MATH1710 Probability and Statistics I

Matthew Aldridge

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Schedule

Week 1 (2–6 October):

- **Lecture 1:** Summary statistics (Monday 2 October)
- **Lecture 2:** Data visualisation (Wednesday 4 October)
  - Map of how to find Chemistry West LT F
- **Problem Sheet 1:** Work through the short and long questions in preparation for your tutorial in Week 2. Deadline for assessed questions: Monday 17 October.

About MATH1710

Organisation of MATH1710

This module is **MATH1710 Probability and Statistics I**. (A small number of second-year scientists are taking this module as the first half of **MATH2700 Probability and Statistics for Scientists**.)

This module lasts for 11 weeks from 2 October to 15 December 2023. The exam will take place between 15 and 26 January 2024.

The module leader, the lecturer, and the main author of these notes is Dr Matthew Aldridge (you can call me "Matt", "Matthew", or "Dr Aldridge", pronounced "*old*-ridge").

Lectures

The main way you will learn new material for this module is by attending lectures. There are two lectures per week. Because this is a very large class each lecture will be delivered twice:

- **Mondays** at 1200 or at 1400, in Chemistry West LT F
- **Wednesdays** at 1500 in Chemistry West LT F or at 1600 in Roger Stevens LT 20

Check your timetable to see which lecture you are assigned to each day (or click this link for a map of how to get to Chemistry West LT F.

I recommend taking your own notes during the lecture. I will put brief summary notes from the lectures on this website, but will not reflect all the details I say and write during the lectures. Lectures will go through material quite quickly and the material may be quite difficult, so it's likely you'll want to spend time reading through your notes after the lecture.

You are probably reading the web version of the notes. If you want a PDF copy (to read offline or to print out), it can be downloaded via the top ribbon of the page. (Warning: I have not made as much effort to make the PDF as neat and tidy as I have the web version, and there may be formatting errors.) I am very keen to hear about errors in the notes, mathematical, typographical or otherwise. Please email me if think you may have found any.

Attendance at lectures is compulsory.

Problem Sheets

There will be 5 problem sheets. Each problem sheet has a number of short and long questions, for you to work on in your own time to help you learn the material, and two assessed questions, which you should submit for marking. The assessed questions on each problem sheet make up 3% of your mark on this module, for a total of 15%. Deadlines are 2pm on Mondays, although I'd recommend completing and submitting the work in the previous week.

Problem Sheet	Lectures covered	Deadline for assessed work
1	1 and 2	Monday 16 October (Week 3)
2	3–6	Monday 30 October (Week 5)
3	7–10	Monday 13 November (Week 7)
4	11–14	Monday 27 November (Week 9)
5	15–18	Monday 11 December (Week 11)

An informal Problem Sheet 6 covering material from Lectures 19 and 20 will be available. Lectures 21 and 22 are revision lectures with no new material.

Assessed questions should be submitted in online through the Gradescope platform. Most students choose to hand-write their solutions on paper and then scan and submit on their phone using the Gradescope app. Further Gradescope details to follow nearer the first deadline.

Tutorials

Tutorials are small groups of about a dozen students. You have been assigned to one of 34 tutorial groups, each with a member of staff as the tutor. Your tutorial group will meet five times, in Weeks 2, 4, 6, 8, and 10; you should check your timetable to see when and where your tutorial group meets.

The tutorials are an interactive session, where the main goal will be to go over your answers to the non-assessed questions on the problems sheets, which you will have worked on in advance of the tutorial. In this smaller group, you will be able to ask detailed questions of your tutor, and have the chance to discuss your answers to the problem sheet. Your tutor may ask you to present some of your work to your fellow students, or may give you the opportunity to work together with others during the tutorial. Your tutor may be willing to give you a hint on the assessed questions if you've made a first attempt but have got stuck. Because of the much smaller groups, the tutorials are the most valuable type of teaching on the module; you should make sure you attend, and you should be well prepared to ensure you make the most of the opportunity.

My recommended approach to problem sheets and tutorials is the following:

- Work through the problem sheet before the tutorial, spending plenty of time on it, and making multiple efforts at questions you get stuck on. I recommend spending *at least 4 hours per problem sheet*. This is a long time, but you shouldn't expect to be able to answer the hardest questions on a problem sheet without making multiple attempts. You don't have to wait until all lectures in a section are complete until starting to work on some of the questions. Collaboration is encouraged when working through the non-assessed problems, but I recommend writing up your work on your own; answers to assessed questions must be solely your own work.
- Take advantage of the small group setting of the tutorial to ask for help or clarification on questions you weren't able to complete.
- After the tutorial, attempt again the questions you were previously stuck on.
- If you're still unable to complete a question after this second round of attempts, *then* consult the solutions.

Your tutor will also be the marker of your answers to the assessed questions on the problem sheets.

Attendance at tutorials is compulsory.

R Worksheets and Practicals

R is a programming language that is particularly good at working with probability and statistics. Learning to use R is an important part of this module, and is used in many other modules in the University, including MATH1712 Probability and Statistics II. R is used by statisticians throughout academia and increasingly in industry too. Learning to program is a valuable skill for all students, and learning to use R is particularly valuable for students interested in statistics and related topics like actuarial science.

You will learn R by working through one R worksheet each week in your own time, starting from Week 2. Even-numbered worksheets will also contain a few questions for assessment, which will be due by 2pm Monday the following week (except the last one). Each of these is worth 3% of your mark for a total of 15%. You will submit your answers through a Microsoft Form (details to follow later). I recommend spending one hour per week on the week's R worksheet, plus one extra hour if there are assessed questions that week.

Week	Worksheet	Deadline for assessed work
2	1: R basics	—
3	2: Vectors	Monday 23 October (Week 4)
4	3: Data in R	—
5	4: Plots I – Making plots	Monday 6 November (Week 6)
6	5: Plots II – Making plots better	—
7	6: Discrete distributions	Monday 20 November (Week 8)
8	7: Discrete random variables	—
9	8: Normal distribution	Monday 4 December (Week 10)
10	9: Law of large numbers	—

Week	Worksheet	Deadline for assessed work
11	10: Recap	Thursday 14 December (Week 11)

R Practical sessions: You will be introduced you to R in your first Practical session, in Week 2. You will first see how to use R on University computers (these sessions will take place in computer "clusters"). There will then be an opportunity to install R on your own device – if you have a laptop on which you want to install R, bring it along to the practical session. A second practical, in Week 3, will allow you to get help on the R Worksheet 2, which is the first worksheet with assessed questions.

There are 11 R practical session groups – check your timetable for Weeks 2 and 3 to see when and where your group meets.

Attendance at the first R practical session (Week 2) is compulsory.

"Office hours" drop-in sessions

If you there is something in the module you wish to discuss one-on-one with the module leader, the place for the is the optional weekly "office hours", which will operate as drop-in sessions. These sessions are an optional opportunity for you to ask questions you have to me; these are particularly useful if there's something on the module that you are stuck on or confused about, but I'm happy to discuss any statistics-related issues or questions you have.

I currently plan two "office hours" drop-in sessions per week:

- Fridays 1100–1200 and 1300–1400 in the Mathematics Boardroom (map).

I may change arrangements as term continues – if attendance levels are low, I will move office hours to be actual office.

If neither time is possible, you may email me to arrange an alternative time to talk to me.

Attendance at "office hours" sessions is optional.

Time management

It is, of course, up to you how you choose to spend your time on this module. But my recommendations for your work would be something like this:

- **Lectures:** 2 hours per week, plus 1 hour per week reading through notes.
- **Problem sheets:** 4 hours per problem sheet, plus 1 extra hour for writing up and submitting answers to assessed questions.

- **R worksheets:** 1 hour per week, plus 1 extra hour if there are assessed questions.
- **Tutorials:** 1 hour every other week.
- **Revision:** 16 hours total at the end of the module.
- **Exam:** 2 hours.

That makes about 100 hours in total. (MATH1710 is a 10-credit module, so is supposed to represent 100 hours work. MATH2700 students are expected to be able to use their greater experience to get through the material in just 75 hours, so should scale these recommendations accordingly.)

Exam

There will be an exam in January, which makes up the remaining 70% of your mark. The exam will consist of 20 short and 2 long questions, and will be time-limited to 2 hours. We'll talk more about the exam format near the end of the module.

Who should I ask about...?

There are over 440 students registered for this module. If each student emails me once a week, and if each email takes me 10 minutes to read and respond, that will take more than 15 hours of my time every day! Generally, it's much better to come to speak to me at the "office hours" drop-in session or, if it will be very quick, before or after a lecture.

- *I don't understand something in the notes or on a problem sheet*: Come to office hours, or ask your tutor in your next tutorial.
- *I'm having difficulties with R:* In Weeks 2 or 3, you should ask at your R practical session; at other times, come to office hours.
- *I have an admin question about arrangements for the module:* Come to office hours or talk to me before/after lectures.
- *I have an admin question about arrangements for my tutorial:* Contact your tutor.
- *I have an admin question about general arrangements for my programme as a whole:* Contact the Student Information Service or speak to your personal tutor.
- *I have a question about the marking of my assessed work on the Problem Sheets:* First, check your feedback on Gradescope; if you still have questions, contact your tutor.
- *I have a question about the marking of my assessed work on the R Worksheets:* You can email me about this.
- *Due to truly exceptional and unforeseeable personal circumstances I require an extension on or exemption from assessed work:* You can apply by filling in the mitigating circumstances form at this link. Neither I nor your tutor can unilaterally offer an extension or exemption, so please don't ask. (Extensions of up to 4 days are available for Problem Sheets. Only exemptions are available for R Worksheets.)

Content of MATH1710

Prerequisites

The formal prerequisite for MATH1710 is "Grade B in A-level Mathematics or equivalent". I'll assume you have some basic school-level maths knowledge, but I won't assume you've studied probability or statistics in detail before (although I recognise that many of you will have). If you have studied probability and/or statistics at A-level (or post-16 equivalent) level, you'll recognise some of the material in this module; however you should find that we go deeper in many areas, and that we treat the material through with a greater deal of mathematical formality and rigour. "Rigour" here means precisely stating our assumptions, and carefully *proving* how other statements follow from those assumptions.

Syllabus

The module has three parts: a short first part on "exploratory data analysis", a long middle part on probability theory, and a short final part on a statistical framework called "Bayesian statistics". There's also the weekly R worksheets, which you could count as a fourth part running in parallel, but which will connect with the other parts too.

An outline plan of the topics covered is the following.

- **Exploratory data analysis** [2 lectures]: Summary statistics, data visualisation
- **Probability** [16 lectures]:
  - Probability with events: Probability spaces, probability axioms, examples and properties of probability, "classical probability" of equally likely events, independence, conditional probability, Bayes' theorem [6 lectures]
  - Probability with random variables: Discrete random variables, expectation and variance, binomial distribution, geometric distribution, Poisson distribution, multiple random variables, law of large numbers, continuous random variables, exponential distribution, normal distribution, central limit theorem [10 lectures]
- **Bayesian statistics** [2 lectures]: Bayesian framework, Beta prior, normal–normal model
- Summary and revision [2 lectures]

You'll notice that this module is heavier on the "Probability" than the "Statistics" of its title. MATH1712 Probability and Statistics II, on the other hand, which many students on this module will take next semester, is almost entirely "Statistics", but uses probabilistic techniques developed here.

Books

You can do well on this module by attending the lectures and tutorials, and working on the problem sheets and R worksheets, without needing to do any further reading beyond this. However, students can benefit from optional pre-reading in advance, extra background reading, or an alternative view on the material, especially in the parts of the module on probability. These books are also a good place to look if you want extra exercises and problems for revision.

For exploratory data analysis, you can stick to Wikipedia, but if you really want a book, I'd recommend:

- GM Clarke and D Cooke, *A Basic Course in Statistics*, 5th edition, Edward Arnold, 2004.

For the probability section, any book with a title like "Introduction to Probability" would do. Some of my favourites are:

- JK Blitzstein and J Hwang, *Introduction to Probability*, 2nd edition, CRC Press, 2019.
- G Grimmett and D Welsh, *Probability: An Introduction*, 2nd edition, Oxford University Press, 2014. (The library has online access.)
- SM Ross, *A First Course in Probability*, 10th edition, Pearson, 2020.
- RL Scheaffer and LJ Young, *Introduction to Probability and Its Applications*, 3rd edition, Cengage, 2010.
- D Stirzaker, *Elementary Probability*, 2nd edition, Cambridge University Press, 2003. (The library has online access.)

I also found lecture notes by Prof Oliver Johnson (University of Bristol) and Prof Richard Weber (University of Cambridge) to be useful.

On Bayesian statistics, we will only taste a brief introduction, but if you want a book, I recommend:

- JV Stone, *Bayes' Rule: A Tutorial Introduction to Bayesian Analysis*, Sebtel Press, 2013.

For R, there are many excellent resources online.

(For all these books I've listed the newest editions, but older editions are usually fine too.)

About these notes

These notes were written by Matthew Aldridge in 2021, and were edited and updated a lot in 2022 and a little bit in 2023. They are based in part on previous notes by Dr Robert G Aykroyd and Prof Wally Gilks. Dr Jason Susanna

Anquandah and Dr Aykroyd advised on the R worksheets. Dr Aykroyd's help and advice on many aspects of the module was particularly valuable.

These notes (in the web format) should be accessible by screenreaders. If you have accessibility difficulties with these notes, contact me.

Part I: Exploratory data analysis

Chapter 1

Summary statistics

1.1 What is EDA?

Statistics is the study of data. **Exploratory data analysis** (or **EDA**, for short) is the part of statistics concerned with taking a "first look" at some data. Later, toward the end of this module, we will see more detailed and complex ways of building models for data, and in MATH1712 Probability and Statistics II (for those who take it) you will see many other statistical techniques – in particular, ways of testing formal hypotheses for data. But here we're just interested in first impressions and brief summaries.

In this section, we will concentrate on two aspects of EDA:

- **Summary statistics:** That is, calculating numbers that briefly summarise the data. A summary statistic might tell us what "central" or "typical" values of the data are, how spread out the data is, or about the relationship between two different variables.
- **Data visualisation:** Drawing a picture based on the data is an another way to show the shape (centrality and spread) of data, or the relationship between different variables.

Even before calculating summary statistics or drawing a plot, however, there are other questions it is important to ask about the data:

- *What is the data?* What variables have been measured? How were they measured? How many datapoints are there? What is the possible range of responses?
- *How was the data collected?* Was data collected on the whole population or just a smaller sample? If a sample: How was that sample chosen? Is that sample representative of the population?
- *Are there any outliers?* "Outliers" are datapoints that seem to be very different from the other datapoints – for example, are much larger or much smaller than the others. Each outlier should be investigated to seek

the reason for it. Perhaps it is a genuine-but-unusual datapoint (which is useful for understanding the extremes of the data), or perhaps there is an extraordinary explanation (a measurement or recording error, for example) meaning the data is not relevant. Once the reason for an outlier is understood, it then *might* be appropriate to exclude it from analysis (for example, the incorrectly recorded measurement). It's usually bad practice to exclude an outlier merely for being an outlier before understanding what caused it.

- *Ethical questions:* Was the data collected ethically and, where necessary, with the informed consent of the subjects? Has it been stored properly? Are their privacy issues with the collection and storage of the data? What ethical issues should be considered before publishing (or not publishing) results of the analysis? Should the data be kept confidential, or should it be openly shared with other researchers for the betterment of science?

1.2 What is R?

R is a programming language that is particularly good at working with probability and statistics. A convenient way to use the language R is through the program **RStudio**. An important part of this module is learning to use R, by completing weekly worksheets – you can read more in the R section of these notes.

R can easily and quickly perform all the calculations and draw all the plots in this section of notes on exploratory data analysis. In this text, we'll show the relevant R code. Code will appear like this:

```
data <- c(4, 7, 6, 7, 4, 5, 5)
mean(data)
```

```
[1] 5.428571
```

Here, the code in the first shaded box is the R commands that are typed into RStudio, which you can type in next to the `>` arrow in the RStudio "console". The numerical answers that R returns are shown here in the second unshaded box. The `[1]` can be ignored (this is just R's way of saying that this is the first part of the answer – but the answer here only has one part anyway). Plots produced by R are displayed in these notes as pictures.

Most importantly for now, *you are not expected to understand the R code in this section yet.* The code is included so that, in the future, as you work through the R worksheets week by week, you can look back at the code in the section, and it will start to make sense. By the time you have finished R Worksheet 5 in Week 6, you should be able understand most of the R code in this section.

1.3 Statistics of centrality

Suppose we have collected some data on a certain variable. We will assume here that we have n datapoints, each of which is a single real number. We can write this data as a vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n).$$

A **statistic** is a calculation from the data $\mathbf{x}$, which is (usually) also a real number. In this section we will look at two types of "summary statistics", which are statistics that we feel will give us useful information about the data.

We'll look here at two types of summary statistic:

- **Statistics of centrality**, which tell us where the "middle" of the data is.
- **Statistics of spread**, which tell us how far the data typically spreads out from that middle.

Some measures of centrality are the following.

Definition 1.1. Consider some real-valued data $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

- The **mode** is the most common value of x_i. (If there are multiple joint-most common values, they are all modes.)
- Suppose the data is ordered as $x_1 \leq x_2 \leq \cdots \leq x_n$. Then the **median** is the central value in the ordered list. If n is odd, this is $x_{(n+1)/2}$; if n is even, we normally take halfway between the two central points, $\frac{1}{2}(x_{n/2} + x_{n/2+1})$.
- The **mean** $\bar{x}$ is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

In that last expression, we've made use of Sigma notation to write down the sum. (If Sigma notation is new to you, I recommend this PDF from MathCentre, or Section 2.4 of Clarke and Cooke, *A Basic Course in Statistics*.)

Example 1.1. Some packets of Skittles (a small fruit-flavoured sweet) were opened, and the number of Skittles in each packet counted. There were 13 packets, and the number of sweets (sorted from smallest to largest) were:

$$59,\ 59,\ 59,\ 59,\ 60,\ 60,\ 60,\ 61,\ 62,\ 62,\ 62,\ 63,\ 63.$$

The mode is 59, because there were 4 packets containing 59 sweets; more than any other number.

Since there are $n = 13$ packets, the middle packet is number $i = 7$, so the median is $x_7 = 60$.

The mean is

$$\bar{x} = \frac{1}{13}(59 + 59 + \cdots + 63) = \frac{789}{13} = 60.7.$$

The median is one example of a "quantile" of the data. Suppose our data is increasing order again. For $0 \leq \alpha \leq 1$, the **α-quantile** $q(\alpha)$ of the data is the datapoint α of the way along the list. Generally, $q(\alpha)$ is equal to $x_{1+\alpha(n-1)}$ when $1 + \alpha(n-1)$ is an integer. (If $1 + \alpha(n-1)$ isn't an integer, there are various conventions of how to choose that we won't go into here. R has *nine* different settings for choosing quantiles! – we will always just use R's default choice.)

- The **median** is the $\frac{1}{2}$-quantile $q(\frac{1}{2})$, which is $q(\frac{1}{2}) = x_7 = 60$ for this data.
- The **minimum** is the 0-quantile $q(0)$, which is $q(0) = x_1 = 59$ for this data.
- The **maximum** is the 1-quantile $q(1)$, which is $q(1) = x_{13} = 63$ for this data
- The **lower quartile** (that's "quartile", as in "quarter" – not "quantile") is the $\frac{1}{4}$-quantile $q(\frac{1}{4})$, which is $q(\frac{1}{4}) = x_4 = 59$ for this data.
- The **upper quartile** is the $\frac{3}{4}$-quantile $q(\frac{3}{4})$, which is $q(\frac{3}{4}) = x_{10} = 62$ for this data.

The following R code reads in some data which has the daily average temperature in Leeds in 2020, divided into months. We can find, for example, the mean October temperature or the lower quartile of the July temperature.

```
temperature <- read.csv("https://mpaldridge.github.io/math1710/data/temperature.csv")
jul <- temperature[temperature$month == "jul", ]
oct <- temperature[temperature$month == "oct", ]

mean(oct$temp)
```

```
[1] 11.93548
```

```
quantile(jul$temp, probs = 1 / 4)
```

```
25%
 15
```

1.4 Statistics of spread

Some measures of spread are:

Definition 1.2. The **number of distinct observations** is precisely that: the number of different datapoints we have after removing any repeats.

The **interquartile range** is the difference between the upper and lower quartiles $\text{IQR} = q(\frac{3}{4}) - q(\frac{1}{4})$.

The **sample variance** is

$$s_x^2 = \frac{1}{n-1}\left((x_1 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2\right) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2,$$

where $\bar{x}$ is the sample mean from before. The **standard deviation** $s_x = \sqrt{s_x^2}$ is the square-root of the sample variance.

Example 1.2. We continue with the Skittles data.

The number of distinct observation is 5. (These are 59, 60, 61, 62, and 63.)

The interquartile range is $x_{10} - x_4 = 62 - 59 = 3$.

You will calculate the sample variance on Problem Sheet 1.

The formula we've given for sample variance is sometimes called the "definitional formula", as it's the formula used to *define* the sample variance. We can rearrange that formula as follows:

$$\begin{aligned}
s_x^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}2x_i\bar{x} + \sum_{i=1}^{n}\bar{x}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - 2\bar{x}\sum_{i=1}^{n}x_i + \bar{x}^2\sum_{i=1}^{n}1\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - 2n\bar{x}^2 + n\bar{x}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right).
\end{aligned}$$

Here, the first line is the definitional formula; the second line is from expanding out the bracket; the third line is taking the sum term-by-term; the fourth line takes any constants (things not involving i) outside the sums; the fifth line uses $\sum_{i=1}^{n} x_i = n\bar{x}$, from the definition of the mean, and $\sum_{i=1}^{n} 1 = 1 + 1 + \cdots 1 = n$; and the sixth line simplifies the final two terms.

This has left us with

$$s_x^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right).$$

This is sometimes called the "computational formula"; this is because it usually takes fewer presses of calculator buttons to compute the sample variance with this formula rather than the definitional formula. (But make sure you keep enough decimal points in $\bar{x}^2$.)

Going back to our weather data in R, we can find the sample variance of the October weather or the interquartile range of the July weather.

```
var(oct$temp)
```

```
[1] 2.862366
```

```
IQR(jul$temp)
```

```
[1] 3
```

Summary

- Exploratory data analysis is about taking a first look at data.
- Summary statistics are numbers calculated from data that give us useful information about the data.
- Summary statistics that measure the centre of the data include the mode, median, and mean.
- Summary statistics that measure the spread of the data include the number of distinct outcomes, the interquartile range, and the sample variance.

Recommended reading:

- Wikipedia: Exploratory data analysis, Mode (statistics), Median, Arithmetic mean, Quantile, Interquartile range.
- Clarke and Cooke, *A Basic Course in Statistics*, Sections 2.1–2.4, 2.7, 4.1–4.4, 4.6, 4.7.

On Problem Sheet 1, you should now be able to complete Questions A1, B1, B2, B4, C1.

Chapter 2

Data visualisations

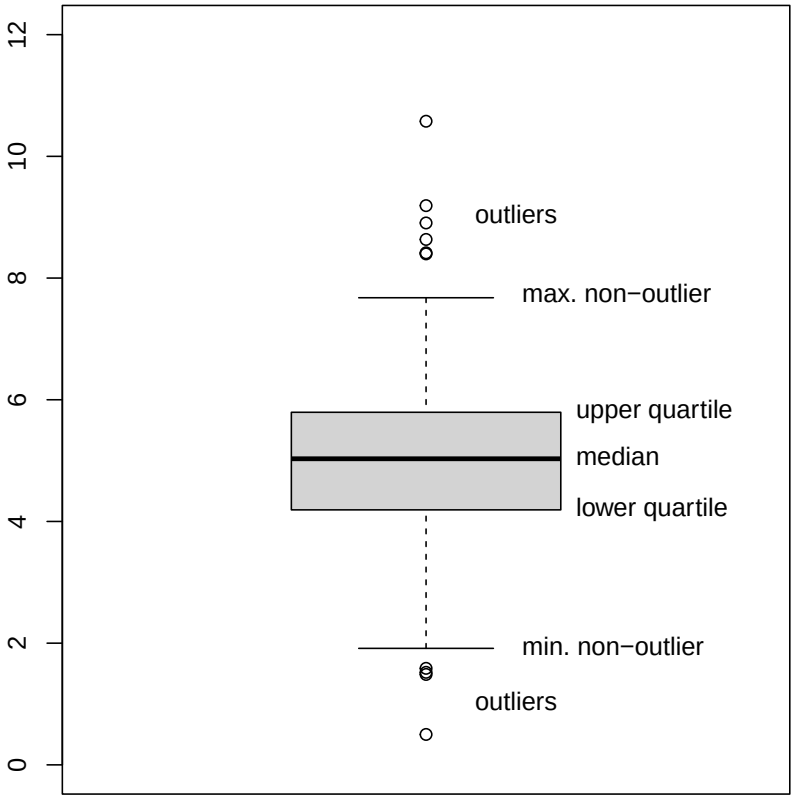
Data visualisations – drawings or graphs based on data – can help us to understand the "shape" of a dataset as part of exploratory data anlaysis. In this lecture, we'll look at three types of data visualisation.

2.1 Boxplots

A **boxplot** is a useful way to illustrate numerical data. It can be easier to tell the difference between different data sets "by eye" when looking at a boxplot, rather than examining raw summary statistics.

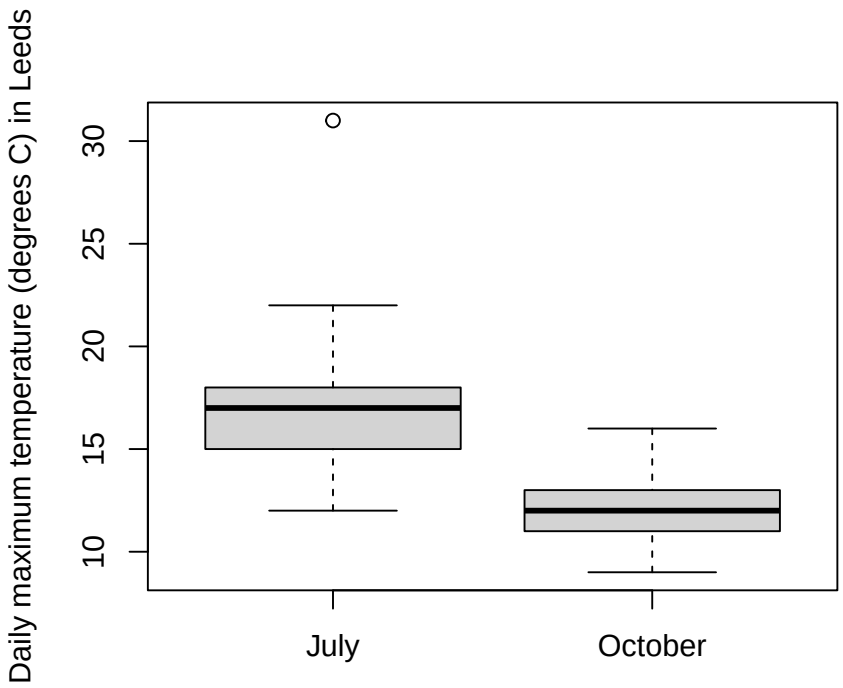
A boxplot is drawn as follows:

- The vertical axis represents the data values.
- Draw a box from the lower quartile $q(\frac{1}{4})$ to the median $q(\frac{1}{2})$.
- Draw another box on top of this from the median $q(\frac{1}{2})$ to the upper quartile $q(\frac{3}{4})$. Note that size of these two boxes put together is the interquartile range.
- Decide which datapoints are outliers, and plot these with circles. (The R default is that any data point less than $q(\frac{1}{4}) - 1.5 \times \text{IQR}$ or greater than $q(\frac{3}{4}) + 1.5 \times \text{IQR}$ is an outlier.)
- Out from the two previous boxes, draw "whiskers" to the minimum and maximum non-outlier datapoints.



When we have multiple datasets, drawing boxplots next to each other can help us to compare the datasets. Here are two boxplots from the July and October temperature data we used in the last lecture. What do you conclude about the data from these boxplots?

```
boxplot(jul$temp, oct$temp,
        names = c("July", "October"),
        ylab = "Daily maximum temperature (degrees C) in Leeds"
)
```



(And yes, I did check the outlier to make sure it was a genuine datapoint.)

2.2 Histograms

Often when collecting data, we don't collect exact data, but rather collect data clumped into "bins". For example, suppose a student wished to use a questionnaire to collect data on how long it takes people to reach campus from home; they might not ask "Exactly how long does it take?", but rather give a choice of tick boxes: "0–5 minutes", "5–10 minutes", and so on.

Consider the following binned data, from $n = 100$ students:

Time	Frequency	Relative frequency
0–5 minutes	4	0.04
5–10 minutes	8	0.08
10–15 minutes	21	0.21
15–30 minutes	42	0.42
30–45 minutes	15	0.15
45–60 minutes	8	0.08
60–120 minutes	2	0.02

Time	Frequency	Relative frequency
Total	100	1

Here the **frequency** f_j of bin j is simply the number of observations in that bin; so, for example, 42 students had journey lengths of between 15 and 30 minutes. The **relative frequency** of bin j is f_j/n; that is, the proportion of the observations in that bin.

Which bin would you say is the most popular – that is, the "modal" bin? The bin with the most observations in it is the "15–30 minute" bin. But this bin covers 15 minutes, while some of the other bins only cover 5 minutes. It would be a fairer comparison to look at the **frequency density**: the relative frequency divided by the size of the bin.

Time	Frequency	Relative frequency	Frequency density
0–5 minutes	4	0.04	0.008
5–10 minutes	8	0.08	0.016
10–15 minutes	21	0.21	0.042
15–30 minutes	42	0.42	0.028
30–45 minutes	15	0.15	0.010
45–60 minutes	8	0.08	0.005
60–120 minutes	2	0.02	0.0003
Total	100	1	

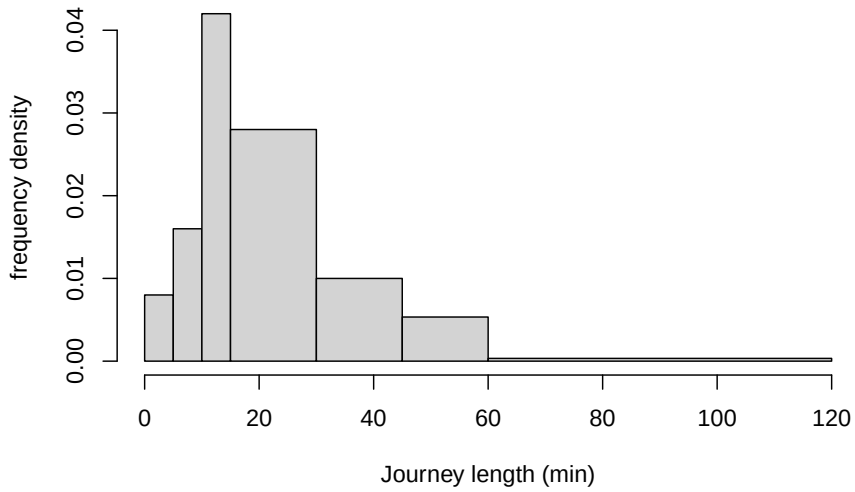
In the first row, for example, the relative frequency is 0.04 and the size of the bin is 5 minutes, so the frequency density is $0.04/5 = 0.008$. We now see that the modal bin – the bin with the highest frequency *density* – is in fact the "10–15 minutes" bin. This bin has somewhat fewer datapoints that the "15–30 minutes" bin, but they're squashed into a much smaller bin.

Data in bins can be illustrated with a **histogram**. A histogram has the measurement on the x-axis, with one bar across the width of each bin, where bars are drawn up to the height of the corresponding frequency density. Note that this means that the area of the bar is exactly the relative frequency of the corresponding bin.

If all the bins are the same width, frequency density is directly proportional to frequency and to relative frequency, so it can be clearer use one of those as the y-axis instead in the equal-width-bins case.

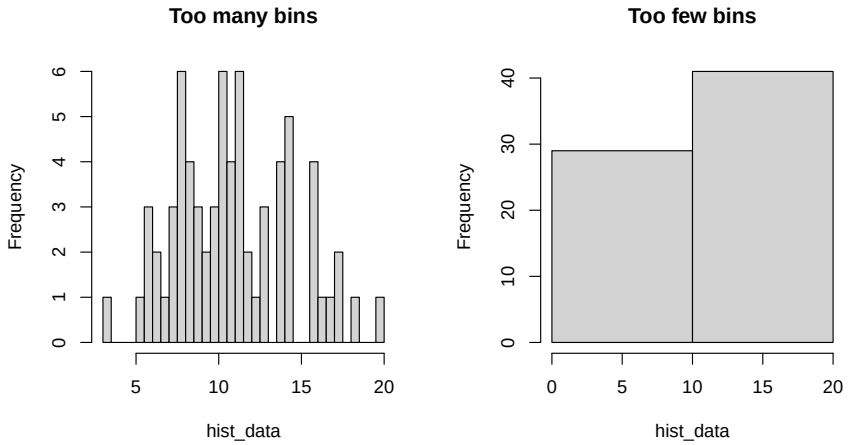
Here is a histogram for our journey-time data:

```
journeys <- read.csv("https://mpaldridge.github.io/math1710/data/journeys.csv")
bins <- c(0, 5, 10, 15, 30, 45, 60, 120)

hist(journeys$midpoint, breaks = bins,
     xlab = "Journey length (min)", ylab = "frequency density", main = ""
)
```



Often we draw histograms because the data was collected in bins in the first place. But even when we have exact data, we might *choose* to divide it into bins for the purposes of drawing a histogram. In this case we have to decide where to put the "breaks" between the bins. Too many breaks too close together, and the small number of observations in each bin will give "noisy" results (see left); too few breaks too far apart, and the wide bins will mean we lose detail (see right).

```
set.seed(2172)
hist_data <- c(rnorm(30, 8, 2), rnorm(40, 12, 3))  # Some fake data

hist(hist_data, breaks = 40, main = "Too many bins")
hist(hist_data, breaks = 2,  main = "Too few bins")
```



We can also calculate some summary statistics even when we have binned data. We mentioned the mode earlier, where the modal bin is the bin of highest frequency density.

What is the median journey length? Well, we don't know exactly, but $0.04 + 0.08 + 0.21$ (the first three bins) is less than 0.5, while $0.04 + 0.08 + 0.21 + 0.42$ (including the fourth bin) is greater than 0.5. So we know that the median student is in the fourth bin, the "15–30 minute" bin, and we can say that the median journey length is between 15 and 30 minutes.

Since we don't have the exact data, it's not possible to exactly calculate the mean and variance. However, we can often get a good estimate by assuming that each observation was in fact right in the centre of its bin. So, for example, we could assume that all 4 observations in the "0–5 minutes" bin were journeys of exactly 2.5 minutes. Of course, this isn't true (or is highly unlikely to be true), but we can often get a good approximation this way.

For our journey-time data, our approximation of the mean would be

$$\bar{x} = \frac{1}{100}(4 \times 2.5 + 8 \times 7.5 + \cdots + 2 \times 90) = 24.4.$$

More generally, if m_j is the midpoint of bin j and f_j its frequency, then we can calculate the binned mean and binned variance by

$$\bar{x} = \frac{1}{n} \sum_j f_j m_j$$
$$s_x^2 = \frac{1}{n-1} \sum_j f_j (m_j - \bar{x})^2$$

2.3 Scatterplots

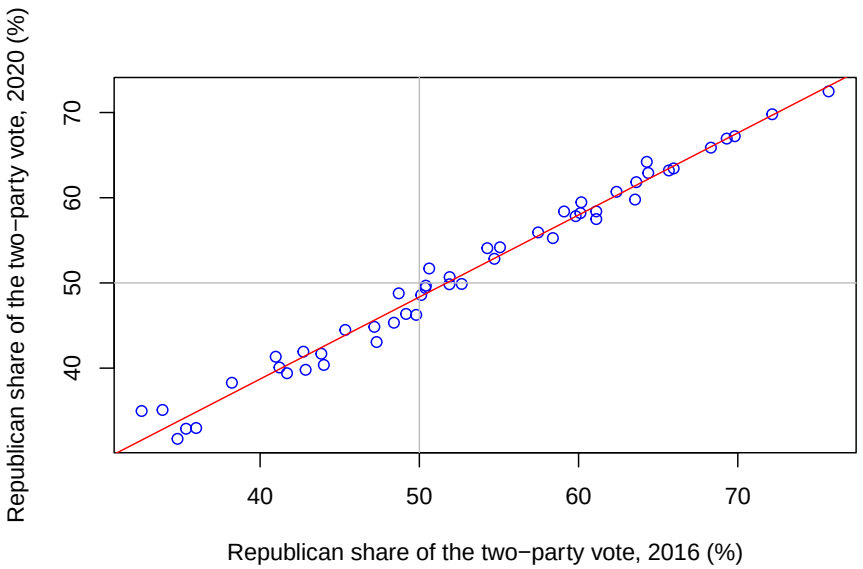
Often, more than one piece of data is collected from each subject, and we wish to compare that data, to see if there is a relationship between the variables.

For example, we could take n second-year maths students, and for each student i, collect their mark x_i in MATH1710 and their mark y_i in MATH1712. This gives is two "paired" datasets, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. We can calculate sample statistics of draw plots for $\mathbf{x}$ and for $\mathbf{y}$ individually. But we might also want to see if there is a relationship *between* $\mathbf{x}$ and $\mathbf{y}$: Do students with high marks in MATH1710 also get high marks in MATH1712?

A good way to visualise the relationship between two variables is to use a **scatterplot**. In a scatterplot, the ith data pair (x_i, y_i) is illustrated with a mark (such as a circle or cross) whose x-coordinate has the value x_i and whose y-coordinate has the value y_i.

In the following scatterplot, we have $n = 50$ datapoints for the 50 US states; for each state i, x_i is the Republican share of the vote in that state in the 2016 Trump–Clinton presidential election, and y_i is the Republican share of the vote in that state in the 2020 Trump–Biden election.

```
elections <- read.csv("https://mpaldridge.github.io/math1710/data/elections.csv")

plot(elections$X2016, elections$X2020,
     col = "blue",
     xlab = "Republican share of the two-party vote, 2016 (%)",
     ylab = "Republican share of the two-party vote, 2020 (%)")

abline(h = 50, col = "grey")
abline(v = 50, col = "grey")
abline(0.195, 0.963, col = "red")
```



We see that there is a strong relationship between $\mathbf{x}$ and $\mathbf{y}$, with high values of x corresponding to high values of y and vice versa. Further, the points on the scatterplot lie very close to a straight line.

A useful summary statistic here is the **correlation**

$$r_{xy} = \frac{s_{xy}}{s_x s_y},$$

where s_{xy} is the **sample covariance**

$$\begin{aligned} s_{xy} &= \frac{1}{n-1}((x_1 - \bar{x})(y_1 - \bar{y}) + \cdots + (x_n - \bar{x})(y_n - \bar{y})) \\ &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}), \end{aligned}$$

and $s_x = \sqrt{s_x^2}$ and $s_y = \sqrt{s_y^2}$ are the standard deviations.

The correlation r_{xy} is always between -1 and $+1$. Values of r_{xy} near $+1$ indicate that the scatterpoints are close to a straight line with an upward slope (big x

= big y); values of r_{xy} near -1 indicate that the scatterpoints are close to a straight line with a downward slope (big x = small y); and values of r_{xy} near 0 indicate that there is a weak linear relationship between x and y.

For the elections data, the correlation is

```
cor(elections$X2016, elections$X2020)
```

```
[1] 0.9919659
```

which, as we expected, is extremely high.

Summary

- Boxplots show the shape of numerical data, and can compare different datasets.
- Histograms show the shape of binned data.
- Scatterplots show the relationship between two datasets.

Recommended reading:

- Wikipedia: Box plot, Histogram, Grouped data, Scatter plot, Pearson correlation coefficient.
- Clarke and Cooke, *A Basic Course in Statistics*, Sections 1.2, 2.5, 4.5, 4.6, 21.2, 21.3.

On Problem Sheet 1, you should now be able to complete all questions.

Problem Sheet 1

You can download this problem sheet as a PDF file

This is Problem Sheet 1, which covers material from Lectures 1 and 2 of the notes. You should work through all the questions on this problem sheet in advance of your tutorial in Week 2. Questions C1 and C2 are assessed questions, and are due in by **2pm on Monday 16 October**. I recommend spending about 4 hours on this problem sheet, plus 1 extra hour to neatly write up and submit your answers to the assessed questions.

A: Short questions

The first three questions are **short questions**, which are intended to be mostly not too difficult. Short questions usually follow directly from the material in the lectures. Here, you should clearly state your final answer, and give enough working-out (or a short written explanation) for it to be clear how you reached that answer. You can check your answers with the solutions-without-working at the bottom of this sheet; solutions-with-working will be available after Friday 13 October. If you get stuck on any of these questions, you might want to ask for guidance in your tutorial.

A1. Consider again the "number of Skittles in each packet" data from Example 1.1.

$$59,\ 59,\ 59,\ 59,\ 60,\ 60,\ 60,\ 61,\ 62,\ 62,\ 62,\ 63,\ 63.$$

(a) Calculate the mean number of Skittles in each packet.

(b) Calculate the sample variance using the definitional formula.

(b) Calculate the sample variance using the computational formula.

(d) Out of (b) and (c), which calculation did you find easier, and why?

A2. Consider the following data sets of the age of elected politicians on a local council. (The "18–30" bin, for example, means from one's 18th birthday to the moment before one's 30th birthday, so lasts 12 years.)

Age (years)	Frequency	Relative frequency	Frequency density
18–30	1		

Age (years)	Frequency	Relative frequency	Frequency density
30–40	2		
40–45	4		
45–50	5		
50–60	6		
60–80	2		
Total	20	1	—

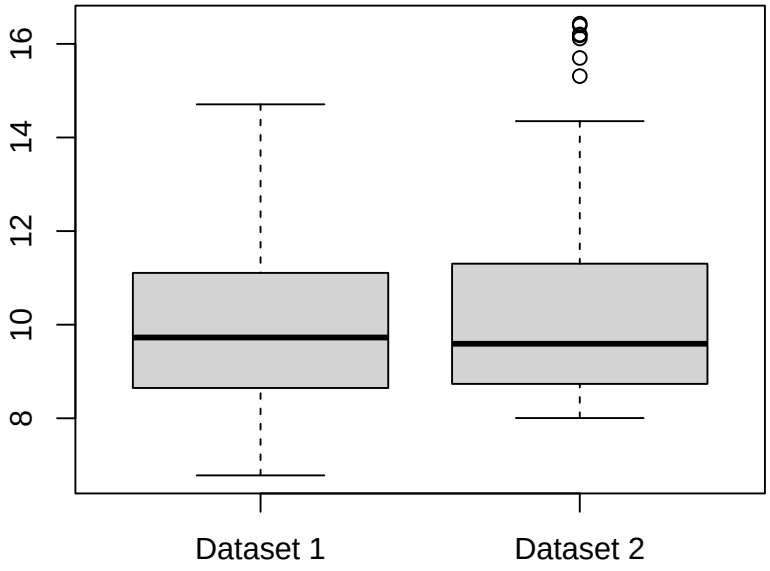
(a) Complete the table by filling in the relative frequency and frequency densities.

(b) What is the median age bin?

(c) What is the modal age bin?

(d) Calculate (the standard approximation of) the mean age of the politicians.

A3. Consider the two datasets illustrated by the boxplots below. Write down some differences between the two datasets.



B: Long questions

The next four questions are **long questions**, which are intended to be harder. Long questions often require you to think originally for yourself, not just directly follow procedures from the notes. You may not be able to solve all of these questions, although you should make multiple attempts to do so. Here, your answers should be written in complete sentences, and you should carefully explain in words each step of your working. Your answers to these questions – not only their mathematical content, but also how to write good, clear solutions – are likely to be the main topic for discussion in your tutorial. Solutions will be available after Friday 13 October.

B1. For each of the two datasets below, calculate the following summary statistics, or explain why it is not possible to do so: mode; median; mean; number of distinct outcomes; inter-quartile range; and sample variance.

(a) Shirt sizes for the $n = 16$ members of a university football squad:

Colour	Xtra Small	Small	Medium	Large	Xtra Large
Number of shirts	0	1	6	4	5

(b) Six packets of Skittles are opened together, a total of $n = 361$ sweets. The colours of these sweets is recorded as follows:

Colour	Red	Orange	Yellow	Green	Purple
Number of Skittles	67	71	87	74	62

B2. A summary statistic is informally said to be "robust" if it typically doesn't change much if a small number of outliers are introduced to a large dataset, or "sensitive" if it often changes a lot when a small number of outliers are introduced. Briefly discuss the robustness or sensitivity of the following summary statistics: **(a)** mode; **(b)** median; **(c)** mean; **(d)** number of distinct outcomes; **(e)** inter-quartile range; and **(f)** sample variance.

B3. Let $\mathbf{a} = (a_1, a_2, \dots a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be two real-valued vectors of the same length. Then the *Cauchy–Schwarz inequality* says that

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

(a) By making a clever choice of (a_i) and (b_i) in the Cauchy–Schwarz inequality, show that $s_{xy}^2 \leq s_x^2 s_y^2$.

(b) Hence, show that the correlation r_{xy} satisfies $-1 \leq r_{xy} \leq 1$.

B4. A researcher wishes to study the effect of mental health on academic achievement. The researcher will collect data on the mental health of a cohort

of students by asking them to fill in a questionnaire, and will measure academic achievement via the students' scores on their university exams. Discuss some of the ethical issues associated with the collection, storage, and analysis of this data, and with the publication of the results of the analysis. Are there ways to mitigate these issues?

(It's not necessary to write an essay for this question – a few short bulletpoints will suffice. There may be an opportunity to discuss these issues in more detail in your tutorial.)

C: Assessed questions

The last two questions are **assessed questions**. This means you will submit your answers, and your answers will be marked by your tutor. These two questions count for 3% of your final mark for this module. If you get stuck, your tutor may be willing to give you a small hint in your tutorial.

The deadline for submitting your solutions is **2pm on Monday 16 October** at the beginning of Week 3. Submission will be via Gradescope, which you can access via Minerva or on the Gradescope mobile app. You should submit your answers as a single PDF file. Most students choose to hand-write their work on paper, then scan-and-submit it to using the Gradescope app on their phone. Your work will be marked by your tutor and returned on Monday 23 October, when solutions will also be made available.

Question C1 is a "short question", where brief explanations or working are sufficient; Question C2 is a "long question", where the marks are not only for mathematical accuracy but also for the clarity and completeness of your explanations.

You should not collaborate with others on the assessed questions: your answers must represent solely your own work. The University's rules on academic integrity – and the related punishments for violating them – apply to your work on the assessed questions.

C1. The monthly average exchange rate for US dollars into British pounds over a 12-month period was:

$$1.306,\ 1.301,\ 1.290,\ 1.266,\ 1.268,\ 1.302,$$
$$1.317,\ 1.304,\ 1.284,\ 1.268,\ 1.247,\ 1.215.$$

(a) Calculate the median for this data.

(b) Calculate the mean for this data.

(c) Calculate the sample variance for this data.

(d) Is the mode an appropriate summary statistic for this sort of data? Why/why not?

C2. **(a)** Suppose that a dataset $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ (with $n \geq 2$) has sample variance $s_x^2 = 0$. Show that all the datapoints are in fact equal.

(b) Prove the following computational formula for the sample covariance:

$$s_{xy} = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}\right).$$

Solutions to short questions

A1. (a) 60.7, (b) 2.40, (c) 2.40, (d) —.
A2. (a) —, (b) 45–50, (c) 45–50, (c) 47.3.
A3. —

R

About R and RStudio

- **R** is a *programming language* that is particularly useful for working with probability and statistics. The R language is very widely used in universities and increasingly widely used in industry. Learning to use R is a mandatory part of this module, and exercises requiring use of R make up at least 15% of your module mark. Many other statistics-related modules at the University also use R.
- **RStudio** is a *program* (or *app*) that gives a convenient way to work with the language R. The RStudio program is made by the company Posit. RStudio is the most common way to use the language R, and learning to use RStudio is strongly recommended.

R and RStudio are free/open-source software.

There are a number ways you can use R and RStudio:

1. *On University computers.* You will learn how to use R and RStudio on University computers in your first practical session, in Week 2.
2. *On your own computer.* R and RStudio can easily be installed on Windows and Mac laptops (and on Intel-based Chromebooks with considerable difficulty). Bring your laptop along to the first practical session to learn how to install R and RStudio.
3. *Using the Posit Cloud.* The Posit Cloud is a way to use R and RStudio online – sort of like a "Google Docs for R". You can use it free for 25 hours a month, which should be plenty for this module, or pay for more. I recommend the Posit Cloud for using R/RStudio with Chromebooks, tablet computers, or when borrowing someone else's device.

R practical sessions

We will introduce you to R and RStudio in your first practical session, in Week 2. We will first show you how to use R on University computers (these sessions will take place in computer "clusters"). There will then be an opportunity to

install R on your own device – if you have a laptop on which you want to install R, bring it along to the practical session.

A second practical, in Week 3, will allow you to get help on the R Worksheet 2, which is the first worksheet with assessed questions.

R worksheets

Each week (starting in Week 2) there will be an R worksheet to work through in your own time. I recommend spending about one hour on each worksheet, plus one extra hour for even-numbered worksheets with assessed questions, for checking and submitting your solutions.

Week	Worksheet	Solutions	Deadline for assessed work
2	**1: R basics**	Solutions	—