About This Document

This is the examples document for MTH 299. Basically it is a loosely organized "universe" of questions (examples) that we think are interesting, helpful, useful for practice, and serve as a preview of what may appear on homework and exams. Here is how we expect you to interact with this document:

- You will have a paper copy of this document with you during all class meetings.
- Each day, your instructor will choose a list of examples from this document that will be the focus of that day's practice activities. This list is dynamic and is designed to address the mixture of needs presented by: new topics, your understanding of old topics, your past homework performance, your upcoming homework assignment. You should always record each day's suggested list of examples. Those lists are suggestive that your instructor thinks those examples are particularly interesting.
- You will work examples in class, with your group-mates.
- You will go home and examine the solutions for these examples (and any other ones about which you are curious). The solutions are posted on your class D2L page.
- You will look for examples that are similar to homework questions in order to practice techniques that will help you do your homework.
- You can generally use the solutions to the examples as a guide for how you will write your solutions. I say "generally" because not all of the solutions in the document have been brought up to the our current presentation standards. When in doubt, ask your instructor. (Note, solutions contain many sentences and phrases intended as extra explanation to the students who are reading them and learning from them. You will obviously modify your homework solutions so that this extra explanation is not there.)

Week 1 **(things related to: sets, set operations, statements, conditional statements)**

Example 1.1. Let $S = \{1, \{2, 3\}, 4\}$. Indicate whether each statement is true or false.

(a) $|S| = 4$

(b) $\{1\} \in S$

(c) $\{2, 3\} \in S$

(d) $\{1, 4\} \subseteq S$

(e) $2 \in S$.

(f) $S = \{1, 4, \{2, 3\}\}$

(g) $\emptyset \subseteq S$

Example 1.2. Write each of the following sets by listing its elements within braces. Find their cardinalities.

(a) $A = \{n \in \mathbb{Z} : -4 < n \leq 4\}$

(b) $B = \{n \in \mathbb{Z} : n^2 < 5\}$

(c) $C = \{n \in \mathbb{N} : n^3 < 100\}$

(d) $D = \{x \in \mathbb{R} : x^2 - x = 0\}$

(e) $E = \{x \in \mathbb{R} : x^2 + 1 = 0\}$

Example 1.3. Find the cardinalities of the following sets.

(a) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$

(b) $\{n \in \mathbb{Z} \mid -2014 \leq n \leq 2014\}$

(c) $\{n \in \mathbb{N} \mid n \text{ has exactly 3 digits}\}$
For example, 7 has one digit and 107 has three digits.

Example 1.4. Compute the cardinality of the set, E, where E is defined as

$$E = \{x \in \mathbb{R} \;:\; \sin(x) = 1/2 \text{ and } |x| \leq 5\}$$

Example 1.5. List all the proper subsets for each of the following sets:

(a) $\{0, 1, 2\}$

(b) $\{0, \{1, 2\}\}$

(c) $\{1, \{2, \{3\}\}\}$

Example 1.6. Suppose $A = \{0, 2, 4, 6, 8\}, B = \{1, 3, 5, 7\}$ and $C = \{2, 8, 4\}$. Find:
(a) $A \cup B$ (b) $A \setminus C$ (c) $B \setminus A$ (d) $B \cap C$ (e) $C \setminus B$

Example 1.7. Let $U = \{1, 2, 3, ..., 14, 15\}$ and $E^c = U \setminus E$. Let $A = \{1, 5, 9, 13\}$ and $B = \{3, 9, 13\}$. Determine the following:

(a) $A \cup B$

(b) $A \cap B$

(c) $A \setminus B$

(d) $B \setminus A$

(e) A^c

(f) $A \cap B^c$.

Example 1.8. Let $U = \{1, 2, 3, 4, 5\}$ and $E^c = U \setminus E$. Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 3\}$. Determine the following:

(a) $(A \cup B) \setminus (B \cap C)$

(b) A^c

(c) $(B \cup C)^c$

(d) $A \times B$.

Example 1.9. Describe each set below in the form

$$\{x \in X : \ x \text{ satisfies property } P\}$$

(a) $D = \{3, 5, 7, 9, \dots\}$

(b) $E = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$

(c) $F = \{2, -4, 8, -16, \dots\}$

Example 1.10. Find the union of of $\{2k \,|\, k \in \mathbb{N}\}$ and $\{2k + 1 \,|\, k \in \mathbb{N}\}$. Prove your answer.
Recall if $E = A \cup B$, you need to show $E \subseteq A \cup B$ and $A \cup B \subseteq E$.

Example 1.11. Define $E = \{z \in \mathbb{Z} \mid z = 2k \text{ for some } k \in \mathbb{N}\}$. Show that $E \subset \mathbb{N}$.

Example 1.12. Prove that $X = Y$, where $X = \{x \in \mathbb{N} : x^2 < 30\}$ and $Y = \{1, 2, 3, 4, 5\}$.

Example 1.13. Prove that $\{x \in \mathbb{Z} : x \text{ is divisible by } 18\} \subset \{x \in \mathbb{Z} : x \text{ is divisible by } 6\}$.

Example 1.14. Prove that $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$, but $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$.

Example 1.15. Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Example 1.16. Define the sets

$$A = \{n \in \mathbb{Z} \ : \ n = 2k, \text{ for some } k \in \mathbb{Z}\}$$

and

$$B = \{m \in \mathbb{Z} \ : \ m = 2^i, \text{ for some } i \in \mathbb{N}\}.$$

Give a careful explanation of why each of the following two statements are true: (i) $B \subseteq A$ and (ii) $A \not\subseteq B$. (Note that this means that B is a *proper* subset of A, see p. 7.)

Example 1.17. Find the intersection of $A = \{2k \,|\, k \in \mathbb{N}\}$ and $B = \{3k \,|\, k \in \mathbb{N}\}$.

Example 1.18. Suppose $A = \{\pi, e, 0\}$ and $B = \{0, 1\}$. Write out the indicated sets by listing their elements.
(a) $A \times B$ (b) $B \times A$ (c) $A \times A$ (d) $B \times B$ (e) $A \times \emptyset$ (f) $A \times B \times B$

Example 1.19. For $A = \{-1, 0, 1\}$ and $B = \{x, y\}$, determine $A \times B$.

Example 1.20. Give a brief justification for the theorem:

Theorem: If $|A| = n$ and $|B| = m$ then $|A \times B| = nm$

Example 1.21. Let $A = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and
$B = \{(x, y) \mid (x-1)^2 + y^2 \leq 1\}$ inside of $\mathbb{R}^2$

(a) Sketch the regions A and B. (Both are closed discs: circles together with the points they enclose.)

(b) Sketch the region $(A \cup B)^c$

(c) Sketch the region $A^c \cap B^c$

(d) Formulate a conjecture which explains your observations in the previous two steps.

Example 1.22. For $A = \{a \in \mathbb{R} : |a| \leq 1\}$ and $B = \{b \in \mathbb{R} : |b| = 1\}$, give a geometric description of the points in the xy-plane belonging to $(A \times B) \cup (B \times A)$.

Example 1.23. Which of the following sentences are statements? For those that are, indicate the truth value.

(i) The integer 123 is prime.

(ii) The integer 0 is even.

(iii) Is $5 \times 2 = 10$??

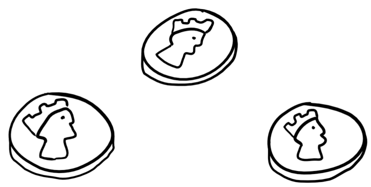
(iv) $x^2 - 4 = 0$.

(v) Multiply $5x + 2$ by 3.

(vi) $5x + 3$ is an odd integer.

Example 1.24. Which of the following statements are true? Give an explanation for each false statement.
(a) $\emptyset \in \emptyset$ (b) $\emptyset \in \{\emptyset\}$ (c) $\{1, 3\} = \{3, 1\}$ (d) $\emptyset = \{\emptyset\}$ (e) $\emptyset \subset \{\emptyset\}$ (f) $1 \subseteq \{1\}$.

Example 1.25. Imagine that there are exactly three coins on a table, as below.



Which of these statements are true?

(a) There are four coins on the table.

(b) There are two coins on the table.

(c) There are three coins on the table.

(d) There is a coin on the table.

Example 1.26. Decide whether or not the following are statements. In the case of a statement, say if it is true or false.

(i) Every real number is an even integer.

(ii) If x and y are real numbers and $5x = 5y$, then $x = y$.

(iii) The integer x is a multiple of 7.

(iv) The sets $\mathbb{Z}$ and $\mathbb{N}$ are infinite.

(v) The derivative of any polynomial of degree 5 is a polynomial of degree 6.

Example 1.27. Consider the sets A, B, C and D below. Which of the following statements are true? Give an explanation for each false statement.

$$A = \{1, 4, 7, 10, 13, 16, \dots\} \qquad C = \{x \in \mathbb{Z} : x \text{ is prime and } x \neq 2\}$$

$$B = \{x \in \mathbb{Z} : x \text{ is odd}\} \qquad D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$$

(a) $25 \in A$ (b) $33 \in D$ (c) $22 \notin A \cup D$ (d) $C \subseteq B$ (e) $\emptyset \in B \cap D$ (f) $53 \notin C$.

Example 1.28. Decide if the following statements are true or false. Explain.

(a) $\mathbb{R}^2 \subseteq \mathbb{R}^3$

(b) $\{(x, y) \in \mathbb{R}^2 : x - 1 = 0\} \subseteq \{(x, y) \in \mathbb{R}^2 : x^2 - x = 0\}$

Example 1.29. For a conditional statement $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}(\{1, 2, 4\})$ determine:

(a) all $A \in S$ for which $P(A)$ is true.

(b) all $A \in S$ for which $P(A)$ is false.

(c) all $A \in S$ for which $A \cap \{1, 2, 3\} = \emptyset$.

Example 1.30. Write a conditional statement $P(n)$ over the domain $S = \{3, 5, 7, 9\}$ such that $P(n)$ is true for half of the elements of S and false for the other half.

Example 1.31. Let the domain of a conditional statement be $\mathbb{R}$ and the conditional statement to be $P(x) : x(x - 1) = 6$

(a) For what values of x is $P(x)$ a true statement?

(b) For what values of x is $P(x)$ a false statement?

Example 1.32. Let the domain of a conditional statement be $\mathbb{Z}$ and the conditional statement to be $P(x) : 3x - 2 > 4$.

(a) For what values of x is $P(x)$ a true statement?

(b) For what values of x is $P(x)$ a false statement?

Example 1.33. For $A = \{x \in \mathbb{R} : |x-1| \le 2\}$ and $B = \{y \in \mathbb{R} : |y-4| \le 2\}$, give a geometric description of the points in the xy-plane belonging to $A \times B$.

Example 1.34. Define the sets

$$E := \{-4, -3, -2, -1, \mathbb{N}\}$$

and

$$F := \mathbb{N} \cup \{0\} \cup \{\mathbb{Z}\}.$$

(i) Solve the equation $2x^3 + 4x^2 = 0$ for $x \in E$.

(ii) Solve the equation $2x^3 + 4x^2 = 0$ for $x \in F$.

(iii) Now, express F in a more elementary form, such as (just for example!)

$$\{A, \{0, 1\}, a, b, c, \dots\},$$

and then go back and do (ii) over again.

Example 1.35. List the elements of the set $S = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \,:\, |x| + |y| = 3\}$. Plot the corresponding points in the Euclidean xy-plane.

Example 1.36. Sketch the set $X = \{(x, y) \in \mathbb{R}^2 \,:\, 1 \leq x^2 + y^2 \leq 4\}$ on the plane $\mathbb{R}^2$. On a separate drawing, shade in the set X^c.

Example 1.37. List the elements of each set below and then sketch a Venn diagram to illustrate how they are related. For this problem, $\mathcal{U} = \{1, 2, 3, \ldots, 10\}$.

$$A = \{n \in \mathcal{U} \mid n \text{ is even }\}, \quad B = \{n \in \mathcal{U} \mid n \text{ is the square of an integer}\}, \quad C = \{n \in \mathcal{U} \mid n^2 - 3n + 2 = 0\}$$

Shade each of the following sets:

(i) $\mathcal{U} \setminus (A \cup B)$

(ii) $(\mathcal{U} \setminus A) \cap (\mathcal{U} \setminus B)$

(iii) $\mathcal{U} \setminus (A \cap B)$

(iv) $(\mathcal{U} \setminus A) \cup (\mathcal{U} \setminus B)$.

Week 2 (things related to: operations on statements, truth tables, quantifiers)

Example 2.1. Express each statement as one of the forms $P \wedge Q, P \vee Q$, or $\neg P$. Be sure to also state exactly what statements P and Q stand for.

(i) The number 8 is both even and a power of 2.

(ii) The number x equals zero, but the number y does not.

(iii) $x \in A \setminus B$.

(iv) $x \neq y$.

(v) $y \geq x$.

(vi) $A \in \{X \in \mathcal{P}(\mathbb{N}) : |X^c| < \infty\}$.

Example 2.2. Complete the following truth table

P	Q	$\neg P$	$\neg Q$
T	T		
T	F		
F	T		
F	F		

Example 2.3. Let $P:$ 15 is odd. and $Q:$ 21 is prime. State each of the following in words, and determine whether they are true or false.
(a) $P \vee Q$ (b) $P \wedge Q$ (c) $(\neg P) \vee Q$ (d) $P \wedge (\neg Q)$

Example 2.4. Let $S = \{1, 2, \dots, 6\}$ and let

$$P(A): \ A \cap \{2, 4, 6\} = \emptyset. \quad \text{and} \quad Q(A): \ A \neq \emptyset.$$

be conditional statements over the domain $\mathcal{P}(S)$.

(a) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \wedge Q(A)$ is true.

(b) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \vee (\neg Q(A))$ is true.

(c) Determine all $A \in \mathcal{P}(S)$ for which $(\neg P(A)) \wedge (\neg Q(A))$ is true.

Example 2.5. For the sets $A = \{1, 2, \dots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, consider the statements

$$P : A \subseteq B. \qquad Q : |A \setminus B| = 6.$$

Determine which of the following statements are true.
(a) $P \vee Q$ (b) $P \vee (\neg Q)$ (c) $P \wedge Q$ (d) $(\neg P) \wedge Q$ (e) $(\neg P) \vee (\neg Q)$.

Example 2.6. Complete the following truth table

P	Q	$\neg Q$	$P \wedge (\neg Q)$
T	T		
T	F		
F	T		
F	F		

Example 2.7. Write a truth table for the following:

(i) $(P \wedge \neg P) \vee Q$,

(ii) $\neg(\neg P \wedge \neg Q)$.

Example 2.8. Decide whether the following pairs of statements are logically equivalent.

(i) $P \wedge Q$ and $\neg(\neg P \vee \neg Q)$

(ii) $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge R$

Example 2.9. Use truth tables to show that the following statements are logically equivalent.

(i) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(ii) $\neg(P \vee Q \vee R) \equiv (\neg P) \wedge (\neg Q) \wedge (\neg R)$

(iii) $\neg(P \wedge Q \wedge R) \equiv (\neg P) \vee (\neg Q) \vee (\neg R)$

(iv) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

Example 2.10. Let the sets A and B be given as $A = \{3, 5, 8\}$ and $B = \{3, 6, 10\}$. Show that the following quantified statement is indeed true:

$$\exists b \in B, \forall a \in A, a - b < 0.$$

Example 2.11. Write the following as English sentences. Say whether they are true or false.

(a) $\forall x \in \mathbb{R}, x^2 > 0$

(b) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, x^n \geq 0$

(c) $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}, ax = x$

Example 2.12. Rewrite each statement using $\forall, \exists$, as appropriate and determine if the statement is true or false.

(a) There exists a positive number x such that $x^2 = 5$.

(b) For every positive number M, there is a positive number N such that $N < 1/M$.

(c) No positive number x satisfies the equation $f(x) = 5$.

Example 2.13. Rewrite the following using $\forall$ and $\exists$. For each statement, determine if it is true or false.

(a) There exists a real number r such that $r = \cos(r)$.

(b) For all $n \in \mathbb{N}$, either n or $n+1$ is even.

(c) For all $x \in \mathbb{R}$, x^2 is nonnegative.

(d) There exists a natural number n satisfying $2n^2 = n^3$.

(e) If x is a real number with $-1 < x < 0$, then $x < x^3$.

(f) If a and ϵ are real numbers with $\epsilon > 0$, then there exists a natural number n such that $|a^n| < \epsilon$.

(g) If f is continuous and $f(0) < 0 < f(1)$, then there exists a real number $c \in (0,1)$ such that $f(c) = 0$. (Look up the intermediate value theorem to try to figure out this one!)

Example 2.14. Decide whether the following are true or false. Provide a proof to justify your answer.

(a) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 1$.

(b) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 1$.

(c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = x$.

(d) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, xy = x$.

Example 2.15. Determine the truth value of each of the following statements, and explain/justify your answer.

(a) $\exists x \in \mathbb{R}, x^2 - x = 0$.

(b) $\forall x \in \mathbb{R}, \sqrt{x^2} = x$.

Example 2.16. Decide whether the following are true or false. Explain your answers.

(a) $\forall p \in \mathbb{P}_2,\ \exists x \in \mathbb{R},\ p(x) < 0$.

(b) $\forall x \in \mathbb{R},\ \exists p \in \mathbb{P}_2,\ p(x) < 0$.

(c) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ (x^2 = y)$.

(d) $\forall y \in \mathbb{Z},\ \exists x \in \mathbb{Z},\ (x^2 = y)$.

Example 2.17. Decide whether the following are true or false. Explain your answers.

(i) $\exists n \in \mathbb{N}$ such that $n^2 \leq n$.

(j) $\exists n \in \mathbb{Z}$ such that $n^2 < n$.

Example 2.18. Determine if the following statement is true or false, and justify your answer.

$$\forall q \in \mathbb{Q},\ \exists x \in \mathbb{R},\ e^{qx} \in \mathbb{Q}.$$

Example 2.19. Determine if the following statement is true or false, and justify your answer.

$$\forall q \in \mathbb{Q},\ \exists x \in \mathbb{R} \setminus \{0\},\ e^{qx} \in \mathbb{Q}.$$

Example 2.20. Determine if the following statement is true or false, and justify your answer.

$$\forall (a,b) \in \left\{(x,y) \in \mathbb{R}^2 \,:\, x < y\right\},\ \exists q \in \mathbb{P}_3,\ q(a) = q(b) = 0.$$

Example 2.21. Determine if the following statement is true or false, and justify your answer.

$$\exists (a,b) \in \left\{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\right\},\ \forall \theta \in [0,\pi],\ (a,b) \neq (\cos\theta, \sin\theta).$$

Example 2.22. Determine if the following statement is true or false, and justify your answer.

$$\forall p \in \mathbb{P}_3,\ \exists q \in \mathbb{P}_5,\ q' = p.$$

Example 2.23. Determine if the following statement is true or false, and justify your answer.

$$\exists p \in \mathbb{P}_3,\ \left[(\exists x \in \mathbb{R},\ (p'(x) \neq 0)) \text{ and } \int_{-1}^{2} p(s)\ ds \neq 0\right].$$

(Note to reader: you could interpret this as: $\exists p \in \mathbb{P}_3,\ \left[p \neq \text{constant and } \int_{-1}^{2} p(s)\ ds \neq 0\right]$.)

Example 2.24. Determine if the following statement is true or false, and justify your answer.

$$\exists p \in \mathbb{P}_3,\ \left(\deg(p) = 3 \text{ and } \int_{-1}^{2} p(s)\ ds = 0\right).$$

Example 2.25. Assume that you know for any real numbers, $a, b, c \in \mathbb{R}$ that if $a \geq 0$ and $b \leq c$, then it holds that $ab \leq ac$. Carefully justify the statement: for any real number, x, it holds that $x^2 \geq 0$.

Example 2.26. Assume that you know that $x < y$. Carefully justify the statement that

$$x < \frac{x+y}{2} < y.$$

Example 2.27. Show that there exists a positive even integer m such that for every positive integer n,

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{2}.$$

Week 3, (things related to: implications, negation, contradiction)

Example 3.1. Consider the statements P : 17 is even. and Q : 19 is prime. Write each of the following statements in words and indicate whether it is true or false.
(a) $\neg P$ (b) $P \vee Q$ (c) $P \wedge Q$ (d) $P \Rightarrow Q$

Example 3.2. Without changing their meanings, convert each of the following sentences into a sentence having the form "If P, then Q."

(a) The quadratic equation $ax^2 + bx + c = 0$ has real roots provided that $b^2 - 4ac \geq 0$.

(b) A function is rational if it is a polynomial.

(c) An integer is divisible by 8 only if it is divisible by 4.

(d) Whenever a circle, C, has a circumference of 4π, it has an area of 4π.

(e) The integer n^3 is even only if n is even.

Example 3.3. Let P : 18 is odd, and Q : 25 is even. State $P \iff Q$ in words. Is $P \iff Q$ true or false?

Example 3.4. Write a truth table for the logical statements:

(i) $P \vee (Q \Rightarrow R)$

(ii) $(P \wedge \neg P) \Rightarrow Q$

(iii) $\neg(P \Rightarrow Q)$

Example 3.5. Use truth tables to show that the following statements are logically equivalent.

(i) $P \Rightarrow Q \equiv (\neg P) \vee Q$

(ii) $P \Rightarrow Q \equiv (P \wedge \neg Q) \Rightarrow (Q \wedge \neg Q)$

Example 3.6. Use truth tables to determine the following questions.

(i) Is the statement $(\neg P) \wedge (P \Rightarrow Q)$ logically equivalent to the statement $\neg(Q \Rightarrow P)$?

(ii) Is the statement $P \Rightarrow Q$ logically equivalent to the statement $(\neg P) \vee Q$?

Example 3.7. Let $A = \{x \in \mathbb{Z} \ : \ 6|x\}$, $B = \{x \in \mathbb{Z} \ : \ 2|x\}$, $C = \{x \in \mathbb{Z} \ : \ 3|x\}$. Prove the following statement.

$$x \in A \iff (\exists y \in B \text{ and } \exists z \in C \text{ such that } x = yz)$$

Example 3.8. Prove that if $x < -4$ and $y > 2$, then the distance from (x, y) to $(1, -2)$ is at least 6. Note that the distance between two points (x_1, y_1) and (x_2, y_2) is given by
$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Example 3.9. Prove that if an isosceles triangle has sides of length x, y, and z, where $x = y$ and $z = \sqrt{2xy}$, then it is a right triangle using The Law of Cosines.

Example 3.10. Prove that if $x^2 \leq 1$, then $x^2 - 7x > -10$.

Example 3.11. Show that $A \Rightarrow B$ is logically equivalent to not(A and not B)
This is the idea behind a proof by contradiction.

(a) Using truth tables

(b) Using other rules of logic we have developed in this course.

Example 3.12. Fill in the blank with **necessary**, **sufficient** or **necessary and sufficient**.

(a) The number $x > 1$ is ____________________ for $x^2 > 1$.

(b) The number $x \in \mathbb{N}$ is ____________________ for $x \geq 0$.

(c) The number $|x| > 1$ is ____________________ for $x^2 > 1$.

Example 3.13. Consider the implication "If $m^2 > 0$, then $m > 0$". Is it true or false?
Write down its converse and determine its truth value.

Example 3.14. Prove the statement:
If $n \in \mathbb{Z}$ is even, and $m \in \mathbb{Z}$ is odd, then $n + m$ is odd.

Example 3.15. Prove that:

(a) If $A \subseteq B$ then $A \cup B = B$.

(b) If $A \cup B = B$ then $A \subseteq B$.

Example 3.16. Let P, Q, and R be statements. Prove that $(P \implies Q) \implies R$ is equivalent to $(P \wedge (\neg Q)) \vee R$.

Example 3.17. Prove that if $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$, then $A = B$ and $B = C$.

Example 3.18. Prove that if $x, y \in \mathbb{R}$, then $\frac{3}{4}x^2 + \frac{1}{3}y^2 \geq xy$.

Example 3.19. If n is an integer, show that $n^2 + n^3$ is an even number.

Example 3.20. Consider the statements P : $\sqrt{2}$ is rational. and Q : $22/7$ is rational. Write each of the following statements in words and indicate whether it is true or false.
(a) $P \Rightarrow Q$ (b) $Q \Rightarrow P$ (c) $(\neg P) \Rightarrow (\neg Q)$ (d) $(\neg Q) \Rightarrow (\neg P)$.

Example 3.21. In each of the following, two conditional statements $P(x)$ and $Q(x)$ over a domain S are given. Determine the truth values of $P(x) \Rightarrow Q(x)$ for each $x \in S$.

(a) $P(x) : |x| = 4;\ Q(x) : x = 4;\ S = \{-4, -3, 1, 4, 5\}$.

(b) $P(x) : x^2 = 16$; $Q(x) : |x| = 4$; $S = \{-6, -4, 0, 3, 4, 8\}$.

(c) $P(x) : x > 3$; $Q(x) : 4x - 1 > 12$; $S = \{0, 2, 3, 4, 6\}$.

Example 3.22. Let $S = \{1, 2, 3, 4\}$. Consider the following conditional statements over the domain S:

$$P(n): \ \frac{n(n-1)}{2} \text{ is even.}$$
$$Q(n): \ 2^{n-2} - (-2)^{n-2} \text{ is even.}$$
$$R(n): 5^{n-1} + 2^n \text{ is prime.}$$

Determine the distinct elements a, b, c, d in S such that
(i) $P(a) \implies Q(a)$ is false; (ii) $Q(b) \implies P(b)$ is true;
(iii) $P(c) \iff R(c)$ is true; (iv) $Q(d) \iff R(d)$ is false;

Example 3.23. For statements P and Q, construct a truth table for $(P \Rightarrow Q) \Rightarrow (\neg P)$.

Example 3.24. Show that if $n \in \mathbb{N}$, then $1 + (-1)^n(2n - 1)$ is a multiple of 4. Next, show the converse, i.e. that if $k \in \mathbb{Z}$ is a multiple of 4, then $k = 1 + (-1)^m(2m - 1)$ for some $m \in \mathbb{N}$.

Example 3.25. Define the *Euclidean norm* of $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ by $||\mathbf{x}|| = \sqrt{x_1^2 + ... + x_n^2}$. Prove that $||\mathbf{x}|| = 0$ if and only if $(x_1, x_2, ..., x_n) = (0, 0, ..., 0)$.

Example 3.26. Decide whether the following pairs of statements are logically equivalent.

(i) $\neg(P \Rightarrow Q)$ and $P \wedge \neg Q$

(ii) $(P \Rightarrow Q) \vee R$ and $\neg((P \wedge \neg Q) \wedge \neg R)$

(iii) $(\neg P) \wedge (P \Rightarrow Q)$ and $\neg(Q \Rightarrow P)$

(iv) $P \wedge (Q \vee \neg Q)$ and $(\neg P) \Rightarrow (Q \wedge \neg Q)$

Example 3.27. Suppose the statement $((P \text{ and } Q) \text{ or } R) \implies (R \text{ or } S)$ is false. Find the truth values of P, Q, R and S. (This can be done without a truth table.)

Example 3.28. State the negation of each of the following statements.

(a) $\sqrt{2}$ is a rational number.

(b) 0 is not a negative number.

(c) 111 is a prime number.

Example 3.29. State the negation of each of the following statements.

(a) The real number r is at most $\sqrt{2}$.

(b) The absolute value of the real number a is less than 3.

(c) Two angles of the triangle are 45°.

(d) The area of the circle is at least 9π.

(e) Two sides of the triangle have the same length.

(f) The point P in the plane lies outside of the circle C.

Example 3.30. State the negations of the following quantified statements:

(a) For every rational number r, the number $1/r$ is rational.

(b) There exists a rational number r such that $r^2 = 2$.

Example 3.31. Negate the following implications:

(a) If you are good I will buy you ice cream.

(b) In order to earn a 4.0 in MTH 299, it is sufficient that one scores 95% on all exams and assignments.

Example 3.32. Negate the following statements and determine which is true, the original or the negation. Justify your answers.

(a) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 1$.

(b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = x$.

Example 3.33. Negate the following. State whether each *original* statement is true or false. If the original is a true statement, prove it. If the original is false, prove that the negated statement is true (this is one of the great uses of negation... of the original statement and its negation, *exactly one of them* is always true).

(a) $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z},\ m \cdot n = 1.$

(b) $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\quad x \cdot y = y.$

Example 3.34. Of the following two statements, one is true and one is false. Determine which one is which, prove why the true one is true, and why the false one is false. (Note, to prove a statement is false, it is equivalent to prove that its negation is true.)

(i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, such that $y^2 \geq 3x + 5$.

(ii) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}$, such that $y^2 \geq 3x + 5$.

Example 3.35. Of the following two statements, one is true and one is false. Determine which one is which, prove why the true one is true, and why the false one is false. (Note, to prove a statement is false, it is equivalent to prove that its negation is true.)

(i) $\forall\ c \in \mathbb{R},\ \exists p \in \{f \in \mathbb{P}_2\ :\ \deg(f) \geq 1\},\ cxp'(x) = p(x)$

(ii) $\forall\ c \in \mathbb{R},\ \exists p \in \mathbb{P}_2,\ xp'(x) + c = p(x)$

Example 3.36. Use the method of proof by contradiction to prove the following statement:

$$\text{If } A \text{ and } B \text{ are sets, then } A \cap (B \setminus A) = \emptyset.$$

Example 3.37. Let n be an integer. Show that n^2 is even if and only if n is even.

Example 3.38. Recall the definition:

> **Definition(s):** A real number x is **rational** if $x = \dfrac{a}{b}$ for some $a, b \in \mathbb{Z}$ where a and b have no common factors and $b \neq 0$. Also, x is **irrational** if it is not rational.

Use contradiction to prove that: The number $\sqrt{2}$ is irrational.

Example 3.39. Prove that $\sqrt{6}$ is an irrational number. Further, show that there are infinitely many positive integers n such that $\sqrt{n}$ is irrational.

Example 3.40. Prove the following:

(a) (Instructor) There are infinitely many prime numbers.

(b) There are no rational number solutions to the equation $x^3 + x + 1 = 0$.

Example 3.41. Prove that there is no largest even integer.

Example 3.42. Suppose a circle has center $(2, 4)$.

(a) Prove that $(-1, 5)$ and $(5, 1)$ are not both on the circle.

(b) Prove that if the radius is less than 5, the circle does not intersect the line $y = x - 6$.

(c) Prove that if $(0, 3)$ is not inside the circle, then $(3, 1)$ is not inside the circle.

Example 3.43. Let n be an integer. Show that n^2 is odd if and only if n is odd.

Example 3.44 (Prove using Contradiction)**.** Suppose $a \in \mathbb{Z}$. Prove that if a^2 is even, then a is even.

Example 3.45. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Example 3.46. Prove that If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Example 3.47. Use the method of proof by contradiction to prove the following:

$$\forall x \in [\pi/2, \pi], \quad \sin x - \cos x \geq 1.$$

(**Hint:** *Consider* $(\sin x - \cos x)^2$.)

Example 3.48. Prove: For every real number $x \in [0, \pi/2]$, we have $\sin x + \cos x \geq 1$.

Example 3.49. Suppose $a, b \in R$. If a is rational and ab is irrational, then b is irrational.

Example 3.50. Prove that if p is prime and $p > 2$, then p is odd.

Week 4 (things related to contrapositive, induction, and divisibility)

Example 4.1. Find the contrapositive of the following statements.

(a) If Jane has grandchildren, then she has children.

(b) If $x = 1$, then x is a solution to $x^2 - 3x + 2 = 0$.

(c) If x is a solution to $x^2 - 3x + 2 = 0$, then $x = 1$ or $x = 2$.

Example 4.2. Write down the contrapositive of the following:

(a) If n is an integer, then $2n$ is an even integer.

(b) You can work here only if you have a college degree.

(c) The car will not run whenever you are out of gas.

(d) Continuity is a necessary condition for differentiability.

Example 4.3. Consider the conditional statements

$$P(x) : x \text{ is prime}$$
$$Q(x) : x \text{ is odd}$$

over the domain $\mathbb{N}$.

(a) Determine the set S whose elements make the statement $P \Rightarrow Q$ **false**. Explain your answer.

(b) Write down in words the contrapositive of $P \Rightarrow Q$ and determine the set T for which the contrapositive is a **true** statement.

(c) Is there any relation between the sets S and T? Explain your answer.

Example 4.4. Write down the contrapositive of the implication "If $7m$ is an odd number, then m is an odd number". Then prove the original implication is true by showing that the contrapositive implication is true.

Example 4.5. Assume that $x \in \mathbb{Z}$. Prove that if $x^2 - 6x + 5$ is even, then x is odd.
(***Hint**: Use the fact that an implication and its contrapositive are logically equivalent. You may want to write the desired implication as $A \implies B$ so that you do the contrapositive correctly.*)

Example 4.6. Suppose $x, y \in \{z \in \mathbb{R} : z \geq 0\}$. Prove if $xy > 100$ then $x > 10$ or $y > 10$.

Example 4.7. Let x be a positive real number. Prove that if $x - \frac{2}{x} > 1$, then $x > 2$ by

(a) a direct proof,

(b) a proof by contrapositive and

(c) a proof by contradiction.

Example 4.8. Give a proof for the claim:
If x and y are odd integers, then xy is odd using:

(a) A direct proof.

(b) Proof by Contrapositive.

(c) Proof by Contradiction.

Example 4.9. Prove by induction that for all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$.

Example 4.10. Prove the following: If $n \in \mathbb{N}$, then

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Example 4.11. If n is a non-negative integer, use mathematical induction to show that $5 \mid (n^5 - n)$.

Example 4.12. Prove the following using mathematical induction:

$$\text{For every integer } n \in \mathbb{N}, \quad 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Example 4.13. Prove the following statement

$$\text{For all } n \in \mathbb{N}, \sum_{i=1}^{n} i \cdot i! = (n+1)! - 1.$$

Example 4.14. Prove that for each $n \in \mathbb{N}$, 9 divides $4^{3n} - 1$. If you do not know what "divides" means, you can look it up in the text– it is in the index.

Example 4.15. Assume that $f : \mathbb{R} \to \mathbb{R}$ is defined via the assignment rule $f(x) = e^{2x}$. We will use the notation that $f^{(n)}(x)$ is the n-th derivative of f. Prove, using induction that for all $n \in \mathbb{N}$, $f^{(n)}(x) = 2^n e^{2x}$. You are allowed to assume the standard differentiation rules for $g(x) = e^x$ and $h(x) = cx$ (for a constant, c), as well as the chain rule.

Example 4.16. Prove that $5 \mid (3^{3n+1} + 2^{n+1})$ for every positive integer n.

Example 4.17. Use mathematical induction to prove the following:

$$\text{For any integer } n \geq 0, \text{ it follows that } 3 \mid (n^3 + 5n + 6).$$

Week 5 (things about functions, injective, surjective)

Example 5.1. There are four different functions $f : \{a, b\} \to \{0, 1\}$. List them all.

Example 5.2. For each of the following, determine the largest set $A \subseteq \mathbb{R}$, such that $f : A \to \mathbb{R}$ defines a function. Next, determine the range, $f(A) := \{y \in \mathbb{R} : f(x) = y, \text{ for some } x \in A\}$.

(i) $f(x) = 1 + x^2$,

(ii) $f(x) = 1 - \frac{1}{x}$,

(iii) $f(x) = \sqrt{3x - 1}$,

(iv) $f(x) = x^3 - 8$,

(v) $f(x) = \frac{x}{x-3}$.

Example 5.3. Suppose $A = \{a, b, c, d, e, f, g\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and the *assignment rule* for f is given by

$$f(a) = 2, \quad f(b) = 3, \quad f(c) = 4, \quad f(d) = 5.$$

State the domain and a choice of co-domain of f so that the assignment rule, f, given, defines a function. As a followup, determine the range of f (range is defined in the supplementary document). (Note to student, there are infinitely many correct choices for the co-domain so that the resulting triple gives a function, but there is exactly one correct choice for the range of this function.)

Example 5.4. Suppose $A = \{-101, -10, -5, 0, 1, 2, 3, 4, 5, 6, 10\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and the *assignment rule* for f is defined by the table:

x	$f(x)$
-10	3
-5	3
2	4
3	2
4	2.

First, evaluate $f(2)$ and $f(-5)$.

Then, state the domain and a choice of co-domain of f so that the assignment rule, f, given, defines a function. As a followup, determine the range of f (range is defined in the supplementary document). (Note to student, there are infinitely many correct choices for the co-domain so that the resulting triple gives a function, but there is exactly one correct choice for the range of this function.)

Example 5.5. Find the largest possible domain, $X \subseteq \mathbb{R}$, so that the formula,

$$f(x) = \frac{3x + 1}{5x - 2}$$

defines a function $f : X \to \mathbb{R}$. Justify your answer.

Example 5.6. Does the formula,

$$f(x) = \frac{3x+1}{5x-2}$$

define a function $f : \mathbb{Z} \to \mathbb{Q}$? Justify your answer.

Example 5.7. *Attempt* to define an assignment for a real number, $x \in \mathbb{R}$, as

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Does this assignment make sense for $x \in \mathbb{R}$? Justify your answer. Does this assignment define a function $f : \mathbb{R} \to \mathbb{R}$? Justify your answer.

Example 5.8. Let $f : \mathbb{R} \to \mathbb{Z}$ be defined via the assignment

$$f(x) = \lfloor x \rfloor := \max\{z \in \mathbb{Z} \ : \ z \leq x\}.$$

(Note, we call f the "integer floor" function.)

(i) evaluate $f(0.1)$, $f(4.7)$, $f(\pi)$, $f(-e)$.

(ii) does this assignment define a function? Justify your answer.

Example 5.9. Let $A = \{x \in \mathbb{R} \mid x > 0\}$. The function $f : A \to B$ (with $B \subseteq \mathbb{R}$) is defined by the assignment rule $f(x) = x^2 - 4x + 5$. What is the largest codomain– i.e. choice of the set, B– so that f is surjective? Carefully justify your answer.

Example 5.10. Define the function, $f : \mathbb{P}_3 \to \mathbb{R}$ via the operation

$$f(p) := \int_0^1 p(x)dx.$$

Is f injective and or surjective from $\mathbb{P}_3$ to $\mathbb{R}$? Justify your answer.

Example 5.11. The following statements give certain properties of functions. You are to do two things:

(a) rewrite the defining conditions using quantifiers and logical connectives, as appropriate

(b) write the negation of part (a) using the same symbolism.

Example: A function f is *odd* if for every x, $f(-x) = -f(x)$.
(a) defining condition: $\forall x, f(-x) = -f(x)$.
(b) negation: $\exists x$ such that $f(-x) \neq -f(x)$.
It is not necessary that you understand precisely what each term means.

(i) A function f is *strictly decreasing* if for every x and y, if $x < y$, then $f(x) > f(y)$.

(ii) A function $f : A \to B$ is *injective* if for every x and y in A, if $f(x) = f(y)$, then $x = y$.

(iii) A function $f : A \to B$ is *surjective* if for every y in B there exists an x in A such that $f(x) = y$.

Example 5.12. The following statements give certain properties of functions. You are to do two things:

(a) rewrite the defining conditions using quantifiers and logical connectives, as appropriate

(b) write the negation of part (a) using the same symbolism.

> Example: A function f is *odd* if for every x, $f(-x) = -f(x)$.
> (a) defining condition: $\forall x, f(-x) = -f(x)$.
> (b) negation: $\exists x$ such that $f(-x) \neq -f(x)$.
> It is not necessary that you understand precisely what each term means.

(i) A function f is *even* if for every x, $f(-x) = f(x)$.

(ii) A function f is *periodic* if there exists a $k > 0$ such that for every x, $f(x + k) = f(x)$.

(iii) A function f is *increasing* if for every x and y, if $x \leq y$, then $f(x) \leq f(y)$.

Example 5.13. Which of the following best identifies $f : \mathbb{R} \to \mathbb{R}$ as a constant function, where x and y are real numbers.

(a) $\exists x, \forall y, f(x) = y$.

(b) $\forall x, \exists y, f(x) = y$.

(c) $\exists y, \forall x, f(x) = y$.

(d) $\forall y, \exists x, f(x) = y$.

Example 5.14. Define the set $G = \{1, 2, 3\}$ and the function $f : \{1, 2, 3\} \to \mathbb{Z}$ be given by $f(n) = n^2$. Define the following sets:

$$\begin{aligned} A &= \{n \in \mathbb{Z} \ : \ \exists m \in G \text{ with } f(m) < n\} \\ B &= \{n \in \mathbb{Z} \ : \ \exists m \in G \text{ with } f(m) > n\} \\ C &= \{n \in \mathbb{Z} \ : \ \forall m \in G, f(m) < n\} \\ D &= \{n \in \mathbb{Z} \ : \ \forall m \in G, f(m) > n\} \end{aligned}$$

Give a basic description of the sets A, B, C, and D, just using subsets of the integers combined with inequalities, e.g. for some fictitious set, E, we have $E = \{x \in \mathbb{Z} \ : \ x \geq 7029\}$.

Example 5.15. Carefully justify both of these two claims under the assumption that x and y are real numbers.

(i) $\max\{x, -x\} = |x|$.

(ii) $\max\{x, y\} = \frac{1}{2}(|x - y| + x + y)$

Note, it may be useful to recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Example 5.16. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Give an example of a function $f : A \to B$ that is neither injective nor surjective.

Example 5.17. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4\}$.

(a) Give an example of a function
$f : A \to B$ such that $\forall y \in B\ \exists x \in A, f(x) = y$.

(b) Give an example of a function $g : A \to B$ such that $\exists y \in B\ \forall x \in A, g(x) = y$.

(c) Give an example of a function $h : B \to A$ such that $\forall x, y \in B, h(x) = h(y) \implies x = y$.

Example 5.18. Negate each statement in Question 5.17 and give an example of a function f, g, h satisfying the corresponding negation, if possible. If no example can be provided explain why.

Example 5.19. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f((m, n)) = 2n - 4m$. Verify whether this function is injective and whether it is surjective.

Example 5.20. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 1$ is injective.

Example 5.21. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$ is injective.
Hint: A graph can help, but a graph is not a proof.

Example 5.22. Let the function $f : [-1, 1] \to [0, 1]$ be defined by $f(x) = |x|$.
Is this function injective? How about if its domain is $[0, 1]$, instead of $[-1, 1]$?

Example 5.23. Consider the cosine function $\cos : \mathbb{R} \to \mathbb{R}$. Decide whether this function is injective and whether it is surjective. What if it had been defined as $\cos : \mathbb{R} \to [-1, 1]$?

Example 5.24. Let $A = \{x \in \mathbb{R} \mid x \neq 2\}$ and the function $f : A \to \mathbb{R}$ be defined by $f(x) = \frac{4x}{x-2}$. Prove that the function f is injective.

Example 5.25. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

Example 5.26. Prove or disprove that the function f is surjective

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} x^2 + 10 & \text{if } x \geq 0 \\ x + 10 & \text{if } x < 0, \end{cases}$$

(ii) $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ defined by $f(x) = |x|$.

(iii) $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x + y, x - y)$.

(iv) Let $A = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 5\}$ and $B = \{x \in \mathbb{R} \,|\, 2 \leq x \leq 8\}$. Let $f : A \to B$ be defined by $f(x) = x + 2$.

Example 5.27. Consider the following function:

$$f : \mathbb{Z} \to \{-1, 1\}$$
$$f(n) = \begin{cases} 1 & n \text{ is odd.} \\ -1 & n \text{ is even.} \end{cases}$$

a) Show that f is surjective. (just give a brief justification)

b) Show that f is not injective. (just give a brief justification)

Example 5.28. For $k \in \mathbb{R}$, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} kx + 5 & x < 0 \\ x^2 + 5 & x \geq 0. \end{cases}$$

For what values of k is f a bijection?

Example 5.29. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the assignment defined by the relationship

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

(a) Does this description of f define a function? Justify your answer.

(b) Is f injective? Justify your answer.

(c) Is f surjective? Justify your answer.

Example 5.30. Define the operation
$$f(p) := \frac{d}{dx} p.$$
Does f define a function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is f an injective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is f a surjective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. ($\mathbb{P}_n$ is the collection of polynomials or order at most n, see the supplementary document.)

Example 5.31. Let Int be the set of intervals,

$$\text{Int} = \{[a, b] \mid a, b \in \mathbb{R},\ a < b\}$$

and let Tri be a set that contains a class of triangles,

$$\begin{aligned}\text{Tri} = \{(\Delta ABC) \,|\, \Delta ABC & \text{ is the unique triangle with vertices} \\ & A = (a, 0), B = (b, 0), C = (x, y) \text{ and with } a < b, y > 0\}.\end{aligned}$$

We can think of Int as the set of closed intervals in $\mathbb{R}$ and Tri as the triangles in $\mathbb{R}^2$ that have one of their sides as an interval in Int (sitting on the x-axis) and the third vertex somewhere above the x-axis. Now, we define a function from Int to Tri as

$$h : \text{Int} \to \text{Tri}$$

using the assignment rule

$$\begin{aligned}\text{input} &= I = [a, b],\\ \text{output} &= h(I) = \text{the unique equilateral triangle whose base is } I \text{ with its 3rd vertex above the x-axis.}\end{aligned}$$

(i) Evaluate $h([1, 5])$.

(ii) Is h an injective function?

(iii) Is h a surjective function?

(Note, it may be useful for you to recognize that because of the equilateral shape of $h(I)$, you can write an explicit formula for the three vertices of $h(I)$, in terms of a and b– when $I = [a, b]$).

Example 5.32. Let the "alphabet set" be

$$A = \{\text{“}a\text{”}, \text{“}b\text{”}, \text{“}c\text{”}, \ldots \text{“}z\text{”}\},$$

and for $n \in \mathbb{N}$, define, A_n as the set of words – or more formally, strings – consisting of at most n letters from the alphabet set A. Formally, this could be

$$A_n = \{\text{string} = \theta_1 \cdots \theta_m \mid m \leq n \text{ and } \theta_i \in A \text{ for } i = 1, \ldots, m\}.$$

Notice that both "eleven" and "xjastr" are strings of length 6 in A_6, so here we don't care if our strings are valid words in the English dictionary.
Consider the function $s : A_6 \to A_6$ that takes a string of length at most 6 to the string with all of the same letters sorted alphabetically. For example, s("number") = "bemnru", and s("eleven") = "eeelnv".
Determine whether s is injective and whether it is surjective. If s is not surjective, how can we redefine the codomain of s to make it surjective?

Example 5.33. Define the set $D \subseteq \mathcal{P}(\mathbb{N})$, as

$$D = \{A \subseteq \mathbb{N} \ :\ A \text{ contains a finite number of elements}\},$$

and the set

$$\text{Even} = \{x \in \mathbb{Z} \ :\ x = 2k \text{ for some } k \in \mathbb{Z}\}.$$

Define a function, $c : D \to \mathbb{N} \cup \{0\}$, via the assignment

$$c(A) = \left| A \bigcap \text{Even} \right|.$$

(Recall that the cardinality, $|E|$, is defined in the text, and the power set, $\mathcal{P}(\mathbb{N})$, is defined in the supplementary material.) Answer the questions about the function c.

(a) Let $E = \{1, 3, 4, 6, 17, 20, 21\}$. Evaluate $c(E)$.

(b) Let $B = \{x \in \mathbb{N} \ : \ 100 < x < 141 \text{ and } x \text{ is divisible by } 7\}$. Evaluate $c(B)$.

(c) Let $H = \{x \in \mathbb{N} \ : \ x \text{ is divisible by } 20\}$. Can you evaluate $c(H)$?

(d) Prove that c is not an injective function.

(e) Prove that c is a surjective function.

Example 5.34. Consider the function $G : \mathbb{P}_4 \to \mathbb{P}_2$ given by taking the second derivative, that is, for a polynomial $f \in \mathbb{P}_4$, we have $G(f) = \dfrac{d^2 f}{dx^2}$

(a) Prove that G is <u>not</u> injective.

(b) Prove that G is surjective.

(c) Let $X = \{p \in \mathbb{P}_4 : 1 + a_2x^2 + a_3x^3 + a_4x^4\}$, and define H, similarly to G as $H : X \to \mathbb{P}_2$, via $H(f) = \dfrac{d^2 f}{dx^2}$. Prove that H <u>is</u> injective.

Example 5.35. Let the function, $R : \mathbb{P}_2 \to \mathbb{P}_4$, be defined via the assignment rule

$$[R(p)](x) := p(x)^2.$$

(a) For $q(x) = x^2 + 3x$, evaluate the output, $R(q)$.

(b) Is R an injective function? Justify your answer.

(c) Is R a surjective function? Justify your answer.

Example 5.36. Give an example of a function $G : \mathbb{P}_2 \to \mathbb{P}_1$ such that when the input is $p(x) = x^2$, the output is $[G(p)](x) = 10x$

Example 5.37. Given an example of a function $g : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective. Justify your answer.

Example 5.38. Give an example of $p : \mathbb{R} \to \mathbb{R}$ such that p is a polynomial of degree 3 that is surjective but not injective. You don't need to justify your answer.

Example 5.39. Define the function $T : \mathbb{P}_2 \to \mathbb{P}_3$ via the assignment

$$[T(f)](x) := \int_0^x f(s)ds, \quad \text{for all } x \in \mathbb{R}.$$

That is to say, the input to T is a polynomial $f \in \mathbb{P}_2$ and the output from T is the new polynomial made by antidifferenting f with a zero constant of integration.

(a) Evaluate $T(f)$ for each of the three instances of f as: $f(s) = 1$, $f(s) = s$, and $f(s) = s^2$. (note, since $T(f)$ is a polynomial, you must show which polynomial it is by plugging in a real variable, x. So, really, you must show what is $[T(f)](x)$ for $x \in \mathbb{R}$.)

(b) Prove that T is an injective function.

(c) Prove that T is not a surjective function.

NOTE! The fundamental theorem of calculus may be incredibly helpful to you here. Also note, you probably do not have to bother to write $f(x) = a_0 + a_1x + a_2x^2$, etc... you can just leave your functions as p, q, f, etc...

Example 5.40. Let the function, $R : \mathbb{P}_3 \to \mathbb{R}$, be defined via the assignment rule

$$R(p) := \frac{p(4) - p(0)}{4}.$$

(i) is R an injective function? Justify your answer.

(ii) is R a surjective function? Justify your answer.

(iii) Describe the function, R, in terms of the graph of p; of the following list of phrases, include the correct and relevant ones in your description: “derivative”, “secant line”, “antiderivative”, “integral”, “slope”, “area under the curve”, “gravitational singularity”.

Week 6 (things related to bijections and inverse functions)

Example 6.1. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined via the assignment

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ 3n+1 & n \text{ is odd} \end{cases}$$

(i) Prove that f is not an injective function.

(ii) Prove that f is a surjective function.

(iii) Does there exist a function $g : \mathbb{N} \to \mathbb{N}$ that satisfies $g(f(x)) = x$ for all $x \in \mathbb{N}$?

(iv) Does there exist a function $h : \mathbb{N} \to \mathbb{N}$ that satisfies $f(h(y)) = y$ for all $y \in \mathbb{N}$?

Example 6.2. The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \pi x - e$ is bijective. Find its inverse.

Example 6.3. Prove that the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective, and find the assignment rule for f^{-1} (you don't have to prove why the assignment rule is what it is).

Example 6.4. Check that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 6 - n$ is bijective. Then find the assignment rule for f^{-1}.

Example 6.5. Let $A = \mathbb{R} \setminus \{1\}$ and define $f : A \to A$ by $f(x) = \frac{x}{x-1}$ for all $x \in A$.

(i) Prove that f is bijective.

(ii) Determine f^{-1}.

Example 6.6. Let $F : (-\infty, \ln(6) - 4) \to (-3, 3)$ be the function defined via $F(x) = e^{x+4} - 3$

(a) Show that F is a bijection.

(b) Find the assignment rule for F^{-1} and state the domain and range of F^{-1}.

Example 6.7. Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, n) = (n, 3xn)$. Check that this is bijective; find its inverse function. Carefully justify that your answer does indeed yield the inverse function.

Example 6.8. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(x, y) = (2x - 3y, x + 1)$$

(a) Show that f is a bijection.

(b) Find the assignment rule for the inverse, f^{-1}, of f.

Example 6.9. Suppose $f : A \to B$ and $g : X \to Y$ are bijective functions. Define a new function $h : A \times X \to B \times Y$ by $h(a, x) = (f(a), g(x))$. Prove that h is bijective.

Example 6.10. (i) Prove the following: If $f : \mathbb{N} \to \mathbb{N}$ and f is strictly increasing, then f is injective.

(ii) Give an example of a function, $f : \mathbb{Z} \to \mathbb{Z}$ such that f is strictly increasing, but f is not bijective.

Example 6.11. Consider the function$f : \mathbb{Z} \to \mathbb{Z}$ by the rule $f(x) = \begin{cases} x & \text{if } x < 0 \\ 1 - x^2 & \text{if } x \geq 0 \end{cases}$
Determine and justify if f is injective and/or surjective. If f is bijective find an inverse function.

Example 6.12. Consider the function $f : \mathbb{R} \to B$ given by the assignment rule

$$f(x) = \begin{cases} x + 13 & \text{if } x < 0 \\ 14 + x^2 & \text{if } x \geq 0. \end{cases}$$

Find the largest set $B \subseteq \mathbb{R}$ such that f has an inverse. Find the assignment rule for the inverse function (you do not need to justify that it is the inverse).

Example 6.13. Suppose X is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define a function $M : X \to X$ via the assignment

$$M(f)(x) := 1 - f(x - 1), \quad \text{for all } x \in \mathbb{R}$$

Show that:

(a) M is a bijection.

(b) Find M^{-1}.

Example 6.14. Define the function $T : \mathbb{P}_2 \times \mathbb{R} \to \mathbb{P}_3$ via the assignment

$$[T(f, a)](x) := \int_0^x f(s)ds + a, \quad \text{for all } x \in \mathbb{R}.$$

That is to say, the input to T is the pair $(f, a) \in \mathbb{P}_2 \times \mathbb{R}$, of a polynomial and a real number. The output from T is the new function made by anti-differentiating f with a zero constant of integration, and then adding the constant a to it.

(a) Evaluate $T(f, 6)$ when f is the polynomial given by $f(x) = 2x^2 - x$.

(b) Prove that T is an injective function.

(c) Prove that T is a surjective function. (Note, if $p \in \mathbb{P}_3$ is given, and you claim to have found some pair (q, a) such that $T(q, a) = p$ please don't forget to check that indeed $[T(q, a)](x) = p(x)$ for all x *and* that q in indeed an element of $\mathbb{P}_2$!)

(d) Find the assignment rule for the function, T^{-1}.

NOTE! The fundamental theorem of calculus may be incredibly helpful to you here. Also note, you probably do not have to bother to write $f(x) = a_0 + a_1x + a_2x^2$, etc... you can just leave your functions as p, q, f, etc...

Example 6.15. Let $\mathbb{P}_n$ be the set of real polynomials of order at most n. For each of the following functions, determine if:

(i) the function is surjective,

(ii) the function is injective, and

(iii) if the function is not injective, and if its domain is X and its codomain is Y, can you give an example of a set $A \subseteq X$ such that the function is injective from $A \to Y$?

(a) Given a polynomial $p \in \mathbb{P}_3$, such that

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

define the function $D : \mathbb{P}_3 \to \mathbb{P}_3$ with the assignment rule

$$[D(p)](x) = p'(x) + a_0 x^3$$

Note that D is a function that takes a real polynomial p as an input and returns another real polynomial, $D(p)$.

(b) Given a polynomial $p \in \mathbb{P}_3$, define the function $I : \mathbb{P}_3 \to \mathbb{R}$ with the assignment rule

$$I(p) = \int_0^1 p(x)\, dx$$

Note that I is a function that takes a real polynomial p as an input and returns a real number.

(c) Given a polynomial $p \in \mathbb{P}_3$, define the function $S : \mathbb{P}_3 \to \mathbb{P}_6$ with the assignment rule

$$S(p) = p^2$$

Note that S is a function that takes a real polynomial p as an input and returns another real polynomial, $S(p)$. When we plug in a real variable x into the polynomial $S(p)$, we get that

$$[S(p)](x) = [p(x)]^2$$

.

(d) Given a polynomial $p \in \mathbb{P}_3$, such that

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

define the function $C : \mathbb{P}_3 \to \mathbb{R}^4$ with the assignment rule

$$[C(p)](x) = (a_0, a_1, a_2, a_3)$$

Note that C is a function that takes a real polynomial p as an input and returns a real 4-vector.

Week 7

Example 7.1. This is just a placeholder example that exists simply to have an easy way to keep the numbering in order.

Week 8

Example 8.1. Let $B_1 = \{1,2\}, B_2 = \{2,3\}, \ldots, B_{10} = \{10,11\}$; that is, $B_i = \{i, i+1\}$ for some $i = 1, 2, \ldots, 10$. Determine the following:

(i) $\displaystyle\bigcup_{i=1}^{5} B_i$

(ii) $\displaystyle\bigcup_{i=1}^{10} B_i$

(iii) $\displaystyle\bigcup_{i=3}^{7} B_i$

(iv) $\displaystyle\bigcup_{i=j}^{k} B_i,$ where $1 \le j \le k \le 10$.

(v) $\displaystyle\bigcap_{i=1}^{10} B_i$

(vi) $B_i \cap B_{i+1}$

(vii) $\displaystyle\bigcap_{i=j}^{j+1} B_i,$ where $1 \le j < 10$

(viii) $\displaystyle\bigcap_{i=j}^{k} B_i,$ where $1 \le j \le k \le 10$.

(Note, see section 1 of the supplementary material for a definition of this strange notation you are most likely encountering for the first time here.)

Example 8.2. For each $n \in \mathbb{N}$, define $I_n = [n, n+1]$. Determine the following sets:

1. $\displaystyle\bigcup_{j=4}^{40} I_n$
2. $\displaystyle\bigcup_{j=7}^{\infty} I_n$
3. $I_k \cap I_{k+1}$
4. $\displaystyle\bigcap_{k=4}^{6} I_k$
5. $I_j \cap I_k$

Example 8.3. Compute the following sets:

1. $\displaystyle\bigcup_{p \in \mathbb{P}_2} \{x \in \mathbb{R} : p(x) < 0\}$
2. $\displaystyle\bigcap_{p \in \mathbb{P}_2} \{x \in \mathbb{R} : p(x) > 0\}$

Example 8.4. Let A_1 and A_2 both be subsets of X. Prove the following two statements.

(a) $\left(\bigcup_{i\in\{1,2\}} A_i\right)^c = \bigcap_{i\in\{1,2\}} A_i^c$

(b) $\left(\bigcap_{i\in\{1,2\}} A_i\right)^c = \bigcup_{i\in\{1,2\}} A_i^c$

Example 8.5. For each $n \in \mathbb{N}$, define A_n to be the closed interval $[-1/n, 1/n]$ of real numbers; that is,

$$A_n = \left\{x \in \mathbb{R} : -\frac{1}{n} \leq x \leq \frac{1}{n}\right\}.$$

Determine:

$$(i) \bigcup_{n\in\mathbb{N}} A_n \qquad (ii) \bigcap_{n\in\mathbb{N}} A_n.$$

Example 8.6. For each $n \in \mathbb{N}$ define A_n to be $A_n = \left(\left(5 - \frac{1}{2n}, 5 + \frac{1}{2n}\right) \setminus \mathbb{Q}\right)$. Find the following sets and justify your answers with proofs.

(i) $\displaystyle\bigcup_{n=1}^{\infty} A_n$

(ii) $\displaystyle\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c$

Example 8.7. Assume that $X = \displaystyle\bigcup_{k\in\mathbb{N}} \left(-\frac{1}{k^2}, 1\right]$. Give a different, more simplified description of the set, X. Justify your answer with a proof.

Example 8.8. Prove that $\displaystyle\bigcap_{x\in\mathbb{N}} [3 - (1/x)^2, 5 + (1/x)^2] = [3, 5]$. See section 1 of the supplementary material for this notation with intersection of sets. (Note, the Archimedean property of $\mathbb{R}$ will be helpful.)

Example 8.9. For each $n \in \mathbb{N} \cup \{0\}$, let $I_n = [\arctan(n), \arctan(n + 1)]$. Show that:

1. $\displaystyle\bigcup_{n=0}^{\infty} I_n = \left[0, \frac{\pi}{2}\right)$

2. $\displaystyle\bigcap_{n=0}^{\infty} I_n = \emptyset$

Example 8.10. Define the sets $A = \displaystyle\bigcup_{n\in\mathbb{N}} \left\{x \in \mathbb{R} : x^2 + \frac{4n}{3} - 4n^2 < \frac{1}{9}\right\}$ and $B = \displaystyle\bigcup_{n\in\mathbb{N}} \left(-2n + \frac{1}{3}, 2n - \frac{1}{3}\right)$. Show that $A = B$.

Example 8.11. State whether the following sets are bounded or unbounded and prove your answer.

(a) $A = \{-2, -1, \frac{1}{2}\}$.

(b) $B = (-\infty, \sqrt{2})$.

(c) $C = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \dots\} = \{\frac{2n-1}{2} \mid n \in \mathbb{N}\}$. (Note, the Archimedean property of $\mathbb{R}$ could be useful.)

(d) $D = \left\{\frac{(-1)^n}{n} \ : \ n \in \mathbb{N}\right\}$

(e) $E = \left\{\frac{-1}{n} \ : \ n \in \mathbb{Q} \setminus \{0\}\right\}$ (Note, the Archimedean property of $\mathbb{R}$ could be useful.)

Example 8.12. Define the function, $f : \mathbb{R} \to \mathbb{R}$ via the assignment

$$f(x) = -(x-5)^2 + 9,$$

and define the set E as

$$E := \{x \in \mathbb{R} \ : \ f(x) \geq -5\}.$$

Prove that the set E is bounded.

Example 8.13. State whether the following sets are bounded or unbounded and prove your answer.

(a) $A = \{n \in \mathbb{N} : n^2 < 10\}$

(b) $B = \{2k \mid k \in \mathbb{N}\}$

(c) $C = \{n/(n+m) : m, n \in \mathbb{N}\}$

(d) $D = \{1 + (-1)^n(2n-1) : n \in \mathbb{N}\}$

Example 8.14. Are the given sets bounded or unbounded? Justify your answer with a proof.

1. $\mathbb{Q} \cap (-\infty, \sqrt{5})$
2. $\left\{\dfrac{r}{s} : r \in \mathbb{Q}, s \in \mathbb{R} \setminus \mathbb{Q}\right\}$

Example 8.15. State whether the following sets are bounded or unbounded and justify your answer with a proof.

1. $\left\{\dfrac{1}{2^k} : k \in \mathbb{Z}\right\}$
2. $\{2^n : n \in \mathbb{Z} \setminus \mathbb{N}\}$
3. $\left\{\dfrac{3n^2}{1-2n^2} : n \in \mathbb{Z}\right\}$
4. $\{1 + (-1)^n : n \in \mathbb{N}\}$

Example 8.16. Discuss whether the following sets are bounded or unbounded and justify your answer with a proof.

1. $A = \left\{\dfrac{1}{1+x} : x \in \mathbb{R}, |x| > 1\right\}$
2. $B = \left\{2 - \dfrac{3}{1+y} : y \in \mathbb{R} \setminus \{-1\} \text{ and } |y| \leq 3\right\}$

Example 8.17. Discuss whether the following sets are bounded or unbounded and justify your answer with a proof.

1. Let $n \in \mathbb{N}$ be fixed and $A = \left\{\dfrac{2m-1}{n} : m \in \mathbb{Z}\right\}$.
2. Let $q \in \mathbb{Q} \setminus \{0\}$ be fixed and $B = \left\{\dfrac{p}{q} : p \in \mathbb{R},\ p \in (-20, 40)\right\}$.

Example 8.18. Are the given sets bounded or unbounded? Justify your answer with a proof.

1. $X = \{\cos x : x \in \mathbb{R}\}$
2. $Y = \left\{\tan x : x \in \mathbb{R} \setminus \{\frac{\pi}{2} + \pi k : k \in \mathbb{Z}\}\right\}$
3. $Z = \{x \in \mathbb{R} : \tan x \leq 1\}$

Example 8.19. $\forall r \in \mathbb{R}^+$ consider $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$. (Recall: $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$)

1. For a fixed $r \in \mathbb{R}^+$, give a geometric description of the set C_r.
2. What is $D = \displaystyle\bigcup_{0 \leq r \leq 10} C_r$? Write down the set, and give a geometric description of D.
3. Consider the set $Y = \{y \in \mathbb{R} : \exists x \in \mathbb{R}, (x, y) \in D\}$. Is Y a bounded set? Justify.

Example 8.20. $\forall s \in \mathbb{R}$ consider $L_s = \{(x, y) \in \mathbb{R}^2 : y = sx\}$.

1. For a fixed $s \in \mathbb{R}$, give a geometric description of the set L_s.
2. What is $P = \bigcup_{s \in \mathbb{R}} L_s$? Write down the set, and give a geometric description of P.
3. Consider the set $Y = \{y \in \mathbb{R} : (0, y) \in P\}$. Is Y a bounded set? Justify.

Example 8.21. $\forall p \in [-1, 1]$ consider $X_p = \{x \in \mathbb{R} : p = \cos x\}$.

1. Is X_p a bounded set?
2. Give a description of $L = \bigcup_{p \in \mathbb{R}} X_p$. Is L a bounded set?

Week 9

Example 9.1. Write out the first five terms of the following sequences:

(a) $\{2n-5\}_{n\in\mathbb{N}}$

(b) $\left\{\frac{1}{2n+5}\right\}_{n\in\mathbb{N}}$

(c) $\left\{\frac{2^n-1}{2^n}\right\}_{n\in\mathbb{N}}$

(d) $\{a_n : a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}, n \geq 3\}_{n\in\mathbb{N}}$

(e) $\left\{1+\left(-\frac{1}{2}\right)^n\right\}_{n\in\mathbb{N}}$

You may assume that each sequence is a function from $\mathbb{N}$ to $\mathbb{R}$.

Example 9.2. For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

(a) $\left\{\frac{3n+1}{7n-4}\right\}_{n\in\mathbb{N}}$

(b) $\left\{\sin(\frac{n\pi}{4})\right\}_{n\in\mathbb{N}}$

(c) $\{(1+1/n)^2\}_{n\in\mathbb{N}}$

(d) $\{(-1)^n n\}_{n\in\mathbb{N}}$

(e) $\left\{\sqrt{n^2+1}-n\right\}_{n\in\mathbb{N}}$

Example 9.3. For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

(a) $\left\{3+\frac{(-1)^n 2}{n}\right\}_{n\in\mathbb{N}}$

(b) $\left\{\frac{n^2-2n+1}{n-1}\right\}_{n\in\mathbb{N}\setminus\{1\}}$

(c) $\left\{\frac{n}{n+1}\right\}_{n\in\mathbb{N}}$

Example 9.4. Using the definition of convergence, that is, an $\epsilon - N$ argument, prove that the following sequences converge to the indicated number:

(a) $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0.$

(b) $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0.$

(c) $\displaystyle\lim_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}.$

Example 9.5. Using the definition of convergence, that is, an $\epsilon - N$ argument, prove that the following sequences converge to the indicated number:

(a) $\lim_{n\to\infty}\left(3+\frac{2}{n^2}\right)=3.$

(b) $\lim_{n\to\infty}\frac{\sin(n)}{2n+1}=0.$

Example 9.6. Find an example of a convergent sequence $\{a_n\}_{n\in\mathbb{N}}$ so that $a_{n+1} > a_n$.

Example 9.7. Find an example of a convergent sequence $\{s_n\}_{n\in\mathbb{N}}$ of irrational numbers that has a rational number as a limit.

Example 9.8. Use the formal definition of a limit to prove that $\left\{2+\frac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$ is a convergent sequence.

Example 9.9. Use the formal definition of a limit to prove that $\lim_{n\to\infty}\frac{1}{n^{1/3}}=0.$

Example 9.10. Using the formal definition of the limit of a sequence, prove $\lim_{n\to\infty}\frac{5}{n^2}=0.$

Example 9.11. Prove that $\sqrt{n+1}-\sqrt{n}$ is convergent using the definition of convergence.

Example 9.12. Use the formal definition of a limit of a sequence to prove that

$$\lim_{n\to\infty}\frac{2n-3}{1-5n}=-\frac{2}{5}.$$

Example 9.13. Use the formal definition of the limit of a sequence to prove that

$$\lim_{n\to\infty}\frac{2n-1}{3n+2}=\frac{2}{3}.$$

Example 9.14. Let (a_n) be a sequence with positive terms (i.e. $\forall n, a_n > 0$) such that $\lim_{n\to\infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

$$\lim_{n\to\infty}\frac{3a_n+1}{2}=2.$$

Example 9.15. Prove that the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ does not converge.

Example 9.16. Prove that for $a_k = \ln(k)$, $\{a_k\}_{k\in\mathbb{N}}$ is not a convergent sequence.

(*You will need to recall from calculus that* ln *is the natural logarithm, and it is the unique function that is the inverse of the exponential function. That is,* ln *is the unique function such that*

$$e^{\ln(y)} = y \ \forall y \in (0,\infty) \ \textit{and} \ \ \ln(e^x) = x \ \ \forall x \in \mathbb{R}.$$

When trying to simplify expressions involving inequalities, you will most likely need to use the fact that both e^x *and* $\ln(x)$ *are increasing functions. See section 6 of the additional notes for a couple of things about increasing functions.*)

Example 9.17. For the sequence whose terms are given by

$$a_n = \frac{(-1)^n 3n + 1}{n - 2},$$

prove that $\{a_n\}_1^\infty$ is not a convergent sequence.

Example 9.18. Prove that $\{\sin\left(\frac{n\pi}{4}\right)\}_{n\in\mathbb{N}}$ does not converge.

Week 10

Example 10.1. For a and b given below, use the division lemma to find the unique quotient and remainder when a is divided by b.

(a) $a = 302, b = 19$

(b) $a = 0, b = 19$

(c) $a = -302, b = 19$

(d) $a = 2002, b = 19$

Example 10.2. Prove that if a divides b and c divides d, then ac divides bd.

Example 10.3. Let $a, b, c, d \in \mathbb{Z}$ with $a, c \neq 0$. Prove that if $a \mid b$ and $c \mid d$, then $ac \mid (ad + bc)$.

Example 10.4. Answer true or false, and give a complete justification. If a divides bc, then a divides b or a divides c.

Example 10.5. Prove or disprove: If $a \mid b$ and $a \mid c$ then $a \mid (xb + yc)$ for any $x, y \in \mathbb{Z}$

Example 10.6. Assume that $a, b \in \mathbb{Z}$. Prove that if r is an integer solution of $r^2 + ar + b = 0$, then $r \mid b$.

Example 10.7. Prove that if $n \in \mathbb{N}$, then $4^{2n} + 10n - 1$ is divisible by 25.

Example 10.8. Compute $|E|$, where

$$E = \{n \in \mathbb{N} : n \leq 3076 \text{ and } n \text{ is divisible by } 19\}.$$

Example 10.9. Compute $|E|$, where

$$E = \{n \in \mathbb{N} : n \leq 3076 \text{ and } n \text{ is not divisible by } 17\}.$$

Example 10.10. Prove that the square of every odd integer is of the form $4k + 1$, where $k \in \mathbb{Z}$ (that is, for each odd integer $a \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $a^2 = 4k + 1$).

Example 10.11. Prove that the square of any integer a is either of the form $3k$ or $3k + 1$, for some choice of $k \in \mathbb{Z}$.

Example 10.12. Prove the following statement in two ways: (i) use the contrapositive; and (ii) a proof by contradiction.

$$\text{Let } m \in \mathbb{Z}. \text{ If } 3 \nmid (m^2 - 1), \text{ then } 3 \mid m.$$

Example 10.13. Prove that if $n \in \mathbb{N}$, then $n^5 - n$ is divisible by 30.

Example 10.14. Define what it means for a natural number p to be prime. Answer whether the following statement is true or false, and give a complete justification: If p is prime and $p > 2$, then $p^2 + 2$ is prime.

Example 10.15. Prove that if $x \equiv 1 \mod 4$ and $y \equiv 1 \mod 4$, then $xy \equiv 1 \mod 4$.

Example 10.16. Let a and b be given integers. Prove $a \equiv b \mod 5$ if and only if $9a + b \equiv 0 \mod 5$.

Example 10.17. Provide the two different proofs below for the following statement, assuming that $x, y \in \mathbb{Z}$. If $y \equiv 2 \mod 4$, then $xy \not\equiv 1 \mod 4$.

(i) A proof by cases, using the distinct and unique possibilities for $n \mod 4$, via the division lemma.

(ii) A proof by contradiction.

Example 10.18. .

1. Prove or disprove: For all positive integers n and for all integers a and b, if $a \equiv b \mod n$ then $a^2 \equiv b^2 \mod n$.
2. Prove or disprove: For all positive integers n and for all integers a and b, if $a^2 \equiv b^2 \mod n$ then $a \equiv b \mod n$.

Example 10.19. Note, in this example, we will demonstrate that there are many ways to phrase mathematical sentences. Both of what follow say the same thing, but they read rather differently.

Presentation #1: Prove that if for $n \geq 2$, $a_1, a_2, \ldots, a_n$ are integers such that $a_i \equiv 1 \mod 3$ for every index, i, $1 \leq i \leq n$, then for all $n \in \mathbb{N} \setminus \{1\}$, $a_1 a_2 \ldots a_n \equiv 1 \mod 3$.

Presentation #2: Assume that for each $n \geq 2$ with $n \in \mathbb{N}$, that $a_1, a_2, \ldots, a_n$ are integers and for all $i \in \{1, \ldots, n\}$, $a_i \equiv 1 \mod 3$. Prove that for all $n \in \mathbb{N} \setminus \{1\}$, $a_1 a_2 \ldots a_n \equiv 1 \mod 3$.

Example 10.20. Assume $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $m \geq 2$. Prove that the following statement is true: If $a \equiv b \mod m$, then $a^n \equiv b^n \mod m$ for every positive integer n. Give two different proofs.

(a) A proof by induction.

(b) A direct proof that uses a factoring rule for $x^n - y^n$ for $n \in \mathbb{N}$.

Example 10.21. (Challenge!) Show that if x, y, z are integers such that $x^2 + y^2 = z^2$, then at least one of them is divisible by 2, at least one is divisible by 3, and at least one is divisible by 5.

Example 10.22. Assume that $a, b \in \mathbb{Z}$. Is the following implication true? If $ab \equiv 0 \mod 5$, then $a \equiv 0 \mod 5$ or $b \equiv 0 \mod 5$. If you believe it is true, then prove it. If you believe it is false, then give an example that demonstrates that it is false.

Example 10.23. Prove that if $x \not\equiv 0 \mod 5$ and $y \not\equiv 0 \mod 5$, then $xy \not\equiv 0 \mod 5$. For this example, give a direct proof, using the cases arising from the remainder in the division lemma with a divisor of 5. (A proof by contrapositive is possible, using the previous question, but you should not do that here.)

Example 10.24. Suppose $a, b \in \mathbb{Z}$ such that $ab \equiv 0 \mod 15$. Is it necessarily true that either $a \equiv 0 \mod 15$ or $b \equiv 0 \mod 15$? Justify your answer.

Example 10.25. Prove the following: Let $a, b, c, m, n \in \mathbb{Z}$, where $m, n \geq 2$. If $a \equiv b \pmod{m}$ and $a \equiv c \pmod{n}$, then $b \equiv c \pmod{d}$, where $d = \gcd(m, n)$.

Example 10.26. Prove $\gcd(m+1, n+1) \mid (mn-1)$ for all $m, n \in \mathbb{Z}$. Prove it by using direct arguments and definitions.

Example 10.27. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $c \mid b$ then $\gcd(a, c) = 1$. (Hint: Use proof by contradiction)

Example 10.28. Prove that if $a, p \in \mathbb{Z}$, p is prime, and p does not divide a, then $\gcd(a, p) = 1$.

Example 10.29. Answer true or false and give a complete justification. If $p > 2$ is prime, then $\gcd(p, p+2) = 1$.

Example 10.30. Answer true or false and give a complete justification. If p is prime and p divides ab, then p divides a or p divides b.

Example 10.31. Prove that there are infinitely many integers x and y such that $x + y = 100$ and $\gcd(x, y) = 5$.

Example 10.32. Assume that $a, b \in \mathbb{N}$ and that $\gcd(a, b) = d$. Prove that $a|b$ if and only if $d = a$.

Example 10.33. Assume that $a, b \in \mathbb{N}$, and we will define the number $\mathrm{lcm}(a, b)$ as

$$\mathrm{lcm}(a, b) = \frac{ab}{\gcd(a, b)}.$$

Let m be given as $m = \mathrm{lcm}(a, b)$. Prove that if c is a common multiple of a and b, then $m|c$.

Example 10.34. Prove that any collection of 87 consecutive integers will contain at least one that is divisible by 87. (Hint, apply the division lemma to the first of the list of these integers.)

Example 10.35. Assume that $a, b \in \mathbb{Z}$. Prove or disprove the following statement: if a and b are odd, then $4|(a-b)$ or $4|a+b$.

Example 10.36. Let $a, b \in \mathbb{Z}$, where both $a \neq 0$ and $b \neq 0$. Prove that if $d = \gcd(a, b)$, $a = a_1 d$ and $b = b_1 d$, then $\gcd(a_1, b_1) = 1$.

Example 10.37. Assume that $a, b \in \mathbb{Z}$ and that $\gcd(a, b) = d$. Prove that if $k \in \mathbb{Z}$ and $k > 0$, then $\gcd(ak, bk) = kd$.

You are expected to read Theorem 28.7 of the textbook and invoke it for a portion of this proof. (Hint, define $g = \gcd(ak, bk)$ and try to show that both $g \leq kd$ and $g \geq kd$.)

Example 10.38. Define the function, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Prove that if f has a root of the form $y = \frac{p}{q}$ with the property that $\gcd(p, q) = 1$, then $p|a_0$ and $q|a_3$. In order to give a proof for this example, you are expected to read and understand Euclid's Lemma, which appears as Corollary 28.9 in the textbook, and then subsequently apply it to this situation.

You may also want to use the unique prime factorization (Fundamental Theorem of Arithmetic) that states that if $x \in \mathbb{N}$, then there exist prime numbers, $p_1, \ldots, p_k$, and integer exponents, $a_1, \ldots, a_k$, so that

$$x = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$

Example 10.39. Assume that $n \in \mathbb{Z}$. Prove that n is divisible by 3 if and only if the sum of the digits of n is divisible by 3. We note that writing

$$n = \sum_{i=0}^{i=K} a_i \cdot 10^i$$

means that the "digits" of n are a_0, a_1, ..., a_K (here K will depend on n). (As a hint, you should try the trick that $10^i = (3^2+1)^i$, and you can easily confirm what is $(3^2+1)^i \mod 3$.)

Example 10.40. Prove that if $n \equiv 5 \mod 6$, then $n \equiv 2 \mod 3$.

Example 10.41. Assume that $m \in \mathbb{Z}$. Prove that at least one of m, $m+2$, $m+4$ is divisible by 3.

Example 10.42. Prove that $\forall\ a \in \mathbb{Z}$, $a^2 \equiv a \mod 2$.

Example 10.43. Prove that $\forall\ n \in \mathbb{Z}$, $\gcd(n, n+1) = 1$.

Example 10.44. Prove that $\forall\ n \in \mathbb{N}$, $\exists\ k, m \in \mathbb{Z}$, $\gcd(k, m) = n$.

Example 10.45. Prove that if $p \in \mathbb{Z}$ and p is odd, then $\left(\sum_{i=1}^{p} i\right) \equiv 0 \mod p$.

Week 11

This section contains examples involving relations and equivalence relations.

Example 11.1. Let $\sim$ be the relation on $\mathbb{R}$ given by $x \sim y$ if $2x < y < 3x + 6$. State whether this relation is reflexive, symmetric, and/or transitive and justify your answer.

Example 11.2. Fill in the following table with Y/N answering whether the given relations on $\mathbb{N}$ are reflexive, symmetric, and/or transitive.

Relation	$<$	$>$	$=$	$\neq$	$\mid$	$\nmid$	$\geq$
Reflexive							
Symmetric							
Transitive							

Example 11.3. A relation on $\mathbb{R}$ is given by $x \sim y$ if and only if $x = -y$. Is the given relation reflexive, symmetric, and/or transitive? Is it an equivalence relation? Justify all your answers with proofs or counterexamples.

Example 11.4. Consider the set $A = \{3, 4, m\}$. Give an example (if possible) of a relation on the set A that satisfies each of the following conditions. Are there any cases where it is not possible to come up with such an example? Why not?

(a) Reflexive, but neither transitive nor symmetric

(b) Symmetric, but neither reflexive nor transitive

(c) Transitive, but neither symmetric not reflexive

(d) Reflexive and transitive but not symmetric

(e) Transitive and symmetric but not reflexive

(f) Reflexive and symmetric but not transitive

(g) Transitive, reflexive, and symmetric

(h) Neither transitive, nor reflexive, nor symmetric

Example 11.5. Repeat example 11.3 with the set $A = \mathbb{Z}$.

Example 11.6. Let $\sim$ be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that the true values of $\sim$ at least contain: $a \sim c$, $c \sim d$, $d \sim g$ and $b \sim f$ (but possibly more). If you know there are three distinct equivalence classes resulting from $\sim$, then determine these three equivalence classes and determine all elements for which the equivalence relation returns a value of "true".

Example 11.7. Consider the set $A = \{a, b, c, d\}$, and assume that $\sim$ is an equivalence relation on A. If $\sim$ returns a "true" output on (a, b) and (b, d), for which other elements must $\sim$ also have a "true" output?

Example 11.8. Define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $3x - 5y \equiv 0 \mod 2$. Prove $\sim$ is an equivalence relation. Describe its equivalence classes.

Example 11.9. Define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $4 \mid (x + 3y)$. Prove $\sim$ is an equivalence relation. Describe its equivalence classes.

Example 11.10. Consider the relation $\sim$ on $\mathbb{N}$ defined by $a \sim b$ if $ab \equiv 3 \mod 7$. Is this relation reflexive, symmetric, and/or transitive? Justify your answers.

Example 11.11. Define a relation $\sim$ on $\mathbb{Z} \times \mathbb{N}$ by $(j, k) \sim (m, n)$ if and only if $jn = km$. Show that $\sim$ is an equivalence relation. Give a brief description of the equivalence classes under $\sim$.

Example 11.12. For $(a, b), (c, d) \in \mathbb{R}^2$ define $(a, b) \sim (c, d)$ to mean that $2a - b = 2c - d$. Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

Example 11.13. Define a relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \qquad \text{means} \qquad |x| + |y| = |x + y|.$$

Is $\sim$ an equivalence relation? Justify your answer.
(**Hint:** *the answer is no.*)

Example 11.14. Let $X = \mathbb{R}^2$, the xy-plane. Define $(x_1, y_1) \sim (x_2, y_2)$ to mean

$$x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Is $\sim$ an equivalence relation? Justify your answer. Give a geometric interpretation of the equivalence classes of $\sim$.

Example 11.15. Define a relation $\sim$ on $\mathbb{R}$ as $x \sim y$ if and only if $\lfloor x \rfloor \leq y < \lceil x \rceil$. Is this an equivalence relation? If yes, justify and list the equivalence classes. If no, give counterexamples for all the properties that fail.
Note that the functions $\lfloor x \rfloor$ and $\lceil x \rceil$ are given by

$$\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$$

and

$$\lceil x \rceil = \min\{z \in \mathbb{Z} : z > x\}$$

Example 11.16. Define a relation $\sim$ on $\mathbb{R}$ as $x \sim y$ if and only if $y = kx$ for some $k \in \mathbb{R} \setminus \{0\}$. Is this an equivalence relation? If yes, justify.

Example 11.17. For $x, y \in \mathbb{R}$, define a relation $\sim$ as $x \sim y$ if and only if $\sin x = \cos\left(\frac{\pi}{2} - y\right)$. Is the given relation an equivalence relation? If yes, justify and list the equivalence classes. If no, given counterexamples for all the properties that fail.

Example 11.18. A relation $\sim$ is defined on $\mathbb{N}$ by $a \sim b$ if $a^2 + b^2$ is even. Prove that $\sim$ is an equivalence relation. Determine the distinct equivalence classes.

Example 11.19. The relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b$ if $a^2 \equiv b^2 \mod 4$ is known to be an equivalence relation. Determine the distinct equivalence classes.

Example 11.20. The relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b$ if $a^2 + 1 \equiv (b^2 + 1) \mod 5$ is known to be an equivalence relation. Determine the distinct equivalence classes.

Example 11.21. Show that the relation $\sim$ on $\mathbb{Z}$ defined by $m \sim n$ if $(m-3)^2 \equiv (n-3)^2 \mod 6$ is an equivalence relation. Determine the distinct equivalence classes.

Example 11.22. Give an example of an equivalence relation that could result in the following equivalence classes.

(a) Every set of antipodal points on the unit circle centered at the origin forms an equivalence class. (Antipodal points are opposite end points of a diameter of a circle.)

(b) $\{b \in \mathbb{Z} : b = 5k, \text{ for some } k \in \mathbb{Z}\}, \{b \in \mathbb{Z} : b = 5k+1 \text{ or } b = 5k+4, \text{ for some } k \in \mathbb{Z}\}, \{b \in \mathbb{Z} : b = 5k+2 \text{ or } b = 5k+3, \text{ for some } k \in \mathbb{Z}\}$

Example 11.23. Let the "alphabet set" be

$$A = \{``a", ``b", ``c", \ldots ``z"\},$$

and for $n \in \mathbb{N}$, define, A_n as the set of words – or more formally, strings – consisting of n letters from the alphabet set A. Formally, this could be

$$A_n = \{\text{string} = \theta_1 \cdots \theta_n \mid \theta_i \in A \text{ for } i = 1, \ldots, m\}.$$

Notice that both "eleven" and "xjastr" are strings of length 6 in A_6, so here we don't care if our strings are valid words in the English dictionary.
Consider the relation on A_5 where a string of 5 letters is related to another string of 5 letters if they have the same letters counted the same number of times. If a letter is repeated, the related strings have the same number of the letter repeated. For example, "burnt" $\sim$ "trubn", and "weave" $\sim$ "eewav", but "weave" $\nsim$ 'vawae". Is this an equivalence relation? If yes, prove your claim and state the equivalence classes. If no, justify.

Example 11.24. For $p, q \in \mathbb{P}_4$, consider the following relations $\sim$ on $\mathbb{P}_4$. Are they reflexive, transitive, and/or symmetric? Justify with proofs or counterexamples. If any of them are equivalence relations, state the equivalence classes.

(a) $p \sim q$ if and only if $p' = q'$

(b) $p \sim q$ if and only if $\int_0^x p(s)ds = \int_0^x q(s)ds$

(c) $p \sim q$ if and only if $p(0) = q(0)$

Example 11.25. Let $\mathbb{P}$ be the set of all polynomials with real coefficients. Define a relation $\sim$ on $\mathbb{P}$ as follows. Given $f, g \in \mathbb{P}$, let $f \sim g$ mean that f and g have the same degree. Thus $(x^2 + 3x - 4) \sim (3x^2 - 2)$ and $(x^3 + 3x^2 - 4) \not\sim (3x^2 - 2)$, for example, where $x \in \mathbb{R}$. Is $\sim$ an equivalence relation? Justify your answer. Now write down the equivalence class $[3x^2 + 2]$.

Week 12

This section contains examples related to partitions.

Example 12.1. Let $\sim$ be an equivalence relation on a nonempty set A and let a and b be elements of A. Then, prove that

$$[a] = [b] \text{ if and only if } a \sim b.$$

Use the above result to show the following:
Let $\sim$ be an equivalence relation defined on a nonempty set A. Show that the set

$$\{[a] : a \in A\}$$

of equivalence classes resulting from $\sim$ is a partition of A.

Example 12.2. Which of the following are partitions of $A = \{1, 2, 3, 4, 5\}$? For each collection of subsets that is not a partition of A, explain your answer.

(a) $S_1 = \{\{1, 3\}, \{2, 5\}\}$.

(b) $S_2 = \{\{1, 2\}, \{3, 4, 5\}\}$.

(c) $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$.

(d) $S_4 = A$.

Example 12.3. Give an example of a partition of $\mathbb{Q}$ into three subsets.

Example 12.4. Which if the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.

(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$.

(b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$.

(c) $S_3 = \{A\}$.

(d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$.

(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$.

Example 12.5. Is the following is a partition of $E = [0, \infty)$? If it is not a partition, explain why. If it is a partition, prove that it is.

$$\{A_i \mid A_{-1} = (1, \infty),\ A_0 = \{0\}, A_n = (\frac{1}{n+1}, \frac{1}{n}] \text{ for } n \in \mathbb{N}\}.$$

Example 12.6. Give an example of a partition of $\mathbb{P}_2$ that contains exactly 4 sets.

Example 12.7. Give an example of a partition of $\mathbb{N}$ into three subsets.

Example 12.8. Decide if the following is a partition of $\mathbb{R}$:

$$S = \{[m, m+1)\}_{m \in \mathbb{Z}} .$$

Justify your answer.

Example 12.9. Explain why the following is *not* a partition of $\mathbb{Z}$:

$$S = \{[m, m+1) \ : \ m \in \mathbb{Z} \ \} .$$

Example 12.10. Prove that for

$$E_n = \{10n, 10n+1, 10n+2 \ldots, 10n+9\},$$

$\{E_n\}_{n\in\mathbb{Z}}$ is a partition of $\mathbb{Z}$. Your answer should include a mention or usage of the Division Lemma (it is in the textbook). (Please use the definition for partition that appears as Definition 1.7 in the supplementary document.)

Example 12.11. Give a partition of $\mathbb{Z}$ that uses the equivalence classes of $\sim$, where $\sim$ is defined as $m \sim n$ is true if $(m-3)^2 \equiv (n-3)^2 \mod 6$ (and is defined to be false otherwise).

Example 12.12. Let $A_1 = \{1^k \mid k \in \mathbb{N}\}$, $A_2 = \{2^k \mid k \in \mathbb{N}\}$, and in general for $l \in \mathbb{N}$ let $A_l = \{l^k \mid k \in \mathbb{N}\}$. Is $\mathcal{T} = \{A_l \mid l \in \mathbb{N}\}$ a partition of $\mathbb{N}$. Justify your answer.

Example 12.13. Let $A_l = \{l, -l\}$ for each $l \in \mathbb{N} \cup \{0\}$. Prove that $\mathcal{T} = \{A_l \mid l \in \mathbb{N} \cup \{0\}\}$ is a partition of $\mathbb{Z}$.

Example 12.14. Let $A = \{x \in \mathbb{R} \mid x^2 > 1\}$. Let $B = \{x \in \mathbb{R} \mid x^2 \leq 1\}$. Is $\{A, B\}$ a partition of $\mathbb{R}$? Justify your answer.

Example 12.15. Let $A = \{x \in \mathbb{R} \mid \sqrt{x} > 1\}$. Let $B = \{x \in \mathbb{R} \ \mid \sqrt{x} \leq 1\}$. Is $\{A, B\}$ a partition of $\mathbb{R}$? If it is prove it, if it isn't give a subset of $\mathbb{R}$ where this is a partition.

Example 12.16. Let $A = \{a, b, c, d, e\}$.

(a) Find a partition of A into exactly two subsets.

(b) Can you find a partition of A into exactly two subsets where each of the subsets has the same cardinality. Justify your answer.

(c) Find a partition of A into exactly three subsets.

(d) Can you find a partition of A into exactly three subsets where each of the subsets has a different cardinality. Justify your answer.

Example 12.17. Let $m \in \mathbb{N}$ and let $A_m = \{n \in \mathbb{N} \mid n \leq m\}$. Prove that the following statement is true: m is even if and only if there exists a partition of A_m into exactly two subsets which have the same cardinality.

Week 13

this section is outdated as of sept 25 and needs to be fixed

Example 13.1. Use the Euclidean Algorithm to find the greatest common divisor for each of the following pairs of integers:

$$(a)\ 51 \text{ and } 288 \qquad (b)\ 357 \text{ and } 629 \qquad (c)\ 180 \text{ and } 252.$$

Example 13.2. Illustrate the Euclidean Algorithm for:

(a) $a = 22, b = -17$

(b) $a = 15, b = 98$