MATH 3740 Final Exam Matt Pattok

1. $xy' + 3y = 2x^5 \quad y(2) = 1$

$y' + \frac{3}{x}y = 2x^4$

$I(x) = e^{\int 3/x\,dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$x^3y' + 3x^2y = 2x^7$

$(x^3y)' = 2x^7$

$x^3y = \frac{1}{4}x^8 + C$

$y = \frac{1}{4}x^5 + \frac{C}{x^3}$

$1 = \frac{32}{4} + \frac{C}{8}$

$1 = 8 + \frac{C}{8}$

$8 = 64 + C$

$-56 = C$

$y = \frac{1}{4}x^5 - \frac{56}{x^3}$

MATH 3740 Final Exam Matt Pattok

2. $A=\begin{bmatrix}1 & 2 & -3\\ 0 & -2 & 2\\ 0 & 0 & 2\end{bmatrix}$ $B=\begin{bmatrix}-1 & 0 & 0\\ 3 & 2 & 0\\ -7 & -5 & 3\end{bmatrix}$

$\det(A)=\begin{vmatrix}-2 & 2\\ 0 & 2\end{vmatrix}=-2(2)-2(0)=-4$

$C=B^T$ $\det(C)=\det(B)$ $\det(C)=-1\begin{vmatrix}2 & 0\\ -5 & 3\end{vmatrix}$

$=-(2(3)-0(-5))=-6$

$\det(A^TB^{-1}A)=\det(A^T)\det(B^{-1})\det(A)$

$\det(A^T)=\det(A)=-4$

$\det(B^{-1})=1/\det(B)=-1/6$

$\det(A^TB^{-1}A)=(-4)(-1/6)(-4)=-8/3$

MATH 3740 Final Exam Matt Pattok

3. $x_1 + 2x_2 + 7x_3 - 9x_4 + 31x_5 = 0$
$2x_1 + 4x_2 + 7x_3 - 11x_4 + 13x_5 = 0$
$3x_1 + 6x_2 + 5x_3 - 11x_4 - 19x_5 = 0$

$$\begin{bmatrix} 1 & 2 & 7 & -9 & 31 \\ 2 & 4 & 7 & -11 & 13 \\ 3 & 6 & 5 & -11 & -19 \end{bmatrix} \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 2 & 7 & -9 & 31 \\ 0 & 0 & -7 & 7 & -49 \\ 0 & 0 & -16 & 16 & -112 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{7}R_2 \to R_2} \begin{bmatrix} 1 & 2 & 7 & -9 & 31 \\ 0 & 0 & 1 & -1 & 7 \\ 0 & 0 & -16 & 16 & -112 \end{bmatrix} \xrightarrow[R_3 + 16R_2 \to R_3]{R_1 - 7R_2 \to R_1} \begin{bmatrix} 1 & 2 & 0 & -2 & -18 \\ 0 & 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 - 2x_4 - 18x_5 = 0$
$x_1 = -2x_2 + 2x_4 + 18x_5$
$x_1 = -2r + 2s + 18t$

$x_3 - x_4 + 7x_5 = 0$
$x_3 = x_4 - 7x_5$
$x_3 = s - 7t$

$(-2r + 2s + 18t, r, s - 7t, s, t)$

$= r(-2, 1, 0, 0, 0) + s(2, 0, 1, 1, 0) + t(18, 0, -7, 0, 1)$

A basis for the solution space is

$\{(-2, 1, 0, 0, 0), (2, 0, 1, 1, 0), (18, 0, -7, 0, 1)\}$.

The dimension of the solution space is 3

MATH 3740 Final Exam Matt Pattok

4. $y^{(4)} - 3y''' + 3y'' - 3y' + 2y = 0$ $y(0)=4$ $y'(0)=-1$ $y''(0)=-3$ $y'''(0)=-3$

$r^4 - 3r^3 + 3r^2 - 3r + 2 = 0$

$(r^2+1)(r-1)(r-2)=0$

$r^2+1=0$ $r=1$ $r=2$

$r^2=-1$

$r=\pm i$

$r_1=1,\ r_2=2,\ r_3=-i,\ r_4=i$

$y = C_1e^x + C_2e^{2x} + e^x(C_3\cos x + C_4\sin x)$

$y' = C_1e^x + 2C_2e^{2x} + e^x(C_3\cos x + C_4\sin x - C_3\sin x + C_4\cos x)$

$y'' = C_1e^x + 4C_2e^{2x} + e^x(2C_4\cos x - 2C_3\sin x)$

$y''' = C_1e^x + 8C_2e^{2x} + e^x(2C_4\cos x - 2C_4\sin x - 2C_3\cos x - 2C_3\sin x$

$4 = C_1 + C_2 + C_3$ $-1 = C_1 + 2C_2 + C_3 + C_4$

$-3 = C_1 + 4C_2 + 2C_4$ $-3 = C_1 + 8C_2 - 2C_3 + 2C_4$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 4 \\ 1 & 2 & 1 & 1 & -1 \\ 1 & 4 & 0 & 2 & -3 \\ 1 & 8 & -2 & 2 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & 13 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$C_1 = 13$ $C_2 = -3$ $C_3 = -6$ $C_4 = -2$

$$y = 13e^x - 3e^{2x} + e^x(-6\cos x - 2\sin x)$$

MATH 3740 Final Exam Matt Pattok

5. $A=\begin{bmatrix}-3&1&-2\\-10&4&-6\\-2&1&-1\end{bmatrix}$ $B=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

$$\det(A-I\lambda)=\begin{vmatrix}-3-\lambda&1&-2\\-10&4-\lambda&-6\\-2&1&-1-\lambda\end{vmatrix}=-\lambda^3+\lambda=0$$

$$-\lambda(\lambda^2-1)=0$$

$$-\lambda(\lambda+1)(\lambda-1)=0$$

$\lambda_1=+1$ $\lambda_2=-1$ $\lambda_3=0$

$\lambda=-1$

$A+I=\begin{bmatrix}-2&1&-2\\-10&5&-6\\-2&1&0\end{bmatrix}\xrightarrow{RREF}\begin{bmatrix}1&-\frac{1}{2}&0\\0&0&1\\0&0&0\end{bmatrix}$ $x=+/2$ $z=0$

$(+/2,+,0)$

$(+/2)(1,2,0)$

$\vec{v}_1=(1,2,0)$

$\lambda=1$

$A-I=\begin{bmatrix}-4&1&-2\\-10&3&-6\\-2&1&-2\end{bmatrix}\xrightarrow{RREF}\begin{bmatrix}1&0&0\\0&1&-2\\0&0&0\end{bmatrix}$ $x=0$, $y=2t$

$(0,2t,t)=t(0,2,1)$ $\vec{v}_2=(0,2,1)$

$\lambda=0$

$A=\begin{bmatrix}-3&1&-2\\-10&4&-6\\-2&1&-1\end{bmatrix}\xrightarrow{RREF}\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}$ $x=-t$ $y=-t$

$(-t,-t,t)=t(-1,-1,1)$

$\vec{v}_3=(-1,-1,1)$

$$X^{-1}=\begin{bmatrix}1&0&-1\\2&2&-1\\0&1&1\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\2&2&-1&0&1&0\\0&1&1&0&0&1\end{array}\right]\xrightarrow{R_2-2R_1\to R_2}\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&2&1&-2&1&0\\0&1&1&0&0&1\end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&1&0&0&1\\0&2&1&-2&1&0\end{array}\right]\xrightarrow{R_3-2R_2\to R_3}\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&1&0&0&1\\0&0&-1&-2&1&-2\end{array}\right]$$

$$\xrightarrow{-R_3\to R_3}\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&1&0&0&1\\0&0&1&2&-1&2\end{array}\right]\begin{matrix}R_1+R_3\to R_1\\R_2-R_3\to R_2\end{matrix}\left[\begin{array}{ccc|ccc}1&0&0&3&-1&2\\0&1&0&-2&1&-1\\0&0&1&2&-1&2\end{array}\right]$$

$$X=\begin{bmatrix}3&-1&2\\-2&1&-1\\2&-1&2\end{bmatrix}$$

MATH 3740 Final Exam Matt Patton

6. $\vec{x}' = \begin{bmatrix} -2 & -2 \\ 2 & -6 \end{bmatrix}\vec{x}$ $\quad \vec{x}(0) = (1,0)$

$A = \begin{bmatrix} -2 & -2 \\ 2 & -6 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -2 \\ 2 & -6-\lambda \end{vmatrix} = 0$

$\lambda^2 + 8\lambda + 16 = 0$

$(\lambda+4)^2 = 0$

$\lambda_1 = \lambda_2 = -4$

$\lambda = -4$

$A + 4I = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x = t$

$(t,t) = t(1,1)$

$\vec{v}_1 = (1,1)$

$\vec{v}_2 = (1,0)$

$\vec{v}_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

$\vec{x}(t) = c_1 e^{-4t}(1,1) + c_2 e^{-4t}((1,1)t + (1,0))$

$(1,0) = c_1(1,1) + c_2(1,0)$

$(1,0) = c_2(1,0)$ $\quad c_1 = 0$ $\quad c_2 = 1$

$\vec{x}(t) = e^{-4t}((1,1)t + (1,0))$

MATH 3740 Final Exam Matt Patton

7. $A=\begin{bmatrix}-3 & 7 & 2\\ -2 & 6 & 2\\ -3 & 3 & 2\end{bmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix}-3-\lambda & 7 & 2\\ -2 & 6-\lambda & 2\\ -3 & 3 & 2-\lambda\end{vmatrix}=0$$

$$-\lambda^3+5\lambda^2-2\lambda-8=0$$

$$-(\lambda-4)(\lambda-2)(\lambda+1)=0$$

$$\lambda_1=-1,\ \lambda_2=2,\ \lambda_3=4$$

$\lambda=-1$

$$A+I=\begin{bmatrix}-2 & 7 & 2\\ -2 & 7 & 2\\ -3 & 3 & 3\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$x=t \quad y=0$

$(t,0,t)=t(1,0,1)$

$\vec{v}_1=(1,0,1)$

$\lambda=2$

$$A-2I=\begin{bmatrix}-5 & 7 & 2\\ -2 & 4 & 2\\ -3 & 3 & 0\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$x=-t \quad y=-t$

$(-t,-t,t)=-t(1,1,-1)$

$\vec{v}_2=(1,1,-1)$

$\lambda=4$

$$A-4I=\begin{bmatrix}-7 & 7 & 2\\ -2 & 2 & 2\\ -3 & 3 & -2\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$x=t \quad z=0$

$(t,t,0)=t(1,1,0)$

$\vec{v}_3=(1,1,0)$

a)

$$\Phi(t)=\begin{bmatrix}e^{-t} & 0 & e^{-t}\\ e^{2t} & e^{2t} & -e^{2t}\\ e^{4t} & e^{4t} & 0\end{bmatrix}^T$$

$$\Phi(0)^{-1}=\begin{bmatrix}1/2 & 1/2 & 0\\ -1/2 & -1/2 & 1\\ 1/2 & -1/2 & 0\end{bmatrix}$$

b)

$$e^{At}=\Phi(t)\Phi(0)^{-1}=\begin{bmatrix}e^{-t} & 0 & e^{-t}\\ e^{2t} & e^{2t} & -e^{2t}\\ e^{4t} & e^{4t} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ -\frac{1}{2} & -\frac{1}{2} & 1\\ -\frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ \frac{1}{2}e^{2t} & \frac{1}{2}e^{2t} & e^{2t}\\ 0 & 0 & e^{4t}\end{bmatrix}$$

MATH 3740 Final Exam Matt Patton

8. $y'' + y = \cos t \quad y(0) = -1 \quad y'(0) = 0$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\cos t\}$

$s^2Y - sy(0) - y'(0) + Y = \frac{s}{s^2+1}$

$s^2Y + s + Y = \frac{s}{s^2+1}$

$Y(s^2+1) + s = \frac{s}{s^2+1}$

$Y(s^2+1) = \frac{s}{s^2+1} - s$

$Y = \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s^2+1}\right) - \frac{s}{s^2+1}$

$Y = \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s^2+1}\right) - \frac{s}{s^2+1}$

$\mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$

$y = \cos t * \sin t - \cos t$

$y = \int_0^t \cos\tau \sin(t-\tau)\, d\tau - \cos t$

$y = \frac{t}{2}\sin t - \cos t$

$\frac{s}{(s^2+1)^2} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+1}$

$s = (s^2+1)(As+B+Cs+D)$

$A = A+C \qquad B = B+D$

$s = (s^2+1)(As+B)$

$s = As^3 + Bs^2 + As + B$

0

MATH 3740 Final Exam Matt Patton

9. $(2x+1)y' + 3y = 0$

$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$(2x+1)\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^n = 0$

$\sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} 3 c_n x^n = 0$

$\sum_{n=0}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 3 c_n x^n = 0$

For $n \geq 0$, $2n c_n + (n+1)c_{n+1} + 3c_n = 0$

$c_{n+1} = \frac{-2n c_n - 3c_n}{n+1}$

$c_{n+1} = -\frac{(2n+3)c_n}{n+1}$

$c_1 = -\frac{3c_0}{1} = -3c_0$

$c_2 = -\frac{5c_1}{2} = \frac{15}{2}c_0$

$c_3 = -\frac{7c_2}{3} = -\frac{35}{2}c_0$

$c_4 = -\frac{9c_3}{4} = \frac{315}{8}c_0$

$y = c_0\left(1 + \frac{15}{2}x - \frac{35}{2}x^3 + \frac{315}{8}x^4 + \cdots\right)$

MATH 3740 Final Exam Matt Pattok

10. $y' = x + 3y \qquad y(1) = 3$

$y_1 = 3 + \int_1^x (t + 9)\,dt = 3 + \left[\frac{t^2}{2} + 9t\right]_1^x$

$= 3 + \frac{x^2}{2} + 9x - \frac{1}{2} - 9 = \frac{1}{2}x^2 + 9x - \frac{13}{2}$ ✓

$y_2 = 3 + \int_1^x \left(t + 3\left(\frac{1}{2}t^2 + 9t - \frac{13}{2}\right)\right)dt$

$= 3 + \int_1^x \left(\frac{3}{2}t^2 + 28t - \frac{39}{2}\right)dt$

$= 3 + \left[\frac{1}{2}t^3 + 14t^2 - \frac{39}{2}t\right]_1^x$

$= 3 + \frac{1}{2}x^3 + 14x^2 - \frac{39}{2}x - \frac{1}{2} - 14 + \frac{39}{2}$

$= \frac{1}{2}x^3 + 14x^2 - \frac{39}{2}x + 8$ ✓

$y_3 = 3 + \int_1^x \left(t + 3\left(\frac{1}{2}t^3 + 14t^2 - \frac{39}{2}t + 8\right)\right)dt$

$= 3 + \int_1^x \left(\frac{3}{2}t^3 + 42t^2 - \frac{115}{2}t + 24\right)dt$

$= 3 + \left[\frac{3}{8}t^4 + 14t^3 - \frac{115}{4}t^2 + 24t\right]_1^x$

$= 3 + \frac{3}{8}x^4 + 14x^3 - \frac{115}{4}x^2 + 24x - \frac{3}{8} - 14 + \frac{115}{4} - 24$

$= \frac{3}{8}x^4 + 14x^3 - \frac{115}{4}x^2 + 24x - \frac{13}{2}$

$y_4 = 3 + \int_1^x \left(t + 3\left(\frac{3}{8}t^4 + 14t^3 - \frac{115}{4}t^2 - 9t - \frac{13}{2}\right)\right)dt$

$= 3 + \int_1^x \left(\frac{9}{8}t^4 + 42t^3 - \frac{345}{4}t^2 - 26t - \frac{39}{2}\right)dt$

$= 3 + \left[\frac{9}{40}t^5 + \frac{21}{2}t^4 - \frac{115}{4}t^3 - 13t^2 - \frac{39}{2}t\right]_1^x$

$= 3 + \frac{9}{40}t^5 + \frac{21}{2}t^4 - \frac{115}{4}t^3 - 13t^2 - \frac{39}{2}t - \frac{9}{40} - \frac{21}{2} + \frac{115}{4} + 13 + \frac{39}{2}$

$= \frac{9}{40}t^5 + \frac{21}{2}t^4 - \frac{115}{4}t^3 - 13t^2 + \frac{201}{8}t + \frac{2140}{40}$