Problem Sheet 2

MATH1710 Probability and Statistics I

University of Leeds, 2022-23

*This is Problem Sheet 2. This problem sheet covers material from Lectures 3 to 6. You should work through all the questions on this problem sheet in preparation for your tutorial in Week 4. The problem sheet contains two assessed questions, which are due in by **2pm on Monday 31 October**.*

A: Short questions

A1. Suppose you toss a coin 4 times.

(a) What would you suggest for a sample space Ω **(i)** if you only care about the total number of heads; **(ii)** if you care about the result of each coin toss?

(b) For each of the cases in part (a), what is $|\Omega|$?

A2. Let A, B and C be events in a sample space Ω. Write the following events using only A, B, C and the complement, intersection, and union operations.

(a) C happens but A doesn't.

(b) At least one of A, B and C happens.

(c) Exactly one of B or C happens.

(d) Exactly two of A, B and C happens.

A3. What is the value of the following expressions?

(a) $6!$

(b) 8^4

(c) $8^{\underline{4}}$

(d) $\displaystyle\binom{10}{4}$

A4. An urn contains 4 red balls and 6 blue balls. Two balls are drawn from the urn. What is the probability that both balls are red, if the balls are drawn **(a)** with replacement; **(b)** without replacement?

B: Long questions

B1. Starting from just the three probability axioms, prove the following statements:

(a) $\mathbb{P}(\varnothing) = 0$.

(b) $\mathbb{P}(A^\mathsf{c}) = 1 - \mathbb{P}(A)$.

B2. In this question, you will have to use the standard two-event form of the addition rule for unions

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(a) Using the two-event addition rule, show that

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D \cup E) - \mathbb{P}\big(C \cap (D \cup E)\big).$$

(b) Using your result from part (a), the two-event addition rule, the distributive law, and the two-event addition rule again, prove the three-event form of the addition rule for unions:

$$\mathbb{P}(C \cup D \cup E) = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E) - \mathbb{P}(C \cap D) - \mathbb{P}(C \cap E) - \mathbb{P}(D \cap E) + \mathbb{P}(C \cap D \cap E).$$

B3. Suppose we pick a number at random from the set $\{1, 2, \dots, 2022\}$.

(a) What is the probability that the number is divisible by 5?

(b) What is the probability the number is divisible by 5 or by 7?

B4. Eight friends are about to sit down at random at a round table. Find the probability that

(a) Ashley and Brook sit next to each other, with Chris directly opposite Brook;

(b) neither Ashley, Brook nor Chris sit next to each other.

B5. A "random digit" is a number chosen at random from $\{0, 1, \dots, 9\}$, each with equal probability. A statistician chooses n random digits (with replacement).

(a) For $k = 0, 1, \dots, 9$, let A_k be the event that all the digits are k or smaller. What is the probability of A_k, as a function of k and n?

(b) Let B_k be the event that the largest digit chosen is equal to k. By finding a relationship between B_k, A_{k-1} and A_k, or otherwise, show that

$$\mathbb{P}(B_k) = \frac{(k+1)^n - k^n}{10^n}.$$

C: Assessed questions

*The last two questions are **assessed questions**. These two questions count for 3% of your final mark for this module.*

*The deadline for submitting your solutions is **2pm on Monday 31 October** at the beginning of Week 5. Submission is via Gradescope. Your work will be marked by your tutor and returned on Monday 7 November, when solutions will also be made available.*

Both questions are "long questions", where the marks are not only for mathematical accuracy but also for the clarity and completeness of your explanations.

You should not collaborate with others on the assessed questions: your answers must represent solely your own work. The University's rules on academic integrity – and the related punishments for violating them – apply to your work on the assessed questions.

C1. Let Ω be a sample space with a probability measure $\mathbb{P}$, and let $A, B \subset \Omega$ be events. For each of the following statements, state whether the statement is true or false (that is, always true or sometimes false). If it is true, briefly justify the statement; if it is false, give a counterexample.

(a) If $\mathbb{P}(A) \leq \mathbb{P}(B)$, then $A \subset B$.

(b) $\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^{\mathsf{c}}) = \mathbb{P}(A)$.

(c) $\mathbb{P}(A \cup B) \leq \mathbb{P}(A)$

(d) If A and B are disjoint, then $\mathbb{P}((A \cup B)^{\mathsf{c}}) = 1 - \mathbb{P}(A) - \mathbb{P}(B)$.

C2. An urn contains 15 balls: 4 red balls, 5 blue balls, and 6 green balls.

(a) If three balls are drawn *with* replacement, what is the probability that all three balls are the *same* colour?

(b) If three balls are drawn *without* replacement, what is the probability that all three balls are *different* colours?