MATH2750 Introduction to Markov Processes

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Schedule

Week 1 (25–29 January)

- About the module
- Section 1: Stochastic processes and the Markov property
- Section 2: Random walks
- Problem Sheet 1
- Lecture: Tuesday at 1400
- Drop-in sessions: Tuesday or Wednesday

About MATH2750

This module is **MATH2750 Introduction to Markov Processes**. The module manager and lecturer is Dr Matthew Aldridge, and my email address is m.aldridge@leeds.ac.uk.

Organisation of MATH2750

This module lasts for 11 weeks. The first nine weeks run from 25 January to 26 March, then we break for Easter, and then the final two weeks run from 26 April to 7 May.

It is my intention, if possible, to try to make the Week 10 and 11 material available at the beginning of the Easter break, so that students who want to study the material in a single 11-week block are able to do so.

Notes and videos

The main way I expect you to learn the material for this course is by reading these notes and by watching the accompanying videos. I will set two sections of notes each week, for a total of 22 sections.

Each section roughly corresponds to one lecture last year, so I recommend spending about an hour on each section. Reading mathematics is a slow process. If you find yourself getting through sections much quicker than an hour, you're probably not reading carefully enough through each sentence of explanation and each line of mathematics, including understanding the motivation as well as just checking the accuracy. (Or you might just be a genius.)

It is possible (but not recommended) to learn the material by only reading the notes and not watching the videos.

Since we will all be relying heavily on these notes, I'm even more keen than usual to hear about errors mathematical, typographical or otherwise. Please, please email me if think you may have found any.

Problem sheets

There will be 10 problem sheets; Problem Sheet n covers the material from the two sections from week n (Sections $2n - 1$ and $2n$), and will be discussed in your workshop in week $n + 1$.

Lectures

There will be one online synchronous "lecture" session each week, on Tuesdays at 1400, with me, run through ?????.

This will not be a "lecture" in the traditional sense of the term, but will be an opportunity to re-emphasise material you have already learned from notes and videos, to give extra examples, and to answer common student questions, with some degree of interactivity.

I will assume you have completed all the work for the previous week by the time of the lecture, but I will not assume you've started the work for that week itself.

Workshops

There will be 10 workshops, starting in the second week. The main goal of the workshops will be to go over your answers to the problems sheets in smaller classes. You will have been assigned to one of three workshop groups, meeting on Mondays or Tuesdays, led by Jason Klebes, Dr Jochen Voss, or me. Your workshop will probably be run through ?????; your workshop leader will contact you before the end of this week if there are different arrangements.

My recommended approach to problem sheets and workshops is the following:

- Work through the problem sheet before the workshop, spending plenty of time on it, and making multiple efforts at questions you get stuck on. I recommend that three hours per problem sheet is a good amount of time to spend, preferably in more than one block. Collaboration is encouraged when working through the problems, but I recommend writing up your work on your own.
- Take advantage of the smaller group setting of the workshop to ask for help or clarification on questions you weren't able to complete.
- After the workshop, attempt again the questions you were previously stuck on.
- If you're still unable to complete a question after this second round of attempts, *then* consult the solutions.

Assessments

There will be four pieces of assessed coursework, making up a total of 15% of your mark for the module. Assessments 1, 2 and 4 will involve writing up answers to a few problems, in a similar style to the problem sheets, and are worth 4% each. (In response to previous student feedback, there are fewer questions per assessment.) Assessment 3 will be a report on some computational work (see below) and is worth 3%. While you may want to discuss an assessment with others in advance of completing it by yourself, copying is not allowed and will be dealt with in accordance with University rules.

The assessments deadlines are:

- Assessment 1: Thursday ??? 1400 (week 3)
- Assessment 2: Thursday ??? 1400 (week 7)
- Assessment 3 (Computing Worksheet 2): Thursday ??? 1400 (week 8)
- Assessment 4: Thursday ??? 1400 (week 11)

Work will be submitted via Gradescope.

Computing worksheets

There will be two computing worksheets, which will look at the material in the course through simulations in R. This material is examinable. You should be able to work through the worksheets in your own time, but there will be optional online drop-in sessions in the weeks 4 and 7 if you need help. (Your session will be listed as "Practical" on your timetable.)

In response to student feedback, some computational work will be assessed. The first computing worksheet will be a practice run, while a report on the second computing worksheet will be the third assessed piece of work.

Drop-in sessions

If you there is something in the course you wish to discuss in detail, the place for the is the optional weekly drop-in session on Tuesdays or Wednesdays with Nikita Merkulov or me. This would be an excellent place to go if you are having trouble understanding something in the written notes, or if you're struggling on a problem sheet question.

Microsoft Team

?????

Time management

It is, of course, up to you how you choose to spend your time on this module. But, if you're interested, my recommendations would be something like this:

- **Every week:** 7 hours per week
    - **Notes and videos:** 2 sections, 1 hour each
    - **Problem sheet:** 3 hours per week
    - **Lecture:** 1 hour per week
    - **Workshop:** 1 hour per week
- **When required:**
    - **Assessments 1, 2 and 4:** 3 hours each
    - **Computer worksheets:** 2 hours each
    - **Revision**: 14 hours
- **Total:** 100 hours

Exam

There will be an exam – or, rather, a final "online time-limited assessment" – after the end of the module, making up the remaining 85% of your mark. The exam will consist of four questions, and you are expected to answer all of them. You will have 48 hours to complete the exam, although the exam itself should represent half a day to a day's work. Further details to follow nearer the time.

Content of MATH2750

Prerequisites

Some students have asked what background you'll be expected to know for this course.

It's essential that you're very comfortable with the basics of probability theory: events, probability, discrete and continuous random variables, expectation, variance, approximations with the normal distribution, etc. Conditional probability and independence are particularly important concepts in this course. This course will use the binomial, geometric, Poisson, normal and exponential distributions, although the notes will usually remind you about them first, in case you've forgotten.

Many students on the module will have studied these topics in MATH1710 Probability and Statistics 1; others will have covered these in different modules.

Syllabus

The course has two major parts: the first part will cover processes in discrete time and the second part processes in continuous time.

An outline plan of the topics covered is the following. (Remember that one week's work is two sections of notes.)

- **Discrete time Markov chains** [12 sections]
    - Introduction to stochastic processes [1 section]
    - Important examples: Random walk, gambler's ruin, linear difference equations, examples from actuarial science [4 sections]
    - General theory: transition probabilities, n-step transition probabilities, class structure, periodicity, hitting times, recurrence and transience, stationary distributions, long-term behaviour [6 sections]
    - Revision [1 section]
- **Continuous time Markov jump processes** [10 sections]
    - Important examples: Poisson process, counting processes, queues [5 sections]
    - General theory: holding times and jump chains, forward and backward equations, class structure, hitting times, stationary distributions, long-term behaviour [4 sections]
    - Revision [1 section]

Books

You can do well on this module by reading the notes and watching the videos, attending the lectures and workshops, and working on the problem sheets, assignments and practicals, without any further reading. However, students can benefit from optional extra background reading or an alternative view on the material.

My favourite book on Markov chains, which I used a lot while planning this course and writing these notes, is:

- J.R. Norris, *Markov Chains*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 1997. Chapters 1-3.

This a whole book just on Markov processes, including some more detailed material that goes beyond this module. Its coverage of of both discrete and continuous time Markov processes is very thorough. Chapter 1 on discrete time Markov chains is available online.

Other good books with sections on Markov processes that I have used include:

- G.R. Grimmett and D.R. Stirzaker, *Probability and Random Processes*, 4th edition, Oxford University Press, 2020. Chapter 6.
- G. Grimmet and D. Walsh, *Probability: an introduction*, 2nd edition, Oxford University Press, 2014. Chapter 12.
- D.R. Stirzaker, *Elementary Probability*, 2nd edition, Cambridge University Press, 2003. Chapter 9.

Grimmett and Stirzaker is an excellent handbook that covers most of undergraduate probability – I bought a copy when I was a second-year undergraduate and still keep it next to my desk. The whole of Stirzaker is available online.

A gentler introduction with plenty of examples is provided by:

- P.W. Jones and P. Smith, *Stochastic Processes: an introduction*, 3nd edition, Texts in Statistical Science, CRC Press, 2018. Chapters 2-7.

although it doesn't cover everything in this module. The whole book is available online.

(I've listed the newest editions of these books, but older editions will usually be fine too.)

And finally...

These notes were mostly written by Matthew Aldridge in 2018–19, and have received updates (mostly in Sections 9–11) and reformatting this year. Some of the material (especially Section 1, Section 6, and numerous diagrams) follows closely previous notes by Dr Graham Murphy, and I also benefited from reading earlier notes by Dr Robert Aykroyd and Prof Alexander Veretennikov. Dr Murphy's general help and advice was also very valuable. Many thanks to students in previous runnings of the module for spotting errors and suggesting improvements.

Part I: Discrete time Markov chains

1   Stochastic processes and the Markov property

- Stochastic processes with discrete or continuous state space and discrete or continuous time
- The Markov "memoryless" property

1.1 Deterministic and random models

A **model** is an imitation of a real-world system. For example, you might want to have a model to imitate the world's population, the level of water in a reservoir, cashflows of a pension scheme, or stock prices. Models allow us to try to understand and predict what might happen in the real world in a low risk, cost effective and fast way.

To design a model requires a set of assumptions about how it will work and suitable parameters need to be determined, perhaps based on past collected data.

An important distinction is between **deterministic** models and **random** (or **stochastic**) models. Deterministic models do not contain any random components, so the output is completely determined by the inputs and any parameters. Random models have variable outcomes to account for uncertainty and unpredictability, so they can be run many times to give a sense of the range of possible outcomes.

Consider models for:

- the future position of the Moon as it orbits the Earth,
- the future price of shares in Apple.

For the moon, the random components – for example, the effect of small meteorites striking the Moon's surface – are not very significant and a deterministic model based on physical laws is good enough for most purposes. For Apple shares, the price changes from day to day are highly uncertain, so a random model can account for the variability and unpredictability in a useful way.

In this module we will see many examples of stochastic models. Lots of the applications we will consider come from financial mathematics and actuarial science where the use of models that take into account uncertainty is very important, but the principles apply in many areas.

1.2 Stochastic processes

If we want to model, for example, the total number of claims to an insurance company in the whole of 2020, we can use a random variable X to model this – perhaps a Poisson distribution with an appropriate mean. However, if we want to track how the number of claims changes over the course of the year 2021, we will need to use a **stochastic process**. (The word "stochastic" means the same thing as "random".)

A stochastic process, which we will usually write as (X_n), is an indexed sequence of random variables that are (usually) dependent on each other.

Each random variable X_n takes a value in a **state space** $\mathcal{S}$ which is the set of possible values for the process. As with usual random variables, the state space $\mathcal{S}$ can be **discrete** or **continuous**. A discrete state space denotes a set of distinct possible outcomes, which can be finite or countably infinite. For example, $\mathcal{S} = \{\text{Heads}, \text{Tails}\}$ is the state space for a single coin flip, while in the case of counting insurance claims, the state space would be the nonnegative integers $\mathcal{S} = \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. A continuous state space denotes an uncountably infinite continuum of gradually varying outcomes. For example, the nonnegative real line $\mathcal{S} = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ is the state space for the amount of rainfall on a given day, while some bounded subset of $\mathbb{R}^3$ would be the state space for the position of a gas particle in a box.

Further, the process has an **index set** that puts the random variables that make up the process in order. The index set is usually interpreted as a **time** variable, telling us when the process will be measured. The index set for time can also be discrete or continuous. Discrete time denotes a process sampled at distinct points, often denoted by $n = 0, 1, 2, \dots$, while continuous time denotes a process monitored constantly over time, often denoted by $t \in \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. In the insurance example, we might count up the number of claims each day – then the discrete index set will be the days of the year, which we could denote $\{1, 2, \dots, 365\}$. Alternatively, we might want to keep a constant tally that we update after every claim, requiring a continuous time index t representing time across the whole year. In discrete time, we can write down the first few steps of the process as $(X_0, X_1, X_2, \dots)$.

This gives us four possibilities in total:

- **Discrete time, discrete space**

- Example: Number of students attending each lecture of maths module.
- **Markov chains** – discrete time, discrete space stochastic processes with a certain "Markov property" – are the main topic of the first half of this module.

- **Discrete time, continuous space**

    - Example: Daily maximum temperature in Leeds.
    - We will briefly mention continuous space Markov chains in the first half of the course, but these are not as important.

- **Continuous time, discrete space**

    - Example: Number of visitors to a webpage over time.
    - **Markov jump processes** – continuous time, discrete space stochastic processes with the "Markov property" – are the main topic of the second half of this module.

- **Continuous time, continuous space**

    - Example: Level of the FTSE 100 share index over time.
    - Such processes, especially the famous Brownian motion – another process with the Markov property – are very important, but outside the scope of this course. See MATH3734 Stochastic Calculus for Finance next year, for example.

1.3 Markov property

Because stochastic processes consist of a large number – even infinitely many; even uncountably infinitely many – random variables that could all be dependent on each other, they can get extremely complicated. The Markov property is a crucial property that restricts the type of dependencies in a process, to make the process easier to study, yet still leaves most of the useful and interesting examples intact. (Although particular examples of Markov processes go back further, the first general study was by the Russian mathematician Andrey Andreyevich Markov, published in 1906.)

Think of a simple board game where we roll a dice and move that many squares forward on the board. Suppose we are currently on the square X_n. Then which square X_{n+1} we move to on our next turn:

- is random, since it depends on the roll of the dice;
- depends on where we are now X_n, since the score of dice will be added onto the number our current square;
- given the square we are now X_n, it doesn't depend any further on which sequence of squares $X_0, X_1, \ldots, X_{n-1}$ we used to get here.

It is this third point that is the crucial property of the stochastic processes we will study in this course, and it is called the **Markov property** or **memoryless property**. We say "memoryless", because it's as if the process forgot how it got here – we only need to remember what square we've reached, not which squares we used to get here. The stochastic process before this moment has no bearing on the future, given where we are now. A mathematical way to say this is that "the past and the future are conditionally independent given the present."

To write this down formally, we need to recall **conditional probability**: the conditional probability of an event A given another event B is written $\mathbb{P}(A \mid B)$, and is the probability that A occurs *given* that B definitely occurs. You may remember the definition

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)},$$

although is often more useful to reason directly about conditional probabilities than use this formula.

(You may also remember that the definition of conditional probability requires that $\mathbb{P}(B) > 0$. Whenever we write down a conditional probability, we implicitly assume the conditioning event has strictly positive probability without explicitly saying so.)

Definition 1.1. Let $(X_n) = (X_0, X_1, X_2, \ldots)$ be a stochastic process in discrete time $n = 0, 1, 2, \ldots$ and discrete space $\mathcal{S}$. Then we say that (X_n) has the **Markov property** if, for all times n and all states $x_0, x_1, \ldots, x_n, x_{n+1} \in \mathcal{S}$ we have

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

Here, the left hand side is the probability we go to state x_{n+1} next conditioned on the entire history of the process, while the right hand side is the probability we go to state x_{n+1} next conditioned only on where we are now x_n. So the Markov property tells us that it only matters where we are now and not how we got here.

(There's also a similar definition for continuous time processes, which we'll come to later in the course.)

Stochastic processes that have the Markov property are much easier to study than general processes, as we only have to keep track of where we are now and we can forget about the entire history that came before.

In the next section, we'll see our first example of a discrete time Markov chain, called the "random walk".

2 Random walk

- Definition of the simple random walk and the exact binomial distribution
- Expectation and variance of general random walks

2.1 Simple random walk

Consider the following **simple random walk** on the integers $\mathbb{Z}$: We start at 0, then at each time step, we go up by one with probability p and down by one with probability $q = 1 - p$. When $p = q = \frac{1}{2}$, we're equally as likely to go up as down, and we call this the **simple symmetric random walk**.

The simple random walk is a simple but very useful model for lots of processes, like stock prices, sizes of populations, or positions of gas particles. (In many modern models, however, these have been replaced by more complicated continuous time and space models.) The simple random walk is sometimes called the "drunkard's walk", suggesting it could model a drunk person trying to stagger home.

Figure 1: Two simulations of random walks.

We can write this as a stochastic process (X_n) with discrete time $n = \{0, 1, 2, \dots\} = \mathbb{Z}_+$ and discrete state space $\mathcal{S} = \mathbb{Z}$, where $X_0 = 0$ and, for $n \geq 0$, we have

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p, \\ X_n - 1 & \text{with probability } q. \end{cases}$$

It's clear from this definition that X_{n+1} (the future) depends on X_n (the present), but, given X_n, does not depend on $X_{n-1}, \dots, X_1, X_0$ (the past). Thus the Markov property holds, and the simple random walk is a **discrete time Markov process** or **Markov chain**.

Example 2.1. *What is the probability that after two steps a simple random walk has reached $X_2 = 2$?*

To achieve this, the walk must go upwards in both time steps, so $\mathbb{P}(X_2 = 2) = pp = p^2$.

Example 2.2. *What is the probability that after three steps a simple random walk has reached $X_3 = -1$?*

There are three ways to reach -1 after three steps: up–down–down, down–up–down, or down–down–up. So

$$\mathbb{P}(X_3 = -1) = pqq + qpq + qqp = 3pq^2.$$

2.2 General random walks

An alternative way to write the simple random walk is to put

$$X_n = X_0 + \sum_{i=1}^{n} Z_i, \tag{1}$$

where the starting point is $X_0 = 0$ and the **increments** $Z_1, Z_2, \ldots$ are independent and identically distributed (IID) random variables with distribution given by $\mathbb{P}(Z_i = 1) = p$ and $\mathbb{P}(Z_i = -1) = q$. You can check that (1) means that $X_{n+1} = X_n + Z_{n+1}$, and that this property defines the simple random walk.

Any stochastic process with the form (1) for some X_0 and some distribution for the IID Z_is is called a **random walk** (without the word "simple").

Random walks often have state space $\mathcal{S} = \mathbb{Z}$, like the simple random walk, but they could be defined on other state spaces. We could look at higher dimensional simple random walks: in $\mathbb{Z}^2$, for example, we could step up, down, left or right each with probability $\frac{1}{4}$. We could even have a continuous state space like $\mathbb{R}$, if, for example, the Z_is had a normal distribution.

We can use this structure to calculate the expectation or variance of any random walk (including the simple random walk).

Let's start with the expectation. For a random walk (X_n) we have

$$\mathbb{E}X_n = \mathbb{E}\left(X_0 + \sum_{i=1}^{n} Z_i \right) = \mathbb{E}X_0 + \sum_{i=1}^{n} \mathbb{E}Z_i = \mathbb{E}X_0 + n\mathbb{E}Z_1,$$

where we've used the linearity of expectation, and that the Z_is are identically distributed.

In the case of the simple random walk, we have $\mathbb{E}X_0 = 0$, since we start from 0 with certainty, and

$$\mathbb{E}Z_1 = \sum_{z \in \mathbb{Z}} z\mathbb{P}(Z_1 = z) = 1 \times p + (-1) \times q = p - q.$$

Hence, for the simple random walk, $\mathbb{E}X_n = n(p - q)$.

If $p > \frac{1}{2}$, then $p > q$, so $\mathbb{E}X_n$ grows ever bigger over time, while if $p < \frac{1}{2}$, then $\mathbb{E}X_n$ grows ever smaller (that is, negative with larger absolute value) over time. If $p = \frac{1}{2} = q$, which is the case of the simple symmetric random walk, then then the expectation $\mathbb{E}X_n = 0$ is zero for all time.

Now the variance of a random walk. We have

$$\mathrm{Var}(X_n) = \mathrm{Var}\left(X_0 + \sum_{i=1}^{n} Z_i \right) = \mathrm{Var}\, X_0 + \sum_{i=1}^{n} \mathrm{Var}\, Z_i = \mathrm{Var}\, X_0 + n\, \mathrm{Var}\, Z_1,$$

where it was crucial that X_0 and all the Z_is were independent (so we had no covariance terms).

For the simple random walk we have $\mathrm{Var}\, X_0 = 0$, since we always start from 0 with certainty. To calculate the variance of the increments, we write

$$\begin{aligned}
\mathrm{Var}(Z_1) &= \mathbb{E}(Z_1 - \mathbb{E}Z_1)^2 \\
&= p\big(1 - (p-q)\big)^2 + q\big(-1 - (p-q)\big)^2 \\
&= p(2q)^2 + q(-2p)^2 \\
&= 4pq^2 + 4p^2q \\
&= 4pq(p + q) \\
&= 4pq.
\end{aligned}$$

Here we've used that $1 - p = q$, $1 - q = p$, and $p + q = 1$; you should take a few moments to check you've followed the algebra here. Hence the variance of the simple random walk is $4pqn$. We see that (unless p is 0 or 1) the variance grows over time, so it becomes harder and harder to predict where the random walk will be.

The variance of the simple symmetric random walk is $4\frac{1}{2}\frac{1}{2}n = n$.

For large n, we can use a normal approximation for a random walk. Suppose the increments process (Z_n) has mean μ and variance σ^2, and that the walk starts from $X_0 = 0$. Then we have $\mathbb{E}X_n = \mu n$ and $\text{Var}(X_n) = \sigma^2 n$, so for large n we can use the normal approximation $X_n \approx \text{N}(\mu n, \sigma^2 n)$. (Remember, of course, that the X_n are not independent.) To be more formal, the central limit theorem tells us that, as $n \to \infty$, we have

$$\frac{X_n - n\mu}{\sigma\sqrt{n}} \to \text{N}(0, 1).$$

2.3   Exact distribution of the simple random walk

In the case of the simple random walk, we can in fact give the exact distribution by writing down an exact formula for $\mathbb{P}(X_n = i)$ for any time n and any state i.

Recall that, at each of the first n times, we independently take an upward step with probability p, and otherwise take a downward step. So if we let Y_n be the number of upward steps over the first n time periods, we see that Y_n has a binomial distribution $Y_n \sim \text{Bin}(n, p)$.

Recall that the binomial distribution has probability

$$\mathbb{P}(Y_n = k) = \binom{n}{k}p^k(1-p)^{n-k} = \binom{n}{k}p^k q^{n-k},$$

for $k = 0, 1, \dots, n$, where $\binom{n}{k}$ is the binomial coefficient "n choose k".

If $Y_n = k$, that means we've taken k upward steps and $n - k$ downward steps, leaving us at position $k - (n - k) = 2k - n$. Thus we have that

$$\mathbb{P}(X_n = 2k - n) = \mathbb{P}(Y_n = k) = \binom{n}{k}p^k q^{n-k}. \tag{2}$$

Note that after an odd number of time steps n we're always at an odd-numbered state, since $2k - \text{odd} = \text{odd}$, while after an even number of time steps n we're always at an even-numbered state, since $2k - \text{even} = \text{even}$.

Writing $i = 2k - n$ gives $k = (n + i)/2$ and $n - k = (n - i)/2$. So we can rearrange (2) to see that the distribution for the simple random walk is

$$\mathbb{P}(X_n = i) = \binom{n}{(n+i)/2}p^{(n+i)/2}q^{(n-i)/2},$$

when n and i have the same parity with $-n \leq i \leq n$, and is 0 otherwise.

In the special case of the simple symmetric random walk, we have

$$\mathbb{P}(X_n = i) = \binom{n}{(n+i)/2}\left(\frac{1}{2}\right)^{(n+i)/2}\left(\frac{1}{2}\right)^{(n-i)/2} = \binom{n}{(n+i)/2}2^{-n}.$$

In the next section, we look at a gambling problem based on the simple random walk.

Problem Sheet 1

You should attempt all these questions and write up your solutions in advance of your workshop in week 2 (??????? ??? or Tuesday ????? ???) where the answers will be discussed.

1. When designing a model for a quantity that changes over time, one has many decisions to make:

- Discrete or continuous state space?
- Discrete or continuous index set for time?
- Deterministic or stochastic model?
- If a stochastic model is chosen, is it reasonable to assume that the Markov property holds?

What would you decide for the following scenarios:

(a) The percentage of US voters with a positive opinion of Donald Trump in the Gallup weekly tracking poll.

(b) The number of points won by a football league club throughout the season.

(c) The temperature of a bowl of water placed in an oven.

(d) The number of people inside the Chemistry building.

Suggestions. This question is meant to inspire discussion, so there are not necessarily right and wrong answers. If I were designing the models, however, my choices might be these:

(a) Discrete space if percentages are given to nearest 1%, otherwise continuous; discrete time (weekly); stochastic; the Markov property might be appropriate, perhaps using some sort of random walk.

(b) Discrete space (number of points); discrete time (update after each game); stochastic; the Markov property might be appropriate, depending on if you think teams can have non-Markovian "winning streaks" (or losing streaks) that don't reflect underlying performance.

(c) Continuous space (temperature); continuous time; if the oven is reliable and the experiment carried out carefully, a deterministic model might be sufficient.

(d) Discrete space (number of people); continuous time; stochastic; Markov property might not be appropriate due to 'bursts' of people leaving during fire drills or lectures finishing early.

2. A fair six-sided dice is rolled twice, resulting in the values $X_1, X_2 \in \{1, 2, \ldots, 6\}$. Let $Y = X_1 + X_2$ be the total score. Calculate:

(a) the probability $\mathbb{P}(Y = 10)$;

Solution. The following table illustrates the possible outcomes Y of the experiment. Each cell of the table is an equally probable outcome.

X_1

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X_2

There are 3 possible ways to get $Y = 10$ (the grey cells in the table) out of the 36 possible outcomes, so we have $\mathbb{P}(Y = 10) = 3/36 = 1/12$.

(b) the conditional probability $\mathbb{P}(Y = 10 \mid X_1 = x)$ for $x = 1, 2, \ldots, 6$;

Solution. Conditioning on $X_1 = x$ means restricting our attention only to column x of the table. Each column has 6 equally probably cells. For $x = 1, 2, 3$, none of the entries equal 10, so $\mathbb{P}(Y = 10 \mid X_1 = x) = 0/6 = 0$. For each of $x = 4, 5, 6$, one of the entries equals 10, so $\mathbb{P}(Y = 10 \mid X_1 = x) = 1/6$.

(c) the conditional probability $\mathbb{P}(X_1 = x \mid Y = 10)$ for $x = 1, 2, \ldots, 6$.

Solution. Conditioning on $Y = 10$ means restricting our attention only to the 3 shaded cells, which are each equally likely. For $x = 1, 2, 3$, none of the shaded cells are in column x, so $\mathbb{P}(X_1 = x \mid Y = 10) = 0/3 = 0$. For each of $x = 4, 5, 6$, one of the shaded cells is in column x, so $\mathbb{P}(X_1 = x \mid Y = 10) = 1/3$.

3. Let (X_n) be a simple random walk starting from $X_0 = 0$ and that at each step goes up one with probability p or down one with probability $q = 1 - p$. What are:

(a) $\mathbb{P}(X_5 = 3)$,

Solution. To get $X_5 = 3$, we must take 4 steps up and 1 step down. The down step can be at any of the 5 time steps. Therefore we have $\mathbb{P}(X_5 = 3) = 5p^4 q$.

(b) $\mathbb{P}(X_5 = 3 \mid X_2 = 2)$,

Solution. Once we're at $X_2 = 2$, we must take 2 steps up and 1 step down over the next 3 time steps. So $\mathbb{P}(X_5 = 3 \mid X_2 = 2) = 3p^2 q$.

(c) $\mathbb{P}(X_n = n - 2)$,

Solution. This requires $n - 1$ steps up and 1 step down, and the down step can be at any of the n time steps. So $\mathbb{P}(X_n = n - 2) = np^{n-1} q$.

(d) $\mathbb{E}X_4$,

Solution. The increments $Z_n = X_n - X_{n-1}$ have expectation $1p + (-1)q = p - q$, so $\mathbb{E}X_4 = 4(p - q)$.

(e) $\mathbb{E}(X_6 \mid X_4 = 2)$,

Solution. We are already at 2, then another two increments will take us up $2(p-q)$ on average. Therefore $\mathbb{E}(X_6 \mid X_4 = 2) = 2 + 2(p - q)$.

4. The price X_n of a stock at the close of day n is modelled as a Gaussian random walk, where the increments (Z_n) have a normal distribution $Z_n \sim \mathrm{N}(\mu, \sigma^2)$. The model assumes a drift of $\mu = 0.7$ and a volatility of $\sigma = 2.2$. The initial price is $X_0 = 42.3$.

(a) Calculate the mean and variance of the price of the stock at the close of day 5.

Solution. The mean and variance are

$$\mathbb{E}X_5 = \mathbb{E}X_0 + n\mathbb{E}Z_1 = 42.3 + 5 \cdot 0.7 = 45.8,$$
$$\mathrm{Var}\, X_5 = \mathrm{Var}\, X_0 + n\, \mathrm{Var}\, Z_1 = 0 + 5(2.2)^2 = 24.2.$$

(b) Give a 95% prediction interval for the price at the close of day 5. (You might find it useful to recall that, if $W \sim \mathrm{N}(0, 1)$ is a standard normal random variable, then $\mathbb{P}(W \leq 1.96) = 0.975$.)

Solution. Note that X_5 itself is normally distributed, so $X_5 \sim \mathrm{N}(45.8, 24.2)$. The 95% prediction interval for a normal distribution $\mathrm{N}(\mu, \sigma^2)$ is $(\mu - 1.96\sigma, \mu + 1.96\sigma)$, so the prediction interval for X_5 is

$$(45.8 - 1.96\sqrt{24.2}, 45.8 + 1.96\sqrt{24.2}) = (36.16, 55.44).$$

(c) After day 4, the prices at the end of each of the first four days have been recorded as $X_1 = 44.4, X_2 = 44.0, X_3 = 47.1, X_4 = 47.8$. Update your prediction interval for the price at the close of day 5, and comment on how it differs from the earlier prediction interval.

Solution. By the Markov property, X_5 depends on X_4, but given X_4 does not depend on the other values, which we can therefore ignore. Since $X_5 = X_4 + Z_5$, we have

$$\mathbb{E}(X_5 \mid X_4) = X_4 + \mathbb{E}Z_5 = 47.8 + 0.7 = 48.5$$
$$\mathrm{Var}(X_5 \mid X_4) = 0 + \mathrm{Var}\, Z_5 = 0 + (2.2)^2 = 4.84.$$

The desired prediction interval is

$$(48.5 - 1.96\sqrt{4.84}, 48.5 + 1.96\sqrt{4.84}) = (44.19, 52.81).$$

Compared to before, the centre of the prediction interval is slightly higher, because the stock has out-performed expectations so far, and the interval is much narrower, because as we get closer to day 5 we become less uncertain.

5. A gambler decides to model her total winnings as a simple random walk starting from $X_0 = 0$ that at each time goes up one with probability p and down one with probability $1 - p$, but where p is unknown. The first 10 recordings, X_1 to X_{10}, are

$$(1, 2, 1, 2, 3, 4, 5, 6, 5, 6).$$

(a) What would you guess for the value of p, given this data?

Solution. In 10 time steps, the process went up $k = 8$ times and down $n - k = 2$ times. So it seems reasonable to guess that p has the value $\hat{p} = 8/10 = 0.8$.

(b) More generally, how would you estimate p from the data $X_0 = 0, X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$?

Solution. We will estimate $\hat{p} = k/n$, where k is the number of upward steps. We saw in lectures that $k = (n + x_n)/2$, so our estimate is

$$\hat{p} = \frac{n + x_n}{2n} = \frac{1}{2} + \frac{x_n}{2n}.$$

(c) Show that your estimate is in fact the maximum likelihood estimate of p.

Solution. You could answer this question by using what you know about maximum likelihood estimation for the binomial distribution from MATH1712, but we'll reason from first principles here.

If we let k be the number of upward steps, then the likelihood is

$$f(\mathbf{x}; p) = p^k (1 - p)^{n-k},$$

making the the log-likelihood

$$\ell(\mathbf{x}; p) = \ln f(\mathbf{x}; p) = k \ln p + (n - k) \ln(1 - p).$$

We maximise this by differentiating and setting equal to 0. The derivative is

$$\frac{\mathrm{d}}{\mathrm{d}p} \ell(\mathbf{x}; p) = \frac{k}{p} - \frac{n - k}{1 - p},$$

so the maximum likelihood estimate $\hat{p}$ satisfies

$$0 = \frac{k}{\hat{p}} - \frac{n - k}{1 - \hat{p}}.$$

Solving this by clearing denominators we get

$$0 = (k - k\hat{p}) - (n\hat{p} - k\hat{p}) = k - n\hat{p},$$

and rearranging gives $\hat{p} = k/n$ as desired.

3 Gambler's ruin

- The gambler's ruin Markov chain
- Equations for probability of ruin and expected duration of the game by conditioning on the first step

3.1 Gambler's ruin Markov chain

Consider the following gambling problem. Alice is gambling against Bob. Alice starts with £a and Bob starts with £b. It will be convenient to write $m = a + b$ for the total amount of money, so Bob starts with £$(m - a)$. At each step of the game, both players bet £1; Alice wins £1 off Bob with probability p, or Bob wins £1 off Alice with probability q. The game continues until one player is out of money (or is "ruined").

Let X_n denote how much money Alice has after n steps of the game. We can write this as a stochastic process with discrete time $n \in \{0, 1, 2, \dots\} = \mathbb{Z}_+$ and discrete state space $\mathcal{S} = \{0, 1, \dots, m\}$. Then $X_0 = a$, and, for $n \geq 0$, we have

$$
X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \text{ if } 1 \leq X_n \leq m - 1, \\ X_n - 1 & \text{with probability } q \text{ if } 1 \leq X_n \leq m - 1, \\ 0 & \text{if } X_n = 0, \\ m & \text{if } X_n = m. \end{cases}
$$

So Alice's money goes up one with probability p or down one with probability q, unless the game is already over with $X_n = 0$ (Alice is ruined) or $X_n = m$ (Alice has won all Bob's money, so Bob in ruined).

We see that the gambler's ruin process (X_n) clearly satisfies the Markov property: the next step X_{n+1} depends on where we are now X_n, but, given that, does not depend on how we got here.

The gambler's ruin process is exactly like a simple random walk started from $X_0 = a$ except that we have **absorbing barriers** and 0 and m, where the random walk stops because one of the players has ruined. (One could also consider random walks with **reflecting barriers**, that bounce the random walk back into the state space, or **mixed barriers** that are absorbing or reflecting at random.)

3.2 Probability of ruin

The gambling game continues until either Alice is ruined ($X_n = 0$) or Bob is ruined ($X_n = m$). A natural question to ask is: What is the probability that the game ends in Alice's ruin?

Let us write r_i for the probability Alice ends up ruined if she currently has £i. Then the probability of ruin for the whole game is r_a, since Alice initially starts with £a. The probability Bob will end up ruined is $1 - r_a$, since one of the players must lose.

What can we say about r_i? Clearly we have $r_0 = 1$ and $r_m = 0$, since this means Alice ($i = 0$) or Bob ($i = m$) is already out of money and is ruined. What about for $1 \leq i \leq m - 1$?

The key is to *condition on the first step*. That is, we can write

$$
\begin{aligned}
\mathbb{P}(\text{ruin}) &= \mathbb{P}(\text{win first round})\,\mathbb{P}(\text{ruin} \mid \text{win first round}) \\
&\quad + \mathbb{P}(\text{lose first round})\,\mathbb{P}(\text{ruin} \mid \text{lose first round}) \\
&= p\,\mathbb{P}(\text{ruin} \mid \text{win first round}) + q\,\mathbb{P}(\text{ruin} \mid \text{lose first round}).
\end{aligned}
$$

Here we have conditioned on whether Alice wins or loses the first round. More formally, we have used the *law of total probability*, which says that if the disjoint events $B_1, \dots, B_k$ cover the whole sample space, then

$$
\mathbb{P}(A) = \sum_{i=1}^{k} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{k} \mathbb{P}(B_i)\,\mathbb{P}(A \mid B_i).
$$

Here, {Alice wins the first round} and {Alice loses the first round} are indeed disjoint events that cover the whole sample space. This idea of "conditioning on the first step" will be the most crucial tool throughout this whole module.

If Alice wins the first round from having £i, she now has £$(i + 1)$. Her probability of ruin is now r_{i+1}, because, by the Markov property, it's as if the game were starting again with Alice having £$(i + 1)$ to start with. The Markov property tells us that it doesn't matter *how* Alice got to having £$(i + 1)$, it only matters how much she has now. Similarly, if Alice loses the first round, she now has £$(i - 1)$, and the ruin probability is r_{i-1}. Hence we have

$$r_i = pr_{i+1} + qr_{i-1}.$$

Rearranging, and including the "boundary conditions", we see that the equation we want to solve is

$$pr_{i+1} - r_i + qr_{i-1} = 0 \qquad \text{subject to} \qquad r_0 = 1,\ r_m = 0.$$

This is a **linear difference equation** – and, because the left-hand side is 0, we call it a **homogeneous** linear difference equation.

We will see how to solve this equation in the next lecture. We will see that, if we set $\rho = q/p$, then the ruin probability is given by

$$r_a = \begin{cases} \dfrac{\rho^a - \rho^m}{1 - \rho^m} & \text{if } \rho \neq 1, \\[2mm] 1 - \dfrac{a}{m} & \text{if } \rho = 1. \end{cases}$$

Note that $\rho = 1$ is the same as the symmetric condition $p = q = \frac{1}{2}$.

Imagine Alice is not playing against her opponent Bob, but rather is up against a large casino. In this case, the casino's capital £$(m - a)$ is typically much bigger than Alice's £a. We can model this by keeping a fixed taking a limit $m \to \infty$. Typically, the casino has "an edge", meaning they have a better than $50 : 50$ chance of winning; this means that $q > p$, so $\rho > 1$. In this case, we see that the ruin probability is

$$\lim_{m \to \infty} r_a = \lim_{m \to \infty} \frac{\rho^a - \rho^m}{1 - \rho^m} = \lim_{m \to \infty} \frac{\rho^a/\rho^m - 1}{1/\rho^m - 1} = \frac{0 - 1}{0 - 1} = 1,$$

so Alice will be ruined with certainty.

Even with a generous casino that offered an exactly fair game with $p = q = \frac{1}{2}$, so $\rho = 1$, we would have

$$\lim_{m \to \infty} r_a = \lim_{m \to \infty} \left(1 - \frac{a}{m}\right) = 1 - 0 = 1,$$

so, even with this fair game, Alice would still be ruined with certainty.

The official advice of the University of Leeds module MATH2750 is that you shouldn't gamble against a casino and expect to win.

3.3 Expected duration of the game

We could also ask for how long we expect the game to last.

We approach this like before. Let d_i be the expected duration of the game when Alice has £i. Our boundary conditions are $d_0 = d_m = 0$, because $X_n = 0$ or $X_n = m$ means that the game is over with Alice or Bob ruined. Again, we proceed by conditioning on the first step, so

$$\mathbb{E}(\text{duration}) = \mathbb{P}(\text{win first round})\,\mathbb{E}(\text{duration} \mid \text{win first round})$$
$$+ \mathbb{P}(\text{lose first round})\,\mathbb{E}(\text{duration} \mid \text{lose first round})$$
$$= p\,\mathbb{E}(\text{duration} \mid \text{win first round}) + q\,\mathbb{E}(\text{duration} \mid \text{lose first round}).$$

More formally, we've used another version of the law of total probability,

$$\mathbb{E}(X) = \sum_{i=1}^{k} \mathbb{P}(B_i)\,\mathbb{E}(X \mid B_i),$$

or, alternatively, the *tower law* for expectations

$$\mathbb{E}(X) = \mathbb{E}_Y \mathbb{E}(X \mid Y) = \sum_y \mathbb{P}(Y = y)\, E(X \mid Y = y),$$

where, in our case, Y was the outcome of the first round.

Now, the expected duration given we win the first round is $1 + d_{i+1}$. This is because the round itself takes 1 time step, and then, by the Markov property, it's as if we are starting again from $i+1$. Similarly, the expected duration given we lose the first round is $1 + d_{i-1}$. Thus we have

$$d_i = p(1 + d_{i+1}) + q(1 + d_{i-1}) = 1 + pd_{i+1} + qd_{i-1}.$$

Don't forget the 1 that counts the current round!

Rearranging, and including the boundary conditions, we have another linear difference equation:

$$pd_{i+1} - d_i + qd_{i-1} = -1 \qquad \text{subject to} \qquad d_0 = 0,\ d_m = 0.$$

Because the right-hand side, -1, is nonzero, we call this an **inhomogeneous** linear difference equation.

Again, we'll see how to solve this in the next lecture, and will find that the solution is given by

$$d_a = \begin{cases} \dfrac{1}{q-p}\left(a - m\dfrac{1-\rho^a}{1-\rho^m}\right) & \text{if } \rho \neq 1, \\ a(m-a) & \text{if } \rho = 1. \end{cases}$$

Thinking again of playing against the casino, with $q > p$, $\rho > 1$, and $m \to \infty$, we see that the expected duration is

$$\lim_{m \to \infty} d_a = \lim_{m \to \infty} \frac{1}{q-p}\left(a - m\frac{1-\rho^a}{1-\rho^m}\right) = \frac{1}{q-p}(a - 0) = \frac{a}{q-p},$$

since ρ^m grows much quicker than m. So Alice ruins with certainty, and it will take time $a/(q-p)$, on average.

In the case of the generous casino, though, with $q = p$, so $\rho = 1$, we have

$$\lim_{m \to \infty} d_a = \lim_{m \to \infty} a(m-a) = \infty.$$

So here, Alice will ruin with certainty, but it may take a very long time until the ruin occurs, since the expected duration is infinite.

In the next section, we see how to solve linear difference equations, in order to find the ruin probability and expected duration of the gambler's ruin.

4 Linear difference equations

- How to solve homogeneous and inhomogeneous linear difference equations
- Solving for probability of ruin and expected duration of the gambler's ruin

Linear difference equations are equations that look like

$$a_k x_{n+k} + a_{k-1} x_{n+k-1} + \cdots + a_1 x_{n+1} + a_0 x_n = f(n) \tag{3}$$

for $n = 0, 1, \ldots$, where the a_i are given constants, $f(n)$ is a given function, and we want to solve for the sequence (x_n). The equation normally comes with some extra conditions, such as the value of the first few x_ns.

When the right-hand side of (3) is zero, so $f(n) = 0$, we say the equation is **homogeneous**; if the right-hand side is nonzero, it is **inhomogeneous**. The number k, where there are $k + 1$ terms on the left-hand side, is called the **degree** of the equation; we are mostly interested in second-degree linear difference equations, which have three terms on the left-hand side.

4.1 Homogeneous linear difference equations

We start with the homogeneous case, which is simpler.

Consider a homogeneous linear difference equation. We shall use the second-degree example

$$x_{n+2} - 5x_{n+1} + 6x_n = 0 \qquad \text{subject to } x_0 = 4, x_1 = 9.$$

Here, the conditions on x_0 and x_1 are **initial conditions**, because they tell us how the sequence (x_n) starts.

For the moment, we shall put the initial conditions to the side and just worry about the equation

$$x_{n+2} - 5x_{n+1} + 6x_n = 0.$$

We start by guessing there might be a solution of the form $x_n = \lambda^n$ for some constant λ. We can find out if there is such a solution by substituting in $x_n = \lambda^n$, and seeing if there's a λ that solves the equation. For our example, we get

$$\lambda^{n+2} - 5\lambda^{n+1} + 6\lambda^n = 0.$$

After cancelling off a common factor of λ^n, we get

$$\lambda^2 - 5\lambda + 6 = 0.$$

This is called the **characteristic equation**. For a general homogeneous linear difference equation (3), the characteristic equation is

$$a_k \lambda^k + a_{k-1} \lambda^{k-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{4}$$

When writing out answers to questions, you can jump straight to the characteristic equation.

We can now solve the characteristic equation for λ. In our example, we can factor the left-hand side to get $(\lambda - 3)(\lambda - 2) = 0$, which has solutions $\lambda = 2$ and $\lambda = 3$. Thus $x_n = 2^n$ and $x_n = 3^n$ both solve our equation. In fact, since the right-hand side of the equation is 0, any linear combination of these two solutions is a solution also, thus we get the **general solution**

$$x_n = A2^n + B3^n,$$

which is a solution for any values of the constants A and B.

For a general characteristic equation with distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_k$, the general solution is

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n + \cdots + C_k \lambda_k^n.$$

If we have a repeated root – say, $\lambda_1 = \lambda_2 = \cdots = \lambda_r$ is repeated r times – than you can check that a solution is given by

$$x_n = (D_0 + D_1 n + \cdots + D_{r-1} n^{r-1}) \lambda_1^n,$$

which should take its place in the general solution.

Once we have the general solution, we can use the extra conditions to find the values of the constants. In our example, we can use the initial conditions to find out the values of A and B. We see that

$$x_0 = A2^0 + B3^0 = A + B = 4,$$
$$x_1 = A2^1 + B3^1 = 2A + 3B = 9.$$

We can now solve this pair of simultaneous equations to solve for A and B. By subtracting twice the first equation from the second we get $B = 1$, and substituting this into the first equation we get $A = 3$. Thus the solution is

$$x_n = 3 \cdot 2^n + 3^n.$$

In conclusion, the process here was:

1. Find the general solution by writing down and solving the characteristic equation.
2. Use the extra conditions to find the values of the constants in the general solution.

If you think you've got the hang of this, you should press on to the next subsection. But if you're still a bit unsure, here are two extra optional examples. (These also give an idea of how I would expect you to set out your answers to similar problems.)

Example 4.1. *Solve the homogeneous linear difference equation*

$$x_{n+2} - x_{n+1} - 6x_n = 0 \qquad \text{subject to} \quad x_0 = 3, \quad x_1 = 4.$$

Step 1. The characteristic equation is

$$\lambda^2 - \lambda - 6 = 0.$$

We can solve this by factorising it as $(\lambda - 3)(\lambda + 2) = 0$, to find the solutions $\lambda_1 = -2$ and $\lambda_2 = 3$. Thus the general solution is

$$x_n = A(-2)^n + B3^n.$$

Step 2. Substituting the initial conditions into the general solution, we have

$$x_0 = A(-2)^0 + B3^0 = A + B = 3$$
$$x_1 = A(-2)^1 + B3^1 = -2A + 3B = 4.$$

We can add twice the first equation from the second to get $5B = 10$, so $B = 2$. We can substitute this into the first equation to get $A = 1$.

The solution is therefore

$$x_n = 1 \cdot (-2)^n + 2 \cdot 3^n = (-2)^n + 2 \cdot 3^n.$$

Example 4.2. *Solve the homogeneous linear difference equation*

$$x_{n+2} + 4x_{n+1} + 4x_n = 0 \qquad \text{subject to} \quad x_0 = 2, \quad x_1 = -6.$$

Step 1. The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0.$$

We can solve this by factorising it as $(\lambda + 2)^2 = 0$, to find a repeated root $\lambda_1 = \lambda_2 = -2$. Thus the general solution is

$$x_n = (A + Bn)(-2)^n.$$

Step 2. Substituting the initial conditions into the general solution, we have

$$x_0 = (A + B0)(-2)^0 = A = 2$$
$$x_1 = (A + B1)(-2)^1 = -2A - 2B = -6.$$

The first immediately gives $A = 2$, and substituting this into the second equation gives $B = 1$.

The solution is therefore

$$x_n = (2 + n)(-2)^n.$$

4.2 Probability of ruin for the gambler's ruin

In the last lecture we saw that probability of ruin for the gambler's ruin process is the solution to

$$pr_{i+1} - r_i + qr_{i-1} = 0 \qquad \text{subject to} \qquad r_0 = 1, \ r_m = 0,$$

where the extra conditions here are **boundary conditions**, because they tell us what happens at the boundaries of the state space.

The characteristic equation is

$$p\lambda^2 - \lambda + q = 0.$$

We can solve the characteristic equation by factorising it as $(p\lambda - q)(\lambda - 1) = 0$. (It might take a moment to check this really is a factorisation of the characteristic equation. Hint: we've used that $p + q = 1$.) So the characteristic equation has roots $\lambda = q/p$, which we called ρ last time, and $\lambda = 1$. Now, if $\rho = 1$ (so $p = q = \frac{1}{2}$) we have a repeated root, while if $\rho \neq 1$ we have distinct roots, so we'll need to deal with the two cases separately.

First, the case $\rho \neq 1$. Since the two roots are distinct, we have the general solution

$$r_i = A\rho^i + B1^i = A\rho^i + B.$$

We can now use the boundary conditions to find A and B. We have

$$r_0 = A\rho^0 + B = A + B = 1,$$
$$r_m = A\rho^m + B = 0.$$

From the first we get $B = 1 - A$, which we substitute into the second to get

$$A\rho^m + 1 - A = 0 \quad \Rightarrow \quad A = \frac{1}{1 - \rho^m},$$

and hence

$$B = 1 - A = 1 - \frac{1}{1 - \rho^m} = -\frac{\rho^m}{1 - \rho^m}.$$

Thus the solution is

$$r_i = \frac{1}{1 - \rho^m}\rho^i - \frac{\rho^m}{1 - \rho^m} = \frac{\rho^i - \rho^m}{1 - \rho^m},$$

as we claimed last time.

Second, the case $\rho = 1$. Now we have a repeated root $\lambda = 1$, so the general solution is

$$r_i = (A + Bi)1^i = A + Bi.$$

Again, we use the boundary conditions, to get

$$r_0 = A + B \cdot 0 = A = 1,$$
$$r_m = A + Bm = 0,$$

and we immediately see that $A = 1$ and $B = -1/m$. Thus the solution is

$$r_i = 1 - \frac{1}{m}i = 1 - \frac{i}{m},$$

as claimed last time.

4.3 Inhomogeneous linear difference equations

Solving inhomogeneous linear difference equations requires three steps:

1. Find the general solution to the *homogeneous* equation by writing down and solving the characteristic equation.
2. Find any solution (a "particular solution") to the inhomogeneous equation. The general solution to the inhomogeneous equation is a particular solution plus the general solution to the homogeneous equation.
3. Use the extra conditions to find the values of the constants in the general solution.

This idea works because adding a solution to the *homogeneous* equation to the left-hand side adds zero to the right-hand side, so maintains a solution to the inhomogeneous equation.

Let's work through the example

$$x_{n+2} - 5x_{n+1} + 6x_n = 2 \qquad \text{subject to } x_0 = 4, x_1 = 9.$$

We already know from earlier that the general solution to the homogeneous equation $x_{n+2} - 5x_{n+1} + 6x_n = 0$ (with a zero on the right-hand side) is

$$x_n = A2^n + B3^n.$$

We now need to find a **particular solution** – that is, any solution – to our new inhomogeneous equation. A general hint here is to guess a solution with the same "shape" as the right-hand side. For example, if the right-hand side is a constant, try a constant for the particular solution. Here our right-hand side is the constant 2, so we should try a constant $x_n = C$. Substituting this into the inhomogeneous equation gives us $C - 5C + 6C = 2$, thus $2C = 2$ and $C = 1$, giving a particular solution $x_n = 1$. The general solution to the inhomogeneous equation is therefore

$$x_n = 1 + A2^n + B3^n,$$

the sum of the particular solution $x_n = 1$ and the general solution to the homogeneous equation.

The right-hand side being a constant, so we guess a constant, is the main case we will deal with. Other cases you could come across include:

- If the right-hand side is a polynomial of degree d, try a polynomial of degree d.
- If the right-hand side is α^n for some α, try $C\alpha^n$.
- If the right-hand side is a constant, but a constant C doesn't work, try Cn. If that still doesn't work, try Cn^2, and so on. We discuss this case further in the next subsection.

Again, we use the initial conditions to get the constants A and B. We have

$$x_0 = 1 + A2^0 + B3^0 = 1 + A + B = 4,$$
$$x_1 = 1 + A2^1 + B3^1 = 1 + 2A + 3B = 9,$$

and we can easily check this gives $A = 1, B = 2$. Thus the solution is

$$x_n = 1 + 1 \cdot 2^n + 2 \cdot 3^n = 1 + 2^n + 2 \cdot 3^n.$$

Again, if you have the hang of this, press on, but we present an extra optional example if things are still unclear.

Example 4.3. *Solve the inhomogeneous linear difference equation*

$$10x_{n+2} - 7x_{n+1} + x_n = 8 \qquad \text{subject to} \quad x_0 = 0, \quad x_1 = \tfrac{13}{10}.$$

Step 1. The characteristic equation is
$$10\lambda^2 - 7\lambda + 1 = 0.$$

We can solve this by factorising it as

$$(2\lambda - 1)(5\lambda - 1) = 0,$$

to find the solutions $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{5}$. Thus the general solution of the homogeneous equation is

$$x_n = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{5}\right)^n.$$

Step 2. Since the right hand side of the inhomogeneous equation is a constant, we guess a constant particular solution with shape $x_n = C$. Substituting in this guess, we get

$$10C - 7C + C = 4C = 8$$

with solution $C = 2$. Thus a particular solution is $x_n = 2$, and the general solution to the inhomogeneous equation is

$$x_n = 2 + A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{5}\right)^n.$$

Step 3. Substituting the initial conditions into the general solution, we have

$$x_0 = 2 + A\left(\frac{1}{2}\right)^0 + B\left(\frac{1}{5}\right)^0 = 2 + A + B = 0 \quad \Rightarrow \quad A + B = -2$$

$$x_1 = 2 + A\left(\frac{1}{2}\right)^1 + B\left(\frac{1}{5}\right)^1 = 2 + A\frac{1}{2} + B\frac{1}{5} = \frac{13}{10} \quad \Rightarrow \quad 5A + 2B = -7.$$

We can subtract twice the first equation from the second to get $3A = -3$, so $A = -1$. We can substitute this into the second equation to get $B = -1$.

The solution is therefore

$$x_n = 2 - \left(\frac{1}{2}\right)^n - \left(\frac{1}{5}\right)^n.$$

4.4 Expected duration for the gambler's ruin

From last time, the expected duration of the gambler's ruin game solves

$$pd_{i+1} - d_i + qd_{i-1} = -1 \qquad \text{subject to} \qquad d_0 = 0, \ d_m = 0.$$

As before, we divide cases based on whether or not $\rho = 1$.

First, the case $\rho \neq 1$. We already know that the general solution to the homogeneous equation is

$$d_i = A\rho^i + B.$$

Now we need a particular solution. It's tempting to guess a constant C for a particular solution, but we know that constants solve the homogeneous equation, since $d_i = B$ is a solution – so a constant will give right-hand side 0, not -1. (You could try out $x_i = C$; you should find that the equation for C doesn't have a solution.) The next best try is to go one degree up: let's guess $x_i = Ci$ instead. This gives

$$\begin{aligned}
-1 &= pC(i+1) - Ci + qC(i-1) \\
&= C(pi + p - i + qi - q) \\
&= C((p + q - 1)i + (p - q)) \\
&= C(p - q),
\end{aligned}$$

since $p + q - 1 = 1 - 1 = 0$. This $C = -1/(p-q) = 1/(q-p)$. Finding a solution for C shows that our guess worked. The general solution to the inhomogeneous equation is

$$d_i = \frac{i}{q - p} + A\rho^i + B.$$

Then to find the constants, we have

$$d_0 = \frac{0}{q-p} + A\rho^0 + B = A + B = 0,$$

$$d_m = \frac{m}{q-p} + A\rho^m + B = 0,$$

which you can check gives

$$A = -B = \frac{1}{q-p} \cdot \frac{m}{1-\rho^m}.$$

Hence, the solution is

$$d_i = \frac{i}{q-p} + \frac{1}{q-p} \cdot \frac{m}{1-\rho^m}\rho^i - \frac{1}{q-p} \cdot \frac{m}{1-\rho^m} = \frac{1}{q-p}\left(i - m\frac{1-\rho^i}{1-\rho^m}\right).$$

Second, the case $\rho = 1$, so $p = q = \frac{1}{2}$. We already know that the general solution to the homogeneous equation is

$$d_i = A + Bi.$$

We need a particular solution. Since both constants $x_i = A$ and linear $x_i = Bi$ terms solve the homogeneous equation (or by checking that guessing $x_i = C$ or $x_i = Ci$ doesn't work), we'll have to go up another degree and try $x_i = Ci^2$. This gives

$$\begin{aligned}
-1 &= \tfrac{1}{2}C(i+1)^2 - Ci^2 + \tfrac{1}{2}C(i-1)^2 \\
&= \tfrac{1}{2}C(i^2 + 2i + 1 - 2i^2 + i^2 - 2i + 1) \\
&= \tfrac{1}{2}C((1-2+1)i^2 + (2-2)i + (1+1)) \\
&= C,
\end{aligned}$$

so the general solution to the inhomogeneous equation is

$$d_i = -i^2 + A + Bi.$$

Then to find the constants, we have

$$d_0 = -0^2 + A + B \cdot 0 = A = 0,$$
$$d_m = -m^2 + A + Bm = 0,$$

giving $A = 0, B = m$. The solution is

$$d_i = -i^2 + 0 + mi = i(m-i).$$

In the next section, we move on to the general theory of discrete time Markov chains.

Problem sheet 2

You should attempt all these questions and write up your solutions in advance of your workshop in week 2 (??????? ??? or Tuesday ????? ???) where the answers will be discussed.

Assessment 1

Assessment 1

5 Discrete time Markov chains

- Definition of time homogeneous discrete time Markov chains
- Calculating n-step transition properties
- The Chapman–Kolomogorov equations

5.1 Time homogeneous discrete time Markov chains

So far we've seen a a few examples of stochastic processes in discrete time and discrete space with the Markov memoryless property. Now we will develop the theory more generally.

To define a so-called "Markov chain", we first need to say where we start from, and second what the probabilities of transitions from one state to another are.

In our examples of the simple random walk and gambler's ruin, we specified the start point X_0 exactly, but we could pick the start point at random according to some distribution $\lambda_i = \mathbb{P}(X_0 = i)$.

After that, we want to know the **transition probabilities** $\mathbb{P}(X_{n+1} = j \mid X_n = i)$ for $i, j \in \mathcal{S}$. Here, because of the Markov property, the transition probability only needs to condition on the state we're in now $X_n = i$, and not on the whole history of the process.

In the case of the simple random walk, for example, we had initial distribution

$$\lambda_i = \mathbb{P}(X_0 = i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and transition probabilities

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

For the random walk (and also the gambler's ruin), the transition probabilities $\mathbb{P}(X_{n+1} = j \mid X_n = i)$ don't depend on n; in other words, the transition probabilities stay the same over time. A Markov process with this property is called **time homogeneous**. We will always consider time homogeneous processes from now on (unless we say otherwise).

Let's write $p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$ for the transition probabilities, which are independent of n. We must have $p_{ij} \geq 0$, since it is a probability, and we must also have $\sum_j p_{ij} = 1$, as this is the sum of the probabilities of all the places you can move to from state i.

Definition 5.1. Let (λ_i) be a probability distribution on a sample space $\mathcal{S}$. Let p_{ij}, where $i, j \in \mathcal{S}$, be such that $p_{ij} > 0$ for all i, j, and $\sum_j p_{ij} = 1$ for all i. Let the time index be $n = 0, 1, 2, \ldots$. Then the **time homogeneous discrete time Markov process** or **Markov chain** (X_n) with initial distribution (λ_i) and transition probabilities (p_{ij}) is defined by

$$\mathbb{P}(X_0 = i) = \lambda_i,$$
$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i) = p_{ij}.$$

When the state space is finite (and even sometimes when it's not), it's convenient to write the transition probabilities (p_{ij}) as a matrix P, called the **transition matrix**, whose (i, j)th entry is p_{ij}. Then the condition that $\sum_j p_{ij} = 1$ is the condition that each of the rows of P add up to 1.

From this, we can calculate, for example,

$$\mathbb{P}(X_0 = i \text{ and } X_1 = j) = \lambda_i p_{ij},$$

as we must start from i and then move to j. Another example would be

$$\mathbb{P}(X_{n+2} = j \text{ and } X_{n+1} = k \mid X_n = i) = p_{ik} p_{kj},$$

as we must first move from i to k, then from k to j.

5.2 A two-state example

Consider a simple two-state Markov chain with state space $\mathcal{S} = \{0, 1\}$ and transition matrix

$$\mathsf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}$$

for some $0 < \alpha, \beta < 1$. Note that the rows of P add up to 1, as they must.

We can illustrate P by a **transition diagram**, where the blobs are the states and the arrows give the transition probabilities. (We don't draw the arrow if $p_{ij} = 0$.) In this case, our transition diagram looks like this:
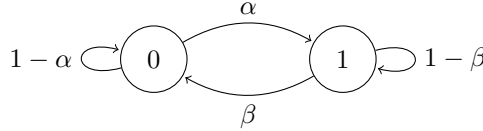

Figure 2: Transition diagram for the two-state Markov chain

We can use this as a simple model of a broken printer, for example. If the printer is broken (state 0) on one day, then with probability α it will be fixed (state 1) by the next day; while if it is working (state 1), then with probability β it will have broken down (state 0) by the next day.

Example 5.1. *If the printer is working on Monday, what's the probability that it also is working on Wednesday?*

If we call Monday day n, then Wednesday is day $n + 2$, and we want to find the two-step transition probability.
$$p_{11}(2) = \mathbb{P}(X_{n+2} = 1 \mid X_n = 1).$$

The key to calculating this is to condition on the first step again – that is, on whether the printer is working on Tuesday. We have

$$\begin{aligned} p_{11}(2) &= \mathbb{P}(X_{n+1} = 0 \mid X_n = 1)\,\mathbb{P}(X_{n+2} = 1 \mid X_{n+1} = 0) \\ &\quad + \mathbb{P}(X_{n+1} = 1 \mid X_n = 1)\,\mathbb{P}(X_{n+2} = 1 \mid X_{n+1} = 1) \\ &= p_{10}p_{01} + p_{11}p_{11} \\ &= \beta\alpha + (1 - \beta)^2. \end{aligned}$$

Another way to think of this as we summing the probabilities of all length-2 paths from 1 to 1, which are $1 \to 0 \to 1$ with probability $\beta\alpha$ and $1 \to 1 \to 1$ with probability $(1 - \beta)^2$

In the above, we used the Markov property to mean the conditional probabilities $\mathbb{P}(X_{n+2} = 1 \mid X_{n+1} = k)$ did not have to depend on X_n.

5.3 n-step transition probabilities

In the above example, we calculated a two-step transition probability $p_{ij}(2) = \mathbb{P}(X_{n+2} = j \mid X_n = i)$ by conditioning on the first step. That is, by considering all the possible intermediate steps k, we have

$$p_{ij}(2) = \sum_{k \in \mathcal{S}} \mathbb{P}(X_{n+1} = k \mid X_n = i)\mathbb{P}(X_{n+2} = j \mid X_{n+1} = k) = \sum_{k \in \mathcal{S}} p_{ik}p_{kj}.$$

But this is exactly the formula for multiplying the matrix P with itself! In other words, $p_{ij}(2) = \sum_k p_{ik}p_{kj}$ is the (i, j)th entry of the matrix $\mathsf{P}^2 = \mathsf{PP}$. If we write $\mathsf{P}(2) = (p_{ij}(2))$ for the matrix of two-step transition probabilities, we have $\mathsf{P}(2) = \mathsf{P}^2$.

More generally, we see that this rule holds over multiple steps, provided we sum over all the possible paths $i \to k_1 \to k_2 \to \cdots \to k_{n-1} \to j$ of length n from i to j.

Theorem 5.1. *Let* (X_n) *be a Markov chain with state space* $\mathcal{S}$ *and transition matrix* $\mathsf{P} = (p_{ij})$. *For* $i, j \in \mathcal{S}$, *write*

$$p_{ij}(n) = \mathbb{P}(X_n = j \mid X_0 = i)$$

for the n-step transition probability. Then

$$p_{ij}(n) = \sum_{k_1, k_2, \ldots, k_{n-1} \in \mathcal{S}} p_{ik_1} p_{k_1 k_2} \cdots p_{k_{n-2} k_{n-1}} p_{k_{n-1} j}.$$

In particular, $p_{ij}(n)$ *is the* (i, j)*th element of the matrix power* P^n, *and the matrix of n-step transition probabilities is given by* $\mathsf{P}(n) = \mathsf{P}^n$.

The so-called **Chapman–Kolmogorov equations** follow immediately from this.

Theorem 5.2 (Chapman–Kolmogorov equations)**.** *Let* (X_n) *be a Markov chain with state space* $\mathcal{S}$ *and transition matrix* $\mathsf{P} = (p_{ij})$. *Then, for non-negative integers* n, m, *we have*

$$p_{ij}(n + m) = \sum_{k \in \mathcal{S}} p_{ik}(n) p_{kj}(m),$$

or, in matrix notation, $\mathsf{P}(n + m) = \mathsf{P}(n)\mathsf{P}(m)$.

In other words, a trip of length $n + m$ from i to j is a trip of length n from i to some other state k, then a trip of length m from k back to j, and this intermediate stop k can be any state, so we have to sum the probabilities.

Of course, once we know that $\mathsf{P}(n) = \mathsf{P}^n$ is given by the matrix power, it's totally obvious that $\mathsf{P}(n+m) = \mathsf{P}^{n+m} = \mathsf{P}^n \mathsf{P}^m = \mathsf{P}(n)\mathsf{P}(m)$.

Example 5.2. In our two-state broken printer example above, the matrix of two-state transition probabilities is given by

$$\begin{aligned}
\mathsf{P}(2) = \mathsf{P}^2 &= \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix} \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix} \\
&= \begin{pmatrix} (1 - \alpha)^2 + \alpha\beta & (1 - \alpha)\alpha + \alpha(1 - \beta) \\ \beta(1 - \alpha) + (1 - \beta)\beta & \beta\alpha + (1 - \beta)^2 \end{pmatrix},
\end{aligned}$$

where the bottom right entry $p_{11}(2)$ is what we calculated earlier.

One final comment. It's also convenient to consider the initial distribution $\lambda = (\lambda_i)$ as a *row* vector. The first-step distribution is given by

$$\mathbb{P}(X_1 = j) = \sum_{i \in \mathcal{S}} \lambda_i p_{ij},$$

by conditioning on the start point. This is exactly the jth element of the vector–matrix multiplication $\lambda \mathsf{P}$. More generally, the row vector of of probabilities after n steps is given by $\lambda \mathsf{P}^n$.

In the next section, we look at how to model some actuarial problems using Markov chains.

6 Examples from actuarial science

- Three Markov chain models for insurance problems

In this lecture we'll set up three simple models for an insurance company that can be analysed using ideas about Markov chains. The first example has a direct Markov chain model. For the second and third examples, we will have to be clever to find a Markov chain associated to the situation.

6.1 A simple no-claims discount model

A motor insurance company puts policy holders into three categories:

- no discount on premiums (state 1)
- 25% discount on premiums (state 2)
- 50% discount on premiums (state 3)

New policy holders start with no discount (state 1). Following a year with no insurance claims, policy holders move up one level of discount. If they start the year in state 3 and make no claim, they remain in state 3. Following a year with at least one claim, they move down one level of discount. If they start the year in state 1 and make at least one claim, they remain in state 1. The insurance company believes that probability that a motorist has a claim free year is $\frac{3}{4}$.

We can model this directly as a Markov chain:

- the state space $\mathcal{S} = \{1, 2, 3\}$ is discrete, as is the time (each year);
- the probability of being in a certain state at a future time is completely determined by the present state (the Markov property);
- the one-step transition probabilities are not time dependent (time homogeneous).

The transition probability and transition diagram of the Markov chain are:

$$\mathsf{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$
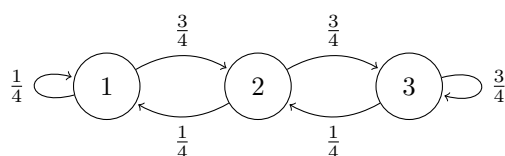
Example 6.1. *What is the probability of having a 50% reduction to your premium three years from now, given that you currently have no reduction on the premium?*

We want to find the three-step transition probability

$$p_{13}(3) = \mathbb{P}(X_3 = 3 \mid X_0 = 1).$$

We can find this by summing over all paths $1 \to k_1 \to k_2 \to 3$. There are two such paths, $1 \to 1 \to 2 \to 3$ and $1 \to 2 \to 3 \to 3$. Thus

$$p_{13}(3) = p_{11}p_{12}p_{23} + p_{12}p_{23}p_{33} = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{36}{64} = \frac{9}{16}.$$

Alternatively, we could directly calculate all the three-step transition probabilities by the matrix method, to get

$$\mathsf{P}(3) = \mathsf{P}^3 = \mathsf{PPP} = \frac{1}{64} \begin{pmatrix} 7 & 21 & 36 \\ 7 & 12 & 45 \\ 4 & 15 & 45 \end{pmatrix}.$$

(You can check this yourself.) The desired $p_{13}(3)$ is the top right entry $36/64 = 9/16$.

6.2 An accident model with memory

According to a different model, a motorist's nth year of driving is either accident free, or has exactly one accident. (The model does not allow for more than one accident in a year.) Let Y_n be a random variable so that,

$$Y_n = \begin{cases} 0 & \text{if the motorist has no accident in year } n, \\ 1 & \text{if the motorist has one accident in year } n. \end{cases}$$

This defines a stochastic process (Y_n) with finite state space $\mathcal{S} = \{0, 1\}$ and discrete time $n = 1, 2, 3, \dots$.

The probability of an accident in year $n + 1$ is modelled as a function of the total number of previous accidents over a function of the number of years in the policy; that is,

$$\mathbb{P}(Y_{n+1} = 1 \mid Y_n = y_n, \dots, Y_2 = y_2, Y_1 = y_1) = \frac{f(y_1 + y_2 + \dots + y_n)}{g(n)},$$

and $Y_{n+1} = 0$ otherwise, where f and g are non-negative increasing functions with $0 \leq f(m) \leq g(m)$ for all m.

Unfortunately (Y_n) is *not* a Markov chain – it's clear that Y_{n+1} depends not only on Y_n, the number accidents this year, but the entire history $Y_1, Y_2, \dots, Y_n$.

However, we have a cunning work-around. Define $X_n = \sum_{i=1}^{n} Y_i$ to be the total number of accidents up to year n. Then (X_n) *is* a Markov chain. In fact, we have

$$\begin{aligned} \mathbb{P}(X_{n+1} = x_n + 1 \mid X_n &= x_n, \dots, X_2 = x_2, X_1 = x_1) \\ &= \mathbb{P}(Y_{n+1} = 1 \mid Y_n = x_n - x_{n-1}, \dots Y_2 = x_2 - x_1, Y_1 = x_1) \\ &= \frac{f((x_n - x_{n-1}) + \dots + (x_2 - x_1) + x_1)}{g(n)} \\ &= \frac{f(x_n)}{g(n)}, \end{aligned}$$

and $X_{n+1} = x_n$ otherwise. This clearly depends only on x_n. Thus we can use Markov chain techniques on (X_n) to lean about the non-Markov process (Y_n)

Let's think about the conditions we placed on this model. First, the transition probabilities should lie in the range $[0, 1]$. Since $\sum_{i=1}^{n} y_i \leq n$, this is where we use $0 \leq f(m) \leq g(m)$. Second, the condition that f is increasing means that between drivers who have been driving the same number of years, we think the more accident-prone in the past is more likely to have an accident in the future. Third, the condition that g is increasing means that between drivers who have had the same number of accidents, we think the one who has spread those accidents over a longer period of time is less likely to have accidents in the future.

6.3 A no-claims discount model with memory

Sometimes, we are presented with a stochastic process which is not a Markov chain, but by altering the state space $\mathcal{S}$ we can sometimes end up with a process which is a Markov chain. As such, it is important to think carefully about choice of state space. To see this we will return to the no-claims discount example.

Suppose now we have an model with four levels of discount:

- no discount (state 1)
- 20% discount (state 2)
- 40% discount (state 3)
- 60% discount (state 4)

If a year is accident free, then the discount increases one level, to a maximum of 60%. If the year has an accident, then the discount decreases, by one level if the year previous to that was accident free, but

by two levels if the previous year had an accident also, to a minimum of no discount. As before, the insurance company believes that probability that a motorist has a claim-free year is $\frac{3}{4} = 0.75$.

We might consider the most natural example of a state space, where the states are discount levels; say, $\mathcal{S} = \{1, 2, 3, 4\}$. But this is not a Markov chain, since if a policy holder has an accident, we may need to know about the past in order to determine probabilities for future states, which violates the Markov property. In particular, if a motorist is in state 3 (40% discount) and has an accident, they will either move down to level 2 (if they had not crashed the previous year, so had previously been in state 2) or to level 1 (if they had crashed the previous year, so had previously been in state 4), depending on their previous level. You can check this is the only violation of the Markov property.

However, we can be clever again, this time in the choice of our state space. Instead, we can split the 40% level into two different states: state "3a" if there was an accident the previous year and state "3b" if there was no accident the previous year. Our states are now:

- no discount (state 1)
- 20% discount (state 2)
- 40% discount, no claim in previous year (state 3a)
- 40% discount, claim in previous year (state 3b)
- 60% discount (state 4)

Now this *is* a Markov chain. Under the assumption of 25% of drivers having an accident each year, the transition matrix is

$$\mathsf{P} = \begin{pmatrix} 0.25 & 0.75 & 0 & 0 & 0 \\ 0.25 & 0 & 0.75 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0.75 \\ 0.25 & 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.75 \end{pmatrix}.$$

The transition diagram is shown below. (Recall that we don't draw arrows with probability 0.)
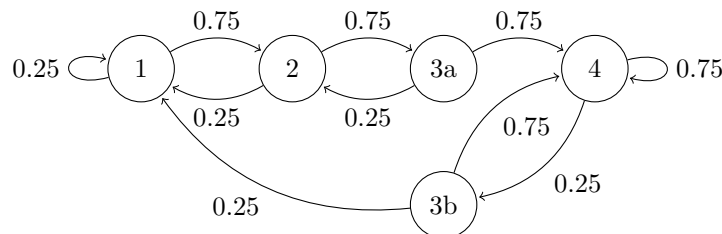
In the next section, we look at how to study large Markov chains by splitting them into smaller pieces called "classes".

Problem sheet 3

Problem Sheet 3

7 Class structure

- Communicating classes and irreducibility
- Period of a state (and class)

7.1 Communicating classes

If we have a large complicated Markov chain, it can be useful to split the state space up into smaller pieces that can be studied separately. The idea is to split to states i and j into different pieces if we can't move from i to j and then back again after some number of steps.

Definition 7.1. Consider a Markov chain on a state space $\mathcal{S}$ with transition matrix P. We say that state $j \in \mathcal{S}$ is **accessible** from state $i \in \mathcal{S}$ and write $i \to j$ if, for some n, $p_{ij}(n) > 0$.

If $i \to j$ and $j \to i$, we say that i **communicates with** j and write $i \leftrightarrow j$.

Here, the condition $p_{ij}(n) > 0$ means that, starting from i, there's a positive chance that we'll get to j at some point in the future – hence the term "accessible".

Theorem 7.1. *Consider a Markov chain on a state space $\mathcal{S}$ with transition matrix P. Then the "communicates with" relation $\leftrightarrow$ is an **equivalence relation**; that is, it has the following properties:*

- ***reflexive:** $i \leftrightarrow i$ for all i;*
- ***symmetric:** if $i \leftrightarrow j$ then $j \leftrightarrow i$;*
- ***transitive:** if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.*

Proof. Reflexivity: Clearly $p_{ii}(0) = 1 > 0$, because in "zero steps" we stay where we are. So $i \leftrightarrow i$ for all i.

Symmetry: The definition of $i \leftrightarrow j$ is symmetric under swapping i and j.

Transitivity. If we can get from i to j and we can get from j to k, then we can get from i to k by going via j. We just need to write that out formally.

Since $i \to j$, we have $p_{ij}(n) > 0$ for some n, and since $j \to k$, we also have $p_{jk}(m) > 0$ for some m. Then, by the Chapman–Kolmogorov equations, we have

$$p_{ik}(n + m) = \sum_{l \in \mathcal{S}} p_{il}(n) p_{lk}(m) \geq p_{ij}(n) p_{jk}(m) > 0,$$

from just picking out the $l = j$ term in the sum. So $i \to k$ too.

The same argument with k and i swapped gives $k \to i$ also, so $i \leftrightarrow k$.

$\square$

A fact you may remember about equivalence relations is that an equivalence relation, like $\leftrightarrow$, partitions the space $\mathcal{S}$ into **equivalence classes**. This means that each state i is in exactly one equivalence class, and that class is the set of states j such that $i \leftrightarrow j$. In this context, we call these **communicating classes**.

Example 7.1. In the simple random walk, provided p is not 0 or 1, every state communicates with every other state, because from i when can get to $j > i$ by going up $j - i$ times, and we can get to $j < i$ by going down $i - j$ times.. Therefore the whole state space $\mathcal{S} = \mathbb{Z}$ is one communicating class.

Example 7.2. Consider the gambler's ruin Markov chain on $\{0, 1, \ldots, m\}$. There are three communicating classes: $\{0\}$ and $\{m\}$, which don't communicate with any other states, and $\{1, 2, \ldots, m-1\}$, which is like the simple random walk.

Example 7.3. Consider the following simple model for an epidemic. We have three states: healthy (H), sick (S), and dead (D). This transition matrix is

$$\mathsf{P} = \begin{pmatrix} p_{\text{HH}} & p_{\text{HS}} & 0 \\ p_{\text{SH}} & p_{\text{SS}} & p_{\text{SD}} \\ 0 & 0 & 1 \end{pmatrix},$$

and the transition diagram is:
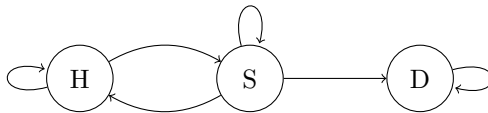
Clearly H and S communicate with each other (you can become infected or recover), while D only communicates with itself (the dead do not recover). Hence, the state space $\mathcal{S} = \{\text{H}, \text{S}, \text{D}\}$ partitions into two communicating classes: $\{\text{H}, \text{S}\}$ and $\{\text{D}\}$.

A few more definitions that will be important later.

Definition 7.2. If the entire state space $\mathcal{S}$ is one communicating class, we say that the Markov chain is **irreducible**.

We say that a communicating class is **closed** if no state outside the class is accessible from any state within the class. That is, class $C \subset \mathcal{S}$ is closed if whenever there exist $i \in C$ and $j \in \mathcal{S}$ with $i \rightarrow j$, then $j \in C$ also. If a class is not closed, we say it is **open**.

If a state i is in a communicating class $\{i\}$ by itself and that class is closed, then we say state i is **absorbing**.

Example 7.4. Going back to the previous examples:

- In the simple random walk, the whole state space is one communicating class which must therefore be closed. The Markov chain has only one class, so is irreducible.
- In the gambler's ruin, classes $\{0\}$ and $\{m\}$ are closed, because the Markov chain stays there forever, and because these closed classes consist of only one state each, 0 and m are absorbing states. The class $\{1, 2, \dots, m-1\}$ is open, as we can escape the class by going to 0 or m. The gambler's ruin chain has multiple classes, so is not irreducible.
- In the "healthy–sick–dead" chain, the class $\{D\}$ is closed, so D is an absorbing state, while the class $\{H, S\}$ is open, as one can leave it by dying. The Markov chain is not irreducible.

7.2 Periodicity

When we discussed the simple random walk, we noted that it alternates between even-numbered and odd-numbered states. This "periodic" behaviour is important to understand if we want to know what will happen to the Markov chain in the future.

The idea is this: List the number of steps for all possible paths starting and ending in the state. Then the period is the greatest common divisor (or "highest common factor") of the integers in this list.

Definition 7.3. Consider a Markov chain with transition matrix P. We say that a state $i \in \mathcal{S}$ has **period** d_i, where

$$d_i = \gcd\{n \in \{1, 2, \dots, \} : p_{ii}(n) > 0\},$$

where gcd denotes the greatest common divisor.

If $d_i > 1$, then the state i is called **periodic**; if $d_i = 1$, then i is called **aperiodic**.

Example 7.5. Consider the simple random walk with $p \neq 0, 1$. We have $p_{ii}(n) = 0$ for odd n, since we swap from odd to even each step. But $p_{ii}(2) = 2pq > 0$. Therefore, all states are periodic with period $\gcd\{2, 4, 6, \dots\} = 2$.

Example 7.6. For the gambler's ruin, states 0 and m are aperiodic (have period 1), since they are absorbing states. The remaining states states $1, 2, \ldots, m-1$ are periodic with period 2, because we swap between odd and even states, as in the simple random walk.

Example 7.7. Consider the Markov chain with transition diagram as shown:
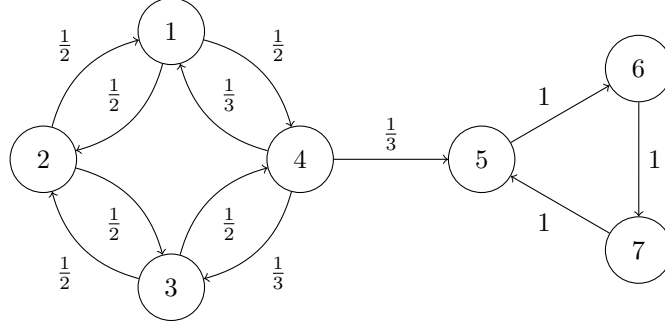
Importantly, we can't return from the triangle side back to the circle side. We thus see there are two communicating classes: $\{1, 2, 3, 4\}$, which is open, and $\{5, 6, 7\}$, which is closed. The Markov chain is not irreducible, and there are no absorbing states.

The circle side swaps between odd and even states (until exiting from 4 to 5), so states 1,2, 3 and 4 all have period 2. The triangle side cycles around with certainty, meaning that states 5, 6, and 7 all have period 3.

You may have noticed in these examples that, within a communicating class, every state has the same period. In fact, it's always the case that states in the same class have the same period.

Theorem 7.2. *All states in a communicating class have the same period.*

Formally: Consider a Markov chain on a state space $\mathcal{S}$ with transition matrix P. If $i, j \in \mathcal{S}$ are such that $i \leftrightarrow j$, then $d_i = d_j$.

In particular, in an irreducible Markov chain, all states have the same period d. We say that an irreducible Markov chain is **periodic** if $d > 1$ and **aperiodic** if $d = 1$.

Proof. Let i, j be such that $i \leftrightarrow j$. We want to show that $d_i = d_j$. First we'll show that $d_i \leq d_j$, and then we'll show that $d_j \leq d_i$, and thus conclude that they're equal.

Since $i \leftrightarrow j$, there exist n, m such that $p_{ij}(n) > 0$ and $p_{ji}(m) > 0$. Then, by the Chapman–Kolmogorov equations,

$$p_{ii}(n + m) = \sum_{k \in \mathcal{S}} p_{ik}(n) p_{ki}(m) \geq p_{ij}(n) p_{ji}(m) > 0.$$

So d_i divides $n + m$.

Let r be such that $p_{jj}(r) > 0$. Then, by the Chapman-Kolmogorov equations again,

$$p_{ii}(n + m + r) \geq p_{ij}(n) p_{jj}(r) p_{ji}(m) > 0.$$

Hence d_i divides $n + m + r$. But since d_i divides $n + m$, it must be that d_i divides r also. So whenever $p_{jj}(r) > 0$, we have that d_i divides r. Since d_i is a common divisor of all the rs with $p_{jj}(r) > 0$, it can't be any bigger that the *greatest* common divisor of all those rs. But that greatest common divisor is by definition d_j, the period of j. So $d_i \leq d_j$.

Repeating the same argument but with i and j swapped over, we get $d_j \leq d_i$ too, and we're done.

$\square$

In the next section, we look at two problems to do with "hitting times": What is the probability we reach a certain state, and how long on average does it take us to get there?

8 Hitting times

- Definitions: Hitting probability, expected hitting time, return probability, expected return time
- Finding these by conditioning on the first step
- Return probability and expected return time for the simple random walk

8.1 Hitting probability and expected hitting time

In Section 3 and Section 4, we used conditioning on the first step to find the ruin probability and expected duration for the gambler's ruin problem. Here, we develop those ideas for general Markov chains.

Definition 8.1. Let (X_n) be a Markov chain on state space $\mathcal{S}$. Let H_A be a random variable representing the hitting time to hit a state in $A \subset \mathcal{S}$, given by

$$H_A = \min\{n \in \mathbb{Z}_+ : X_n \in A\}.$$

The most common case we'll be interested in will be when $A = \{j\}$ is just a single state, so

$$H_j = \min\{n \in \mathbb{Z}_+ : X_n = j\}.$$

(We use the convention that $H_A = \infty$ if $X_n \notin A$ for all n.)

The **hitting probability** h_{iA} of states $A \subset \mathcal{S}$ starting from state $i \in \mathcal{S}$ is

$$h_{iA} = \mathbb{P}(X_n \in A \text{ for some } n \geq 0 \mid X_0 = i) = \mathbb{P}(H_A < \infty \mid X_0 = i).$$

Again, the most common case of interest is h_{ij} the hitting probability of state j starting from state i.

The **expected hitting time** η_{iA} of states $A \subset \mathcal{S}$ starting from state $i \in \mathcal{S}$ is

$$\eta_{iA} = \mathbb{E}(H_A \mid X_0 = i).$$

Clearly η_{iA} can only be finite if $h_{iA} = 1$.

For the gambler's ruin problem, we found equations for hitting probabilities and expected hitting times by conditioning on the first step and solving the resulting equations. We do the same here for other Markov chains.

Example 8.1. *Consider a Markov chain with transition matrix*

$$\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate the probability that the chain is absorbed at state 2 when started from state 1.

This is asking for the hitting probability h_{12}. The key is to find simultaneous equations for all hitting probabilities h_{i2} to state 2. Clearly we have $h_{22} = 1$, as we are already there. Also, $h_{42} = 0$, since 4\$is an absorbing state.

For the other states, we find equations by conditioning on the first step, as we did in the gambler's ruin problem. We have

$$h_{12} = \tfrac{1}{5}h_{12} + \tfrac{1}{5}h_{22} + \tfrac{1}{5}h_{32} + \tfrac{2}{5}h_{42} \qquad \Rightarrow \qquad .$$

This is because, starting from 1, there's a $\frac{1}{5}$ probability of staying at 1; then by the Markov property it's like we're starting again from 1, so the hitting probability is still h_{12}. Similarly, there's a $\frac{1}{5}$ probability of moving 2; then by the Markov property it's like we're starting again from 2, so the hitting probability is still h_{22}. And so on.

We can make a similar equation for h_{32}:

$$h_{32} = \tfrac{1}{2}h_{22} + \tfrac{1}{2}h_{42}$$

We can simplify these two equations, using $h_{22} = 1$ and $h_{42} = 0$, and rearrange them, to give

$$\tfrac{4}{5}\tfrac{4}{5}h_{12} = \tfrac{1}{5} + \tfrac{1}{5}h_{32}$$
$$h_{32} = \tfrac{1}{2}$$

Substituting the second equation into the first, we get $\tfrac{4}{5}h_{12} = \tfrac{1}{5} + \tfrac{1}{5} \cdot \tfrac{1}{2} = \tfrac{3}{10}$, so $h_{12} = \tfrac{5}{4}\tfrac{3}{10} = \tfrac{3}{8}$.

So the technique to find hitting probability h_{ij} from i to j is:

1. Set up equations for all the hitting probabilities h_{kj} to j by conditioning on the first step.
2. Solve the resulting simultaneous equations.

It is recommended to derive equations for hitting probabilities from first principles by conditioning on the first step. However, we can state what the general formula is: by the same conditioning method, we get

$$h_{iA} = \begin{cases} \displaystyle\sum_{j \in \mathcal{S}} p_{ij} h_{jA} & \text{if } i \notin A \\ 1 & \text{if } i \in A. \end{cases}$$

It can be shown that, if these equations have multiple solutions, that the hitting probabilities are in fact the smallest non-negative solutions.

Example 8.2. *Consider the simple no-claims discount chain from Lecture 6, which had transition matrix*

$$\mathsf{P} = \begin{pmatrix} \tfrac{1}{4} & \tfrac{3}{4} & 0 \\ \tfrac{1}{4} & 0 & \tfrac{3}{4} \\ 0 & \tfrac{1}{4} & \tfrac{3}{4} \end{pmatrix}.$$

Given we start in state 1 (no discount), find the expected amount of time until we reach state 3 (50% discount).

This question asks us to find η_{13}. Again, we start by writing down equations for all the η_{i3}s. Clearly we have $\eta_{33} = 0$.

By conditioning on the first step, we have

$$\eta_{13} = 1 + \tfrac{1}{4}\eta_{13} + \tfrac{3}{4}\eta_{23} \qquad \Rightarrow \qquad \tfrac{3}{4}\eta_{13} - \tfrac{3}{4}\eta_{23} = 1,$$
$$\eta_{23} = 1 + \tfrac{1}{4}\eta_{13} + \tfrac{3}{4}\eta_{33} \qquad \Rightarrow \qquad -\tfrac{1}{4}\eta_{13} + \eta_{23} = 1.$$

This is because the first step takes time 1, then we condition on what happens next.

The first equation plus three-quarters times the second gives

$$(\tfrac{3}{4} - \tfrac{3}{4}\tfrac{1}{4})\eta_{13} = \tfrac{9}{16}\eta_{13} = 1 + \tfrac{3}{4} = \tfrac{7}{4} = \tfrac{28}{16},$$

which has solution $\eta_{13} = \tfrac{28}{9} = 3.11$.

Similarly, if we need to, we can give a general formula

$$\eta_{iA} = \begin{cases} 1 + \displaystyle\sum_{j \in \mathcal{S}} p_{ij} \eta_{jA} & \text{if } i \notin A \\ 0 & \text{if } i \in A. \end{cases}$$

Again, if we have multiple solutions, the expected hitting times are the smallest non-negative solutions.

8.2 Return times

Under the definitions above, the hitting probability and expected hitting time to a state from itself are always $h_{ii} = 1$ and $\eta_{ii} = 0$, as we're "already there".

In this case, it can be interesting to look instead at the random variable representing the **return time**,

$$M_i = \min\{n \in \{1, 2, \dots\} : X_n = i\}.$$

Note that this only considers times $n = 1, 2, \dots$ not including $n = 0$, so is the *next* time we come back, after $n = 0$.

We then have the **return probability** and **expected return time**

$$m_i = \mathbb{P}(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = \mathbb{P}(M_i < \infty \mid X_0 = i),$$
$$\mu_i = \mathbb{E}(M_i \mid X_0 = i).$$

Just as before, we can find these by conditioning on the first step. The general equations are

$$m_i = \sum_{j \in \mathcal{S}} p_{ij} h_{ji}, \qquad \mu_i = 1 + \sum_{j \in \mathcal{S}} p_{ij} \eta_{ji},$$

where, if necessary, we take the minimal non-negative solution again.

8.3 Hitting and return times for the simple random walk

We now turn to hitting probabilities for the simple random walk, which goes up with probability p and down with probability $q = 1 - p$. This material is mandatory and is examinable, but is a bit technical; students who are struggling or have fallen behind might make a tactical decision to skip to the summary theorem at the end of the subsection and come back to the details at a later date.

Without loss of generality, we look at h_{i0}, the probability the random walk hits 0. We will assume that $i \geq 0$. For $i < 0$, we can get the desired result by looking at the random walk in the mirror – that is, by swapping the role of p and q, and treating the positive value $-i$.

For an initial condition, it's clear we have $h_{00} = 1$.

For general $i > 0$, we condition on the first step, to get

$$h_{i0} = p h_{i+1\,0} + q h_{i-1\,0}.$$

We recognise this equation from the gambler's ruin problem, and recall that we have to treat the cases $p \neq \frac{1}{2}$ and $p = \frac{1}{2}$ separately.

When $p \neq \frac{1}{2}$, the general solution is $h_{i0} = A + B\rho^i$, where $\rho = q/p \neq 0$, as before. The initial condition $h_{00} = 1$ gives $A = 1 - B$, so we have a family of solutions

$$h_{i0} = 1 + B(\rho^i - 1).$$

In the gambler's ruin problem we had another boundary condition to find B. Here we have no other conditions, but we can use the minimality condition that the hitting probabilities are the smallest non-negative solution to the equation.

When $\rho > 1$, so $p < \frac{1}{2}$, the term ρ^i tends to infinity. Thus the minimal non-negative solution will have to take $B = 0$, since taking $B < 0$ would eventually give a negative solution, so $B = 0$ gives the smallest non-negative solution. Hence the solution is $h_{i0} = 1 + 0\rho^i = 0$, meaning we hit 0 with certainty. This makes sense, because for $p < 1/2$ the random walk "drifts" to the left, and eventually gets back to 0.

When $\rho < 1$, so $p > \frac{1}{2}$, we have that $\rho^i - 1$ is negative and tends to -1. So to get a small solution we want B as large as possible, but keeping the solution non-negative limits us to $B \leq 1$; thus the minimality condition is achieved at $B = 1$. The solution is $h_{i0} = 1 + (\rho^i - 1) = \rho^i$. This is strictly less than 1, because for $p > 1/2$, the random walk drifts to the right, so might drift further and further away from 0 and not come back.

For $p = \frac{1}{2}$, so $\rho = 1$, we recall the general solution $h_{i0} = A + Bi$, and the condition $h_{00} = 1$ gives $A = 1$. Because i is negative, to get the answer to be small we want B to be big, but non-negativity limits us to $B \leq 0$. So the minimal non-negative solution takes $B = 0$, so we have $h_{i0} = 1$.

In conclusion, for $i > 0$, the hitting probabilities are given by

$$h_{i0} = \begin{cases} \left(\dfrac{q}{p}\right)^i & \text{if } p > \frac{1}{2} \\ 1 & \text{if } p \le \frac{1}{2}. \end{cases}$$

(Remember that for $i < 0$, we get the desired result by swapping the role of p and q, and treating the positive value $-i$.)

What about the return time to 0 (or, by symmetry, to any state $i \in \mathbb{Z}$)? By conditioning on the first step, $m_0 = p h_{10} + q h_{-10}$. We then have

$$m_0 = \begin{cases} p \times 1 \ + q \times 1 \ = 1 & \text{if } p = \frac{1}{2} \\ p \times \dfrac{q}{p} + q \times 1 \ = 2q < 1 & \text{if } p > \frac{1}{2} \\ p \times 1 \ + q \times \dfrac{p}{q} = 2p < 1 & \text{if } p < \frac{1}{2}. \end{cases}$$

So for the simple symmetric random walk ($p = \frac{1}{2}$) we have $m_i = 1$ and are certain to return to 0 again and again, while for $p \ne \frac{1}{2}$, we have $m_i < 1$ and we might never return.

One could also calculate expected hitting times η_{i0}. For $p \ne \frac{1}{2}$, we have $\eta_{i0} = \infty$, since $m_0 < 1$ and we might never come back.

For $p = \frac{1}{2}$, conditioning on the first step gives the equation

$$\eta_{i0} = 1 + \tfrac{1}{2}\eta_{i+10} + \tfrac{1}{2}\eta_{i-10},$$

with initial condition $\eta_{00} = 0$. This has the solution $\eta_{i0} = -i^2 + Bi = i(B - i)$. Then non-negativity demands $B = \infty$, as any other value would make the $-i^2$ negative for big enough i. This gives $\eta_{i0} = \infty$, and $\mu_0 = \infty$ also. So for $p = \frac{1}{2}$, while the random walk always return to 0, it may take a very long time to do so.

In conclusion, this is what we found out:

Theorem 8.1. *Consider the simple random walk with $p \ne 0, 1$. Then for all states i we have the following:*

- *For $p \ne \frac{1}{2}$, the return probability m_i is strictly less than 1.*
- *For the simple symmetric random walk with $p = \frac{1}{2}$, the return probability m_i is equal to 1.*
- *For all p, the expected return time μ_i is infinite.*

In the next section, we look at how the return probabilities and expected return times characterise which states continue to be visited by a Markov chain in the long run.

Problem sheet 4

Problem Sheet 4

9 Recurrence and transience

- Definitions of recurrence and transience
- Recurrence and transience as class properties
- Positive and null recurrence

9.1 Recurrence and transience of states

When thinking about the long-run behaviour of Markov chains, we are interested two different types of states:

- **Recurrent states**, which we always keep returning to again and again;
- **Transient states**, which we may return to a few times, but eventually we will leave and never come back.

The official definition is as follows; we will show that other properties follow from this.

Definition 9.1. Let (X_n) be a Markov chain on a state space $\mathcal{S}$. For $i \in \mathcal{S}$, let m_i be the return probability

$$m_i = \mathbb{P}(X_n = i \text{ for some } n \geq 1 \mid X_0 = i).$$

If $m_i = 1$, we say that state i is **recurrent**; if $m_i < 1$, we say that state i is **transient**.

**Theorem 9.1.** *Consider a Markov chain with transition matrix* P *and state space* $\mathcal{S}$.

- *If the state i is recurrent, then $\sum_{n=1}^{\infty} p_{ii}(n) = \infty$, and we return to state i infinitely many times with probability* 1.
- *If the state i is transient, then $\sum_{n=1}^{\infty} p_{ii}(n) < \infty$, and we return to state i infinitely many times with probability* 0.

Here $\sum_{n=1}^{\infty} p_{ii}^{(n)}$ is the expected number of returns to i starting from i.

We'll come to the proof in a moment, but first some examples.

Example 9.1. Consider the simple random walk. In the last section we saw that $m_i = 1$ for the simple symmetric random walk with $p = \frac{1}{2}$, so the simple symmetric random walk is recurrent. But for $p \neq \frac{1}{2}$, we have $m_i < 1$, so all the other simple random walks are transient.

Example 9.2. We saw this example in Lecture 7:
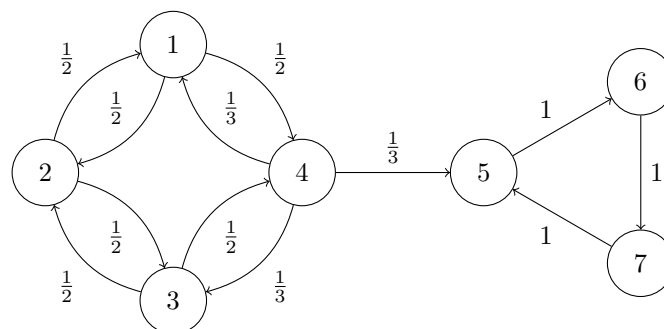
For states 5, 6 and 7, it's clear that the return probability is 1, since the Markov chain cycles around the triangle, so these states are recurrent.

States 1, 2, 3 and 4 are recurrent. In a moment we'll see a very quick way to show this, but in the meantime we can prove it directly by getting our hands dirty.

From state 4, we might go straight to state 5, in which case we can't come back, so $m_4 \leq 1 - p_{45} = \frac{2}{3}$, and state 4 is transient. Similarly, $m_1 \leq 1 - p_{14}p_{45} = \frac{5}{6}$, because if we move from 1 to 4 to 5, we definitely won't come back to 1, so state 1 is transient. By the similar arguments, $m_3 \leq 1 - p_{34}p_{45} = \frac{5}{6}$, and $m_2 \leq 1 - p_{21}p_{14}p_{45} = \frac{11}{12}$, so these states are both transient too.

Notice that the communicating class $\{1, 2, 3, 4\}$ is all transient, while the communicating class $\{5, 6, 7\}$ is all recurrent. We shall return to this point shortly. But first, we've put off the proof for too long.

Proof of Theorem 9.1. Suppose state i is recurrent, so the probability we return to i is $m_i = 1$. By the Markov property, we it's as if we restart the chain from i, so the probability we return to i is again still $m_i = 1$. Repeating this, we visit infinitely often with probability 1. In particular, the if number of visits to i starting from i is infinite, then its expectation is infinite too, and this expectation is $\sum_{n=1}^{\infty} p_{ii}(n) = \infty$.

Suppose state i is transient, so the probability we return to i is $m_i < 1$. Then the probability we return to i exactly r times before never coming back is

$$\mathbb{P}((\text{number of returns to } i) = r) = m_i^r (1 - m_i),$$

since we must return on the first r occasions, but then fail to return any more. This is a geometric distribution $\text{Geom}(1 - m_i)$ (the version with support $\{0, 1, 2, \dots\}$). Since the expectation of a $\text{Geom}(p)$ random variable is $(1 - p)/p$, the expected number of returns is

$$\mathbb{E}(\text{number of returns to } i) = \sum_{n=1}^{\infty} p_{ii}(n) = \frac{1 - (1 - m_i)}{1 - m_i} = \frac{m_i}{1 - m_i}.$$

This is finite, since $m_i < 1$. Since the expected number of returns is finite, the probability we return infinitely many times must be 0.

$\square$

9.2 Recurrence and transience of classes

We could find whether each state is transient or recurrent by calculating (or bounding) all the return probabilities m_i, using the methods in the previous section. But the following two theorems will give some highly convenient short-cuts.

Theorem 9.2. *Within a communicating class, either every state is transient or every state is recurrent.*

Formally: Let $i, j \in \mathcal{S}$ be such that $i \leftrightarrow j$. If i is recurrent, then j is recurrent also; while if i is transient, then j transient also.

For this reason, we can refer to a communicating class as a "recurrent class" or a "transient class". If a Markov chain is irreducible, we can refer to it as a "recurrent Markov chain" or a "transient Markov chain".

Proof. First part. Suppose $i \leftrightarrow j$ and i is recurrent. Then, for some m, n we have $p_{ij}^{(n)}, p_{ji}^{(m)} > 0$. Then, by the Chapman–Kolmogorov equations,

$$\sum_{r=1}^{\infty} p_{jj}^{(n+m+r)} \geq \sum_{r=1}^{\infty} p_{ji}^{(m)} p_{ii}^{(r)} p_{ij}^{(n)} = p_{ji}^{(m)} \left(\sum_{r=1}^{\infty} p_{ii}^{(r)} \right) p_{ij}^{(n)}.$$

If i is recurrent, then $\sum_r p_{ii}^{(r)} = \infty$. Then from the above equation, we also have $\sum_r p_{jj}^{(n+m+r)} = \infty$, meaning $\sum_s p_{jj}^{(s)} = \infty$, and j is recurrent.

Second part. Suppose i is transient. Then j cannot be recurrent, because the previous argument with i and j swapped over would force i to in fact be recurrent also. So j must be transient.

$\square$

Theorem 9.3.

- *Every non-closed communicating class is transient.*
- *Every finite closed communicating class is recurrent.*

This theorem completely classifies the transience and recurrence of classes, with rare exception of infinite closed classes, which can require further examination.

Proof. First part. Suppose i is in a non-closed communicating class, so for some j we have $i \to j$, meaning $p_{ij}^{(n)} > 0$ for some n, but $j \not\to i$, meaning that once we reach j we cannot return to i. We need to show that i is transient.

The probability we return to i after time n is

$$\mathbb{P}(\text{return to } i \text{ after time } n \mid X_0 = i)$$
$$= p_{ij}^{(n)} \, \mathbb{P}(\text{return to } i \text{ after time } n \mid X_n = j, X_0 = i)$$
$$+ \left(1 - p_{ij}^{(n)}\right) \mathbb{P}(\text{return to } i \text{ after time } n \mid X_n \neq j, X_0 = i)$$
$$\leq \mathbb{P}(\text{return to } i \text{ after time } n \mid X_n = j, X_0 = i) + \left(1 - p_{ij}^{(n)}\right)$$
$$\leq 0 + \left(1 - p_{ij}^{(n)}\right)$$
$$< 1,$$

since we can't get from j to i, and since $p_{ij}^{(n)} > 0$. If i were recurrent we would certainly return infinitely often, and in particular certainly return after time n. So i must be transient instead.

Second part. Suppose the class C is finite and closed. Then for some $i \in C$, the probability that we return to i infinitely many times must be strictly positive, as we are going to stay in finitely many states of C for infinitely many time steps. Then that state i is not transient, so it must be recurrent, meaning the whole class is recurrent.

$\square$

Going back to the earlier example, we see that the class $\{5, 6, 7\}$ is closed and finite, and therefore recurrent, while class $\{1, 2, 3, 4\}$ is not closed and therefore transient. This is much less effort than the previous method!

9.3 Positive and null recurrence

It can be useful to further divide recurrent classes, where the return probability $m_i = 1$, by whether the expected return time μ_i is finite or not.

Definition 9.2. Let (X_n) be a Markov chain on a state space $\mathcal{S}$. Let $i \in \mathcal{S}$ be a recurrent state, and let μ_i be the expected return time. If $\mu_i < \infty$, we say that state i is **positive recurrent**; if $\mu_i = \infty$, we say that state i is **null recurrent**.

The following facts are not difficult to prove, in a similar way to the previous results:

1. In a recurrent class, either all states are positive recurrent or all states are null recurrent.
2. All finite closed classes are positive recurrent.

The first result means we can refer to a "positive recurrent class" or a "null recurrent class", and an irreducible Markov chain can be a "positive recurrent Markov chain" or a "null recurrent Markov chain".

Putting everything so far together:

- non-closed classes are transient;
- finite closed classes are positive recurrent;
- infinite closed classes can be positive recurrent, null recurrent, or transient.

We know that the simple symmetric random walk is recurrent. We also saw in [the last section]((#S08-return-rw) that $\mu_i = \infty$, so it is null recurrent.

We can also consider the simple symmetric random walk in d-dimensions, on $\mathbb{Z}^d$. At each step we pick one of the coordinates and increase or decrease it by one; each of the $2d$ possibilities having probability $1/(2d)$. We have seen that for $d = 1$ this is null recurrent. A famous result by the Hungarian mathematician George Pólya from 1921 states the simple symmetric random walk is null recurrent for $d = 1$ and $d = 2$, but is transient for $d \geq 3$. (Perhaps this is why cars often crash into each other, but aeroplanes very rarely do?)

9.4 Strong Markov property

This subsection is optional and nonexaminable.

There was a cheat somewhere in this section – did you notice it? The last two times I lectured this course, I just hoped no one would notice – and no one did. Still, it's a bit naughty, so in this optional and nonexaminable section, I'll come clean.

The Markov property says that, if at some fixed time n we have $X_n = i$, then the Markov chain from that point on is just like starting all over again from the state i. When we applied this in the proof of Theorem 9.1, we were using as n the first return to state i. But that's not a fixed time – it's a random time. Have we messed up?

Actually we're fine. The reason is that the first return to i isn't just any old random time, it's a "stopping time", and the Markov property applies to stopping times too.

A stopping time is a random time where "you know when you get there".

Definition 9.3. Let (X_n) be a stochastic process in discrete time, and let T be a random time. Then T is a **stopping time** if for all n the event $\{T = n\}$ is determined by the random variables $X_0, X_1, \dots, X_n$.

So, for example:

- "The first visit to state i" is stopping time, because as soon as we return, we know the value of T.
- "Three time-steps after the second visit to j" is a stopping time, because after our second visit we count on three more steps and have T.
- "The time-step before the first visit to i" is not a stopping time, because we need to go one step further on to know whether or not we had just been at time T.
- "The final visit to j" is not a stopping time, because at the time of the visit we don't yet know whether we'll come back again or not.

There are lots of places in probability theory and finance when something that is true about a fixed time is also true about a random stopping time. When we use the Markov property with a stopping time, we call it the "strong Markov property".

Theorem 9.4 (Strong Markov property)**.** *Let (X_n) be a Markov chain on a state space $\mathcal{S}$, and let T be a stopping time that is finite with probability 1. Then all states $x_0, \dots, x_{T-1}, i, j \in \mathcal{S}$ we have*

$$\mathbb{P}(X_{T+1} = j \mid X_T = i, X_{T-1} = x_{T-1} \dots, X_0 = x_0) = p_{ij}.$$

Proof. We have

$$\mathbb{P}(X_{T+1} = x_j \mid X_T = i, X_{T-1} = x_{T-1} \dots, X_0 = x_0)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(T = n)\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = x_{n-1} \dots, X_0 = x_0, T = n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(T = n)\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = x_{n-1} \dots, X_0 = x_0)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(T = n)\mathbb{P}(X_{n+1} = j \mid X_n = i)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(T = n)p_{ij}$$

$$= p_{ij} \sum_{n=0}^{\infty} \mathbb{P}(T = n)$$

$$= p_{ij},$$

as desired. The second line was by conditioning on the value of T; in the third line we deleted the superfluous conditioning $T = n$, because T is a stopping time, so the event $T = n$ is entirely decided by

$X_n, X_{n-1}, \ldots, X_0$; the fourth line used the (usual non-strong) Markov property; the fifth line is just the definition of p_{ij}; the sixth line took p_{ij} out of the sum; and the seventh line is because T is finite with probability 1, so $\mathbb{P}(T = n)$ sums to 1.

$\square$

9.5 A useful lemma

This subsection is optional and nonexaminable.

The following lemma will be used in some later optional and nonexaminable proofs.

Lemma 9.1. *Let (X_n) be an irreducible and recurrent Markov chain. Then for any initial distribution and any state j, we will certainly hit j, so the hitting time H_j is finite with probability 1.*

Proof. It suffices to prove the lemma when the initial distribution is "start at i". (We can repeat for all i, then build any initial distribution from a weighted sum of "start at i"s.)

Since the chain is irreducible, we know we can get from i to j, so pick m with $p_{ji}(m) > 0$. Since the chain is recurrent, we know the return probability from j to j is 1, and we return infinitely many times with probability 1. We just need to glue these two facts together.

We have

$$
\begin{aligned}
1 &= \mathbb{P}(X_n = j \text{ for infinitely many } n \mid X_0 = j) \\
&= \mathbb{P}(X_n = j \text{ for some } n > m \mid X_0 = j) \\
&= \sum_k \mathbb{P}(X_m = k \mid X_0 = j)\, \mathbb{P}(X_n = j \text{ for some } n > m \mid X_m = k, X_0 = j) \\
&= \sum_k p_{jk}(m)\, \mathbb{P}(H_j < \infty \mid X_0 = k),
\end{aligned}
$$

where the last line used the Markov property to treat the chain as starting over again when it reaches some state k at time m. Note that $\sum_k p_{jk}(m) = 1$, since that's the sum of the probabilities of going anywhere in m steps. This means we must have $\mathbb{P}(H_j < \infty \mid X_0 = k)$ whenever $p_{jk}(m) > 0$, to ensure the final line does indeed sum to 1. But we stated earlier that $p_{ji}(m) > 0$, so we indeed have $\mathbb{P}(H_j < \infty \mid X_0 = i)$, as required.

$\square$

In the next section, we look at how positive recurrent Markov chains can settle into a stationary distribution and experience long-term stability.

10 Stationary distributions

- Stationary distributions and how to find them
- Conditions for existence and uniqueness of the stationary distribution

10.1 Definition

Consider the two-state "broken printer" Markov chain from Lecture 5.
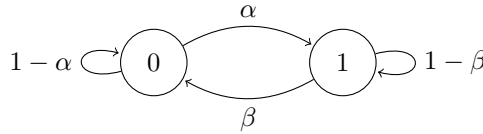
Suppose we start the chain from the initial distribution

$$\lambda_0 = \mathbb{P}(X_0 = 0) = \frac{\beta}{\alpha + \beta} \qquad \lambda_1 = \mathbb{P}(X_0 = 1) = \frac{\alpha}{\alpha + \beta}.$$

(You may recognise this from Question 1 on Problem Sheet 3.) What's the distribution after step 1? By conditioning on the initial state, we have

$$\mathbb{P}(X_1 = 0) = \lambda_0 p_{00} + \lambda_1 p_{10} = \frac{\beta}{\alpha + \beta}(1 - \alpha) + \frac{\alpha}{\alpha + \beta}\beta = \frac{\beta}{\alpha + \beta},$$

$$\mathbb{P}(X_1 = 1) = \lambda_0 p_{01} + \lambda_1 p_{11} = \frac{\beta}{\alpha + \beta}\alpha + \frac{\alpha}{\alpha + \beta}(1 - \beta) = \frac{\alpha}{\alpha + \beta}.$$

So we're still in the same distribution we started in. By repeating the same calculation, we're still going to be in this distribution after step 2, and step 3, and forever.

More generally, if we start from a state given by a distribution $\pi = (\pi_i)$, then after step 1 the probability we're in state j is $\sum_i \pi_i p_{ij}$. So if $\pi_j = \sum_i \pi_i p_{ij}$, we stay in this distribution forever. In matrix form, this is $\pi = \pi \mathsf{P}$. (Remember that π is a *row* vector.) We call such a distribution a stationary distribution.

Definition 10.1. Let (X_n) be a Markov chain on a state space $\mathcal{S}$ with transition matrix P. Let $\pi = (\pi_i)$ be a distribution on $\mathcal{S}$, in that $\pi_i \geq 0$ for all $i \in \mathcal{S}$ and $\sum_{i \in \mathcal{S}} \pi_i = 1$. We call π a **stationary distribution** if

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i p_{ij} \quad \text{for all } j \in \mathcal{S},$$

or, equivalently, if $\pi = \pi \mathsf{P}$.

Note that we're saying the *distribution* $\mathbb{P}(X_n = i)$ stays the same; the Markov chain (X_n) itself will keep moving. One way to think is that if we started off a thousand Markov chains, choosing each starting position to be i with probability π_i, then (roughly) $1000\pi_j$ of them would be in state j at any time in the future – but not necessarily the same ones each time.

10.2 Finding a stationary distribution

Let's try an example. Consider the no-claims discount Markov chain from Lecture 6 with state space $\mathcal{S} = \{1, 2, 3\}$ and transition matrix

$$\mathsf{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

We want to find a stationary distribution π, which must solve the equation $\pi = \pi \mathsf{P}$, which is

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

Writing out the equations coordinate at a time, we have

$$\pi_1 = \tfrac{1}{4}\pi_1 + \tfrac{1}{4}\pi_2,$$
$$\pi_2 = \tfrac{3}{4}\pi_1 + \tfrac{1}{4}\pi_3,$$
$$\pi_3 = \tfrac{3}{4}\pi_2 + \tfrac{3}{4}\pi_3.$$

Since π must be a distribution, we also have the "normalising condition"

$$\pi_1 + \pi_2 + \pi_3 = 1.$$

The way to solve these equations is first to solve for all the variables π_i in terms of a convenient π_j (called the "working variable") and then substitute all of these expressions into the normalising condition to find a value for π_j.

Let's choose π_2 as our working variable. It turns out that $\pi = \pi P$ always gives one more equation than we actually need, so we can discard one of them for free. Let's get rid of the second equation, and the solve the first and third equations in terms of our working variable π_2, to get

$$\pi_1 = \tfrac{1}{3}\pi_2 \qquad \pi_3 = 3\pi_2. \tag{5}$$

Now let's turn to the normalising condition. That gives

$$\pi_1 + \pi_2 + \pi_3 = \tfrac{1}{3}\pi_2 + \pi_2 + 3\pi_2 = \tfrac{13}{3}\pi_2 = 1.$$

So the working variable is solved to be $\pi_2 = \tfrac{3}{13}$. Substituting this back into (5), we have $\pi_1 = \tfrac{1}{3}\pi_2 = \tfrac{1}{13}$ and $\pi_3 = 3\pi_2 = \tfrac{9}{13}$. So the full solution is

$$\pi = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} \tfrac{1}{13} & \tfrac{3}{13} & \tfrac{9}{13} \end{pmatrix}.$$

The method we used here can be summarised as follows:

1. Write out $\pi = \pi P$ coordinate by coordinate. Discard one of the equations.
2. Select one of the π_i as a working variable and treat it as a parameter. Solve the equations in terms of the working variable.
3. Substitute the solution into the normalising condition to find the working variable, and hence the full solution.

It can be good practice to use the equation discarded earlier to check that the calculated solution is indeed correct.

One extra example for further practice and to show how you should present your solutions to such problems:

Example 10.1. *Consider a Markov chain on state space $\mathcal{S} = \{1, 2, 3\}$ with transition matrix*

$$P = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{4} \\ \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4} \\ 0 & \tfrac{1}{4} & \tfrac{3}{4} \end{pmatrix}.$$

Find a stationary distribution for this Markov chain.

Step 1. Writing out $\pi = \pi P$ coordinate-wise, we have

$$\pi_1 = \tfrac{1}{2}\pi_1 + \tfrac{1}{4}\pi_2$$
$$\pi_2 = \tfrac{1}{4}\pi_1 + \tfrac{1}{2}\pi_2 + \tfrac{1}{4}\pi_3$$
$$\pi_3 = \tfrac{1}{4}\pi_1 + \tfrac{1}{4}\pi_2 + \tfrac{3}{4}\pi_3.$$

We choose to discard the third equation.

Step 2. We choose π_1 as our working variable. From the first equation we get $\pi_2 = 2\pi_1$. From the second equation we get $\pi_3 = 2\pi_2 - \pi_1$, and substituting the previous $\pi_2 = 2\pi_1$ into this, we get $\pi_3 = 3\pi_1$.

Step 3. The normalising condition is

$$\pi_1 + \pi_2 + \pi_3 = \pi_1 + 2\pi_1 + 3\pi_1 = 6\pi_1 = 1.$$

Therefore $\pi_1 = \frac{1}{6}$. Substituting this into our previous expressions, we get $\pi_2 = 2\pi_1 = \frac{1}{3}$ and $\pi_3 = 3\pi_1 = \frac{1}{2}$. Thus the solution is

$$\pi = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

We can check our answer with the discarded third equation, just to make sure we didn't make any mistakes. We get

$$\pi_3 = \tfrac{1}{4}\pi_1 + \tfrac{1}{4}\pi_2 + \tfrac{3}{4}\pi_3 = \tfrac{1}{4}\tfrac{1}{6} + \tfrac{1}{4}\tfrac{1}{3} + \tfrac{3}{4}\tfrac{1}{2} = \tfrac{1}{24} + \tfrac{2}{24} + \tfrac{9}{24} = \tfrac{1}{2},$$

which is as it should be.

10.3 Existence and uniqueness

Given a Markov chain its natural to ask:

1. Does a stationary distribution exist?
2. If a stationary distribution does exists, is there only one, or are there be many stationary distributions?

The answer is given by the following very important theorem.

Theorem 10.1. *Consider an irreducible Markov chain.*

- *If the Markov chain is positive recurrent, then a stationary distribution π exists, is unique, and is given by $\pi_i = 1/\mu_i$, where μ_i is the expected return time to state i.*
- *If the Markov chain is null recurrent or transient, then no stationary distribution exists.*

We give an optional and nonexaminable proof to the first part below.

In our no-claims discount example, the chain is irreducible and, like all finite state irreducible chains, it is positive recurrent. Thus the stationary distribution $\pi = (\frac{1}{13}, \frac{3}{13}, \frac{9}{13})$ we found is the unique stationary distribution for that chain. Once we have the stationary distribution π, we get the expected return times $\mu_i = 1/\pi_i$ for free: the expected return times are $\mu_1 = 13$, $\mu_2 = \frac{13}{3} = 4.33$, and $\mu_3 = \frac{13}{9} = 1.44$.

Note the condition in Theorem 10.1 that the Markov chain is irreducible. What if the Markov chain is not irreducible, so has more than one communicating class? We can work out what must happen from the theorem:

- If none of the classes are positive recurrent, then no stationary distribution exists.
- If exactly one of the classes is positive recurrent (and therefore closed), then there exists a unique stationary distribution, supported only on that closed class.
- If more the one of the classes are positive recurrent, then many stationary distributions will exist.

Example 10.2. Consider the simple random walk with $p \neq 0, 1$. This Markov chain is irreducible, and is null recurrent for $p = \frac{1}{2}$ and transient for $p \neq \frac{1}{2}$. Either way, the theorem tells us that no stationary distribution exists.

Example 10.3. Consider the Markov chain with transition matrix

$$\mathsf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

This chain has two closed positive recurrent classes, $\{1, 2\}$ and $\{3, 4\}$.

Solving $\pi = \pi\mathsf{P}$ gives

$$
\begin{aligned}
\pi_1 &= \tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_2 &\Rightarrow&& \pi_1 &= \pi_2 \\
\pi_2 &= \tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_2 &\Rightarrow&& \pi_1 &= \pi_2 \\
\pi_3 &= \tfrac{1}{4}\pi_3 + \tfrac{1}{2}\pi_4 &\Rightarrow&& 3\pi_3 &= 2\pi_4 \\
\pi_4 &= \tfrac{3}{4}\pi_3 + \tfrac{1}{2}\pi_4 &\Rightarrow&& 3\pi_3 &= 2\pi_4,
\end{aligned}
$$

giving us the same two constraints twice each. We also have the normalising condition $\pi_1+\pi_2+\pi_3+\pi_4 = 1$. If we let $\pi_1 + \pi_2 = \alpha$ and $\pi_3 + \pi_4 = 1 - \alpha$, we see that

$$
\pi = \begin{pmatrix} \tfrac{1}{2}\alpha & \tfrac{1}{2}\alpha & \tfrac{2}{5}(1-\alpha) & \tfrac{3}{5}(1-\alpha) \end{pmatrix}
$$

is a stationary distribution for any $0 \le \alpha \le 1$, so we have infinitely many stationary distributions.

10.4 Proof of existence and uniqueness

This subsection is optional and nonexaminable.

It's very important to be able to find the stationary distribution(s) of a Markov chain – you can reasonably expect a question on this to turn up on the exam. You should also know the conditions for existence and uniqueness of the stationary distribution. Being able to *prove* existence and uniqueness is less important, although for completeness we will do so here.

Theorem 10.1 had two points. The more important point was that irreducible, positive recurrent Markov chains have a stationary distribution, that it is unique, and that it is given by $\pi_i = 1/\mu_i$. We give a proof of that below, doing the existence and uniqueness parts separately. The less important point was that null recurrent and transitive Markov chains do not have a stationary distribution, and this is more fiddly. You can find a proof (usually in multiple parts) in books such as Norris, *Markov Chains*, Section 1.7.

Existence: *Every positive recurrent Markov chain has a stationary distribution.*

Before we start, one last definition. Let us call a vector ν a **stationary vector** if $\nu\mathsf{P} = \nu$. This is exactly like a stationary distribution, except without the normalisation condition that it has to sum to 1.

Proof. Suppose that (X_n) is recurrent (either positive or null, for the moment).

Our first task will be to find a stationary vector. Fix an initial state k, and let ν_i be the expected number of visits to i before we return back to k. That is,

$$
\begin{aligned}
\nu_i &= \mathbb{E}\#\{\text{visits to } i \text{ before returning to } k\} \\
&= \mathbb{E}\sum_{n=1}^{M_k} \mathbb{P}(X_n = i \mid X_0 = k) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_n = i \text{ and } n \le M_k \mid X_0 = k),
\end{aligned}
$$

where M_k is the return time, as in Section 8. Let us note for later use that, under this definition, $\nu_k = 1$, because the only visit to k is the return to k itself.

Since ν is counting the number of visits to different states in a certain (random) time, it seems plausible that ν suitably normalised could be a stationary distribution, meaning that ν itself could be a stationary vector. Let's check.

We want to show that $\sum_i \nu_i p_{ij} = \nu_j$. Let's see what we have:

$$
\begin{aligned}
\sum_{i \in \mathcal{S}} \nu_i p_{ij} &= \sum_{i \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{P}(X_n = i \text{ and } n \le M_k \mid X_0 = k)\, p_{ij} \\
&= \sum_{n=1}^{\infty} \sum_{i \in \mathcal{S}} \mathbb{P}(X_n = i \text{ and } X_{n+1} = j \text{ and } n \le M_k \mid X_0 = k) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_{n+1} = j \text{ and } n \le M_k \mid X_0 = k).
\end{aligned}
$$

(Exchanging the order of the sums is legitimate, because recurrence of the chain means that M_k is finite with probability 1.) We can now do a cheeky bit of monkeying around with the index n, by swapping out the visit to k at time M_k with the visit to k at time 0. This means instead of counting the visits from 1 to M_k, we can count the visits from 0 to $M_k - 1$. Shuffling the index about, we get

$$
\begin{aligned}
\sum_{i \in \mathcal{S}} \nu_i p_{ij} &= \sum_{n=0}^{\infty} \mathbb{P}(X_{n+1} = j \text{ and } n \leq M_k - 1 \mid X_0 = k) \\
&= \sum_{n+1=1}^{\infty} \mathbb{P}(X_{n+1} = j \text{ and } n + 1 \leq M_k \mid X_0 = k) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_n = j \text{ and } n \leq M_k \mid X_0 = k) \\
&= \nu_j.
\end{aligned}
$$

So ν is indeed a stationary vector.

We now want normalise ν into a stationary distribution by dividing through by $\sum_i \nu_i$. We can do this if $\sum_i \nu_i$ is finite. But $\sum_i \nu_i$ is the expected total number of visits to all states before return to k, which is precisely the expected return time μ_k. Now we use assume that (X_n) is *positive* recurrent. This means that μ_k is finite, so $\pi = (1/\mu_k)\nu$ is a stationary distribution.

$\square$

Uniqueness: *For an irreducible, positive recurrent Markov chain, the stationary distribution is unique and is given by $\pi_i = 1/\mu_i$.*

I read the following proof in Stirzaker, *Elementary Probability*, Section 9.5.

Proof. Suppose the Markov chain is irreducible and positive recurrent, and suppose π is a stationary distribution. We want to show that $\pi_i = 1/\mu_i$ for all i.

The only equation we have for μ_k is this one from Section 8:

$$
\mu_k = 1 + \sum_j p_{kj} \eta_{jk}. \tag{6}
$$

Since that involves the expected hitting times η_{ik}, let's write down the equation for them too:

$$
\eta_{ik} = 1 + \sum_j p_{ij} \eta_{jk} \qquad \text{for all } i \neq k. \tag{7}
$$

In order to apply the fact that π is a stationary distribution, we'd like to get these into an equation with $\sum_i \pi_i p_{ij}$ in it. Here's a way we can do that: Take (7), multiply it by π_i and sum over all $i \neq k$, to get

$$
\sum_i \pi_i \eta_{ik} = \sum_{i \neq k} \pi_i + \sum_j \sum_{i \neq k} \pi_i p_{ij} \eta_{jk}. \tag{8}
$$

(The sum on the left can be over all i, since $\eta_{kk} = 0$.) Also, take (6) and multiply it by π_k to get

$$
\pi_k \mu_k = \pi_k + \sum_j \pi_k p_{kj} \eta_{jk} \tag{9}
$$

Now add (8) and (9) together to get

$$
\sum_i \pi_i \eta_{ik} + \pi_k \mu_k = \sum_i \pi_i + \sum_j \sum_i \pi_i p_{ij} \eta_{jk}.
$$

We can now use $\sum_i \pi_i p_{ij} = \pi_j$, along with $\sum_i \pi_i = 1$, to get

$$
\sum_i \pi_i \eta_{ik} + \pi_k \mu_k = 1 + \sum_j \pi_j \eta_{jk}.
$$

But the first term on the left and the last term on the right are equal, and because the Markov chain is irreducible and positive recurrent, they are finite. (That was our lemma in the previous section.) Thus we're allowed to subtract them, and we get $\pi_k \mu_k = 1$, which is indeed $\pi_k = 1/\mu_k$. We can repeat the argument for every choice of k.

$\square$

In the next section, we see how the stationary distribution tells us very important things about the long-term behaviour of a Markov chain.

Problem sheet 5

Problem Sheet 5

11 Long-term behaviour of Markov chains

- The limit theorem: convergence to the stationary distribution for irreducible, aperiodic, positive recurrent Markov chains
- The ergodic theorem for the long-run proportion of time spent in each state

11.1 Convergence to equilibrium

In this section we're interested in what happens to a Markov chain (X_n) in the long-run – that is, when n tends to infinity.

One thing that *could* happen over time is that the distribution $\lambda \mathsf{P}^n$ of the Markov chain could gradually settle down towards some "equilibrium distribution". Further, perhaps that long-term equilibrium might not depend on the initial distribution, but the initial behaviour might eventually almost disappear, exhibiting a "lack of memory" of the start of the process.

Just in case that does happen, let's give it a name.

Definition 11.1. Let (X_n) be a Markov chain on a state space $\mathcal{S}$ with transition matrix P. Suppose there exists a distribution $\mathbf{p}^* = (p_i^*)$ on $\mathcal{S}$ (so $p_i^* \geq 0$ and $\sum_i p_i^* = 1$) such that, whatever the initial distribution $\lambda = (\lambda_i)$, we have $\mathbb{P}(X_n = j) \to p_j^*$ as $n \to \infty$ for all $j \in \mathcal{S}$. Then we say that $\mathbf{p}^*$ is an **equilibrium distribution**.

It's clear there can only be at most one equilibrium distribution – but will there be one at all? The following is the most important result in this course. (Recall that an irreducible Markov chain is aperiodic if it has period 1.)

Theorem 11.1 (Limit theorem). *Let (X_n) be an irreducible and aperiodic Markov chain. Then for any initial distribution λ, we have that $\mathbb{P}(X_n = j) \to 1/\mu_j$ as $n \to \infty$, where μ_j is the expected return time to state j.*

In particular:

- *Suppose (X_n) is positive recurrent. Then the unique stationary distribution π given by $\pi_j = 1/\mu_j$ is the equilibrium distribution, so $\mathbb{P}(X_n = j) \to \pi_j$ for all j.*
- *Suppose (X_n) is null recurrent or transient. Then $\mathbb{P}(X_n = j) \to 0$ for all j, and there is no equilibrium distribution.*

I particularly like how this one theorem gathers together all the ideas from the course in one result (Markov chains, irreducibility, periodicity, recurrence/transience, positive/null recurrence, return times, stationary distributions...).

Note the three conditions for convergence to an equilibrium distribution: irreducibility, aperiodicity, and positive recurrence.

Consider a irreducible, aperiodic, positive recurrent Markov chain. Taking the initial distribution to be starting in state i with certainty, the limit theorem tells us that $p_{ij}(n) \to \pi_j$ for all i and j. This means that the n-step transition matrix will have the limiting value

$$\lim_{n \to \infty} \mathsf{P}(n) = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \\ \pi_1 & \pi_2 & \cdots & \pi_N \\ \vdots & \vdots & \vdots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_N \end{pmatrix},$$

where each row is identical.

We give an full proof of the limit theorem below (optional and nonexaminable). However, this result gets part way there and is straightforward.

Theorem 11.2. *If an equilibrium distribution $\mathbf{p}^*$ does exist, then $\mathbf{p}^*$ is a stationary distribution.*

Given this result it's clear that an irreducible Markov chain cannot have an equilibrium distribution if it is null recurrent or transient, as it doesn't even have a stationary distribution. So the positive recurrent case is the hard (nonexaminable) one.

Proof. We need to verify that $\mathbf{p}^*\mathsf{P} = \mathbf{p}^*$. We have

$$\sum_i p_i^* p_{ij} = \sum_i \left(\lim_{n \to \infty} p_{ki}(n) \right) p_{ij} = \lim_{n \to \infty} \sum_i p_{ki}(n) p_{ij} = \lim_{n \to \infty} p_{kj}(n+1) = p_j^*,$$

as desired.

$\square$

(Strictly speaking, swapping the sum and the limit is only formally justified when the state space is finite, although the theorem is true universally.)

11.2   Examples of convergence and non-convergence

Example 11.1. The two-state "broken printer" Markov chain is irreducible, aperiodic, and positive recurrent, so its stationary distribution is also the equilibrium distribution. We proved this from first principles in Question 1 on Problem Sheet 3.

Example 11.2. Recall the simple no-claims discount Markov chain from Lecture 6, which is irreducible, aperiodic, and positive recurrent. We saw last time that is has the unique stationary distribution

$$\pi = \begin{pmatrix} \frac{1}{13} & \frac{3}{13} & \frac{9}{13} \end{pmatrix} = (0.0769 \quad 0.2308 \quad 0.6923).$$

From the limit theorem, we see that the n-step transition probability tends to a limit where every row is equal to π. We can check using a computer: for $n = 12$, say,

$$\mathsf{P}(12) = \mathsf{P}^{12} = \begin{pmatrix} 0.0770 & 0.2308 & 0.6923 \\ 0.0769 & 0.2308 & 0.6923 \\ 0.0769 & 0.2308 & 0.6923 \end{pmatrix},$$

where $p_{ij}(12)$ is equal to π_j up to at least 3 decimal places for all i, j.

Example 11.3. The simple random walk is null recurrent for $p = \frac{1}{2}$ and transient otherwise. Either way, we have $\mathbb{P}(X_n = i) \to 0$ for all states i, and there is no equilibrium distribution.

Example 11.4. Consider a Markov chain (X_n) on state space $\mathcal{S} = \{0, 1\}$ and transition matrix

$$\mathsf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So at each stage we swap from state 0 to state 1 and back again. This chain is irreducible and positive recurrent, so it has a unique stationary distribution, which is clearly $\pi = (\frac{1}{2} \quad \frac{1}{2})$.

However, we don't have convergence to equilibrium. If we start from initial distribution (λ_0, λ_1), then $\mathbb{P}(X_n = 0) = \lambda_0$ for even n and $\mathbb{P}(X_n = 0) = \lambda_1$ for odd n. When $\lambda_0 \neq \frac{1}{2}$, this does not converge.

The point is that this chain is *not* aperiodic: it has period 2, so the limit theorem does not apply.

Example 11.5. Consider a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$ and transition diagram as shown below.
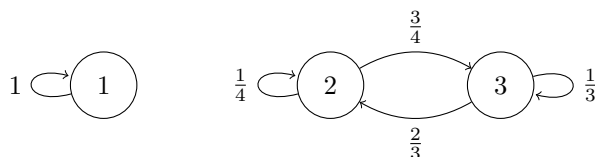
This chain is not irreducible, but has two aperiodic and positive recurrent communicating classes. In particular, it has many stationary distributions including $(1, 0, 0)$ and $(0, \frac{8}{17}, \frac{9}{17})$ (optional exercise for the reader). If we start in state 1, then the limiting distribution is the former, while if we start in states 2 or 3, the limiting distribution is the latter.

In particular, as $n \to \infty$, we have

$$\mathsf{P}^{(n)} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{8}{17} & \frac{9}{17} \\ 0 & \frac{8}{17} & \frac{9}{17} \end{pmatrix}.$$

11.3 Ergodic theorem

The limit theorem looked at the limit of $\mathbb{P}(X_n = j)$, the probability that the Markov chain is in state j at some specific point in time n a long time in the future. We could also look at the *long-run amount of time* spent in state j; that is, averaging the behaviour over a long time period. (The word "ergodic" is used in mathematics to refer to concepts to do with the long-term proportion of time.)

Let us write

$$V_j(N) := \#\{n < N : X_n = j\}$$

for the total number of visits to state j up to time N. Then we can interpret $V_j(n)/n$ as the proportion of time up to time n spent in state j, and its limiting value (if it exists) to be the **long-run proportion of time** spent in state j.

Theorem 11.3 (Ergodic theorem)**.** *Let (X_n) be an irreducible Markov chain. Then for any initial distribution λ we have that $V_j(n)/n \to 1/\mu_j$ almost surely as $n \to \infty$, where μ_j is the expected return time to state j.*

In particular:

- *Suppose (X_n) is positive recurrent. Then there is a unique stationary distribution π given by $\pi_j = 1/\mu_j$, and $V_j(n)/n \to \pi_j$ almost surely for all j.*
- *Suppose (X_n) is null recurrent or transient. Then $V_j(n)/n \to 0$ almost surely for all j.*

For completeness, we should note that "almost sure" convergence means that $\mathbb{P}(V_j(n)/n \to 1/\mu_j) = 1$, although the precise definition is not important for us in this module.

Note that, because we are averaging over a long-time period, we no longer need the condition that the Markov chain is aperiodic. For converge of the long-term proportion of time to the stationary distribution we just need irreducibility and positive recurrence.

Again, we give an optional and nonexaminable proof below.

Example 11.6. Recall the simple no-claims discount Markov chain. Since this chain is irreducible and positive recurrent, we now see that the long-term proportion of time spent in each state corresponds to the stationary distribution $\pi = (\frac{1}{13} \quad \frac{3}{13} \quad \frac{9}{13})$. Therefore, over the lifetime of an insurance policy held for a long period of time, the average discount is approximately

$$\tfrac{1}{13}(0\%) + \tfrac{3}{13}(25\%) + \tfrac{9}{13}(50\%) = \tfrac{21}{52} = 40.4\%.$$

Example 11.7. The two-state "swapping" chain we saw earlier did have a unique stationary distribution $(\frac{1}{2}, \frac{1}{2})$, but did not have an equilibrium distribution, because it was periodic. But it is true that $V_0(n)/n \to \pi_0 = \frac{1}{2}$ and $V_1(n)/n \to \pi_! = \frac{1}{2}$, due to the ergodic theorem. So although where we are at some specific point in the future depends on where started from, in the long run we always spend half our time in each state.

11.4 Proofs of the limit and ergodic theorems

This subsection is optional and nonexaminable.

Again, it's important to be able to use the limit and ergodic theorems, but less important to be able to prove them.

First, the limit theorem. The only bit left is the first part: that for an irreducible, aperiodic, positive recurrent Markov chain, the stationary distribution π is an equilibrium distribution.

This cunning proof uses a technique called "coupling". When looking at two different random objects X and Y (like random variables or stochastic processes), it seems natural to prefer X and Y to be independent. However, **coupling** is the idea that it can sometimes be beneficial to let X and Y actually be dependent on each other.

Proof of Theorem 11.1. Let (X_n) be our irreducible, aperiodic, positive recurrent Markov chain with transition matrix P and initial distribution λ. Let (Y_n) be a Markov chain also with transition matrix P but "in equilibrium" – that is, started from the stationary distribution π, and thus staying in that distribution for ever.

Pick a state $s \in \mathcal{S}$, and let T be the first time the $X_n = Y_n = s$ (or $T = \infty$, if that never happens). Now here's the coupling: after T, when (X_n) and (Y_n) collide at s, then make (X_n) stick to (Y_n), so $X_n = Y_n$ for $n \geq T$. Since a Markov chain has no memory, $(X_{T+n}) = (Y_{T+n})$ is still just a Markov chain with the same transition probabilities from that point on. (Readers of a previous optional subsection will recognise T as a stopping time and will notice we're using the strong Markov property.) Most importantly, thanks to the coupling, from the time T onwards, (X_n) will also always have distribution π, a fact that will obviously be very useful in this proof.

It will be important that T is finite with probability 1. Define $(\mathbf{Z}_n)$ by $\mathbf{Z}_n = (X_n, Y_n)$. So $(\mathbf{Z}_n)$ is a Markov chain on $\mathcal{S} \times \mathcal{S}$, and T is the expected hitting time of $(\mathbf{Z}_n)$ to the state $(s, s) \in \mathcal{S} \times \mathcal{S}$. The transition probabilities for $(\mathbf{Z}_n)$ are $\tilde{\mathsf{P}} = (\tilde{p}_{(i,k)(j,l)})$ where

$$\tilde{p}_{(i,k)(j,l)} = p_{ij}p_{kl}.$$

This is the probability that the joint chain goes from $\mathbf{Z}_n = (X_n, Y_n) = (i, k)$ to $\mathbf{Z}_{n+1} = (X_{n+1}, Y_{n+1}) = (j, l)$.

Since the original Markov chain is irreducible and aperiodic, this means that $p_{ij}(n), p_{kl}(n) > 0$ for all n sufficiently large, so $\tilde{p}_{(i,k)(j,l)}(n) > 0$ for all n sufficiently large also, meaning that $(\mathbf{Z}_n)$ is irreducible (and, although this isn't required, aperiodic). Further, $(\mathbf{Z}_n)$ has a stationary distribution $\tilde{} = (\tilde{\pi}_{(i,k)})$ where

$$\tilde{\pi}_{(i,k)} = \pi_i \pi_k,$$

which means that $(\mathbf{Z}_n)$ is positive recurrent. Thus T is finite with probability 1.

So we can finally prove the limit theorem. We want to show that $\mathbb{P}(X_n = i)$ tends to π_i. The difference between them is

$$\begin{aligned}
\left| \mathbb{P}(X_i = i) - \pi_i \right| &= \mathbb{P}(n \leq T) \times \left| \mathbb{P}(X_i = i \mid n \leq T) - \pi_i \right| + P(n > T) \times |\pi_i - \pi_i| \\
&= \mathbb{P}(n \leq T) \times \left| \mathbb{P}(X_i = i \mid n \leq T) \right| \\
&\leq \mathbb{P}(n \leq T).
\end{aligned}$$

Here, the equality on the first line is because (X_n) follows the stationary distribution exactly once it sticks to (Y_n) after time T, and the inequality on the third line is because the absolute difference between two probabilities is between 0 and 1. But we've already shown that T is finite with probability 1, so $\mathbb{P}(n \leq T) = \mathbb{P}(T \geq n) \to 0$, and we're done.

$\square$

The proof of the ergodic theorem uses the law of large numbers. Recall that the law of large numbers states that if $Y_1, Y_2, \dots$ are IID random variables with mean μ, then

$$\frac{Y_1 + Y_2 + \dots + Y_n}{n} \to \mu \qquad \text{as } n \to \infty.$$

This means it's also true that for any sequence (a_n) with $a_n \to \infty$, we also have

$$\frac{Y_1 + Y_2 + \dots + Y_{a_n}}{a_n} \to \mu \qquad \text{as } n \to \infty.$$

Proof of Theorem 11.3. If (X_n) is transient, then the number of visits to state i is finite with probability 1, so $V_i(n)/n \to 0$, as required.

Suppose instead that (X_n) is recurrent. By our useful lemma we know we will hit i in finite time, so we can ignore that negligible "burn-in" period, and (by the strong Markov property) assume we start from i. Let $M_i^{(r)}$ be the time between the rth and $(r+1)$th visits to i. Note that the $M_i^{(r)}$ are IID with mean μ_i.

The time of the last visit to i before time n is

$$M_i^{(1)} + M_i^{(2)} + \cdots + M_i^{(V_i(n)-1)} < n,$$

and the time of the first visit to i after time n is

$$M_i^{(1)} + M_i^{(2)} + \cdots + M_i^{(V_i(n))} \geq n.$$

Hence

$$\frac{M_i^{(1)} + M_i^{(2)} + \cdots + M_i^{(V_i(n)-1)}}{V_i(n)} < \frac{n}{V_i(n)} \leq \frac{M_i^{(1)} + M_i^{(2)} + \cdots + M_i^{(V_i(n))}}{V_i(n)}. \tag{10}$$

Because (X_n) is recurrent, we keep returning to i, so $V_i(n) \to \infty$ with probability 1. Hence, by the law of large numbers, both the left- and right-hand sides of (10) tend to $\mathbb{E}M_i^{(r)} = \mu_i$. So $n/V_i(n)$ is sandwiched between them, and tends to μ_i too. Finally $n/V_i(n) \to \mu_i$ is equivalent to $V_i(n)/n \to 1/\mu_i$, so we are done.

$\square$

This completes the material on discrete time Markov chains. **In the next section**, we recap what we have learned, and have a little time for some revision.

12 Revision of Part 1: Discrete time Markov chains

- No new material in this section, but a half-week break to catch up and revise what we've learned

12.1 Things to do

We've now finished the material of the Part 1 of the module, on discrete time Markov chains. So this is a good time to take stock, revise what we've learned, and make sure we're completely up to date before starting Part 2 of the module on continuous time processes.

Some things you may want to do in lieu of reading a section of notes:

- Make sure you've completed **Problem Sheets 1 to 6**, and go back to any questions that stumped you before.
- Start working on **Assessment 2** which is due on ?????? ??? (Week 8).
- Start working on **Computer Practical 2** (which doubles as Assessment 3). There are optional computer drop-in sessions in Week 7, and the work is due on ????? ???? (Week 9).
- Re-read any sections of notes you struggled with earlier.
- Take the opportunity to look at the optional non-examinable subsections if you opted out the first time around. (Section 9, Section 10, Section 11)
- Let me know if there's anything from this part of the course you'd like me to go through in next week's lecture.

12.2 Summary of Part 1

The following list is not exhaustive, but if you can do most of the things on the list, you're well placed for the exam.

- Define the simple random walk and other random walks.

- Perform elementary calculations for the simple random walk, including by referring to the exact binomial distribution.

- Calculate the expectation and variance of general random walks.

- Define the gambler's ruin Markov chain.

- Find the ruin probability and expected duration for the gambler's ruin by (i) setting up equations by conditioning on the first step and (ii) solving the resulting linear difference equation.

- Draw a transition diagram of a Markov chain given the transition matrix

- Calculate n-step transitions probabilities by (a) summing the probabilities over all relevant paths or (b) calculating the matrix power

- Find the communicating classes in a Markov chain.

- Calculate the period of communicating class,

- Calculate hitting probabilities and expected hitting times by (i) setting up equations by conditioning on the first step and (ii) solving the resulting simultaneous equations

- Define positive recurrence, null recurrence and transience, and explain their properties

- Find the positive recurrence, null recurrence or transience of communicating classes

- Find the stationary distribution of Markov chain

- Give conditions for a stationary distribution to exist and be unique

- Give conditions for convergence to an equilibrium distribution

- Calculate long-term proportions of time using the ergodic theorem

In the next section, we begin our study of continuous time Markov processes by looking at the most important example: the Poisson process.

Part II: Continuous time Markov jump processes

13   Poisson process with Poisson increments

- Reminder: the Poisson distribution
- The Poisson process has independent Poisson increments
- Summed and marked Poisson processes

13.1   Poisson distribution

In the next three sections we'll be considering the Poisson process, a continuous time discrete space process with the Markov property. Given its name, it's not surprising to hear the Poisson *process* is related to the Poisson *distribution*. Let's start with a reminder of that.

Recall that a discrete random variable X has a **Poisson distribution** with rate λ, written $X \sim \mathrm{Po}(\lambda)$, if its probability mass function is

$$\mathbb{P}(X = n) = \mathrm{e}^{-\lambda} \frac{\lambda^n}{n!} \qquad n = 0, 1, 2, \dots.$$

The Poisson distribution is often used to model the number of "arrivals" in a fixed amount of time – for example, the number of calls to a call centre in one hour, the number of claims to an insurance company in one year, or the number of particles decaying from a large amount of radioactive material in one second.

The Poisson distribution is named after the French mathematician and physicist Siméon Denis Poisson, who studied it in 1837, although it was used by another French mathematician, Abraham de Moivre, more than 100 years earlier.

Recall the following facts about of the Poisson distribution:

1. Its expectation is $\mathbb{E}X = \lambda$ and the variance is $\mathrm{Var}(X) = \lambda$.
2. If $X \sim \mathrm{Po}(\lambda)$ and $Y \sim \mathrm{Po}(\mu)$ are independent, then $X + Y \sim \mathrm{Po}(\lambda + \mu)$.
3. Let $X \sim \mathrm{Po}(\lambda)$ represent some arrivals, and independently "mark" each arrival with probability p. Then the number of marked arrivals Y has distribution $Y = \mathrm{Po}(p\lambda)$, the number of unmarked arrivals Z has distribution $Z \sim \mathrm{Po}((1-p)\lambda)$, and Y and Z are independent.

(When mathematicians say "recall" usually they mean "remember this well-known fact you've been taught before", but sometimes they secretly mean "I'm going to tell you something you probably haven't seen before, but I can't be bothered to prove it". Fact number 3 might be more of the latter kind…)

13.2   Definition 1: Poisson increments

Suppose that, instead of just modelling the number of arrivals in a *fixed* amount of time, we want to continually model the total number of arrivals as it changes over time. This will be a stochastic process with discrete state space $\mathcal{S} = \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and continuous time $\mathbb{R}_+ = [0, \infty)$. In continuous time,

we will normally write stochastic processes as $(X(t))$, with the time variable being a t in brackets, rather than a subscript n as we had in discrete time.

Suppose calls arrive at a call centre at a rate of $\lambda = 100$ an hour. The following assumptions seem reasonable:

- We begin counting with $X(0) = 0$ calls.
- The number of calls in the first hour is $X(1) \sim \text{Po}(100)$. The number of calls in the second hour $X(2) - X(1)$ will also be Po(100), and will be independent of the number of calls in the first hour.
- The number of calls in a two hour period will be $X(t+2) - X(t) \sim \text{Po}(200)$, while the number of calls in a half-hour period will be $X(t + \frac{1}{2}) - X(t) \sim \text{Po}(50)$.

These properties will define the Poisson process.

Definition 13.1. The **Poisson process** with rate λ is defined as followed. It is a stochastic process $(X(t))$ with continuous time $t \in [0, \infty)$ and discrete state space $\mathcal{S} = \mathbb{Z}_+$ with the following properties:

1. $X(0) = 0$;
2. Poisson increments: $X(t+s) - X(t) \sim \text{Po}(\lambda s)$ for all $s, t > 0$;
3. independent increments: $X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are independent for all $t_1 \leq t_2 \leq t_3 \leq t_4$.

Note that the condition $t_1 \leq t_2 \leq t_3 \leq t_4$ means that the time interval from t_1 to t_2 and the time interval from t_3 to t_4 don't overlap. (Overlapping time intervals will not have independent increments, as arrivals in the overlap will count for both.)

The Poisson process was discovered in the first decade 20th century, and the process was named after the distribution. Many people were working on similar things, so it's difficult to say who discovered it first, but important work was done by the Swedish actuary and mathematician Filip Lundberg and the Danish engineer and mathematician AK Erlang,

Example 13.1. *Claims arrive at insurance company at a rate of $\lambda = 8$ per hour, modelled as a Poisson process. What is the probability there are no claims in a given 15 minute period?*

By property 2, the number of claims in 15 minutes is a Poisson distribution with mean $\frac{15}{60}\lambda = 2$. The probability there are no claims is

$$\text{e}^{-2}\frac{2^0}{0!} = \text{e}^{-2} = 0.135.$$

Example 13.2. *A professor receives visitors to her office at a rate of $\lambda = 2.5$ per day, modelled as a Poisson process. What is the probability she gets at least one visitor every day this (5-day) week?*

The probability she gets at least one visitor on any given day is

$$1 - \text{e}^{-2.5}\frac{2.5^0}{0!} = 1 - \text{e}^{-2.5} = 0.918.$$

By property 3, the numbers of visitors on different days are independent, so the probability of getting at least one visitor each day this week is $0.918^5 = 0.652$.

13.3 Summed and marked Poisson processes

The following theorem shows that the sum of two Poisson processes is itself a Poisson process.

Theorem 13.1. *Let $(X(t))$ and $(Y(t))$ be independent Poisson processes with rates λ and μ respectively. Then the process $(Z(t))$ given by $Z(t) = X(t) + Y(t)$ is a Poisson process with rate $\lambda + \mu$.*

Proof. Given fact number 2, that the sum of a Po(λ) and a Po(μ) is a Po($\lambda + \mu$), its easy to check the necessary properties.

$\square$

Example 13.3. *A student receives email to her university mail address at a rate of $\lambda = 4$ emails per hour, and to her personal email address at a rate of $\mu = 2$ per hour. Using a Poisson process model, what is the probability the student receives 3 or fewer emails in a 30 minute period?*

The total number of emails is a sum of Poisson processes with rate $\lambda + \mu = 6$. The total number of emails received in half an hour is Poisson with rate $(\lambda + \mu)/2 = 3$. Thus the probability that 3 or fewer emails are received is

$$\mathrm{e}^{-3}\frac{3^0}{0!} + \mathrm{e}^{-3}\frac{3^1}{1!} + \mathrm{e}^{-3}\frac{3^2}{2!} + \mathrm{e}^{-3}\frac{3^3}{3!} = 13\mathrm{e}^{-3} = 0.647.$$

We also have the **marked Poisson process**, which can be thought of as the opposite to the summed process: the summed process combines two processes together, while the marked process splits one process into two.

Theorem 13.2. *Let $(X(t))$ be a Poisson process with rate λ. Each arrival is independently marked with probability p. Then the marked process $(Y(t))$ is a Poisson process with rate $p\lambda$, the unmarked process $(Z(t))$ is a Poisson process with rate $(1-p)\lambda$, and $(Y(t))$ and $(Z(t))$ are independent.*

Proof. Given the third fact that you were "reminded" of, it's easy to check the necessary properties.

$\square$

Example 13.4. *In the 2017/18 English Premier League football season, an average of $\lambda = 2.81$ goals were scored per game, with $p = 0.56$ of them scored by the home team. If we model this as a Poisson process, what is the probability a game ends 1–1?*

The number of home goals scored is Poisson with rate $p\lambda = 0.56 \times 2.81 = 1.57$, and the number of away goals scored is Poisson with rate $(1-p)\lambda = 1.24$. Since these are independent, the probability the home and the away team both score 1 goal is

$$\mathrm{e}^{-1.57}\frac{1.57^1}{1!} \times \mathrm{e}^{-1.24}\frac{1.24^1}{1!} = 1.57\mathrm{e}^{-1.57} \times 1.24\mathrm{e}^{-1.24} = 0.12,$$

or 12%.

In the next section, we look at the Poisson process in a different way: the times in between arrivals have an exponential distribution.

14 Poisson process with exponential holding times

- Reminder: the exponential distribution
- The Poisson process has exponential holding times
- The Markov property in continuous time

14.1 Exponential distribution

Last time, we introduced the Poisson process by looking at the random number of arrivals in fixed time period, which follows a Poisson distribution. Another way of looking at the Poisson process is to look at the random amount of of time required for a fixed number of arrivals. For this, we'll need the exponential distribution.

We start by recalling the exponential distribution. Recall that we say a continuous random variable T has the **exponential distribution** with rate λ, and write $T \sim \mathrm{Exp}(\lambda)$, if it has the probability distribution function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.

You are reminded of the following facts about the exponential distribution:

- The cumulative distribution function is $F(t) = \mathbb{P}(T \leq t) = 1 - e^{-\lambda t}$; although it is usually more convenient to deal with the tail probability $\mathbb{P}(T > t) = 1 - F(t) = e^{-\lambda t}$.
- The expectation is $\mathbb{E}T = 1/\lambda$ and the variance is $\mathrm{Var}(T) = 1/\lambda^2$.

The following memoryless property will of course be important in a course about memoryless Markov processes.

Theorem 14.1. *Let $T \sim \mathrm{Exp}(\lambda)$. Then, for any $s, t \geq 0$,*

$$\mathbb{P}(T > t + s \mid T > t) = \mathbb{P}(T > s).$$

Suppose we are waiting an exponentially distributed time for an alarm to go off. No matter how long we've been waiting for, the remaining time to wait is still exponentially distributed, with the same parameter – hence "memoryless".

Proof. By standard use of conditional probability, we have

$$\mathbb{P}(T > t + s \mid T > t) = \frac{\mathbb{P}(T > t + s \text{ and } T > t)}{\mathbb{P}(T > t)} = \frac{\mathbb{P}(T > t + s)}{\mathbb{P}(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s},$$

which is still the tail probability of the exponential distribution.

$\square$

The following property will be important later on in the course.

Theorem 14.2. *Let $T_1 \sim \mathrm{Exp}(\lambda_1)$, $T_2 \sim \mathrm{Exp}(\lambda_2)$, ..., $T_n \sim \mathrm{Exp}(\lambda_n)$ be independent exponential distributions, and let T be the minimum of the $T_i s$, so $T = \min\{T_1, T_2, \ldots, T_n\}$. Then*

$$T \sim \mathrm{Exp}(\lambda_1 + \lambda_2 + \cdots + \lambda_n).$$

Further, the probability that T_j is the smallest of all the $T_i s$ is

$$\mathbb{P}(T = T_j) = \frac{\lambda_j}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.$$

The proof of this is an exercise on Problem Sheet 7.

14.2 Definition 2: exponential holding times

Exponential distributions are often used to model "waiting times" – for example, the amount of time until a light bulb breaks, or the times between buses arriving. We often use the term 'holding time' to refer to the time between consecutive arrivals.

When modelling a process $(X(t))$ counting many arrivals at rate λ, we might model the process like this: after waiting an $\text{Exp}(\lambda)$ amount of time, an arrival appears. After another $\text{Exp}(\lambda)$ amount of time, another arrival appears. And so on.

- We start with $X(0) = 0$.
- Let $T_1, T_2, \cdots \sim \text{Exp}(\lambda)$ be the holding times, all independent. Then

$$X(t) = \begin{cases} 0 & \text{for } 0 \leq t < T_1 \\ 1 & \text{for } T_1 \leq t < T_1 + T_2 \\ 2 & \text{for } T_1 + T_2 \leq t < T_1 + T_2 + T_3 \\ \text{and so on.} \end{cases}$$

It turns out (as you've probably already guessed) that the process as described above is also the Poisson process!

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Theorem 14.3. *Let $(X(t))$ be a Poisson process with rate λ, as defined by its independent Poisson increments (Definition **REF REF REF**). Then $(X(t))$ has the exponential holding times structure described above.*

Proof. Let $(X(t))$ be a Poisson process with rate λ. We seek the distribution of the first arrival time T_1. We have

$$\mathbb{P}(T_1 > t_1) = \mathbb{P}(X(t_1) - X(0) = 0) = e^{-\lambda t_1} \frac{(\lambda t_1)^0}{0!} = e^{-\lambda t_1},$$

since the arrival comes after t_1 if there are no arrivals in the interval $[0, t_1]$. But this is precisely the tail probability of an exponential distribution with rate λ.

Now consider the second holding time. We have

$$\mathbb{P}(T_2 > t_2 \mid T_1 = t_1) = \mathbb{P}(X(t_1 + t_2) - X(t_1) = 0) = e^{-\lambda((t_1 + t_2) - t_1)} = e^{-\lambda t_2}$$

by the same argument as before. This is the tail distribution for another $\text{Exp}(\lambda)$ distribution, independent of T_1.

Repeating this argument for all n gives the desired exponential holding time structure.

$\square$

Example 14.1. *Customers visit a second-hand bookshop at a rate of $\lambda = 5$ per hour. Each customer a book with probability $p = 0.4$. What is the expected time to make ten sales, and what is the standard deviation?*

The count of books sold marked Poisson process, which we saw last time is itself a Poisson process with rate $p\lambda = 2$.

The expected time of the tenth sale is

$$\mathbb{E}(T_1 + \cdots + T_{10}) = 10 \, \mathbb{E} T_1 = 10 \times \tfrac{1}{2} = 5 \text{ hours.}$$

The variance is

$$\text{Var}(T_1 + \cdots + T_{10}) = 10 \, \text{Var}(T_1) = 10 \times \frac{1}{2^2} = 2.5,$$

where we used that the holding times are independent, so the standard deviation is $\sqrt{2.5} = 1.58$ hours.

What is the probability it takes more than an hour to sell the first book?

This is the probability the first holding time is longer than an hour, which is

$$\mathbb{P}(T_1 > 1) = \mathrm{e}^{-2 \times 1} = \mathrm{e}^{-2} = 0.135.$$

An alternative way to solve this would be to say that the first book is is sold later than 1 hour in if $X(1) = 0$. So using the Poisson increments definition from last time, we have

$$\mathbb{P}(X(1) = 0) = \mathrm{e}^{-2 \times 1} \frac{(2 \times 1)^0}{0!} = \mathrm{e}^{-2} = 0.135.$$

We get the same answer either way.

14.3   Markov property in continuous time

We previously saw the Markov "memoryless" property in discrete time. The equivalent definition in continuous time is the following.

Definition 14.1. Let $(X(t))$ be a stochastic process on a discrete state space $\mathcal{S}$ and continuous time $t \in [0, \infty)$. We say that $(X(t))$ has the **Markov property** if

$$\mathbb{P}(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, \dots, X(t_1) = x_1, X(t_0) = x_0)$$
$$= \mathbb{P}(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n)$$

for all times $t_0 < t_1 < \dots < t_n < t_{n+1}$ and all states $x_0, x_1, \dots, x_n, x_{n+1} \in \mathcal{S}$.

In other words, the state of the process at some point t_{n+1} in the future depends on where we are now t_n, but, given that, does not depend on any collection of previous times $t_0, t_1, \dots, t_{n-1}$.

The Poisson process does indeed have the Markov property, since, by the property of independent increments, we have that

$$\mathbb{P}(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, \dots, X(t_0) = x_0) = \mathbb{P}(X(t_{n+1}) - X(t_n) = x),$$

where $x = x_{n+1} - x_n$. By Poisson increments, this has probability

$$\mathrm{e}^{-\lambda(t_{n+1} - t_n)} \frac{\left(\lambda(t_{n+1} - t_n)\right)^x}{x!}.$$

Alternatively, by the memoryless property of the exponential distribution, we see that we can restart the holding time from t_n and still have this and all future holding times exponentially distributed.

In the next section, we look at the Poisson process in a third way, by looking at what happens in a very small time period.