MTH 299 Transitions, Fall 2021 Homework 4

Question 1. Prove that the sum of the first n odd numbers is equal to n^2 by induction.

Question 2. Prove that $n^3 + 2n$ is divisible by 3, for all integers n, by induction.

Question 3. Prove the following formulas by induction.

(a) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

(b) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Question 4∗. Reprove the same two formulas via direct proofs, using the hints below.

(a) Write down *twice* the desired sum as

$$\begin{aligned} 1 + 2 + 3 + \cdots\cdots + n \\ +n + \cdots\cdots + 3 + 2 + 1. \end{aligned}$$

and add vertically first.

(b) Notice that the RHS[1] of the second formula is the square of the RHS of the first formula. Consider an increasing sequence of cubes in space, each formed of k^3 unit cubes, as k ranges from 1 to n. Can you rearrange the unit cube blocks into a square configuration (of height 1) in a systematic way that works for all n? The side length of the square should be $1 + 2 + 3 + \cdots + n$.

[1] RHS means "right hand side."

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1. We have some natural number n and will prove by induction that the sum of the first n odd numbers is equal to n^2. Let the function $f(n)$ output the sum of the first n odd numbers. We will use $n=1$ for our base case, which renders $f(n)=1=1^2$, so the claim holds for the base case. Our induction hypothesis is that $f(n)=n^2$ for some natural number n. If $f(n)=n^2$, then $f(n+1)=n^2+k$, where k is the odd number after the nth odd number. Since the nth odd number is 1 below the nth even number, the next odd number is 1 above the nth even number. The nth even number is $2n$, so the odd number after the nth odd number is $2n+1$, so $k=2n+1$, so $f(n+1)=n^2+2n+1$. $(n+1)^2=n^2+2n+1$ too, so $f(n+1)=(n+1)^2$, meaning the claim holds for each successive case. Therefore, by mathematical induction, the sum of the first n odd numbers is equal to n^2. Q.E.D.

2. We have some integer n and will prove by induction that n^3+2n is divisible by 3 for all n. We will use a base case $n=0$, which renders $n^3+2n=0$, and $3|0$, so the claim holds for the base case. Our induction hypothesis is that $3|(n^3+2n)$ for some $n\in\mathbb{Z}$. Since $3|(n^3+2n)$, we can write $n^3+2n=3m$ for some integer n, and then $(n+1)^3+2(n+1)=n^3+3n^2+3n+1+2n+2=n^3+2n+3n^2+3n+3$ $=3m+3n^2+3n+3=3(m+n^2+n+1)$, which is the product of 3 and some integer, so $3|((n+1)^3+2(n+1))$. Then $(n-1)^3+2(n-1)=n^3-3n^2+3n-1+2n-2=n^3+2n-3n^2+3n-3$ $=3m-3n^2+3n-3=3(m-n^2+n-1)$, which is the product of 3 and some integer, so $3|((n-1)^3+2(n-1))$. Since the

claim follows for $n+1$ and $n-1$ from any n for which it is true, and we have a true base case $n=0$, the claim follows for all $n \in \mathbb{Z}$ by mathematical induction, and therefore n^3+2n is divisible by 3 for all integers n. Q.E.D.

3\. a. We are given a natural number n and will prove by induction that $1+2+3+\cdots+n = n(n+1)/2$. Let the function $f(n) = 1+2+3+\cdots+n$. We will take the base case $n=1$. $f(1)=1$, and $1(1+1)/2 = 2/2 = 1$, so the claim holds for the base case. Our induction hypothesis is that $f(n) = n(n+1)/2$ for some natural number n. If it holds, then $f(n+1) = (n+1)(n+1+1)/2$ $= (n+1)(n+2)/2 = (n^2+3n+2)/2$. Since $f(n)$ is the sum $1+2+3+\cdots+n$, $f(n+1) = 1+2+3+\cdots+n+n+1$ $= f(n)+n+1 = n(n+1)/2+n+1 = (n^2+n)/2+n+1$, which can be simplified by multiplying the $n+1$ by $2/2$, as in $(n^2+n)/2+(2n+2)/2$, combining to $(n^2+3n+2)/2$. Thus $f(n+1)$ $= (n+1)(n+1+1)/2$, meaning the claim holds for each successive n, and as we have a base case $n=1$ for which the claim holds, it follows by mathematical induction that the claim holds for all $n \in \mathbb{Z}$. Therefore, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ holds true for all natural numbers n.

b. We are given a natural number n and will prove by induction that $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$. Let the function $f(n) = 1^3+2^3+3^3+\cdots+n^3$. We will take the base case $n=1$. $f(1) = 1^3 = 1$, and $1^2(1+1)^2/4 = 2^2/4 = 4/4 = 1$, so the claim holds for the base case. Our induction hypothesis is that $f(n) = n^2(n+1)^2/4$ for some natural number n.

If it holds, $f(n+1) = (n+1)^2(n+2)^2/4$

$= (n^2+2n+1)(n^2+4n+4)/4 = (n^4+6n^3+13n^2+12n+4)/4.$

Since $f(n) = 1^3+2^3+3^3+\dots+n^3$, $f(n+1) = 1^3+2^3+3^3+\dots+n^3+(n+1)^3$

$= f(n)+(n+1)^3 = n^2(n+1)^2/4+(n+1)^3$

$= n^2(n^2+2n+1)/4 + (n^3+3n^2+3n+1) = (n^4+2n^3+n^2)/4+(n^3+3n^2+3n+1)$

$= (n^4+2n^3+n^2)/4 + (4n^3+12n^2+12n+4)/4$

$= (n^4+6n^3+13n^2+12n+4)/4$. Thus $f(n+1)$

$= (n+1)^2(n+2)^2/4$, meaning the claim holds for each successive n, which includes all $n \in \mathbb{N}$ by induction because the base case $n=1$ holds. Therefore $1^3+2^3+3^3+\dots+n^3 = n^2(n+1)^2/4$ for all natural numbers n. Q.E.D.

4. a. We have some natural number n and will directly prove that $1+2+3+\dots+n = n(n+1)/2$. Denoting the sum as S, we can see that $2S$ could be written as the sum plus itself, rearranged:

$1 + 2 + 3 + \dots n$

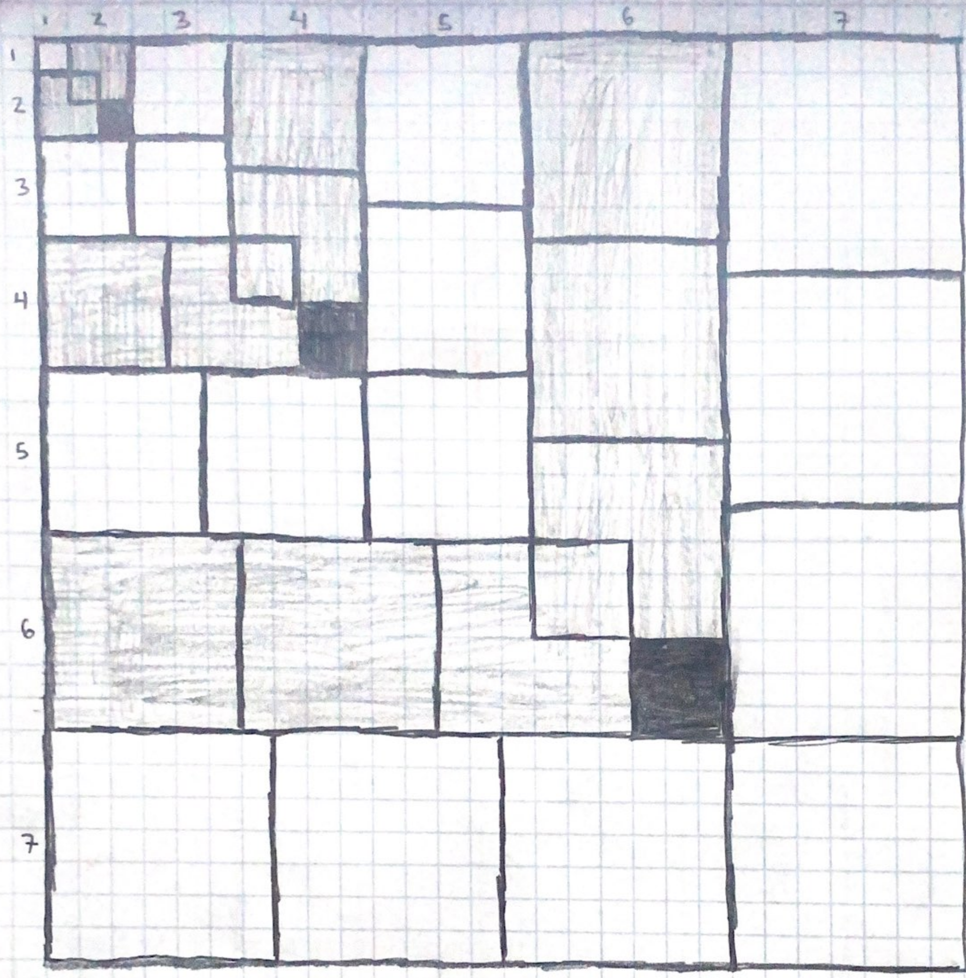
$+ n + (n-1) + (n-2) + \dots + 1$

$= (n+1)+(n+1)+(n+1)+\dots+(n+1).$

Since the sum has n terms, $2S = n(n+1)$, and so $S = n(n+1)/2$. Therefore, for any natural number n,

$1+2+3+\dots+n = n(n+1)/2$. Q.E.D.

4.b.



We are given a natural number n and will prove that $1^3+2^3+3^3+\dots+n^3 = n^2(n+1)^2/4$. Consider the above diagram. Imagine each gridded square as a cube of size 1. The square shape seen could then be formed by taking each "layer" of cubes with ascending side length — note there is 1 1×1 square, 2 2×2 squares, 3 3×3 squares, etc. Adding a cube adds a level to the square. Levels added with cubes of even side length are shaded for ease in examination. Note that even levels have gaps, marked by darker shading, but they also have overlap in equal size to the gap, so the square is indeed perfect. Now note that each layer adds to the side length the same as the length of the added cube. Thus the square's side length with n cubes is $1+2+3+\dots+n$, which we will call S, and its area is S^2. Due to how the square was constructed, its area is also the sum of the first n cubes, $1^3+2^3+3^3+\dots+n^3$, which we will call C. Thus the diagram shows that $C=S^2$ for $1 \le n \le 7$, that is, 7 points of intersection. $S=n(n+1)/2$, a quadratic, so S^2 is quartic, and C is a sum of cubes, so it is a polynomial of degree at most 4. That means 5 points of intersection would verify equality, and we have 7, so $C=S^2=(n(n+1)/2)^2=n^2(n+1)^2/4$. Therefore, for any natural number n, $1^3+2^3+3^3+\dots+n^3=n^2(n+1)^2/4$. Q.E.D.