Old. Likelihood in the paper.

$$\log L(a,v,k) = \sum_{x,t} \{ D_{x,t}(a_x + v_x k) - E^c_{x,t} \exp(a_x + v_t k) \} + \text{constant}$$

New Likelihood function with βx parameter.

$$\log L(\beta, v, k) = \sum_{x,t} \{ D_{x,t}(\beta_x \cdot \log e_{\theta,t} + v_x k) - E_{x,t} \cdot \exp(\beta x \log e_{\theta,t} + v_x k) \} + c.$$

Taking the first and second partial derivatives with respect to βx, leads to

$$\frac{\partial L}{\partial \beta x} = \sum_t \{ D_{x,t} \cdot \log e_{\theta,t} - E_{x,t} \cdot \exp(\beta x \log e_{\theta,t} + v_x k) \cdot \log e_{\theta,t} \}$$

$$= \log e_{\theta,t} \left[\sum_t (D_{x,t} - \hat{D}_{x,t}) \right]$$

$$\frac{\partial^2 L}{\partial^2 \beta x} = \log e_{\theta,t} \left[\sum_t (- \hat{D}_{x,t}) \right] \cdot \log e_{\theta,t}$$

$$= (\log e_{\theta,t})^2 \left[- \sum_t (\hat{D}_{x,t}) \right]$$

Using this and the updating scheme proposed by Brouhns et al. given by

$$\hat{\theta}(v+1) = \hat{\theta}(v) - \frac{\partial L(v) / \partial \theta}{\partial^2 L(v) / \partial^2 \theta},$$

The parameter β_x can be updated as

$$\hat{\beta}_x(w+1) = \hat{\beta}_x(w) - \frac{\sum_t (D_{x,t} - \hat{D}_{x,t})}{-\sum_t \hat{D}_{x,t} \cdot \log e_{o,t}}.$$