Gradient Non-Linearity in B-Tensor Encoding

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1 Gradient Waveform

For a time-varying gradient waveform $\vec{G}(t) = [G_x(t)\, G_y(t)\, G_z(t)]^T$, we have the time-varying q-vector $\vec{q}(t)$

$$\vec{q}(t) = \gamma \int_0^t \vec{G}(t')\, \mathrm{d}t'$$

where γ is the ^{1}H gyromagnetic ratio ($2\pi \cdot 42.577 \times 10^6\, \mathrm{m}^{-1}\, \mathrm{T}^{-1}$).

The gradients waveform has duration t_{tot} and is designed such that we have

$$\vec{G}(t) = \vec{0} \text{ for } t \notin]0, t_{tot}[$$

and

$$\vec{q}(t) = \vec{0} \text{ for } t \notin]0, t_{tot}[.$$

We decompose the time-varying q-vector into a time-varying q-vector norm ($q(t)$) and unit-norm orientation ($\vec{n}(t)$),

$$\vec{q}(t) = q(t) \cdot \vec{n}(t).$$

and we define the B-tensor ($\mathbf{B}$) has

$$\begin{aligned}\mathbf{B} &= \int_0^{t_{tot}} (\vec{q}(t') \otimes \vec{q}(t'))\,\mathrm{d}t' \\ &= \int_0^{t_{tot}} q^2(t')(\vec{n}(t') \otimes \vec{n}(t'))\,\mathrm{d}t'\end{aligned}$$

where $\otimes$ is the outer vector product.

2 Gradient Non-Linearity

We model the gradient non-linearity by a time-constant gradient non-linearity tensor ($\mathbf{L}$)

$$\mathbf{L} = \begin{bmatrix} L_{xx} & L_{xy} & L_{xz} \\ L_{yx} & L_{yy} & L_{yz} \\ L_{zx} & L_{zy} & L_{zz} \end{bmatrix}$$

acting upon the time-varying expected-gradient ($\vec{G}(t)$) to give us the time-varying actual-gradient ($\vec{G_a}(t)$)

$$\vec{G_a}(t) = \mathbf{L} \cdot \vec{G}(t).$$

Similarly, we can compute the time-varying actual-q-vector ($\vec{q_a}(t)$) as

$$\begin{aligned}\vec{q_a}(t) &= \gamma \int_0^t \vec{G_a}(t')\,\mathrm{d}t' \\ &= \gamma \int_0^t \mathbf{L} \cdot \vec{G}(t')\,\mathrm{d}t' \\ &= \mathbf{L} \cdot (\gamma \int_0^t \vec{G}(t')\,\mathrm{d}t') \\ &= \mathbf{L} \cdot \vec{q}(t)\end{aligned}$$

and we maintain the two key properties

$$\vec{G_a}(t) = \mathbf{L} \cdot \vec{G}(t) = \mathbf{L} \cdot \vec{0} = \vec{0} \text{ for } t \notin]0, t_{tot}[$$

and

$$\vec{q_a}(t) = \mathbf{L} \cdot \vec{q}(t) = \mathbf{L} \cdot \vec{0} = \vec{0} \text{ for } t \notin]0, t_{tot}[\,.$$

For a generic B-tensor $\mathbf{B}$ from a generic q-vector $\mathrm{q}\left(t\right)$:

$$\mathbf{B} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{xy} & B_{yy} & B_{yz} \\ B_{xz} & B_{yz} & B_{zz} \end{bmatrix}$$

$$\mathrm{q}\left(t\right) = \begin{bmatrix} \mathrm{q_x}\left(t\right) \\ \mathrm{q_y}\left(t\right) \\ \mathrm{q_z}\left(t\right) \end{bmatrix}$$

We have

$$\begin{aligned}
\mathbf{B} &= \begin{bmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{xy} & B_{yy} & B_{yz} \\ B_{xz} & B_{yz} & B_{zz} \end{bmatrix} \\
&= \int_0^{t_{tot}} \left(\vec{q}(t) \otimes \vec{q}(t)\right) \mathrm{d}t \\
&= \int_0^{t_{tot}} \begin{bmatrix} \mathrm{q_x}^2\left(t\right) & \mathrm{q_x}\left(t\right)\mathrm{q_y}\left(t\right) & \mathrm{q_x}\left(t\right)\mathrm{q_z}\left(t\right) \\ \mathrm{q_x}\left(t\right)\mathrm{q_y}\left(t\right) & \mathrm{q_y}^2\left(t\right) & \mathrm{q_y}\left(t\right)\mathrm{q_z}\left(t\right) \\ \mathrm{q_x}\left(t\right)\mathrm{q_z}\left(t\right) & \mathrm{q_y}\left(t\right)\mathrm{q_z}\left(t\right) & \mathrm{q_z}^2\left(t\right) \end{bmatrix} \mathrm{d}t \\
&= \begin{bmatrix} \int_0^{t_{tot}} \mathrm{q_x}^2\left(t\right) \mathrm{d}t & \int_0^{t_{tot}} \mathrm{q_x}\left(t\right)\mathrm{q_y}\left(t\right) \mathrm{d}t & \int_0^{t_{tot}} \mathrm{q_x}\left(t\right)\mathrm{q_z}\left(t\right) \mathrm{d}t \\ \int_0^{t_{tot}} \mathrm{q_x}\left(t\right)\mathrm{q_y}\left(t\right) \mathrm{d}t & \int_0^{t_{tot}} \mathrm{q_y}^2\left(t\right) \mathrm{d}t & \int_0^{t_{tot}} \mathrm{q_y}\left(t\right)\mathrm{q_z}\left(t\right) \mathrm{d}t \\ \int_0^{t_{tot}} \mathrm{q_x}\left(t\right)\mathrm{q_z}\left(t\right) \mathrm{d}t & \int_0^{t_{tot}} \mathrm{q_y}\left(t\right)\mathrm{q_z}\left(t\right) \mathrm{d}t & \int_0^{t_{tot}} \mathrm{q_z}^2\left(t\right) \mathrm{d}t \end{bmatrix}
\end{aligned}$$

We introduce a generic GNL tensor $\mathbf{L}$

$$\mathbf{L} = \begin{bmatrix} L_{xx} & L_{xy} & L_{xz} \\ L_{yx} & L_{yy} & L_{yz} \\ L_{zx} & L_{zy} & L_{zz} \end{bmatrix}$$

and compute the resulting B-tensor from the distorted q-vector ($\vec{q_a}(t)$)

$$\begin{aligned} \vec{q_a}(t) &= \mathbf{L} \cdot \vec{q}(t) \\ &= \begin{bmatrix} L_{xx}\,\mathrm{q_x}(t) + L_{xy}\,\mathrm{q_y}(t) + L_{xz}\,\mathrm{q_z}(t) \\ L_{yx}\,\mathrm{q_x}(t) + L_{yy}\,\mathrm{q_y}(t) + L_{yz}\,\mathrm{q_z}(t) \\ L_{zx}\,\mathrm{q_x}(t) + L_{zy}\,\mathrm{q_y}(t) + L_{zz}\,\mathrm{q_z}(t) \end{bmatrix} \end{aligned}$$

and we can compute the distorted B-tensor $\mathbf{B_a}$

$$\begin{aligned} \mathbf{B_a} &= \begin{bmatrix} (B_a)_{xx} & (B_a)_{xy} & (B_a)_{xz} \\ (B_a)_{xy} & (B_a)_{yy} & (B_a)_{yz} \\ (B_a)_{xz} & (B_a)_{yz} & (B_a)_{zz} \end{bmatrix} \\ &= \int_0^{t_{tot}} \left(\vec{q_a}(t) \otimes \vec{q_a}(t)\right) \mathrm{d}t \end{aligned}$$

See `proof_helper.py` for the long form computation, here we will only do the term $(\mathbf{B_a})_{xx}$

$$
\begin{aligned}
(\mathbf{B_a})_{xx} &= \int_0^{t_{tot}} \left(L_{xx}\,\mathrm{q_x}\,(t) + L_{xy}\,\mathrm{q_y}\,(t) + L_{xz}\,\mathrm{q_z}\,(t)\right)^2\,\mathrm{d}t \\
&= \int_0^{t_{tot}} L_{xx}^2\,\mathrm{q_x}^2\,(t) + 2L_{xx}L_{xy}\,\mathrm{q_x}\,(t)\,\mathrm{q_y}\,(t) + 2L_{xx}L_{xz}\,\mathrm{q_x}\,(t)\,\mathrm{q_z}\,(t) \\
&\quad + L_{xy}^2\,\mathrm{q_y}^2\,(t) + 2L_{xy}L_{xz}\,\mathrm{q_y}\,(t)\,\mathrm{q_z}\,(t) + L_{xz}^2\,\mathrm{q_z}^2\,(t)\,\mathrm{d}t \\
&= \int_0^{t_{tot}} L_{xx}^2\,\mathrm{q_x}^2\,(t)\,\mathrm{d}t + \int_0^{t_{tot}} 2L_{xx}L_{xy}\,\mathrm{q_x}\,(t)\,\mathrm{q_y}\,(t)\,\mathrm{d}t + \int_0^{t_{tot}} 2L_{xx}L_{xz}\,\mathrm{q_x}\,(t)\,\mathrm{q_z}\,(t)\,\mathrm{d}t \\
&\quad + \int_0^{t_{tot}} L_{xy}^2\,\mathrm{q_y}^2\,(t)\,\mathrm{d}t + \int_0^{t_{tot}} 2L_{xy}L_{xz}\,\mathrm{q_y}\,(t)\,\mathrm{q_z}\,(t)\,\mathrm{d}t + \int_0^{t_{tot}} L_{xz}^2\,\mathrm{q_z}^2\,(t)\,\mathrm{d}t \\
&= L_{xx}^2 \int_0^{t_{tot}} \mathrm{q_x}^2\,(t)\,\mathrm{d}t + 2L_{xx}L_{xy} \int_0^{t_{tot}} \mathrm{q_x}\,(t)\,\mathrm{q_y}\,(t)\,\mathrm{d}t + 2L_{xx}L_{xz} \int_0^{t_{tot}} \mathrm{q_x}\,(t)\,\mathrm{q_z}\,(t)\,\mathrm{d}t \\
&\quad + L_{xy}^2 \int_0^{t_{tot}} \mathrm{q_y}^2\,(t)\,\mathrm{d}t + 2L_{xy}L_{xz} \int_0^{t_{tot}} \mathrm{q_y}\,(t)\,\mathrm{q_z}\,(t)\,\mathrm{d}t + L_{xz}^2 \int_0^{t_{tot}} \mathrm{q_z}^2\,(t)\,\mathrm{d}t \\
&= L_{xx}^2 B_{xx} + 2L_{xx}L_{xy}B_{xy} + 2L_{xx}L_{xz}B_{xz} \\
&\quad + L_{xy}^2 B_{yy} + 2L_{xy}L_{xz}B_{yz} + L_{xz}^2 B_{zz}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{B_a})_{xy} = L_{xx}L_{yx}B_{xx} + (L_{xx}L_{yy} + L_{xy}L_{yx})B_{xy} + (L_{xx}L_{yz} + L_{xz}L_{yx})B_{xz} \\
+ L_{xy}L_{yy}B_{yy} + (L_{xy}L_{yz} + L_{xz}L_{yy})B_{yz} + L_{xz}L_{yz}B_{zz}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{B_a})_{xz} = L_{xx}L_{zx}B_{xx} + (L_{xx}L_{zy} + L_{xy}L_{zx})B_{xy} + (L_{xx}L_{zz} + L_{xz}L_{zx})B_{xz} \\
+ L_{xy}L_{zy}B_{yy} + (L_{xy}L_{zz} + L_{xz}L_{zy})B_{yz} + L_{xz}L_{zz}B_{zz}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{B_a})_{yy} = L_{yx}^2 B_{xx} + 2L_{yx}L_{yy}B_{xy} + 2L_{yx}L_{yz}B_{xz} \\
+ L_{yy}^2 B_{yy} + 2L_{yy}L_{yz}B_{yz} + L_{yz}^2 B_{zz}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{B_a})_{yz} = L_{yx}L_{zx}B_{xx} + (L_{yx}L_{zy} + L_{yy}L_{zx})B_{xy} + (L_{yx}L_{zz} + L_{yz}L_{zx})B_{xz} \\
+ L_{yy}L_{zy}B_{yy} + (L_{yy}L_{zz} + L_{yz}L_{zy})B_{yz} + L_{yz}L_{zz}B_{zz}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{B_a})_{zz} = {} & L_{zx}^2 B_{xx} + 2L_{zx}L_{zy}B_{xy} + 2L_{zx}L_{zz}B_{xz} \\
& + L_{zy}^2 B_{yy} + 2L_{zy}L_{zz}B_{yz} + L_{zz}^2 B_{zz}
\end{aligned}
$$