one way $(1, 1)$ of rolling a total of 2, so $p(2) = \frac{1}{36}$, but there are 5 ways of rolling a 6: $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$; so $p(5) = \frac{5}{36}$.

The PMF p of X is given by

x	2	3	4	5	6	7
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$

x	···	8	9	10	11	12
$p(x)$	···	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



Note that since $p(x) = \mathbb{P}(X = x)$ is a probability, it must be greater than or equal to 0, by Axiom 1. Further, if we add up $p(x)$ we get

$$\sum_x p(x) = \sum_{x \in \text{Range}(X)} \mathbb{P}(X = x) = \mathbb{P}(X \in \text{Range}(X)) = \mathbb{P}(\Omega) = 1,$$

by Axiom 2. Hence we have the following:

Theorem 9.1. *Let X be a discrete random variable, and let p_X be its PMF. Then*

- $p_X(x) \geq 0$ *for all* $x \in \text{Range}(X)$*;*
- $\sum_{x \in \text{Range}(X)} p_X(x) = 1.$

9.3 Cumulative distribution function

Sometimes it is useful to know the probability a random variable X is less or equal to than some value x. This is captured by the cumulative distribution function.

Definition 9.4. Let X be a random variable. Then the **cumulative distribution function** (or **CDF**) F_X of X is given by

$$F_X(x) = \mathbb{P}(X \leq x) \qquad \text{for } x \in \mathbb{R}..$$

When X is a discrete random variable, we can find $F_X(x) = \mathbb{P}(X = x)$ by simply adding up the probabilities of all the outcomes y that are less than or equal to x. That is,

$$F_X(x) = \sum_{y \leq x} \mathbb{P}(X = y) = \sum_{y \leq x} p_X(y).$$

Example 9.3. *Let $X \sim Bern(p)$ be a Bernoulli random variable with success probability p. What is the CDF F_X of X?*

The value $F_X(x)$ of CDF will depend on the number x given as an input.

- If x is a number less than 0 (like $x = -1$, for example), then the event $X \leq x$ cannot happen, as there are no outcomes for X that are that small. So $F_X(x) = 0$.
- If x is a number at least 0 but smaller than 1 (like $x = \frac{1}{2}$, for example), then the event $X \leq x$ can only happen if $X = 0$. So $F_X(x) = \mathbb{P}(X = 0) = p_X(0) = 1 - p$.
- If x is a number at least 1 (like $x = 2$, for example), then event $X \leq x$ will definitely happen, because both outcomes, 0 and 1, are less than or equal to x. So $F_X(x) = 1$.

So the CDF F_X is

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - p & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1. \end{cases}$$

Example 9.4. *Let X be the sum of two dice rolls. What is the CDF F?*

Again, the answer depends on the value of the number x.

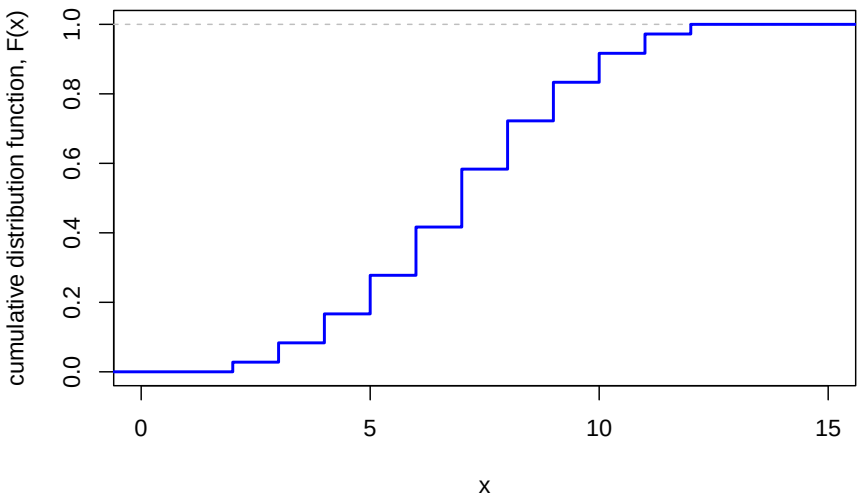
- If $x < 2$, then $X \leq x$ cannot occur. So $F(x) = 0$.
- If $2 \leq x < 3$, then the only outcome of X consistent with $X \leq x$ is the outcome $X = 2$. So $F(x) = p(2) = \frac{1}{36}$.
- If $3 \leq x < 4$, then the only outcomes of X consistent with $X \leq x$ are the outcomes $X = 2$ and $X = 3$. So $F(x) = p(2) + p(3) = \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$.
- If $4 \leq x < 5$, then the only outcomes of X consistent with $X \leq x$ are the outcomes $X = 2$, $X = 3$ and $X = 4$. So $F(x) = p(2) + p(3) + p(4) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36}$.
- ...

- If $x \geq 12$, then all the outcomes in the range of X have $X \leq x$, so $F(x) = 1$.

Hence, the CDF F of X is given by the following step function:

$x \in$	$(-\infty, 2)$	$[2, 3)$	$[3, 4)$	$[4, 5)$	$[5, 6)$	$[6, 7)$
$F(x)$	0	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$

$x \in$	$\cdots$	$[7, 8)$	$[8, 9)$	$[9, 10)$	$[10, 11)$	$[11, 12)$	$[12, \infty)$
$F(x)$	$\cdots$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1



Note that the CDF is a "step function" that starts at 0, then jumps up suddenly at each of the values $2, 3, \ldots, 12$, finally ending up at 1.

For any random variable X with CDF F,

- if x is smaller than everything in the Range(X), then $F(x) = 0$, because X cannot be that small;
- if x is greater than everything in the Range(X), then $F(x) = 1$, because X cannot be any bigger than that;
- $F(x)$ is non-decreasing in x, because the events $\{X \leq x\}$ get bigger as x gets increases.

So $F(x)$ starts at 0 and goes upwards until it gets to 1. For a discrete random variable, there are sudden upward jumps at values of x in the range of X, with constant values in between.

Summary

- A random variable is a numerical summary of a random experiment.
- The probability mass function (PMF) is $p_X(x) = \mathbb{P}(X = x)$.
- The cumulative distribution function (CDF) is $F_X(x) = \mathbb{P}(X \leq x)$.
- A Bernoulli random variable is 0 with probability $1 - p$ and 1 with probability p.

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 4.1 and 4.2.
- Grimmett and Welsh, *Probability*, Section 2.1.

Chapter 10

Expectation and variance

We continue our study of discrete random variables. Recall that the PMF p_X of a discrete random variable X is $p_X(x) = \mathbb{P}(X = x)$.

10.1 Expectation

Often, we will be interested in the "average" value of a random variable – for example, the "average" total from two dice rolls – which represents what the "central" value of the random variable is. This average is called the "expectation".

Definition 10.1. Let Ω be a finite or countably infinite sample space, $\mathbb{P}$ be a probability measure on Ω, and X be a discrete random variable on Ω. Then the **expectation** (or **expected value**) of X is

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\{\omega\}).$$

If p_X is the PMF of X, then a more convenient formula is

$$\mathbb{E}X = \sum_{x \in \text{Range}(X)} x\, p_X(x).$$

We get the second formula from the first by grouping together all outcomes ω that lead to the same value $x = X(\omega)$ of X. It's only this second formula we actually use when calculating expectations.

Note that "expectation" is simply the name that mathematicians give to the value $\mathbb{E}X = \sum_x x\, p(x)$. We don't necessarily "expect" to get the value $\mathbb{E}X$ as the outcome in the normal English-language sense of the word "expect". (Indeed, you might like to check that the expectation of a single dice roll is 3.5, but you certainly don't "expect" to get the number 3.5 in a single roll of the dice!) We will see later in the module that the the expectation can be interpreted as a sort of "long-run mean outcome".

Example 10.1. *Let $X \sim Bern(p)$ be a Bernoulli trial with success probability p. What is the expectation $\mathbb{E}X$?*

We know that the PMF is $p(0) = 1 - p$ and $p(1) = p$. So, using the second formula in the definition, we have

$$\mathbb{E}X = \sum_x x\, p(x) = 0 \times (1-p) + 1 \times p = p.$$

Example 10.2. *What is the expected value of the sum of two dice rolls?*

When X is the total of two dice rolls, we found the PMF of X last time. The expectation is

$$\begin{aligned}\mathbb{E}X &= \sum_{x \in \text{Range}(X)} x\, p(x) \\ &= 2 \times \tfrac{1}{36} + 3 \times \tfrac{2}{36} + \cdots + 12 \times \tfrac{1}{36} \\ &= \tfrac{252}{36} \\ &= 7.\end{aligned}$$

10.2 Functions of random variables

In previous examples, we looked at X being the total of the dice rolls. But we could equally well chosen to have looked at a different random variable that is a function of that total X, like "double the total and add 1" $Y = 2X + 1$, or "the total minus 4, all squared" $Z = (X-4)^2$. (I'm not sure *why* you'd care about these, but you could study them if you wanted to…)

It is possible, although sometimes a bit tricky, to work out the whole PMF of these new random variables that are functions of X – and indeed you may learn how to do this if you take more probability or statistics modules next year. Here, we will stick to the easier problem of just calculating the expectation of the new random variables.

Theorem 10.1 (Law of the unconscious statistician)**.** *Let X be a random variable, and let $Y = g(X)$ be another random variable that is a function g of X. Then*

$$\mathbb{E}Y = \mathbb{E}\, g(X) = \sum_x g(x)\, p_X(x).$$

(The rather cruel name of this theorem is, I think, because this is the formula you might carelessly write down for $\mathbb{E}g(X)$ if you weren't thinking carefully – but it turns out it's correct!)

Proof. (Non-examinable) The PMF of y can be given in terms of the PMF of x – we just need to add up all the xs that lead to the same y. That is,

$$p_Y(y) = \sum_{x\,:\,g(x)=y} p_X(x).$$

(Remember that the colon : here means "such that", so this is a sum over all the x such that $g(x) = y$.)

Using this, and from the definition of expectation, we have

$$\begin{aligned}\mathbb{E}Y &= \sum_y y\, p_Y(y) \\ &= \sum_y y \sum_{x:g(x)=y} p_X(x) \\ &= \sum_y \sum_{x:g(x)=y} y\, p_X(x) \\ &= \sum_y \sum_{x:g(x)=y} g(x)\, p_X(x),\end{aligned}$$

since $y = g(x)$ inside the second sum. But these two sums together are summing over all x, just partitioned by which value of y they lead to, so they can be replaced by just a single sum over x. That gives the theorem. □

There are some functions for which this expression becomes particularly simple.

Theorem 10.2 (Linearity of expectation, 1)**.** *Let X be a random variable. Then*

1. $\mathbb{E}(aX) = a\mathbb{E}X$;
2. $\mathbb{E}(X + b) = \mathbb{E}X + b$.

Proof. We use the law of the unconscious statistician.

For part 1, we can take the a outside the sum, to get

$$\mathbb{E}(aX) = \sum_x ax\, p_X(x) = a \sum_x x\, p_X(x) = a\mathbb{E}X.$$

For part 2, we have

$$\begin{aligned}\mathbb{E}(X + b) &= \sum_x (x + b)\, p_X(x) \\ &= \sum_x \left(x\, p_X(x) + b\, p_X(x)\right) \\ &= \sum_x x\, p_X(x) + \sum_x b\, p_X(x) \\ &= \mathbb{E}(X) + b \sum_x p_X(x) \\ &= \mathbb{E}(X) + b.\end{aligned}$$

The last line was because PMFs always add up to 1, so $\sum_x p_X(x) = 1$. □

So for our "double the dice total and add 1" random variable $Y = 2X + 1$, we have

$$\mathbb{E}Y = \mathbb{E}(2X + 1) = 2\mathbb{E}X + 1 = 2 \times 7 + 1 = 15.$$

10.3 Variance

In the same way as the expectation of a random variable tells us about central values of it, the "variance" of a random variable tells us about the spread of typical values.

Definition 10.2. Let X be a random variable with expectation $\mathbb{E}X = \mu$. Then the **variance** of X is

$$\mathrm{Var}(X) = \mathbb{E}(X - \mu)^2.$$

(To be clear, the notation here means the expectation of $(X - \mu)^2$; and *not* $\mathbb{E}(X - \mu)$ then squared, which would be $0^2 = 0$.)

Note that $(X - \mu)^2 \geq 0$ is a square, so always non-negative, and hence the variance $\mathrm{Var}(X) \geq 0$ is always non-negative also. Sometimes we call the square-root of the variance the **standard deviation**.

It may not surprise you, if you remember Lecture 1 that to go along with this "definitional formula" for the variance, we also have a "computational formula", which can sometimes be more convenient.

Theorem 10.3. *Let X be a random variable with expectation $\mathbb{E}X = \mu$. Then the variance $\mathrm{Var}(X) = \mathbb{E}(X - \mu)^2$ can also be calculated as*

$$\mathrm{Var}(X) = \mathbb{E}X^2 - \mu^2.$$

(Again, $\mathbb{E}X^2$ means the expectation of X^2, not $\mathbb{E}X$ squared, which would make the variance 0.)

Proof. Similar to before, we expand out the brackets, and use linearity of expectation (in the same way we "brought the sum inside" with the sample variance previously). We get

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X - \mu)^2 \\
&= \mathbb{E}(X^2 - 2\mu X + \mu^2) \\
&= \mathbb{E}X^2 - \mathbb{E}(2\mu X) + \mathbb{E}\mu^2 \\
&= \mathbb{E}X^2 - 2\mu\,\mathbb{E}X + \mu^2.
\end{aligned}$$

But we said that $\mathbb{E}X$ would be called μ, so we can substitute in $\mathbb{E}X = \mu$, to get

$$\mathrm{Var}(X) = \mathbb{E}X^2 - 2\mu^2 + \mu^2 = \mathbb{E}X^2 - \mu^2,$$

as required. □

(A brief optional note for pedants: Writing $\mathbb{E}(X^2 - 2\mu X) = \mathbb{E}X^2 - 2\mu X$ is not, strictly speaking, justified by the result that above we called "linearity of expectation, 1". However, you can check that it easily follows from the law of the unconscious statistician, and we will also later see a result we call "linearity of expectation, 2", of which it is a special case.)

Example 10.3. Let $X \sim \text{Bern}(p)$ be a Bernoulli trial, and recall that $\mathbb{E}X = p$.

Using the definitional formula, we have

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}(X-p)^2 \\ &= (0-p)^2\, p_X(0) + (1-p)^2\, p_X(1) \\ &= p^2 \times (1-p) + (1-p)^2 \times p \\ &= p(1-p)(p+(1-p)) \\ &= p(1-p).\end{aligned}$$

Alternatively, using the computational formula, we have

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}X^2 - p^2 \\ &= (0^2\, p_X(0) + 1^2 p_X(1)) - p^2 \\ &= 0 \times (1-p) + 1 \times p - p^2 \\ &= p - p^2 \\ &= p(1-p).\end{aligned}$$

Example 10.4. For the total of two dice, using the computational formula, we have

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}X^2 - \mu^2 \\ &= \left(2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + \cdots + 12^2 \times \frac{1}{36}\right) - 7^2 \\ &= \frac{1974}{36} - 49 \\ &= \frac{70}{12} \approx 5.8.\end{aligned}$$

Finally, a result on what happens to the variance of simple functions of random variables.

Theorem 10.4. *Let X be a random variable. Then*

1. $\text{Var}(aX) = a^2\, \text{Var}(X)$;
2. $\text{Var}(X+b) = \text{Var}(X)$.

You will prove this on the problem sheet.

Summary

- The expectation of a random variable X is $\mathbb{E}X = \sum_x x\, p_X(x)$.
- The variance of a random variable X with expectation μ is $\text{Var}(X) = \mathbb{E}(X-\mu)^2$.
- $\mathbb{E}(aX+b) = a\mathbb{E}X + b$ and $\text{Var}(aX+b) = a^2\, \text{Var}(X)$.

Recommended reading:

- Stirzaker, *Elementary Probability*, Section 4.3.
- Grimmett and Welsh, *Probability*, Sections 2.3 and 2.4.

On Problem Sheet 3, you should now be able to complete all questions.

Problem Sheet 3

This is Problem Sheet 3. This problem sheet covers material from Lectures 7 to 10. You should work through all the questions on this problem sheet in preparation for your tutorial in Week 6. The problem sheet contains two assessed questions, which are due in by **Monday 13 November**.

A: Short questions

A1. Consider dealing two cards (without replacement) from a pack of cards. Which of the following pairs of events are independent?

(a) "The first card is a Heart" and "The first card is Red".

(b) "The first card is a Heart" and "The first card is a Spade".

(c) "The first card is a Heart" and "The first card is an Ace".

(d) "The first card is a Heart" and "The second card is a Heart".

(e) "The first card is a Heart" and "The second card is an Ace".

A2. Consider rolling two dice. Let A be the event that the first roll is even, let B be the event that the second roll is even, and let C be the event that the total score is even. You may assume the dice rolls are independent; so, in particular, events A and B are independent.

(a) Are A and C independent?

(b) Are B and C independent?

(c) Is it true that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$?

A3. Two events A and B have probabilities $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.5$, and the two events are independent. What is $\mathbb{P}(A \cup B)$?

A4. Consider the random variable X with the following PMF:

x	-1	0	0.5	2
$p(x)$	0.1	0.4	0.2	0.3

Find **(a)** the CDF F_X, **(b)** the expectation $\mathbb{E}X$, and **(c)** the variance $\mathrm{Var}(X)$ of X.

A5. Consider the random variable X with the following PMF:

x	1	2	4	5	a
$p(x)$	0.1	0.2	0.1	b	0.1

This random variable has $\mathbb{E}X = 4.3$. Find the values of a and b.

A6. A temperature T_C measured in degrees Celsius can be converted to a temperature T_F in degrees Fahrenheit using the formula $T_F = \frac{9}{5}T_C + 32$.

The average daily maximum temperature in Leeds in July is 19.0 °C. The variance of the daily maximum temperature measured in degrees Celsius is 10.4.

(a) What is the average daily maximum temperature in degrees Fahrenheit?

(b) What is the variance of the daily maximum temperature when measured in degrees Fahrenheit?

B: Long questions

B1. Suppose A and B are independent events. Show that A and B^{c} are also independent events.

B2. You are dealt a hand of 13 cards from a 52-card deck. Let $E_{\mathrm{A}}, E_{\mathrm{K}}, E_{\mathrm{Q}}, E_{\mathrm{J}}$ respectively be the events that your hand contains the Ace, King, Queen and Jack of Spades.

(a) What is $\mathbb{P}(E_{\mathrm{A}})$, the probability that your hand contains the Ace of Spades?

(b) Explain why $\mathbb{P}(E_{\mathrm{K}} \mid E_{\mathrm{A}}) = \frac{12}{51}$.

(c) Using the chain rule, calculate the probability that your hand contains all four of the Ace, King, Queen and Jack of Spades.

(d) Check that your answer agrees with the answer we found by classical probability methods in Example 6.4 in Lecture 6. Which method do you prefer?

B3. Soldiers are asked about their use of illegal drugs, using a so-called "randomised survey". Each soldier is handed a deck of three cards, picks one of the three cards at random, and responds according to what the card says. The three cards say:

1. "Say 'Yes.' "
2. "Say 'No.' "
3. "Truthfully answer the question 'Have you taken any illegal drugs in the past 12 months?' "

(a) What are some advantages or disadvantages of performing the experiment this way?

(b) Suppose that 40% of soldiers respond "Yes". What is the likely proportion of soldiers who have taken illegal drugs in the past 12 months.

(c) If a soldier responds "Yes", what is the probability that the soldier has taken illegal drugs in the past 12 months.

B4. A random variable X_n is said to follow the *discrete uniform distribution* on $\{1, 2, \dots, n\}$ if each of the n values in that set $\{1, 2, \dots, n\}$ is equally likely.

(a) Show that the expectation of X_n is $\mathbb{E}X_n = \dfrac{n+1}{2}$.

(b) Find the variance of X_n.

(c) Let Y be a discrete uniform distribution on $b - a + 1$ values $\{a, a+1, a+2, \dots, b-1, b\}$, for integers a and b with $a < b$. By finding a relationship between Y and X_n, for an appropriate value of n, find the expectation and variance of Y.

You may use without proof the standard results

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2} \qquad \sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}.$$

B5. A gambling game works as follows. You keep tossing a fair coin until you first get a Head. If the first Head comes up on the nth coin toss, then you win 2^n pounds.

(a) What is the probability that the first Head is seen on the nth toss of the coin?

(b) Show that the expected winnings from playing this game are infinite.

(c) The "St Petersburg paradox" refers to the observation that, despite the expected winnings from this game being infinite, few people would be prepared to play this game for, say, £100, and almost no one for £1000. Discuss a few possible "resolutions" to this paradox which could explain why people are unwilling to play this game despite seemingly having infinite expected winnings.

C: Assessed questions

The last two questions are **assessed questions**. These two questions count for 3% of your final mark for this module.

The deadline for submitting your solutions is **2pm on Monday 13 November** at the beginning of Week 7. Submission is via Gradescope. Your work will be marked by your tutor and returned on Monday 20 November, when solutions will also be made available.

Both questions are "long questions", where the marks are not only for mathematical accuracy but also for the clarity and completeness of your explanations.

You should not collaborate with others on the assessed questions: your answers must represent solely your own work. The University's rules on academic integrity – and the related punishments for violating them – apply to your work on the assessed questions.

C1. A computer spam filter is 98% effective at sending spam emails to my junk folder, but will also incorrectly send 1% of legitimate emails to my junk folder. Suppose that 1 in 10 emails are spam. What proportion of emails in my junk folder are actually legitimate emails? Explain your solution fully – you may use any results from the lectures provided you state them clearly.

C2. Let X be a random variable.

(a) Let $Y = aX$ be another random variable. What is $\mathbb{E}Y$, in terms of $\mu = \mathbb{E}X$?

(b) Using part (a), show that $\mathrm{Var}(aX) = a^2 \mathrm{Var}(X)$.

(c) Prove that $\mathrm{Var}(X + b) = \mathrm{Var}(X)$.

Solutions to short questions

A1. (c) and (e) are independent
A2. (a) Yes (b) Yes (c) No
A3. 0.8
A4. (a) — (b) 0.6 (c) 0.99
A5. $a = 9, b = 0.5$
A6 (a) 66.2 °F (b) 33.7 °F^2

Chapter 11

Binomial and geometric distributions

The **mid-semester check-in survey** is now open.

Last week, we developed the idea of random variables, and in particular discrete random variables. We saw that the benefit of random variables is that we can just worry about their distribution, which often allows us to move the sample space Ω and other more technical matters into the background. (Here, we informally use the word "distribution" to refer to the probability mass function of a random variable – or, later, the continuous equivalent, the probability density function).

There are some distributions – or, rather, some families of distributions – that are so useful that we often want to use them for modelling real-world quantities. This week, we will look at a number of useful discrete distributions.

11.1 Binomial distribution

One family of distributions we have already seen is the Bernoulli trial $\text{Bern}(p)$, which is 1 with probability p and 0 with probability $1-p$. We saw that this could model whether or a biased coin lands Heads, or more generally whether an experiment is successful.

Example 11.1. *Suppose we toss 10 independent biased coins, each of which lands Heads with probability 0.7 and Tails with probability 0.3. What is the probability we get exactly 8 Heads altogether?*

The probability that any specific 8 coins land Heads and the other 2 land Tails is $0.7^8 \times 0.3^2$. However, there are $\binom{10}{8}$ choices for which 8 coins are the ones that land Heads. Hence, the probability is

$$\mathbb{P}(8 \text{ Heads}) = \binom{10}{8} \times 0.7^8 \times 0.3^2 = 0.23.$$

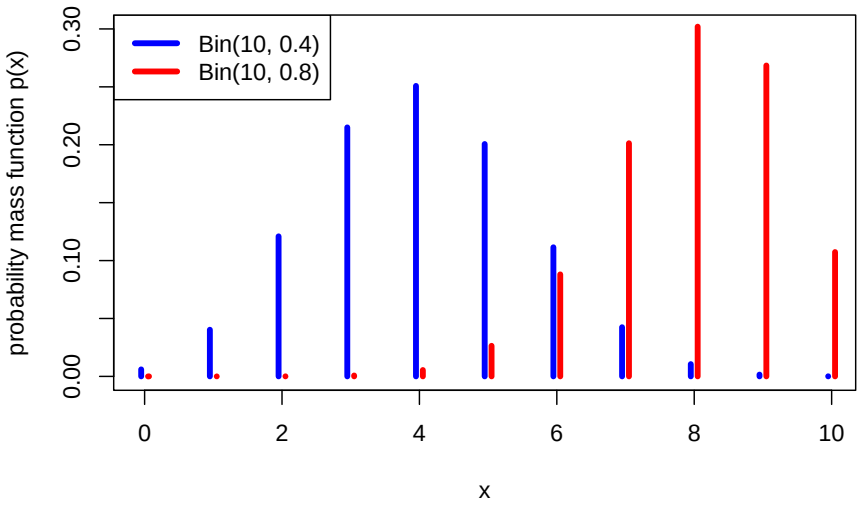
This is a special case of the binomial distribution.

Definition 11.1. Let X be a discrete random variable with range $\{0, 1, 2, \dots, n\}$ and PMF

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

Then we say that X follows the **binomial distribution** with parameters n and k, and write $X \sim \text{Bin}(n, p)$.

So a binomial random variable represents the number of successes in n Bernoulli trials. In our previous example, the number of Heads from the coin tosses was $\text{Bin}(10, 0.7)$.



Example 11.2. *Let $X \sim \text{Bin}(8, 0.2)$. What is (a) $\mathbb{P}(X = 3)$? (b) $\mathbb{P}(X \geq 2)$?*

For (a), we have from the definition

$$\mathbb{P}(X = 3) = \binom{8}{3} 0.2^3 (1-0.2)^{8-3} = 56 \times 0.2^3 \times 0.8^5 = 0.147.$$

For (b), this is an "at least" question, so it's more convenient to look at the complementary event, $\mathbb{P}(X < 2)$. So

$$\begin{aligned}
\mathbb{P}(X \geq 2) &= 1 - \mathbb{P}(X < 2) \\
&= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \\
&= 1 - 0.8^8 - 8 \times 0.2 \times 0.8^7 \\
&= 1 - 0.168 - 0.336 \\
&= 0.497.
\end{aligned}$$

What about the expectation and variance of a binomial random variable?

Theorem 11.1. *Let $X \sim Bin(n, p)$. Then*

- $\mathbb{E}X = np$,
- $\mathrm{Var}(X) = np(1-p)$.

One can prove this by working out the sums – for example, the expectation is the value of the sum

$$\mathbb{E}X = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x},$$

which is a bit tricky to calculate, but not fundamentally difficult mathematics. However, in next section we will see an easier way, so we'll reserve the proof until then instead.

For my 10 biased coins that are each Heads with probability 0.7, the expectation and variance are

$$\mathbb{E}X = 10 \times 0.7 = 7$$
$$\mathrm{Var}(X) = 10 \times 0.7 \times 0.3 = 2.1$$

11.2 Geometric distribution

Example 11.3. *I decide to roll a fair dice until I first roll a six, and then stop. What's the probability I get the first six on my 5th roll of the dice?*

For the first six to be on the 5th attempt, the first 4 rolls have to be non-sixes, and then the fifth roll has to be a six. This has probability

$$\left(\tfrac{5}{6}\right)^4 \times \tfrac{1}{6} = \tfrac{625}{7776} = 0.08.$$

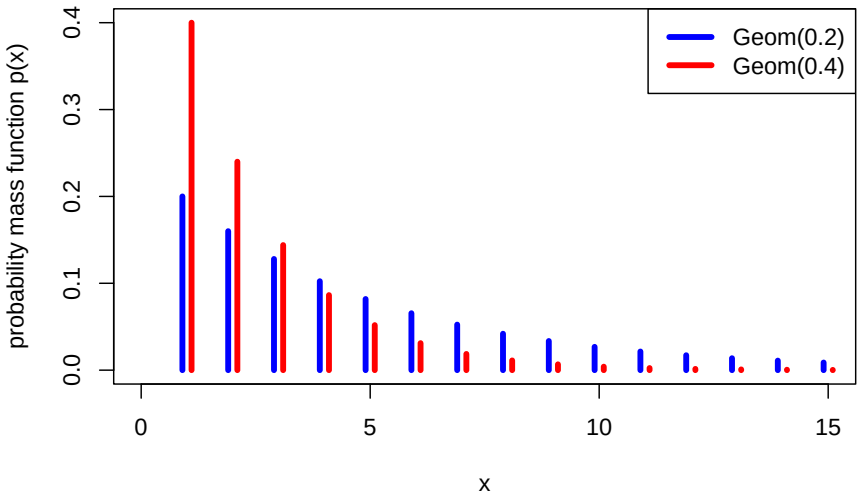
This is a special case of the geometric distribution.

Definition 11.2. Let X be a discrete random variable with range $\{1, 2, \dots\}$ and PMF

$$p(x) = (1-p)^{x-1} p.$$

Then we say that X follows the **geometric distribution** with parameter p, and write $X \sim \mathrm{Geom}(p)$.

So a geometric random variable represents the number of Bernoulli(p) trials until the first success. In our previous example, the number of dice rolls until a six was $\mathrm{Geom}(\frac{1}{6})$.



Example 11.4. *Let* $X \sim \text{Geom}(0.4)$. *What is (a)* $\mathbb{P}(X = 3)$*? (b)* $\mathbb{P}(X \geq 3)$.

For part (a), we have

$$\mathbb{P}(X = 3) = (1 - 0.4)^2 \times 0.4 = 0.144.$$

For part (b), we have

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) = 1 - 0.4 - (1 - 0.4) \times 0.4 = 1 - 0.64 = 0.36.$$

Theorem 11.2. *Let* $X \sim Geom(p)$. *Then*

- $\mathbb{E}X = \dfrac{1}{p}$,
- $\text{Var}(X) = \dfrac{1-p}{p^2}$.

So the expected number of rolls until rolling a six is

$$\mathbb{E}X = \frac{1}{\frac{1}{6}} = 6,$$

with variance

$$\text{Var}(X) = \frac{1 - \frac{1}{6}}{\left(\frac{1}{6}\right)^2} = 30.$$

Proof. (Non-examinable) For the expectation, we want to calculate

$$\mathbb{E}X = \sum_{x=1}^{\infty} x(1-p)^{x-1}p = p\sum_{x=0}^{\infty} x(1-p)^{x-1}.$$

(We can include the $x = 0$ term in the sum since it is equal to 0.)

At this point we will invoke the identity

$$\sum_{x=0}^{\infty} xa^{x-1} = \frac{1}{(1-a)^2},$$

which can be proved by differentiating the standard sum of a geometric progression

$$\sum_{x=0}^{\infty} a^x = \frac{1}{1-a}$$

with respect to a.

Using that identity with $a = 1 - p$, we get

$$\mathbb{E}X = p\sum_{x=0}^{\infty} x(1-p)^{x-1} = p\,\frac{1}{\left(1-(1-p)\right)^2} = \frac{1}{p},$$

as required.

For the variance, we will use a trick that sometimes comes in useful, which is to start by calculating $\mathbb{E}X(X-1)$. Here we get

$$\mathbb{E}X(X-1) = \sum_{x=1}^{\infty} x(x-1)(1-p)^{x-1}p = p(1-p)\sum_{x=0}^{\infty} x(x-1)(1-p)^{x-2}.$$

To calculate the sum, we note that differentiating the geometric progression formula twice gives

$$\sum_{x=0}^{\infty} x(x-1)a^{x-2} = \frac{2}{(1-a)^3},$$

so we get

$$\mathbb{E}X(X-1) = p(1-p)\sum_{x=0}^{\infty} x(x-1)(1-p)^{x-2} = p(1-p)\,\frac{2}{p^3} = \frac{2(1-p)}{p^2}.$$

We now want to use the computational formula $\mathrm{Var}(X) = \mathbb{E}X^2 - \mu^2$ to get the variance. We know $\mu = 1/p$, and from the calculation above, we have

$$\mathbb{E}X(X-1) = \mathbb{E}X^2 - \mathbb{E}X = \mathbb{E}X^2 - \frac{1}{p} = \frac{2(1-p)}{p^2}.$$

So

$$\begin{aligned}\mathrm{Var}(X) = \mathbb{E}X^2 - \mu^2 &= \left(\frac{2(1-p)}{p^2} + \frac{1}{p}\right) - \left(\frac{1}{p}\right)^2 \\ &= \frac{2(1-p)+p-1}{p^2} \\ &= \frac{1-p}{p^2}.\end{aligned}$$

□

Note: Here, we defined a geometric random variable as being the number of trials up to *and including* the first success, which is a number in $\{1, 2, ...\}$. However, some authors define it as the number of failures *before* the first success, which is a number in $\{0, 1, 2, ...\}$. If X is our definition and Y is the second "number of failures" definition, then X and $Y + 1$ have the same distribution. Annoyingly, R uses the "number of failures before success" definition, as we will discuss in a later R worksheet.

Summary

Distribution	Range	PMF	Expectation	Variance
Bernoulli: Bern(p)	$\{0, 1\}$	$p(0) = 1-p$, $p(1) = p$	p	$p(1-p)$
Binomial: Bin(n, p)	$\{0, 1, ..., n\}$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Geometric: Geom(p)	$\{1, 2, ...\}$	$(1-p)^{x-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Recommended reading:

- Stirzaker, *Elementary Probability*, Sections 4.2 and 4.3.
- Grimmett and Welsh, *Probability*, Section 2.2.

Chapter 12

Poisson distribution

We have seen three important families of discrete random variables: the Bernoulli, binomial, and geometric distributions. We now look at our fourth and final discrete distribution: the Poisson distribution.

12.1 Definition and properties

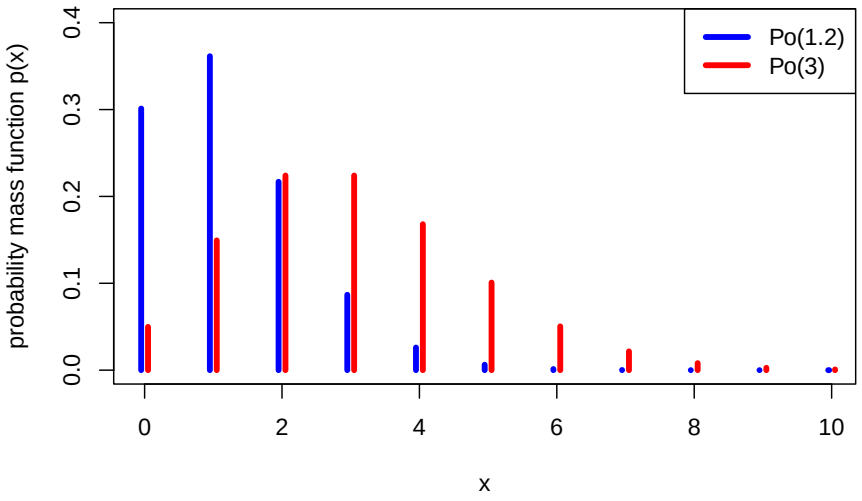
Another important distribution is the Poisson distribution. The Poisson distribution (roughly "*pwa*-song") is typically used to model "the number of times something happens in a set period of time". For example, the number of emails you receive in a day; the number of claims at an insurance company each year; or the number of calls to call centre in one hour. (Famously, one of the first historical datasets modelled using a Poisson distribution was "the number of Prussian soldiers in different cavalry units kicked to death by their own horse between 1875 and 1894".) We'll explain why the Poisson distribution is a good model for this in the next subsection.

Definition 12.1. Let X be a discrete random variable with range $\{0, 1, 2, \dots\}$ and PMF

$$p_X(x) = \mathrm{e}^{-\lambda}\frac{\lambda^x}{x!}.$$

Then we say that X follows the **Poisson distribution** with **rate** λ, and write $X \sim \mathrm{Po}(\lambda)$.

Here, λ is a lower-case Greek letter "lambda". I should also note that we interpret $0! = 1$.



The Poisson distribution is named after the French mathematician Siméon-Denis Poisson who wrote about it in 1837, although the origin of the idea is more than 100 years earlier with another French mathematician, Abraham de Moivre.

Example 12.1. *I receive emails from students at the rate of $\lambda = 3$ per hour, modelled as a Poisson distribution. What is the probability I get (a) two emails in an hour, (b) no emails in an hour?*

The number of emails per hour is $X \sim \text{Po}(3)$.

For (a), we have

$$\mathbb{P}(X = 2) = p(2) = \text{e}^{-3}\frac{3^2}{2!} = \frac{9}{2}\text{e}^{-3} = 0.224.$$

For part (b), and remembering that $0! = 1$, we have

$$\mathbb{P}(X = 0) = p(0) = \text{e}^{-3}\frac{3^0}{0!} = \text{e}^{-3} = 0.050.$$

The parameter λ is called the "rate" because that indeed the number of emails (or insurance claims, or phone calls, or deaths by horse-kicking) that we expect to see.

Theorem 12.1. *Let $X \sim Po(\lambda)$. Then*

1. *$p(x)$ is indeed a PMF, in that $\sum_{x=0}^{\infty} p(x) = 1$.*
2. *$\mathbb{E}X = \lambda$,*
3. *$\text{Var}(X) = \lambda$.*

Proof. We'll do the first two here, then you can do the variance in Problem Sheet 4.

It will be useful to remember the Taylor series for the exponential function,

$$\mathrm{e}^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^k}{x!}.$$

For part 1, to see that the PMF does indeed sum to one, note that the Taylor series gives us

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^x}{x!} = \mathrm{e}^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \mathrm{e}^{-\lambda}\, \mathrm{e}^{\lambda} = 1.$$

For part 2, for the expectation, we have

$$\begin{aligned}
\mathbb{E}X &= \sum_{x=0}^{\infty} x\, \mathrm{e}^{-\lambda} \frac{\lambda^x}{x!} \\
&= \mathrm{e}^{-\lambda} \sum_{x=1}^{\infty} x\, \frac{\lambda^x}{x!} \\
&= \mathrm{e}^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \\
&= \lambda \mathrm{e}^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}
\end{aligned}$$

In the second line, we took $\mathrm{e}^{-\lambda}$ outside the sum, and allowed ourselves to start the sum from 1, since the $x = 0$ term was 0 anyway; in the third line, we cancelled the x from the $x!$ to get $(x-1)!$; and in the fourth line we took one of the λs in λ^x outside the sum, to give ourselves terms in $x-1$ inside the sum. We can now "re-index" the sum by putting $y = x - 1$, to get

$$\mathbb{E}X = \lambda \mathrm{e}^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \lambda \mathrm{e}^{-\lambda} \mathrm{e}^{\lambda} = \lambda,$$

where we used the Taylor series again. □

12.2 Poisson approximation to the binomial

Suppose I own a watch shop in Leeds. My watches are very expensive, so I don't need to sell many each day – in fact, I sell an average of 4.8 watches per day. How should I model the number of watches sold each day as a random variable?

One way could be to say this. There are n people living in Leeds and the surrounding area. On any given day, each of those n people will independently buy a watch from my shop with some probability p. Thus the total number of watches I sell could be modelled as a binomial distribution $\mathrm{Bin}(n, p)$.

But what should n and p be? To make the average $\mathbb{E}X = np = 4.8$, I must take $p = 4.8/n$. But what about n? We know n is a very big number, because Leeds is a big city – so let's take a limit as $n \to \infty$. It turns out, that this distribution $\text{Bin}(n, 4.8/n)$ becomes a Poisson(4.8) distribution!

Theorem 12.2. *Fix $\lambda \geq 0$, and let $X_n \sim Bin(n, \lambda/n)$ for all integers $n \geq \lambda$. Then $X_n \to Po(\lambda)$ in distribution as $n \to \infty$, by which we mean that if $Y \sim Po(\lambda)$, then*

$$p_{X_n}(x) \to p_Y(x) \qquad \text{for all } x \in \{0, 1, \dots\}.$$

A looser way to state the principle of this theorem would be this: *When n is very large and p very small, in such a way that np is a small-ish number, then $Bin(n, p)$ is well approximated by $Po(\lambda)$ where $\lambda = np$.*

This is why a Poisson distribution is a good model for the number of occurrences in a set time period. It applies if there lots of things that could happen (large n), each one is individually unlikely (small p), and on average a few of them will actually happen ($\lambda = np$ small-ish).

Example 12.2. *A lecturer teaches a module with $n = 100$ students, and estimates that each student turns up to office hours drop-in sessions independently with probability $p = 0.035$. What is the probability that (a) exactly 5, (b) 2 or more students turn up to a drop-in session?*

If we let X be the number of students that turn up to a drop-in session, then the exact distribution of X is $X \sim \text{Bin}(100, 0.035)$.

For part (a), we then have

$$\mathbb{P}(X = 5) = \binom{100}{5} 0.035^5 (1 - 0.035)^{100-5} = 0.134.$$

For part (b), we have an "at least" event, so we use the complement rule to get

$$\begin{aligned}
\mathbb{P}(X \geq 2) &= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \\
&= 1 - \binom{100}{0} 0.035^0 (1 - 0.035)^{100-0} + \binom{100}{1} 0.035^1 (1 - 0.035)^{100-1} \\
&= 1 - (1 - 0.035)^{100} + 100 \times 0.035 (1 - 0.035)^{99} \\
&= 1 - 0.028 - 0.103 \\
&= 0.869
\end{aligned}$$

Alternatively, it might be more convenient to approximate X by a Poisson distribution $Y \sim \text{Po}(100 \times 0.035) = \text{Po}(3.5)$.

For part (a), this gives

$$\mathbb{P}(Y = 5) = \mathrm{e}^{-3.5} \frac{3.5^5}{5!} = 0.132,$$

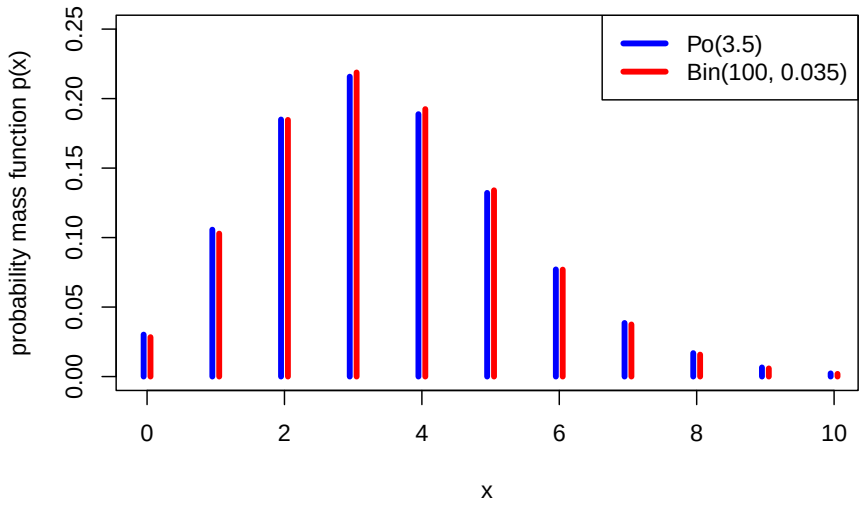
which is very close to the exact answer above of 0.134.

For part (b), the approximation gives

$$\begin{aligned}\mathbb{P}(Y \geq 2) &= 1 - \mathbb{P}(Y = 0) - \mathbb{P}(Y = 1) \\ &= 1 - \mathrm{e}^{-3.5}\frac{3.5^0}{0!} - \mathrm{e}^{-3.5}\frac{3.5^1}{1!} \\ &= 1 - \mathrm{e}^{-3.5} - 3.5\mathrm{e}^{-3.5} \\ &= 1 - 0.030 - 0.106 \\ &= 0.864\end{aligned}$$

which is very close to the exact answer above of 0.869.

The following graph shows how close the Po(3.5) distribution is to a Bin(100, 0.035) distribution – not exact, but pretty good.



For completeness, we include a proof of Theorem 12.2 here, although since it discusses use of limits, it's not examinable material for this module.

Proof. (Non-examinable) We need to show that, as $n \to \infty$,

$$p_X(x) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1 - \frac{\lambda}{n}\right)^{n-x} \to \mathrm{e}^{-\lambda}\frac{\lambda^x}{x!} = p_Y(x).$$

Let's try. The left-hand side is, by some simple rearrangements,

$$\begin{aligned}
&\binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}\\
&\quad=\frac{n(n-1)\cdots(n-x+1)}{x!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\\
&\quad=\frac{\lambda^x}{x!}\frac{n(n-1)\cdots(n-x+1)}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\\
&\quad=\frac{\lambda^x}{x!}\frac{n}{n}\frac{n-1}{n}\cdots\frac{n-x+1}{n}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\\
&\quad=\frac{\lambda^x}{x!}1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}.
\end{aligned}$$

Now let's take each of the terms in turn. First $\lambda^x/x!$ looks very promising, and can stay. Second, each of the terms $1, 1-1/n, \ldots, 1-(x-1)/n$ tend to 1 as $n\to\infty$. Third,

$$\left(1-\frac{\lambda}{n}\right)^n\to \mathrm{e}^{-\lambda};$$

this is from the standard "compound interest" result that

$$\left(1+\frac{a}{n}\right)^n\to \mathrm{e}^a \qquad \text{as } n\to\infty.$$

Finally

$$\left(1-\frac{\lambda}{n}\right)^{-x}\to 1,$$

as $1-\lambda/n\to 1$, and x is fixed. Putting all that together gives the result. □

12.3 Distributions as models for data

Families of distributions – like the Bernoulli, binomial, geometric and Poisson distributions we have seen so far in this module – are very useful for models in statistics. The idea of using families of distributions as models for data will be developed in the topic of Bayesian statistics we will discuss in Lectures 19 and 20 of this module, and, even more so, will be extremely important throughout the whole of MATH1712 Probability and Statistics II.

The families of distributions we have looked at here are sometimes called "parametric families", in that each of the distributions depended on one or more parameters: p for the Bernoulli and geometric distributions, both n and p for the binomial distribution, and λ for the Poisson distribution. (Later in the module we will also see some continuous parametric families: the exponential, normal and beta distributions.) This means we can adopt a model that data comes from one of the distributions within a family, then use data to estimate the value of that parameter.

For example:

- When testing the bias of a coin, you might assume, counting Heads as 1 and Tails as 0, that the outcome of each test is Bernoulli distributed with parameter p, but where the value of the Heads probability p is unknown. You could then toss the coin many times and use this data to estimate p.

- The number of years between severe summer floods in a tropical climate could be modelled as geometrically distributed where the flood risk parameter p is unknown. By look at the gaps between severe floods in historical data, a statistician could try to estimate p.

- A tutor might assume that the number of students that turn up to each tutorial is binomially distributed where n is known to equal 12, the number of students assigned to the group, but p, the "turning-up probability" is unknown. The tutor could then take records of how many students turned up to all the tutorials, and use this to estimate p.

- The number of calls received each day by call centre might be modelled as a Poisson distribution where the rate λ is unknown. The company could exam its records to estimate λ.

There are two main methods statisticians use to estimate parameters:

- **Bayesian statistics:** Here, one starts with a subjective "prior" distribution for the parameters, which represents one's personal belief about which possible values for that parameter are more or less likely before conducting any experiment. After the experiment, one then uses Bayes' theorem to update that belief to a "posterior" distribution of one's beliefs about the parameter *given* the data. The Bayesian approach will be introduced in Lecture 19 of this module.

- **Frequentist statistics:** Frequentist statistics does not involve any subjective prior views. Rather, frequentism is about assessing the extent to which the data is consistent with certain hypotheses.

  - You might try to find the value for the parameter that seems "most consistent" with the data, and use that as an estimate of the parameter: "My best guess of the Heads probability p of the coin is $p = 0.53$.
  - You might find a range of values for the parameter that are all at least somewhat consistent with the data: "I am confident the value of the Heads probability lies in the range $0.49 \leq p \leq 0.57$, as these values are all consistent with the data."
  - You could test if a specific hypothesis is consistent with the data or not: "The data is consistent with the hypothesis that $p = 0.50$, which would mean the coin is fair, so I have no strong evidence for concluding that the coin is biased."

The frequentist approach will pursued in detail throughout MATH1712.

Summary

Distribution	Range	PMF	Expectation	Variance
Bernoulli: Bern(p)	$\{0, 1\}$	$p(0) = 1-p$, $p(1) = p$	p	$p(1-p)$
Binomial: Bin(n, p)	$\{0, 1, \dots, n\}$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Geometric: Geom(p)	$\{1, 2, \dots\}$	$(1-p)^{x-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson: Po(λ)	$\{0, 1, \dots\}$	$\mathrm{e}^{-\lambda} \frac{\lambda^x}{x!}$	λ	λ

- Stirzaker, *Elementary Probability*, Section 4.2.
- Grimmett and Welsh, *Probability*, Section 2.2.

Problem Sheet 4

This is Problem Sheet 4. This problem sheet covers Lectures 11 to 14. You should work through all the questions on this problem sheet in preparation for ~~your~~ **the online** tutorial in Week 8. The problem sheet contains two assessed questions, which are due in by **2pm on** ~~Monday 28~~ **Tuesday 29 November**.

A: Short questions

A1. Let $X \sim \text{Bin}(20, 0.4)$. Calculate

(a) $\mathbb{P}(X = 8)$

(b) $\mathbb{P}(8 \leq X \leq 11)$

(c) $\mathbb{E}X$

A2. Let $X \sim \text{Geom}(0.2)$. Calculate

(a) $\mathbb{P}(X = 2)$

(b) $\mathbb{P}(X \geq 3)$

(c) $\text{Var}(X)$

A3. Let $X \sim \text{Po}(2.5)$. Calculate

(a) $\mathbb{P}(X = 3)$

(b) $\mathbb{P}(X \geq \mathbb{E}X)$

A4. Consider the following joint PMF:

$p_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$y=3$
$x=0$	$2k$	$2k$	k	0
$x=1$	k	$3k$	k	k
$x=2$	0	k	k	$2k$

(a) Find the value of k that makes this a joint PMF.

(b) Find the marginal PMFs of X and Y.

(c) What is the conditional distribution of Y given $X = 1$?

(d) Are X and Y independent?

B: Long questions

B1. Calculate the CDF $F(x) = \mathbb{P}(X \leq x)$ of the geometric distribution…

(a) …by summing the PMF;

(b) …by explaining how the "number of trials until success" definition tells us what $1 - F(x) = \mathbb{P}(X > x)$ must be.

(c) A gambler rolls a pair of dice until he gets a double-six. What is the probability that this takes between 20 and 40 double-rolls?

B2. Let $X \sim \text{Geom}(p)$. Recall that X represents the number of trials up to and including the first success. Recall also that $\mathbb{E}X = 1/p$ and $\text{Var}(X) = (1-p)/p^2$.

Let Y be a geometric distribution with parameter p according to the alternative "number of failures *before* the first success" definition.

(a) Write down the PMF for Y.

(b) Explain why the expectation of Y of

$$\mathbb{E}Y = \frac{1}{p} - 1 = \frac{1-p}{p}.$$

(c) What is the variance of Y?

B3 Let $X \sim \text{Po}(\lambda)$.

(a) Show that $\mathbb{E}X(X-1) = \lambda^2$. You may use the Taylor series for the exponential,

$$\mathrm{e}^{\lambda} = \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}.$$

(b) Hence show that $\text{Var}(X) = \lambda$. You may use the fact that $\mathbb{E}X = \lambda$.

B4. Each week in the UK about 15 million Lotto tickets are sold. As we saw in Lecture 6, the probability of each ticket winning is about 1 in 45 million. Estimate the proportion of weeks when there is **(a)** a roll-over (no jackpot winners), **(b)** a unique jackpot winner, or **(c)** when multiple winners share the jackpot. State any modelling assumptions you make and the approximation that you use.

B5. Let X and Y be Bernoulli($\frac{1}{2}$) random variables.

(a) Write down the table for the joint PMF of X and Y if X and Y are independent.

(b) Write down a table for a joint PMF of X and Y that is consistent with their marginal distributions but that leads to X and Y having a positive correlation.

(c) Write down a table for a joint PMF of X and Y that is consistent with their marginal distributions but that leads to X and Y having a negative correlation.

C: Assessed questions

The last two questions are **assessed questions**. These two questions count for 3% of your final mark for this module.

The deadline for submitting your solutions is **2pm on Monday 27 November** at the beginning of Week 9. Submission will be via Gradescope. Your work will be marked by your tutor and returned on Monday 4 December, when solutions will also be made available.

Both questions are "long questions", where the marks are not only for mathematical accuracy but also for the clarity and completeness of your explanations.

You should not collaborate with others on the assessed questions: your answers must represent solely your own work. The University's rules on academic integrity – and the related punishments for violating them – apply to your work on the assessed questions.

C1. A collector wants to collect football stickers to fill an album. There are n unique stickers to collect. Each time the collector buys a sticker, it is one of the n stickers chosen independently uniformly at random. Unfortunately, it is likely the collector will end up having "swaps", where he has received the same sticker more than once, so he will likely need to buy more than n stickers in total to fill his album. But how many?

(a) Suppose the collector has already got j unique stickers (and some number of swaps), for $j = 0, 1, 2, \ldots, n-1$. Let X_j be the the number of extra stickers he buys until getting a new unique sticker. Explain why X_j is geometrically distributed, and state the parameter $p = p_j$ of the geometric distribution.

(b) Hence, show that the expected number of stickers the collector must buy to fill his album is

$$n \sum_{k=1}^{n} \frac{1}{k}.$$

(c) The World Cup 2020 sticker album required $n = 670$ unique stickers to complete it, and stickers cost 18p each. Using the expression from (b), calculate the expected amount of money needed to fill the album. You should do this calculation in R and include the command you used in your answer.

(d) By approximating the sum in part (b) by an integral, explain why the expected number of stickers required is approximately $n \log n$, where log denotes the natural logarithm to base e.

C2. Let X and Y be random variables, and let a and b be constants.

(a) Starting from the definition of covariance, show that $\mathrm{Cov}(aX, Y) = a\,\mathrm{Cov}(X, Y)$. You may find it helpful to remember that if $\mathbb{E}X = \mu_X$, then $\mathbb{E}aX = a\mu_X$.

(b) Show that $\mathrm{Cov}(X + b, Y) = \mathrm{Cov}(X, Y)$.

Now let X, Y, Z be *independent* random variables with common variance σ^2.

(c) Find the value of $\text{Corr}(2X - 3Y + 4, 2Y - Z - 1)$. You may use any facts about covariance from the notes, including those from parts (a) and (b) of this question, provided you state them clearly.

Solutions to short questions

A1. (a) 0.180 (b) 0.528 (c) 8
A2. (a) 0.16 (b) 0.64 (c) 20
A3. (a) 0.214 (b) 0.456
A4. (a) $\frac{1}{15}$ (d) No

Other stuff

R Worksheets

Each week (starting in Week 2) there will be an R worksheet to work through in your own time. I recommend spending about one hour on each worksheet, plus one extra hour for even-numbered worksheets with assessed questions, for checking and submitting your solutions.

Week	Worksheet	Solutions	Deadline for assessed work
2	**1: R basics**	Solutions	—
3	**2: Vectors**	Solutions	Monday 23 October
4	**3: Data in R**	Solutions	—
5	**4: Plots I** – Making plots		Monday 6 November
6	**5: Plots II** – Making plots better		—

What are R and RStudio?

- **R** is a *programming language* that is particularly useful for working with probability and statistics. The R language is very widely used in universities and increasingly widely used in industry. Learning to use R is a mandatory part of this module, and exercises requiring use of R make up at least 15% of your module mark. Many other statistics-related modules at the University also use R.
- **RStudio** is a *computer program* (or *app*) that gives a convenient way to work with the language R. The RStudio program is made by the company Posit. THe program RStudio is the most common way to use the language R, and learning to use RStudio is strongly recommended.

R and RStudio are free/open-source software.

There are a number ways you can use R and RStudio:

How to use R and RStudio

1. *On University computers.* You will learn how to use R and RStudio on University computers in your first practical session, in Week 2.

2. *On your own computer.* R and RStudio can easily be installed on Windows and Mac laptops. Bring your laptop along to the first practical session to learn how to install R and RStudio.

3. *Using the Posit Cloud.* The Posit Cloud is a way to use R and RStudio online – sort of like a "Google Docs for R". You can use it free for 25 hours a month, which should be plenty for this module, or pay for more. I recommend the Posit Cloud for using R/RStudio with Chromebooks, tablet computers, or when borrowing someone else's device.

Accessing R and RStudio on University computers

R and RStudio can be used on University computers via the AppsAnywhere portal. AppsAnywhere is the University of Leeds system for loading "unusual" programs (common programs like Microsoft Office and web browsers are preloaded).

There are three steps to using R and RStudio on University computers:

1. Open the AppsAnywhere portal.
2. Load the language R onto your computer.
3. Open the program RStudio.

First, open the AppsAnywhere portal by double-clicking on the desktop icon. This will open a web browser, and invite you to "Open AppsAnywhere Launcher" – you should accept and open. AppsAnywhere has loaded properly when the blue "Validation in progress…" box turns into a green "Validation Successful" box.

Second, launch R from AppsAnywhere. R is called "Cran R 4.2.0 x64" on AppsAnywhere, so searching for "Cran" is an easy way to find it. Click "Launch".

This will do two things. First, it will silently load the language R in the background. Second, it will open a program called "RGui". RGui is basically like an older and less good version of RStudio; we do not recommend using the RGui program, so you can close it. (The R language will remain loaded.)

Third, launch RStudio from AppsAnywhere. The most recent version on AppsAnywhere is "RStudio 2023 (03.0.386)". Click "Launch". After a few second, RStudio will launch. (If invited to choose a version of the language R, pick "64-bit". If invited to update R or RStudio, decline.)

You need to repeat all three steps each time you log onto a University computer.

Installing R and RStudio

When you install R and RStudio, it's important that you install the language R first, and only install the program RStudio after the language R has already been installed. This ensures that RStudio can "find" R on your computer.

1. *First*, install R. Go to the Comprehensive R Archive Network (CRAN) and follow the instructions:

- Windows: Click "Download R for Windows", then "Install R for the first time". The main link at the top should be to download the most recent version of R.
- Mac: Click Download R for macOS, and then download the relevant PKG file. (Most modern Macbooks are based on Apple's M1 or M2 processors, so you can choose "Apple silicon arm64 build". Some older Macbooks, mostly 2020 or earlier, have Intel processors; for these you should use the "Intel 64-bit build".)

2. *After* R is installed, *then* install RStudio. Go to the "Download RStudio" page at posit.co and follow the instructions. You want "RStudio Desktop" and you want the free version, if given a choice.

Now, whenever you want to use R and RStudio, simply open program RStudio. (The language R will automatically be loaded on your computer.)

For Chromebooks, we recommend using the Posit Cloud, as mentioned above. However, if you have an Intel-based Chromebook and are feeling brave, then we have had success installing R and RStudio using these instructions, which are long and complicated.

Solutions and group feedback

This page has the solutions to all the non-assessed questions on Problem Sheet 1. Solutions are added after all tutorials on a Problem Sheet have finished.

Solutions to assessed questions are available on Minerva in the "Assessments and Feedback" tab, from one week after the deadline.

There are many ways you get feedback on this module, both group feedback (feedback that is generally relevant to many people) and individual feedback (feedback based specifically on your own approach to the work).

- You will have received both individual and group spoken feedback in your tutorial (the more you speak up in your tutorial, the more individualised the feedback you get in return).
- These solutions include group written feedback on common issues for the class.
- Most importantly, when your work on assessed questions is marked, individual written feedback will be given via the Gradescope site. It is very important that you read that feedback.
- Finally, students who would like even more feedback can discuss their work with me in the "office hours" drop-in sessions.

Problem Sheet 1

A: Short questions

A1. Consider again the "number of Skittles in each packet" data from Example 1.1.

$$59,\ 59,\ 59,\ 59,\ 60,\ 60,\ 60,\ 61,\ 62,\ 62,\ 62,\ 63,\ 63.$$

(a) Calculate the mean number of Skittles in each packet.

Solution. This was in the notes:

$$\bar{x} = \frac{1}{13}(59 + 59 + \cdots + 63) = \frac{789}{13} = 60.6923\cdots \approx 60.7.$$

(b) Calculate the sample variance using the definitional formula.

Solution.

$$\begin{aligned} s_x^2 &= \frac{1}{13-1}\left((59-60.7)^2 + (59-60.7)^2 + \cdots + (63-60.7)^2\right) \\ &= \frac{1}{12}(2.86 + 2.86 + \cdots + 5.33) \\ &= \frac{1}{12} \times 28.77 \\ &= 2.40 \end{aligned}$$

(c) Calculate the sample variance using the computational formula.

Solution.

$$\begin{aligned} s_x^2 &= \frac{1}{13-1}\left((59^2 + 59^2 + \cdots + 63^2) - 13 \times 60.6923^2)\right) \\ &= \frac{1}{12}(47915 - 47886.2) \\ &= 2.40 \end{aligned}$$

Group feedback: With the computational formula, the value $\sum_i x_i^2 - n\bar{x}^2$ is typically a fairly small number given as the difference between two very big numbers $\sum_i x_i^2$ and $n\bar{x}^2$. This means you have to get the two big numbers very precise, to ensure the cancellation happens correctly; in particular, make sure you use plenty of decimal places of accuracy in $\bar{x}$.

(d) Out of (b) and (c), which calculation did you find easier, and why?

Solution. The computational formula required fewer presses of the calculator buttons, because $\sum_i x_i^2$ is fewer button-presses than $\sum_i (x_i - \bar{x})^2$, where you have to subtract the means before squaring.

On the other hand, the expression inside the brackets of the computational formula is a fairly small number given as the difference of two very large numbers, so it was necessary to use lots of decimal places of accuracy in $\bar{x}$ to make sure the second large number was accurate and therefore that the subtraction cancelled correctly.

Group feedback: Many different answers for (d) are fine provided you give a justification.

A2. Consider the following data sets of the age of elected politicians on a local council. (The "18–30" bin, for example, means from one's 18th birthday to the moment before one's 30th birthday, so lasts 12 years.)

Age (years)	Frequency	Relative frequency	Frequency density
18–30	1		
30–40	2		
40–45	4		
45–50	5		
50–60	6		
60–80	2		
Total	20	1	—

(a) Complete the table by filling in the relative frequency and frequency densities.

Solution.

Age (years)	Frequency	Relative frequency	Frequency density
18–30	1	0.05	0.0042
30–40	2	0.1	0.01
40–45	4	0.2	0.04
45–50	5	0.25	0.05
50–60	6	0.3	0.03
60–80	2	0.1	0.005
Total	20	1	—

(b) What is the median age bin?

Solution. The 10th- and 11th-largest observations are both in the 45–50 bin, which is therefore the median bin.

(c) What is the modal age bin?

Solution. The bin with the largest frequency density is 45–50, which is therefore the modal bin.

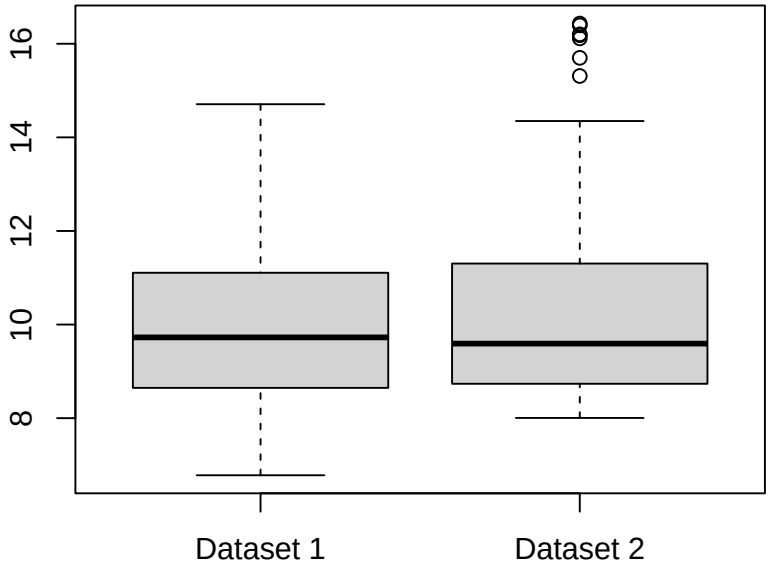
Group feedback: Remember that the modal bin is the one with the largest frequency *density*, not necessarily the bin with the highest frequency.

(d) Calculate (the standard approximation of) the mean age of the politicians.

Solution. Pretending that each person is in the centre of their bin, we have

$$\bar{x} = \frac{1}{20}(1 \times 24 + 2 \times 35 + \cdots + 2 \times 65) = \frac{971.9}{20} = 48.6.$$

A3. Consider the two datasets illustrated by the boxplots below. Write down some differences between the two datasets.



Solution. Some answers could be:

- The median and inter-quartile range of Dataset 2 appear to be very slightly larger than those in Dataset 1, although the differences are very small and might not be important in real life.
- Dataset 2 has a few outliers; Dataset 1 has none.
- While Dataset 1 is fairly "balanced" either side of the median, Dataset 2 shows what statisticians call a "positive skew": the data above the median is much more spread out than the data below the median.

Group feedback: You can probably think of other answers.

B: Long questions

B1. For each of the two datasets below, calculate the following summary statistics, or explain why it is not possible to do so: mode; median; mean; number of distinct outcomes; inter-quartile range; and sample variance.

(a) Shirt sizes for the $n = 16$ members of a university football squad:

Colour	Xtra Small	Small	Medium	Large	Xtra Large
Number of shirts	0	1	6	4	5

Solution. The modal shirt size is medium. The number of distinct outcomes is 4 (we don't quite "Xtra Small", which was not observed in the data).

This time, we can order the data from smallest to largest, even though the data is not numerical. Since $(16+1)/2-8.5$, the median datapoint is the 8th or 9th datapoints, which are Large.

Since $1+0.25(16-1)=4.75$ the lower quartile is the 4th or 5th datapoints, which are Medium. Since $1+0.75(16-1)=12.25$, the upper quartile is the 12th or 13th datapoints, which are Xtra Large. So we can certainly say that the inner quartiles range from Medium to Xtra Large. We could probably also say that the interquartile range is 3 shirt sizes (Medium, Large, Xtra Large).

Again, because the data is not numerical, we can't add it up, so can't calculate a mean or sample variance.

(b) Six packets of Skittles are opened together, a total of $n=361$ sweets. The colours of these sweets is recorded as follows:

Colour	Red	Orange	Yellow	Green	Purple
Number of Skittles	67	71	87	74	62

Solution. The modal colour is Yellow. The number of distinct outcomes is 5.

It's not possible to calculate the median or the quartiles, because, unlike numerical data, the colours can't be put "in order" from smallest to largest.

It's not possible to calculate the mean or sample variance, as these require us to have numerical data that can be "added up", but this can't be done with colours.

Group feedback: Make sure your explanation is clear for why we can't calculate a median for the Skittles data but can for the shirts: the key is whether or not the data can be *ordered*.

B2. A summary statistic is informally said to be "robust" if it typically doesn't change much if a small number of outliers are introduced to a large dataset, or "sensitive" if it often changes a lot when a small number of outliers are introduced. Briefly discuss the robustness or sensitivity of the following summary statistics: **(a)** mode; **(b)** median; **(c)** mean; **(d)** number of distinct outcomes; **(e)** inter-quartile range; and **(f)** sample variance.

Solutions.

(a) An outlier will typically be the only data point with its value, or certainly rare. Therefore, the mode will typically not change at all if a small number of outliers are introduced, so is robust. (The exception is for data where every observation is likely to be different, so any outliers become "joint modes" along

with everything else; but in this case the mode is not a useful statistic in the first place.)

(b) The introduction of outliers will typically only change the median a little bit, by shifting it between different nearby values in the "central mass" of the data. In particular, the *size* of the outliers won't make any difference at all (only whether they are "high outliers" above the median or "low outliers" below the median). So the median is robust.

(c) The mean can change a lot if outliers are introduced, especially if the outlier is enormously far our from the data. So the mean is sensitive.

(d) The number of distinct outcomes will only increase by (at most) 1 for each outlier introduced. This is not typically a relevant increase, so the number of distinct outcomes is robust.

(e) The interquartile range is robust, for the same reason as the median.

(f) The sample variance is sensitive, for the same reason as the mean.

(You might like to think about situations where it's better to use a robust statistic or better to use a sensitive statistic.)

Group feedback: I find it helpful to suppose I was studying the net worth of people in my tutorial group, and calculating summary statistics. How would those statistics changed change if Elon Musk (owner of Tesla and Twitter, net worth roughly $200 billion) joined my tutorial group? The mean and sample variance would change an enormous amount, while the median and interquartile range would barely change at all in comparison.

Remember that "robust" and "sensitive" are general descriptions rather than precise mathematical definitions. So it doesn't matter if you disagree with my opinions provided that you give clear and detailed explanations to back up your opinion.

B3. Let $\mathbf{a} = (a_1, a_2, \ldots a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two real-valued vectors of the same length. Then the *Cauchy–Schwarz inequality* says that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

(a) By making a clever choice of (a_i) and (b_i) in the Cauchy–Schwarz inequality, show that $s_{xy}^2 \leq s_x^2 s_y^2$.

Solutions. Recalling the formulas for s_{xy}, s_x^2, and s_y^2,

$$s_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}),$$

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2,$$

$$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2,$$

and comparing them with the Cauchy–Schwarz inequality, it looks like taking $a_i = x_i - \bar{x}$ and $b_i = y_i - \bar{y}$ might be useful.

Making that substitution, we get

$$\left(\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})\right)^2 \leq \left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)\left(\sum_{i=1}^{n}(y_i - \bar{y})^2\right).$$

These are very close to the formulas for s_{xy}, s_x^2, and s_y^2, but are just missing the "$1/(n-1)$"s; what we in fact have is

$$\big((n-1)s_{xy}\big)^2 \leq (n-1)s_x^2 \cdot (n-1)s_y^2.$$

Cancelling $(n-1)^2$ from each side, we have $s_{xy}^2 \leq s_x^2 s_y^2$, as required.

Group feedback: Keep trying different choices for (a_i) and (b_i); maybe your first attempt won't work, but it pays to be persistent!

A fancier choice is $a_i = (x_i - \bar{x})/\sqrt{n-1}$ and $b_i = (y_i - \bar{y})/\sqrt{n-1}$, to get the exact result without needing a second cancellation step, but I would find that harder to spot.

(b) Hence, show that the correlation r_{xy} satisfies $-1 \leq r_{xy} \leq 1$.

Solutions. Recall the formula for the correlation is

$$r_{xy} = \frac{s_{xy}}{s_x s_y}.$$

We can make part (a) look a bit like this dividing both sides by $s_x^2 s_y^2$, to get

$$\frac{s_{xy}^2}{s_x^2 s_y^2} \leq 1.$$

In fact that's the square of the correlation on the left-hand side, so we've shown that $r_{xy}^2 \leq 1$.

Finally, we note that if a number squared is less than or equal to 1, then the number must be between -1 and +1 inclusive. (Numbers bigger than 1 get bigger still when squared; numbers smaller than -1 become bigger than +1 when squared; numbers between -1 and +1 get closer to 0.) Hence we have shown that $-1 \leq r_{xy} \leq 1$, as required.

Group feedback: In part (b) there's a temptation to "square-root both sides of the inequality". But you have to be extremely careful if you do this – make sure you are properly accounting for the positive and negative square roots on both sides (if necessary), and where that does or doesn't require reversing the direction of the inequality. I recommend leaving the square-root operation until the last possible moment of the proof or, perhaps even better, reasoning through words as I did above.

Remember that you can still attempt part (b) even if you got stuck on part (a).

B4. A researcher wishes to study the effect of mental health on academic achievement. The researcher will collect data on the mental health of a cohort

of students by asking them to fill in a questionnaire, and will measure academic achievement via the students' scores on their university exams. Discuss some of the ethical issues associated with the collection, storage, and analysis of this data, and with the publication of the results of the analysis. Are there ways to mitigate these issues?

(It's not necessary to write an essay for this question – a few short bulletpoints will suffice. There may be an opportunity to discuss these issues in more detail in your tutorial.)

Group feedback: There are no "correct" or "incorrect" answers here, but here are a few things that students in my own tutorials brought up, which may act as a prompt for your own discussions.

- It's important the students/subjects have given their consent for their data to be used this way. It must be "informed consent", where they understand for what purpose the data will be used, how it will be stored, and so on. It must be easy and painless for students to decline to take part.
- Consideration should be given on how to anonymise the data as much as possible – it's not necessary for those analysing the data to know which questionnaire or which exam result belongs to which student, only that the questionnaire and results can be paired up.
- Even if after data is anonymised, care should be taken about whether the students could be worked out from the data. For example, if only one student did a certain combination of modules, their identity could "leak" that way. Perhaps imprecise data, such as classes rather than exact marks, might help maintain privacy while only slightly reducing the usefulness of the data?
- On one hand, it seems like this data should perhaps be deleted once analysis has been carried out, for the privacy of the students. On the other hand, principles of "open science" suggest that the data should be kept – and even publicly made available – for other researchers to check the work. There are competing ethical considerations here.
- If correlations are found in the data, care should be taken when publishing the analysis not to wrongly suggest a causation. (Just because X and Y are positively correlated, it doesn't mean that X *causes* Y – or that Y causes X.)

You can probably think of many other things.

Problem Sheet 2

A: Short questions

A1. Suppose you toss a coin 4 times.

(a) What would you suggest for a sample space Ω **(i)** if you only care about the total number of heads; **(ii)** if you care about the result of each coin toss?

(b) For each of the cases in part (a), what is $|\Omega|$?

Solution.

(i) We can take $\Omega = \{0, 1, 2, 3, 4\}$, with $|\Omega| = 5$.

(ii) Here, $\Omega = \{\text{HHHH}, \text{HHHT}, \text{HHTH}, \ldots, \text{TTTT}\}$ should be the set of all sequences of four "H"s or "T"s. So here, $|\Omega| = 2^4 = 16$.

A2. Let A, B and C be events in a sample space Ω. Write the following events using only A, B, C and the complement, intersection, and union operations.

(a) C happens but A doesn't.

Solution. This is "C and not A": $C \cap A^{\mathsf{c}}$.

(b) At least one of A, B and C happens.

Solution. This is simply the union $A \cup B \cup C$.

(c) Exactly one of B or C happens.

Solution. One way to write this is to split it up as "'B but not C' or 'C but not B'", which is $(B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C)$.

An alternative is to split it up as "'B or C' but not 'both B and C'", which is $(B \cup C) \cap (B \cap C)^{\mathsf{c}}$.

You can check these are equal by (for example) using De Morgan's law and the distributive law to expand out the second version.

(d) Exactly two of A, B and C happens.

Solution. I would split this up into "A and B but not C", "A and C but not B", and "B and C but not A" and take the union. This gives

$$(A \cap B \cap C^{\mathsf{c}}) \cup (A \cap B^{\mathsf{c}} \cap C) \cup (A^{\mathsf{c}} \cap B \cap C).$$

There are other equivalent formulations.

A3. What is the value of the following expressions?

(a) $6!$

Solution.

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

(b) 8^4

Solution.

$$8^4 = 8 \times 8 \times 8 \times 8 = 4096$$

(c) $8^{\underline{4}}$

Solution.

$$8^{\underline{4}} = 8 \times 7 \times 6 \times 5 = 1680$$

(d) $\binom{10}{4}$

Solution.

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

A4. An urn contains 4 red balls and 6 blue balls. Two balls are drawn from the urn. What is the probability that both balls are red, if the balls are drawn **(a)** with replacement; **(b)** without replacement?

Solution.

(a) There are $|\Omega| = 10^2 = 100$ ways to draw two balls with replacement. There are $|A| = 4^2 = 16$ ways to draw two red balls. So $\mathbb{P}(A) = \frac{16}{100} = 0.16$.

(b) There are $|\Omega| = 10^{\underline{2}} = 10 \times 9 = 90$ ways to draw two balls without replacement. There are $|A| = 4^{\underline{2}} = 4 \times 3 = 12$ to draw two red balls. So $\mathbb{P}(A) = \frac{12}{90} = \frac{2}{15} = 0.133$.

B: Long questions

B1. Starting from just the three probability axioms, prove the following statements:

(a) $\mathbb{P}(\emptyset) = 0$.

Solution. Let A be any event (such as $A = \emptyset$ or $A = \Omega$, for example). Then $A \cup \emptyset = A$, and the union is disjoint – since $\emptyset$ contains no sample points, it certainly can't contain any sample points that are also in A. Then applying Axiom 3, we get $\mathbb{P}(A) + \mathbb{P}(\emptyset) = \mathbb{P}(A)$. Subtracting $\mathbb{P}(A)$ from both sides gives the result.

Alternatively, if you prove part (b) first, you can apply that with $A = \emptyset$. Since $\emptyset^{\mathsf{c}} = \Omega$ and Axiom 2 tells us that $\mathbb{P}(\Omega) = 1$, the result follows.

Group feedback: With this, and most "prove from the axioms" questions, the key is to find a relevant disjoint union, which then allows us to use Axiom 3. So if we can find $C = A \cup B$ as a disjoint union (hopefully containing some events relevant to the question at hand), Axiom 3 allows us to write $\mathbb{P}(C) = \mathbb{P}(A) + \mathbb{P}(B)$.

(b) $\mathbb{P}(A^{\mathsf{c}}) = 1 - \mathbb{P}(A)$.

Solution. A very useful and relevant disjoint union is $A \cup A^{\mathsf{c}} = \Omega$. Applying Axiom 3 gives us $\mathbb{P}(A) + \mathbb{P}(A^{\mathsf{c}}) = \mathbb{P}(\Omega)$. But Axiom 2 tells us that $\mathbb{P}(\Omega) = 1$, so $\mathbb{P}(A) + \mathbb{P}(A^{\mathsf{c}}) = 1$. Rearranging gives the result.

B2. In this question, you will have to use the standard two-event form of the addition rule for unions

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(a) Using the two-event addition rule, show that

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D \cup E) - \mathbb{P}(C \cap (D \cup E)).$$

Solution. As with the Cauchy–Schwarz question from Problem Sheet 1, the key is to make a good choice for what A and B should be. This time, $A = C$ and $B = D \cup E$ will work well, since $C \cup (D \cup E) = C \cup D \cup E$. (You can call this "associativity", if you like.) Making that substitution immediately gives us

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D \cup E) - \mathbb{P}(C \cap (D \cup E)),$$

as required.

(b) Using your result from part (a), the two-event addition rule, the distributive law, and the two-event addition rule again, prove the three-event form of the addition rule for unions:

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(C \cap D) - \mathbb{P}(C \cap E) - \mathbb{P}(D \cap E) + \mathbb{P}(C \cap D \cap E).$$

Solution. Let's take the three terms on the right of the equation from part (a) separately.

The first term is $\mathbb{P}(C)$, which is fine as it is.

The second term is $\mathbb{P}(D \cup E)$. This is the probability of the union of two events, so we can use addition rule for the union of two events to get

$$\mathbb{P}(D \cup E) = \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(D \cap E).$$

The third term is $\mathbb{P}(C \cap (D \cup E))$. If we use the distributive law, as suggested in the question, we get $C \cap (D \cup E) = (C \cap D) \cup (C \cap E)$, so we want to find $\mathbb{P}((C \cap D) \cup (C \cap E))$. But this is another union of two events again, this time with $A = C \cap D$ and $B = C \cap E$. So the two-event addition rule gives

$$\mathbb{P}((C \cap D) \cup (C \cap E)) = \mathbb{P}(C \cap D) + \mathbb{P}(C \cap E) - \mathbb{P}(C \cap D \cap E),$$

since $(C \cap D) \cap (C \cap E) = C \cap D \cap E$.

Finally, we put this all together, and get

$$\begin{aligned}
&\mathbb{P}(C \cup D \cup E) \\
&\quad = \mathbb{P}(C) + \big(\mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(D \cap E)\big) - \big(\mathbb{P}(C \cap D) + \mathbb{P}(C \cap E) - \mathbb{P}(C \cap D \cap E)\big) \\
&\quad = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(C \cap D) - \mathbb{P}(C \cap E) - \mathbb{P}(D \cap E) + \mathbb{P}(C \cap D \cap E),
\end{aligned}$$

which is what we wanted.

B3. Suppose we pick a number at random from the set $\{1, 2, \ldots, 2023\}$.

(a) What is the probability that the number is divisible by 5?

Solution. The sample space is $\Omega = \{1, 2, \ldots, 2023\}$. Clearly $|\Omega| = 2023$. The event in question is $A = \{5, 10, \ldots, 2020\}$ of numbers up to 2023 that are divisible by 5. Thus $|A|$ is the largest integer no bigger than $\frac{2023}{5} = 404.6$, which is 404, as this is how many times 5 "goes into" 2023. Hence

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{404}{2023} = 0.1997,$$

just a tiny bit smaller than $\frac{1}{5}$.

Group feedback: With these "classical probability" questions, the steps should always be:

1. State clearly what the sample space Ω is.
2. Count how many outcomes $|\Omega|$ are in the sample space.
3. State clearly what the event A is.
4. Count how many outcomes $|A|$ are in the event.
5. The desired probability is then $\mathbb{P}(A) = |A|/|\Omega|$.

(b) What is the probability the number is divisible by 5 or by 7?

Solution. With the same Ω and A, now let B be the numbers up to 2023 divisible by 7; so we're looking for $\mathbb{P}(A \cup B)$. By the addition rule for unions, this is

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

We already know $\mathbb{P}(A) = \frac{404}{2022}$, so need to find out $\mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$.

This time, 7 goes into 2023 exactly, so $|B|$ is $\frac{2023}{7} = 289$. So

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{289}{2023} = \frac{1}{7}.$$

Now, $A \cap B$ is the set of numbers divisible by both 5 and 7, which is precisely the numbers divisible by their least common multiple $5 \times 7 = 35$. Then $|A \cap B|$ is $\frac{2023}{35} = 57.8$ rounded down, so $\mathbb{P}(A \cap B) = \frac{57}{2023}$.

So finally, we have

$$\mathbb{P}(A \cup B) = \frac{404}{2023} + \frac{289}{2023} - \frac{57}{2023} = \frac{636}{2023} = 0.314,$$

just a tiny bit larger than $\frac{11}{35}$

B4. Eight friends are about to sit down at random at a round table. Find the probability that

(a) Ashley and Brook sit next to each other, with Chris directly opposite Brook;

Solution. Let Ω be the sample space of ways the friends can sit around the table. This is an ordering problem, so $|\Omega| = 8!$.

Let A be the event in the question. What is $|A|$? Well,

- Ashley can sit anywhere, so has 8 choices of seat.
- Brook can sit either directly to Ashley's left or directly to Ashley's right, so has 2 choices of seat.
- Chris must sit directly opposite Brook, so only has 1 choice of seat.
- The remaining five friends can fill up the remaining seats however they like, so have 5, 4, 3, 2, and 1 choices respectively.

Hence $|A| = 8 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1$. Thus we get

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{8 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{2 \times 1}{7 \times 6} = \frac{1}{21}.$$

Group feedback: As we have discussed more recently, after this Problem Sheet was assigned, often "classical probability" problems also can be equivalently

solved by the step-by-step "chain rule" method. Can you use a chain rule argument to find the same answer as

$$\mathbb{P}(A) = 1 \times \frac{2}{7} \times \frac{1}{6} \times 1 \times 1 \times 1 \times 1 \times 1 = \frac{1}{21}?$$

(b) neither Ashley, Brook nor Chris sit next to each other.

Solution. The sample space Ω is as before. Let's count the outcomes in B, the event in the question.

- Ashley can sit anywhere, so has 8 choices of seat.
- Chris's number of choices will depend on where Brook sits, so we'll have to count Brook's and Chris's choices together:
  - Brook cannot sit next to Ashley.
  - If Brook sits next-but-one to Ashley – of which there are 2 choices – then Chris has 3 choices: Chris cannot sit on the seat directly between Ashley and Brook, nor directly next to Ashley on the other side, nor directly next to Brook on the other side, leaving $6 - 3 = 3$ choices.
  - If Brook sits neither next nor next-but-one to Ashley – of which there are 3 choices – then Chris has 2 choices: he cannot sit to the right or left of Ashley, nor to the right or left of Brook, leaving $6 - 4 = 2$ choices.
- The remaining friends have 5, 4, 3, 2, and 1 choices again.

Hence, $|B| = 8 \times (2 \times 3 + 3 \times 2) \times 5 \times 4 \times 3 \times 2 \times 1$. So

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{8 \times (2 \times 3 + 3 \times 2) \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{2 \times 3 + 3 \times 2}{7 \times 6} = \frac{12}{42} = \frac{2}{7}.$$

Alternatively, in a previous year's tutorial, a MATH1710 student suggested to me the following rather elegant solution. Suppose the five other friends are already sat at a round table with five chairs. Ashley, then Brook, then Chris will each bring along their own chair, and push into one of the gaps between the friends.

Ashley has 5 gaps to choose from, then Brook will have 6 gaps (Ashley joining the table will have increased the number of gaps by 1), then Chris will have 7, so the total number of ways they can push in is $|\Omega| = 5 \times 6 \times 7$.

To not sit next to each other, Ashley can push in any of the 5 gaps, Brook only has $6 - 2 = 4$ choices (not in the gap directly to the left or right of Ashley), and Chris only has $7 - 4 = 3$ choices (not in the gaps directly to the left or right of Ashley nor the gaps directly to the left or right of Brook – these four gaps are distinct assuming Brook was not next to Ashley). Hence $|B| = 5 \times 4 \times 3$, and we have

$$\mathbb{P}(B) = \frac{5 \times 4 \times 3}{5 \times 6 \times 7} = \frac{4 \times 3}{6 \times 7} = \frac{12}{42} = \frac{2}{7}.$$

Group feedback: Again, an equivalent answer can be derived using the step-by-step "chain rule" method.

B5. A "random digit" is a number chosen at random from $\{0, 1, \dots, 9\}$, each with equal probability. A statistician chooses n random digits (with replacement).

(a) For $k = 0, 1, \dots, 9$, let A_k be the event that all the digits are k or smaller. What is the probability of A_k, as a function of k and n?

Solution. The sample space is $\Omega = \{0, 1, \dots, 9\}^n$, the set of length-$n$ sequences of digits between 0 and 9. The number of these is $|\Omega| = 10^n$, as there are 10 choices for each of the n digits.

The event A_k is $\{0, 1, \dots, k\}^n$, the set of length-n sequences of digits that are between 0 and k. The number of these is $|A_k| = (k+1)^n$. (Note that it's $k+1$ because we're allowing 0 as well.)

Hence, the probability is

$$\mathbb{P}(A_k) = \frac{|A_k|}{|\Omega|} = \frac{(k+1)^n}{10^n}.$$

(b) Let B_k be the event that the largest digit chosen is equal to k. By finding a relationship between B_k, A_{k-1} and A_k, or otherwise, show that

$$\mathbb{P}(B_k) = \frac{(k+1)^n - k^n}{10^n}.$$

Solution. Consider the event A_k that all the digits are at most k. Within A_k, *either* one or more of the digits equal k, in which case that k is the largest digit and we are in B_k; *or* none of the digits equal k, in which case they are all at most $k-1$, and we are in A_{k-1}. Only one of these two possibilitie can occur, so we have a disjoint union

$$A_k = B_k \cup A_{k-1}.$$

Applying Axiom 3 to the disjoint union gives

$$\mathbb{P}(A_k) = \mathbb{P}(B_k) + \mathbb{P}(A_{k-1}).$$

Rearranging this gives

$$\mathbb{P}(B_k) = \mathbb{P}(A_k) - \mathbb{P}(A_{k-1}).$$

Substituting in the answer from part (a) gives

$$\mathbb{P}(B_k) = \frac{(k+1)^n}{10^n} - \frac{(k-1+1)^n}{10^n} = \frac{(k+1)^n - k^n}{10^n}.$$