Problem Set 8: Complex Numbers

Goal: Become familiar with math operations using complex numbers; see how complex numbers can be used to show the frequency response of an RC circuit.

Note: This PSet will be much easier if you have already watched the lectures on complex numbers.

Deliverables: This worksheet and two plots.

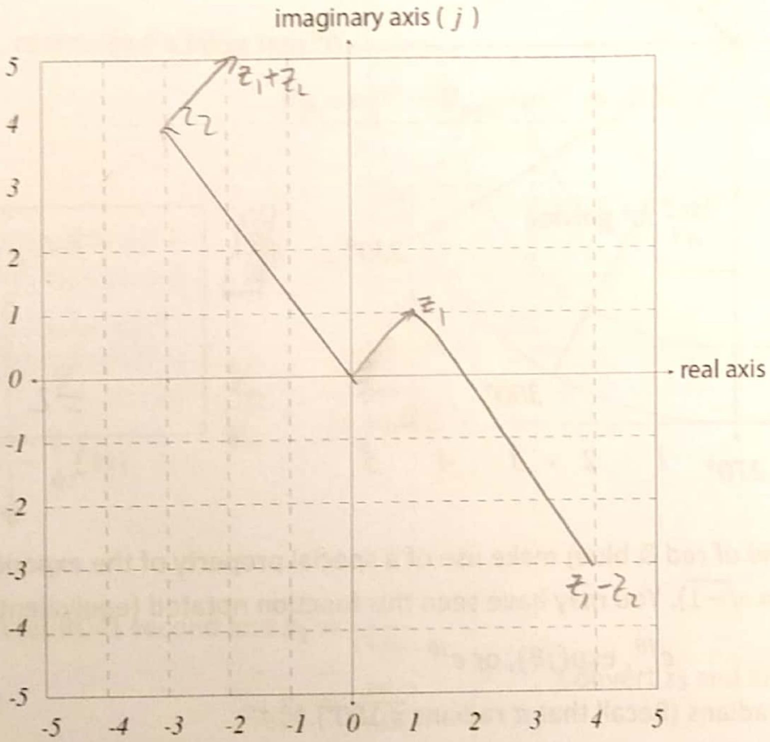
Part I: Basic Operations with complex numbers

For the following, take $z_1 = 1 + j$ and $z_2 = -3 + 4j$.

1. Convert z_1 and z_2 to polar and exponential notation (find r, θ).

   $z_1 = \sqrt{2}\,\text{cis}\,\pi/4$ $\quad \sqrt{2}e^{+i\pi/4}$ $\quad z_2 = 5\,\text{cis}\left(\tan^{-1}\left(-4/3\right)\right)$ $\quad 5e^{-i\tan^{-1}(-4/3)}$

2. Plot z_1 and z_2 on the complex plane below. Use this plane for the next questions.



3. Compute $z_1 + z_2$. Show $z_1 + z_2$. graphically on a plot in the complex plane.

   $-2+5j$

4. Compute $z_1 - z_2$. Show $z_1 - z_2$ graphically on a plot in the complex plane.

   $4 + -3j$ $\quad \cancel{i(\pi/4 - \tan^{-1}(-4/3))}$

5. Compute $z_1 z_2$. Repeat the computation using a different notation.

   $5\sqrt{2}e^{i(\pi/4 + \tan^{-1}(-4/3))}$ $\quad (-1+7j)$

6. Compute z_1/z_2 using complex notation. Compute z_2/z_1 and compare.

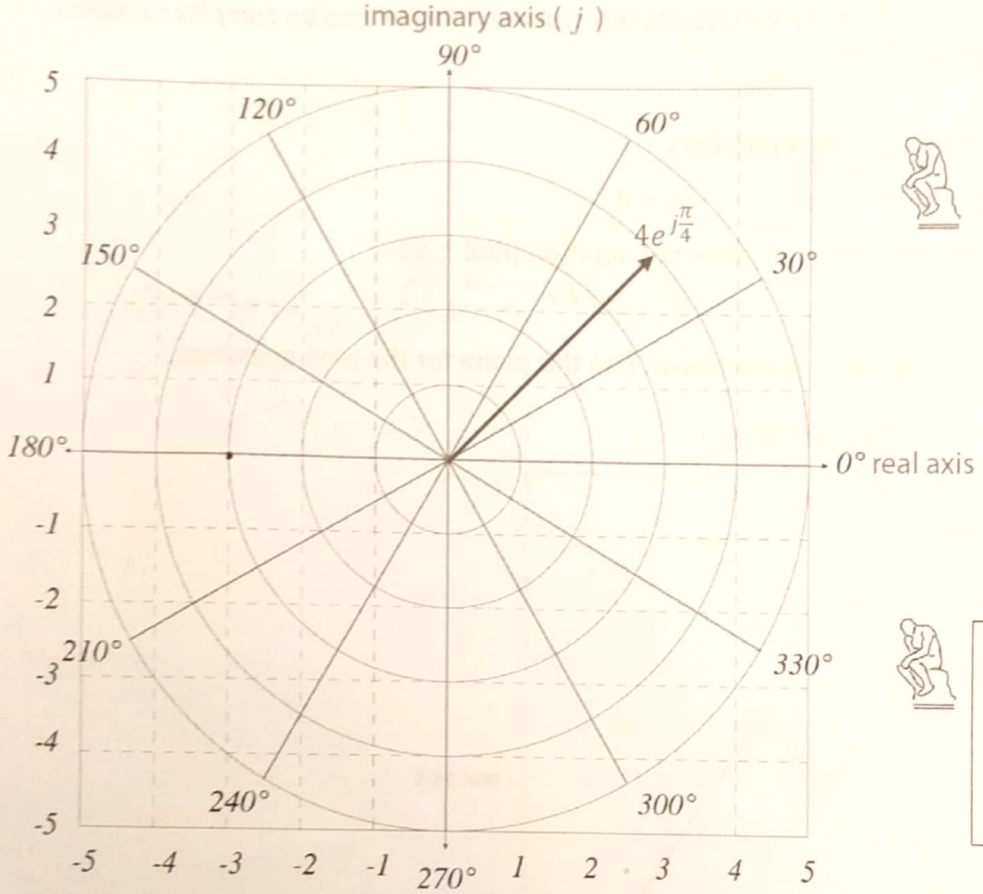
   $\frac{\sqrt{2}}{5}e^{i(\pi/4 + \tan^{-1}(-4/3))}$ $\quad \frac{1-7i}{25}$

7. Compute z_1^4.

   $4e^{i\pi}$ $\quad -4$

Part II: Plotting complex numbers

Complex numbers using **polar notation** are super useful for illustrating how a circuit responds to time-varying signals. The polar axes can be superimposed on the complex plane as shown:



What does $3 \cdot e^{j\pi}$ look like?

A Wavegen input signal is a sine wave, $V_{peak-peak} = 4Volts$. In r, θ notation, what is its r? $=2$

The **polar coordinates** (above grid of red & blue) make use of a special property of the ***exponential function*** when it operates on $j(=\sqrt{-1})$. You may have seen this function notated (equivalently) as:

$$e^{j\theta}, \exp(j\theta), \text{ or } e^{i\theta}$$

where θ represents an angle in radians (Recall that π radians = 180°).

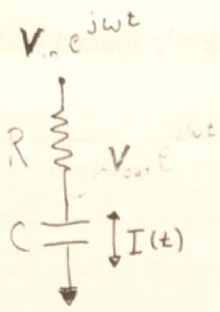
The amazing property of $e^{j\theta}$ is known as Euler's formula (section 6.3 in your book):

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

If θ varies with a frequency, ω, $\theta = \omega \cdot t$, what would $e^{j\omega t}$ look like in time?

Click this link to see. There is more info on page 6 for those who are interested.

Recall from Figure 6.3 that if we represent our cosine voltage input to a **low-pass filter** with polar notation,



$$V_{in}(t) = \boldsymbol{V_{in}} \cdot e^{j\omega t}$$

And $\boldsymbol{V_{in}}$ represents a complex number.

And remember that because the R and C are in series, the time varying current passing through both will be the same, we get,

$$\frac{V_{in}(t) - V_{out}(t)}{R} = C\frac{dV_{out}(t)}{dt}$$

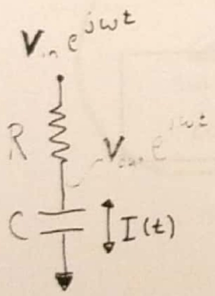
and, rearranged a bit,

$$\boldsymbol{V_{in}} \cdot e^{j\omega t} - \boldsymbol{V_{out}} \cdot e^{j\omega t} = RCj\omega \boldsymbol{V_{out}} \cdot e^{j\omega t}$$

> $\boldsymbol{V_{out}}$, while a complex number, does not vary with time, so $\frac{dV_{out}}{dt}$ treats $\boldsymbol{V_{out}}$ as a constant.

Or

solving for $\frac{\boldsymbol{V_{out}}}{\boldsymbol{V_{in}}}$,



$$\frac{V_{out}}{V_{in}} = \frac{1}{1+j\omega RC}$$

Let's let RC=1 second and $z_3 = \frac{1}{1+j\omega}$

And $z_4 = \frac{j\omega}{1+j\omega}$

Convert z_3 and z_4 to r, θ notation.

$$r = \sqrt{1+w^2}$$

$$z_3 = \sqrt{1+w^2}\, e^{j \arctan(w)}$$

$$z_3 = \frac{1}{\sqrt{1+w^2}} e^{-j \arctan(w)}$$

$$z_4 = \frac{jw}{\sqrt{1+w^2}} e^{-j \arctan(w)}$$

Plot the magnitude of r for z_3 and z_4 as a function of ω on a log-log scale. Let ω* vary from 10^{-3} to 10^{3}.

*In Matlab, you can use the command, y= logspace(-3,3), to generate a logarithmically-spaced vector, y, that spans 10^{-3} to 10^{3}

Knowing that z_3 and z_4 represent the $\frac{V_{out}}{V_{in}}$ of low- and high-pass filters, what do you expect the graphs to look like?

Low Pass  High Pass 

Plot θ in degrees for z_3 and z_4 as a function of ω on a semilog* scale . Let ω vary from 10^{-3} to 10^{3}.

*In MATLAB, use semilogx (x, y) to plot linear values for y and log x.

This plot is the phase angle part of the Bode plot. Which value should be plotted on a log scale, ω or θ?

You expect θ at the natural frequency ("cutoff frequency") to be 45°, where $\cos(\theta) = \sin(\theta)$.

Is your plot what you expect?

Watch $e^{j\omega t}$ vary as θ varies with a frequency, ω: $\theta = \omega \cdot t$

Click this link to see.

These images illustrate for r=1,

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

at different points in time ($\theta = \omega \cdot t$)