

Problem Set 8: Complex Numbers

Goal: Become familiar with math operations using complex numbers; see how complex numbers can be used to show the frequency response of an RC circuit.

Note: This PSet will be much easier if you have already watched the lectures on complex numbers.



Deliverables: This worksheet and two plots.

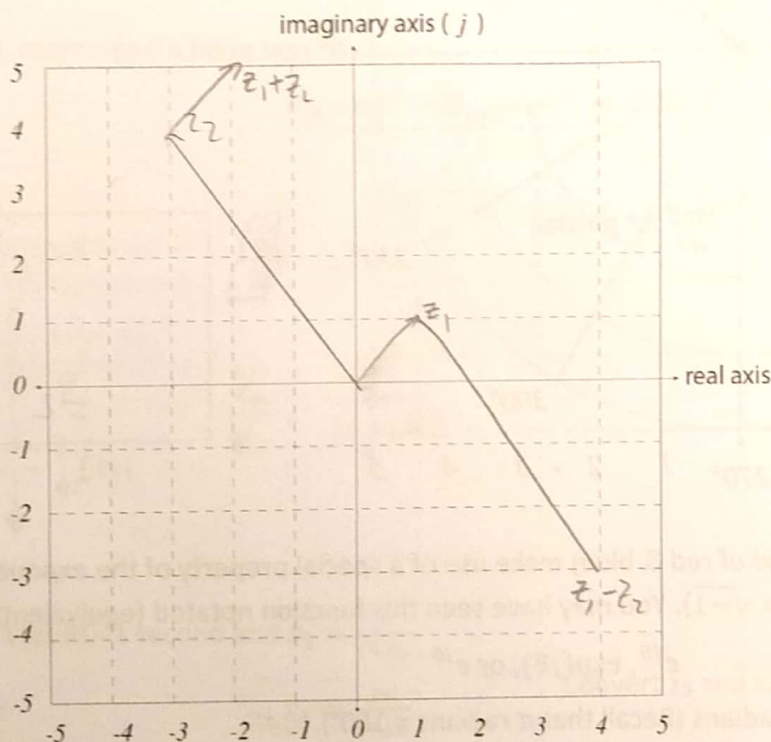
Part I: Basic Operations with complex numbers

For the following, take $z_1 = 1 + j$ and $z_2 = -3 + 4j$.

1. Convert z_1 and z_2 to polar and exponential notation (find r, θ).

$$z_1 = \sqrt{2} \text{cis } \pi/4 = \sqrt{2} e^{i\pi/4} \quad z_2 = 5 \text{cis } (\tan^{-1}(-4/3)) = 5 e^{-i \tan^{-1}(4/3)}$$

2. Plot z_1 and z_2 on the complex plane below. Use this plane for the next questions.



3. Compute $z_1 + z_2$. Show $z_1 + z_2$ graphically on a plot in the complex plane.

$$-2 + 5j$$

4. Compute $z_1 - z_2$. Show $z_1 - z_2$ graphically on a plot in the complex plane.

$$4 - 3j$$

5. Compute $z_1 z_2$. Repeat the computation using a different notation.

$$5\sqrt{2} e^{i(\pi/4 - \tan^{-1}(4/3))}$$

6. Compute z_1/z_2 using complex notation. Compute z_2/z_1 and compare.

$$\frac{\sqrt{2}}{5} e^{i(\pi/4 + \tan^{-1}(4/3))}$$

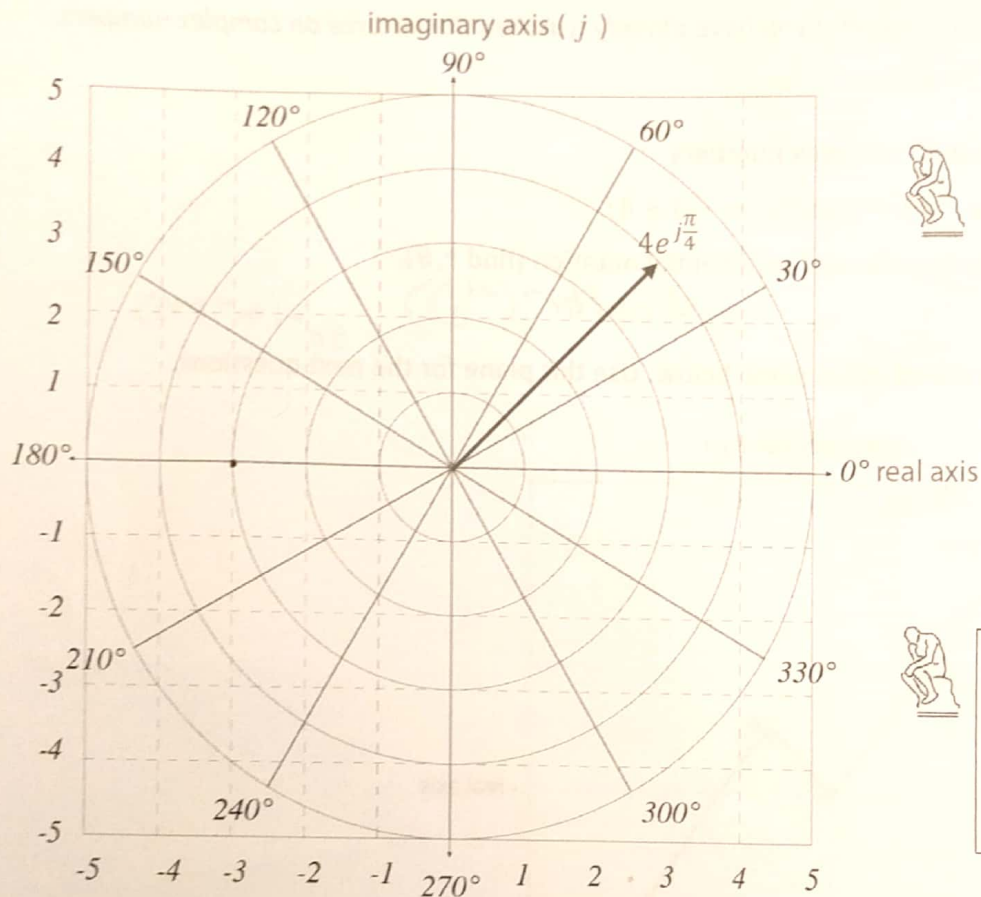
$$\frac{(-1 + 7j)}{25}$$

7. Compute z_1^4 .

$$4e^{i\pi} = -4$$

Part II: Plotting complex numbers

Complex numbers using **polar notation** are super useful for illustrating how a circuit responds to time-varying signals. The polar axes can be superimposed on the complex plane as shown:



What does $3 \cdot e^{j\pi}$ look like?



A Wavegen input signal is a sine wave, $V_{\text{peak-peak}} = 4\text{Volts}$. In r, θ notation, what is its r ? $\Rightarrow 2$

The **polar coordinates** (above grid of red & blue) make use of a special property of the **exponential function** when it operates on $j (= \sqrt{-1})$. You may have seen this function notated (equivalently) as:

$$e^{j\theta}, \exp(j\theta), \text{ or } e^{i\theta}$$

where θ represents an angle in radians (Recall that π radians = 180°).

The amazing property of $e^{j\theta}$ is known as Euler's formula (section 6.3 in your book):

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$



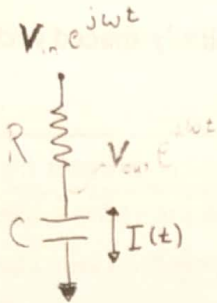
If θ varies with a frequency, ω , $\theta = \omega \cdot t$, what would $e^{j\omega t}$ look like in time?

Click this [link](#) to see. There is more info on page 6 for those who are interested.

Recall from Figure 6.3 that if we represent our cosine voltage input to a **low-pass filter** with polar notation,

$$V_{in}(t) = V_{in} \cdot e^{j\omega t}$$

And V_{in} represents a complex number.



And remember that because the R and C are in series, the time varying current passing through both will be the same, we get,

$$\frac{V_{in}(t) - V_{out}(t)}{R} = C \frac{dV_{out}(t)}{dt}$$

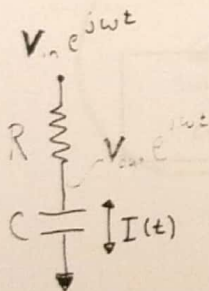
and, rearranged a bit,

$$V_{in} \cdot e^{j\omega t} - V_{out} \cdot e^{j\omega t} = RCj\omega V_{out} \cdot e^{j\omega t}$$

Or

$$\text{solving for } \frac{V_{out}}{V_{in}},$$

V_{out} , while a complex number, does not vary with time, so $\frac{dV_{out}}{dt}$ treats V_{out} as a constant.



$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}$$


Let's let $RC=1$ second and $z_3 = \frac{1}{1+j\omega}$

And

$$z_4 = \frac{j\omega}{1+j\omega}$$

Convert z_3 and z_4 to r, θ notation.



$$\begin{aligned} r &= \sqrt{1+\omega^2} \\ z_3 &= \sqrt{1+\omega^2} e^{j \arctan(\omega)} \\ z_4 &= \frac{1}{\sqrt{1+\omega^2}} e^{j \arctan(\omega)} \\ z_4 &= \frac{j\omega}{\sqrt{1+\omega^2}} e^{-j \arctan(\omega)} \end{aligned}$$


 Plot the magnitude of r for z_3 and z_4 as a function of ω on a log-log scale. Let ω^* vary from 10^{-3} to 10^3 .

*In Matlab, you can use the command, $y = \text{logspace}(-3,3)$, to generate a logarithmically-spaced vector, y , that spans 10^{-3} to 10^3



Knowing that z_3 and z_4 represent the $\frac{V_{out}}{V_{in}}$ of low- and high-pass filters, what do you expect the graphs to look like?

Low Pass  High Pass 

 Plot θ in degrees for z_3 and z_4 as a function of ω on a semilog* scale. Let ω vary from 10^{-3} to 10^3 .

*In MATLAB, use `semilogx(x, y)` to plot linear values for y and $\log x$.




This plot is the phase angle part of the Bode plot. Which value should be plotted on a log scale, ω or θ ?



You expect θ at the natural frequency ("cutoff frequency") to be 45° , where $\cos(\theta) = \sin(\theta)$.

Is your plot what you expect?

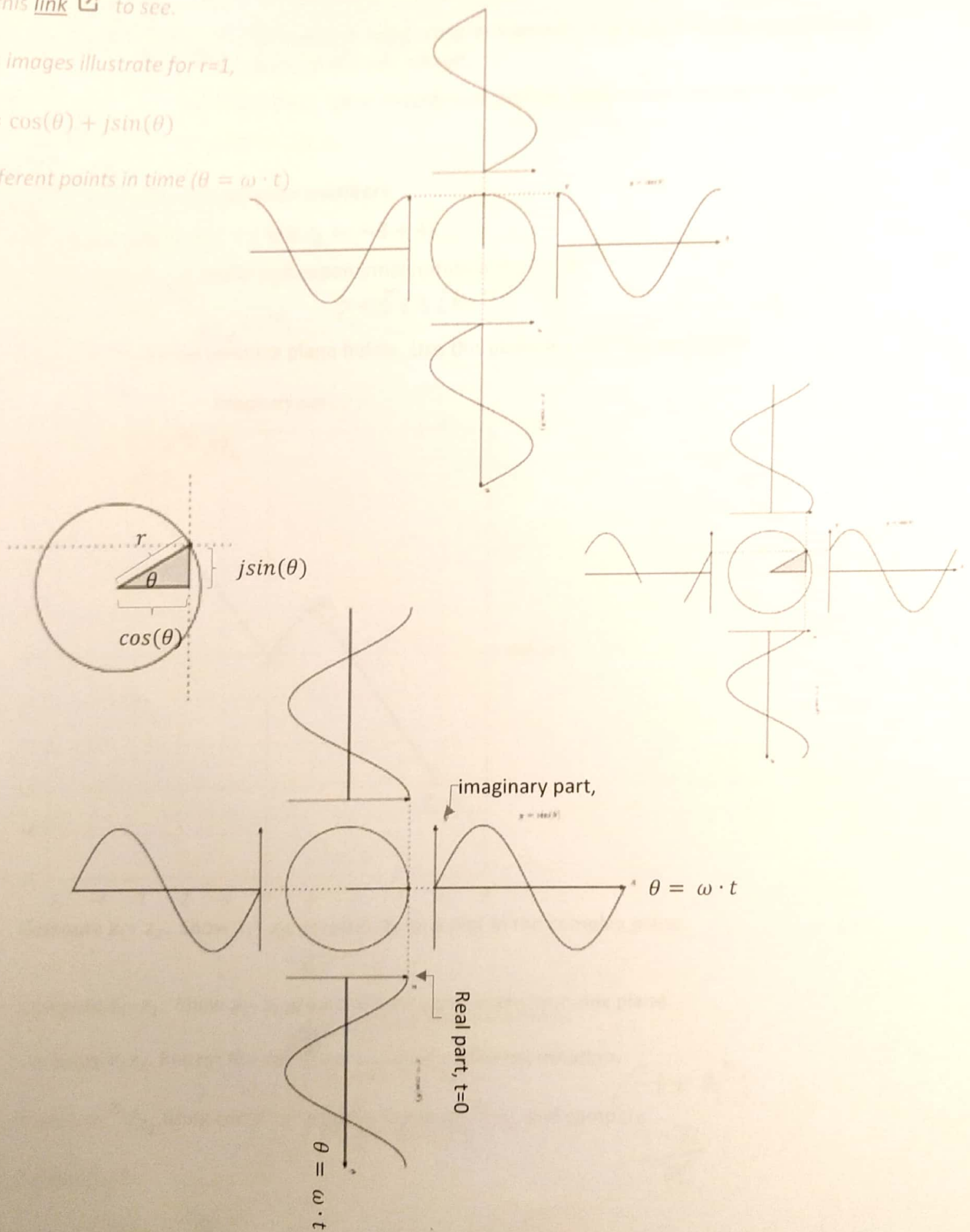
Watch $e^{j\omega t}$ vary as θ varies with a frequency, ω : $\theta = \omega \cdot t$

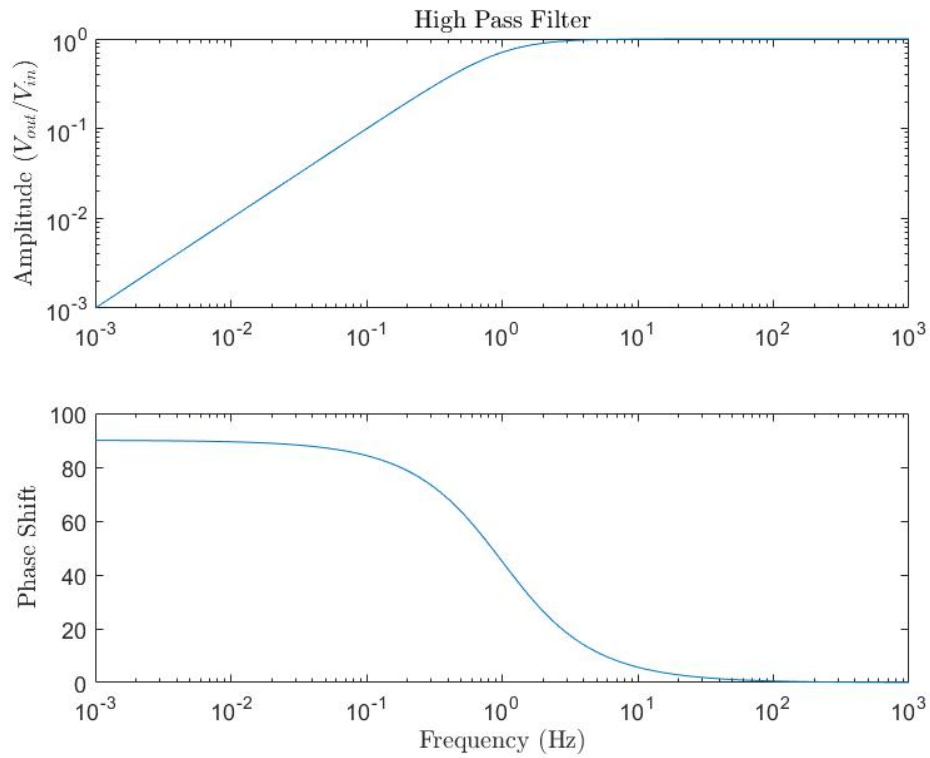
Click this [link](#)  to see.

These images illustrate for $r=1$,

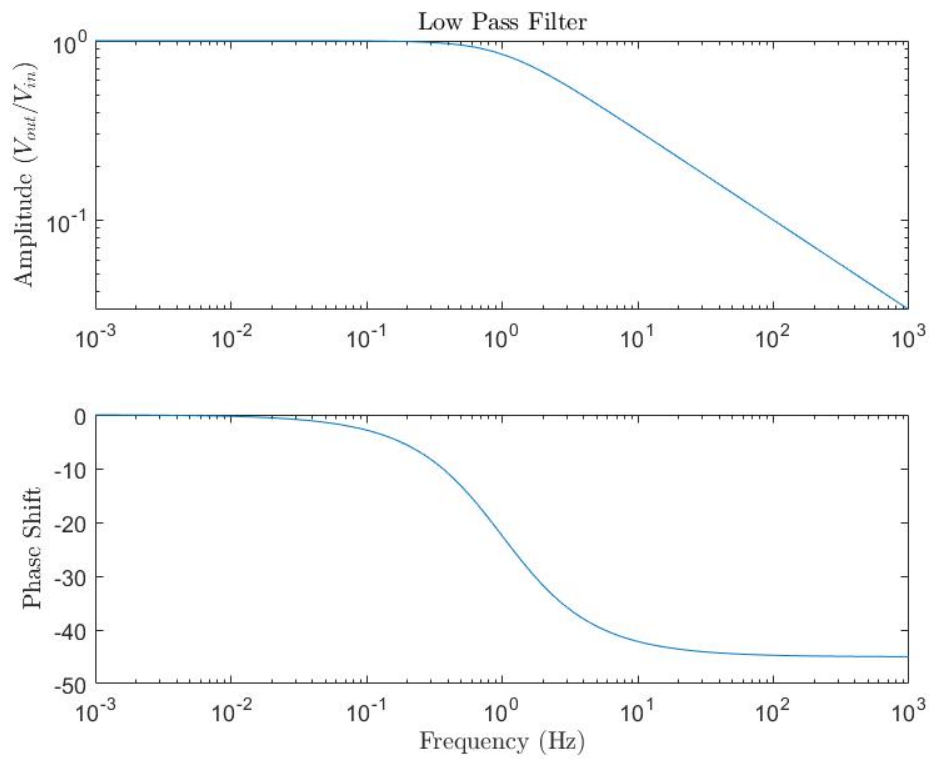
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

at different points in time ($\theta = \omega \cdot t$)





The above plot shows Magnitude vs. Frequency and Angle vs. Frequency for z_3 .



The above plot shows Magnitude vs. Frequency and Angle vs. Frequency for z_4 .