Feuille TD 4

L2 Math, DL2 Math-éco ①
2016-17
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Ex 1.

1) B.T. appliqué à $\bar{Z}_n$.

$E(\bar{Z}_n) = p$

$V(\bar{Z}_n) = \frac{p(1-p)}{n}$

$$P(|\bar{Z}_n - p| \geq d) \leq \frac{V(\bar{Z}_n)}{d^2} = \frac{p(1-p)}{nd^2} \leq \frac{1}{4nd^2}$$

On prend d tq $\frac{1}{4nd^2} \leq 0{,}05$.

c.à.d $d^2 \geq \frac{1}{4n \times 0{,}05}$

ie $d \geq \sqrt{\frac{1}{4n \times 0{,}05}} = \frac{1}{2\sqrt{n \times 0{,}05}}$

$n = 550 \quad d = 0{,}095$

2) $P(|\bar{Z}_n - p| \geq d) \leq \frac{1}{4nd^2} \leq \alpha$ si $d \geq \frac{1}{2\sqrt{n\alpha}}$

$P(|\bar{Z}_n - p| \geq d) \leq \alpha \iff 1 - P(|\bar{Z}_n - p| < d) \leq \alpha$.

$\iff P(|\bar{Z}_n - p| < d) \geq 1 - \alpha$.

Dc $P(-d \leq \bar{Z}_n - p \leq d) \geq 1 - \alpha$

$\iff P(\bar{Z}_n - d \leq p \leq \bar{Z}_n + d) \geq 1 - \alpha$

$$IC_{1-\alpha} = \left[\bar{Z}_n - \frac{1}{2\sqrt{n\alpha}} \; ; \; \bar{Z}_n + \frac{1}{2\sqrt{n\alpha}} \right]$$

$\alpha = 10\%$. $\left[\frac{42}{550} - \frac{1}{2\sqrt{550 \times 0{,}1}} \; ; \; \frac{42}{550} + \frac{1}{2\sqrt{550 \times 0{,}1}} \right]$

$$= [\, 0{,}0089 \; ; \; 0{,}1438 \,]$$

$\alpha = 5\%$ $\quad [\, 0 \; ; \; 0{,}1717 \,]$

$\uparrow$

$-0{,}019$

$\alpha = 1\%$ $\quad [\, 0 \; ; \; 0{,}2896 \,]$.

$-0{,}1368$

3) $\quad P(|\bar{Z}_n - p| \geq d) \leq 2\exp(-2nd^2)$

On cherche d tq $2\exp(-2nd^2) \leq \alpha$

$$2\exp(-2nd^2) \leq \alpha$$

$$\Leftrightarrow \quad \exp(-2nd^2) \leq \frac{\alpha}{2}$$

$$\Leftrightarrow \quad -2nd^2 \leq \ln\frac{\alpha}{2}.$$

$$\Leftrightarrow \quad d^2 \geq -\frac{\ln \alpha/2}{2n}. \qquad \Leftrightarrow \quad d \geq \sqrt{-\frac{\ln \alpha/2}{2n}}$$

$P(|\bar{Z}_n - p| \geq d) \leq \alpha$ si $d \geq \sqrt{-\frac{\ln \alpha/2}{2n}}$

$$\left[\bar{Z}_n - d \ ; \bar{Z}_n + d\right] = \left[\bar{Z}_n - \sqrt{-\frac{\ln \alpha/2}{2n}} \ ; \bar{Z}_n + \sqrt{-\frac{\ln \alpha/2}{2n}}\right]$$

$\alpha = 10\%$ $\left[\frac{42}{550} - \sqrt{-\frac{\ln(0{,}1/2)}{2 \times 550}} \ ; \frac{42}{550} + \sqrt{-\frac{\ln(0{,}1/2)}{2 \times 550}}\right]$

$[0{,}0241775 \ ; 0{,}1285498]$

$\alpha = 5\%$ $\left[\frac{42}{550} - \sqrt{-\frac{\ln(0{,}05/2)}{2 \times 550}} \ ; \frac{42}{550} + \sqrt{-\frac{\ln(0{,}05/2)}{2 \times 550}}\right]$

$[0{,}01845399 \ ; 0{,}1342733]$

$\alpha = 1\%$

$[0{,}00696153 2 \ ; 0{,}145765$

4) $\sqrt{n} \frac{\bar{Z}_n - p}{\sqrt{p(1-p)}} \sim \mathcal{N}(0,1)$ approx si n est grand

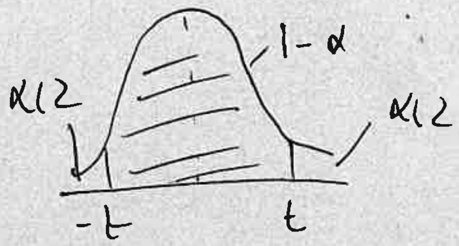
et $\hat{m}$.

$\sqrt{n} \frac{\bar{Z}_n - p}{\sqrt{\bar{Z}_n(1-\bar{Z}_n)}} \sim \mathcal{N}(0,1)$ approx si n est gd.

Dc $P\left(-t \leq \sqrt{n} \frac{\bar{Z}_n - p}{\sqrt{\bar{Z}_n(1-\bar{Z}_n)}} \leq t\right)$

$\approx P(-t \leq Z \leq t)$ où $Z \sim \mathcal{N}(0,1)$

On choisit donc t tq



$$P(-t \le Z \le t) = 1-\alpha.$$

cad $P(Z \le t) = 1 - \frac{\alpha}{2}$.

t est le quantile d'ordre $1-\frac{\alpha}{2}$ de la loi $\mathcal{N}(0,1)$.

Et on obtient

$$P\left(\bar{Z}_n - t\sqrt{\frac{\bar{Z}_n(1-\bar{Z}_n)}{n}} \le p \le \bar{Z}_n + t\sqrt{\frac{\bar{Z}_n(1-\bar{Z}_n)}{n}}\right) \approx 1-\alpha.$$

$\alpha = 10\%$ $\quad t = 1,645$. $[0,0577 ; 0,09499]$

$\alpha = 5\%$ $\quad t = 1,96$. $[0,0542 ; 0,0985]$

$\alpha = 1\%$ $\quad t = 2,575$. $[0,0472 ; 0,1055]$ $\quad P(Z \le t) = 1 - \frac{0,01}{2} = 0,995$.

Ex 2.

X resultat d'un dosage en mg/litre

$X \sim \mathcal{N}(\mu, \sigma_0^2)$ $\sigma_0^2 = 1$

$X_1 \dots X_n$ iid $n = 5$.

1) $1-\alpha = 95\%$.

St $\bar{X}_n$. $\bar{X}_n \sim \mathcal{N}(\mu, \frac{1}{n})$

$$\sqrt{n}(\bar{X}_n - \mu) \sim \mathcal{N}(0,1).$$

$$P(-1{,}96 \leq \sqrt{n}(\bar{X}_n - \mu) \leq 1{,}96) = 0{,}95.$$

$$\Leftrightarrow P(\bar{X}_n - 1{,}96/\sqrt{n} \leq \mu \leq \bar{X}_n + 1{,}96/\sqrt{n}) = 0{,}95.$$

$$IC = \left[\bar{X}_n - \frac{1{,}96}{\sqrt{n}} ; \bar{X}_n + \frac{1{,}96}{\sqrt{n}}\right].$$

I.C. observé $n = 5$. $\bar{x} = 73{,}1$ $\Rightarrow [72{,}22346 ; 73{,}97654]$

2) $$l = \frac{2 \times 1{,}96}{\sqrt{n}}$$

On cherche n tq $l \leq 0{,}1$.

ie n tq $\frac{2 \times 1{,}96}{\sqrt{n}} \leq 0{,}1$

$$n \geq (2 \times 1{,}96/0{,}1)^2 = 1536{,}64 \quad \boxed{1537}$$

Ex 3

X tps de réaction en sec d'un conducteur dans un état normal.

$E(X) = \mu$

$V(X) = \sigma^2$

$X_1 \dots X_n$ iid $n = 307$. On observe $\bar{x} = 1{,}05$ s.

1) $X \sim \mathcal{N}(\mu ; 0{,}2)$

(a) $\sqrt{n}\,\dfrac{\bar{X}_n - \mu}{\sqrt{0{,}2}} \sim \mathcal{N}(0,1)$

Au niveau 5%,

$$\left[\bar{X}_n - 1{,}96\sqrt{\frac{0{,}2}{n}} \; ; \; \bar{X}_n + 1{,}96\sqrt{\frac{0{,}2}{n}}\right]$$

I.C. observé : $[0{,}99998 ; 1{,}1000 3]$ $[1 ; 1{,}1]$

(b) D distance de réaction

$D = X \times \dfrac{130}{3600}$

$E(D) = \mu \times \dfrac{130}{3600}$

$D = X \times \dfrac{130}{3600} \times 1000$

$= X \dfrac{13000\cancel{0}}{360\cancel{0}} = X \dfrac{325}{9}$

$P(I_i \leq \mu \leq I_s) = 0{,}95$

Et $P(I_i \dfrac{130}{3600} \leq E(D) \leq I_s \dfrac{130}{3600}) = 0{,}95$.

En km :

$[0{,}03611 ; 0{,}03972]$

En mètres :

$[36{,}11 ; 39{,}72]$

2) (a) $\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$

consistant

(b) $$\sqrt{n}\, \frac{\bar{X}_n - \mu}{\hat{\sigma}_n} \sim \mathcal{N}(0,1) \text{ approx si } n \text{ gd}.$$

$$P\left(\bar{X}_n - 1{,}96\, \frac{\hat{\sigma}_n}{\sqrt{n}} \leq \mu \leq \bar{X}_n + 1{,}96\, \frac{\hat{\sigma}_n}{\sqrt{n}}\right) \simeq 0{,}95.$$

I.C. observé au niveau de confiance asympt 95%

$[0{,}99635 \; ; \; 1{,}10365]$

(c) niveau asymptotique

Non : IC pour la distance de réaction

$[35{,}98 \; ; \; 39{,}85]$

Ex 4.

X_i nb de clients téléphonant au central un jour i

$X_1 \ldots X_n$ iid $X_i \sim \mathcal{P}(\lambda)$. $n = 100$

1) $$\sqrt{n}\, \frac{\bar{X}_n - \lambda}{\sqrt{\bar{X}_n}} \sim \mathcal{N}(0,1) \text{ approx si } n \text{ gd}$$

$$P\left(\bar{X}_n - t_\alpha \sqrt{\frac{\bar{X}_n}{n}} \leq \lambda \leq \bar{X}_n + t_\alpha \sqrt{\frac{\bar{X}_n}{n}}\right) \simeq 1 - \alpha$$

si t_α quantile d'ordre $1 - \frac{\alpha}{2}$ de la $\mathcal{N}(0,1)$

2)

$\alpha = 10\%$ $\quad t_\alpha = 1,645$

$\alpha = 5\%$ $\quad t_\alpha = 1,96$

$\alpha = 1\%$ $\quad t_\alpha = 2,575$

$n = 100$ $\quad \bar{x} = 2,89$

$\alpha = 10\%$ $\quad [2,61035 \ ; \ 3,16965]$

$\alpha = 5\%$ $\quad [2,5568 \ ; \ 3,2232]$

$\alpha = 1\%$ $\quad [2,45225 \ ; \ 3,32775]$

3) augmenter n.

Ex 5 $\quad X_i$ resultat du dosage i

$X_i \sim \mathcal{N}(m, \sigma^2)$

$X_1 \ldots X_n$ iid $\quad n = 10$

1) $\hat{m}_{obs} = \bar{X}_{n\,obs} = 1,053$

2) $t = 2,262$

$$\bar{X}_n - 2,262 \frac{S}{\sqrt{n}} \ ; \ \bar{X}_n + 2,262 \frac{S}{\sqrt{n}}$$

$\Sigma x_i^2 = 11,1791$

$$s = \sqrt{1,12 - 1,053^2} = 0,095$$