Deriving Geodesics on a 3D Mesh from the Heat Equation

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Geodesics

Figure – An application of our method : a dromedary stepping on a hot rock

Varadhan's Formula I

Remember the Heat Equation :

$$\boxed{\Delta H = \frac{\partial}{\partial t} H} \tag{Heat}$$

We will use it to compute the geodesics, following [CWW12].

Varadhan's Formula II

Théorème 1.1: Varadhan's Formula [Var67]

If (M, g) is a complete Riemannian manifold, and $k_{t,x}(y)$ is the associated heat kernel, converging to $\delta_x(y)$ at small times, then :

$$\boxed{-4t \log k_{t,x}(y) \xrightarrow[t \to 0]{} d\left(x, y\right)^2} \tag{VF}$$

where d is the distance function on the manifolds.

Related Work I

The most widely used method is solving :

$$\|\nabla \varphi\| = 1 \text{ subject to boundary conditions } \varphi_{|\gamma} = 0 \qquad (1)$$

but it needs $\mathcal{O}(n \log n)$ and $\mathcal{O}(n)$ time per step in the Gauss-Seidel algorithm and loses information.

Related Work II

Another method based on Schrödinger's equation has been proposed in [GR14] which ressembles ours, but with more limitations and a need for arbitrary precision on $\hbar$ needed :

$$H\psi(x) = E\psi(x) \text{ where } H = -\frac{\hbar^2}{2m}\Delta + V(x) \tag{2}$$

Applications

- Image Analysis : [LB81] by Lantuejoul and Beucher
- Heuristically-driven Propagation : [PC09]
- Solidworks

The Algorithm

Algorithm The Heat Method

1. Integrate $\dot{u} = \Delta u$ for some fixed time t.
2. Evaluate the vector field $X = \frac{-\nabla u}{\|\nabla u\|}$.
3. Solve the Poisson equation $\Delta \varphi = \nabla \cdot X$.

Initial conditions allow for a single source point or any piecewise submanifold.

Discretizing Time I

We use a single backward Euler step :

$$(\mathrm{id} - t\Delta)\, u_t = u_0 \tag{3}$$

our problem is then solving :

$$\begin{aligned} (\mathrm{id} - t\Delta)\, v_t = 0 \quad &\text{on} \quad M \setminus \{x\} \\ v_t = 1 \quad &\text{on} \quad x \end{aligned} \tag{BP}$$

Discretizing Time II

The correction of the algorithm comes from the proof of Theorem 1.1 in [Var67] :

$$-\frac{\sqrt{t}}{2}\log v_t \underset{t\to 0}{=} \varphi \qquad (\delta t)$$

Discretizing Space I

We only need a gradient ∇, divergence $\nabla\cdot$ and laplacian Δ. We consider a triangulation $\left(\mathcal{V}, \mathcal{F} \subseteq \mathcal{V}^3\right)$.

Discretizing Space II

The gradient of u in a given triangle f is :

$$\nabla_f u = \frac{1}{2A_f} \sum_{i \in f} u_i \left(N \wedge e_{j_i} \right) \quad (4)$$

and, as a matrix of size $3\,|\mathcal{F}| \times |\mathcal{V}|$:

$$\begin{aligned} \left(\tilde{\nabla}\right)_{i,j} &= \left(J\vec{e}_{j_i}\right)_1 \\ \left(\tilde{\nabla}\right)_{i+|\mathcal{F}|,j} &= \left(J\vec{e}_{j_i}\right)_2 \\ \left(\tilde{\nabla}\right)_{i+2|\mathcal{F}|,j} &= \left(J\vec{e}_{j_i}\right)_3 \end{aligned} \quad (5)$$

Discretizing Space III

Then, we can compute the divergence operator's matrix as the transpose of the gradient for the face area dot product :

$$\nabla \cdot = \nabla^T A \tag{6}$$

where A is defined as previously as a $3\,|\mathcal{F}|$ diagonal matrix containing the areas of the faces.

Discretizing Space IV

Finally :

$$\Delta = \nabla \cdot \nabla = \nabla^T A \nabla \tag{7}$$

which is coherent with :

$$(Lu)_i = \frac{1}{2VA_i} \sum_{j \in \mathcal{N}(i)} \left(\cot \alpha_{i,j} + \cot \beta_{i,j}\right)\left(u_j - u_i\right) \tag{8}$$

We will call L_C the sum part of this operator. Then, retrieving our distance function amounts to solving : $\left(VA - tL_C\right) u = u_0$ and then $L_C \varphi = \nabla \cdot \left(\frac{-\nabla u}{\|\nabla u\|}\right)$.

Time Step I

Théorème 3.1: Graph Distance

Let $G = (V, E)$ be the graph induced by any real symmetric matrix A, and consider :

$$(\mathrm{Id} - tA)\, u_t = \delta$$

where δ is a Kronecker delta at a source vertex $u \in V$ and $t > 0$. Then :

$$\boxed{\varphi_G = \lim_{t \to 0} \frac{\log u_t}{\log t}}$$

Time Step II

For $t < \frac{1}{\sigma}$:

$$u_t = \sum_{k=0}^{\infty} t^k A^k \delta$$

Let $v \in V$ be n edges away from u and consider $r_t = |R_v| / |s_v|$ where :

$$s_v = (t^n A^n \delta)_v \neq 0, \qquad R_v = \left(\sum_{k=n+1}^{\infty} t^k A^k \delta \right)_v$$

Time Step III

Since :

$$|s| \leq \sum_{k \geq n+1} t^k \left\| A^k \delta \right\| \leq \sum_{k \geq n+1} t^k \sigma^k$$

and thus :

$$r_t \leq \frac{t^{n+1}\sigma^{n+1} \sum_{k=0}^{+\infty} t^k \sigma^k}{t^n \left(a^n \delta\right)_v} = c \frac{t}{1 - t\sigma}$$

Finally :

$$\log s_0 = n \log t + \log \left(A^n \delta\right)_v$$

Complexity

Our time complexity is :

$$\mathcal{O}\left(\left|\mathcal{F}\right|^2\right)$$

and our space complexity is :

$$\mathcal{O}\left(\left|\mathcal{F}\right|\left|\mathcal{V}\right|\right)$$

when taking our matrix representation as sparse as possible, while not increasing our time complexity too much.

Benchmarks I

We used the following mesh for the $[0,1] \times [0,1] \times 0$ square embedded in the 3-dimensional euclidean space :

$$\mathcal{V} = \left\{ \left(k\varepsilon, k'\varepsilon\right) \;\middle|\; 0 \leq k, k' \leq \frac{1}{\varepsilon} \right\}$$

$$\mathcal{F} = \left\{ \left(i, i+1, i+n\right), \left(i, i+n, i+n-1\right) \;\middle|\; i \leq \frac{1}{\varepsilon^2} \right\} \cap \mathcal{V}^3$$

Benchmarks II

Here, the average distance between two connected points is thus given by :

$$h = \frac{4}{6}\varepsilon + \frac{2}{6}\sqrt{2}\varepsilon = \frac{2+\sqrt{2}}{3}\varepsilon \simeq 1.14 \times \varepsilon$$

and thus $t = 1.3\varepsilon^2$.

Results in time Complexity I

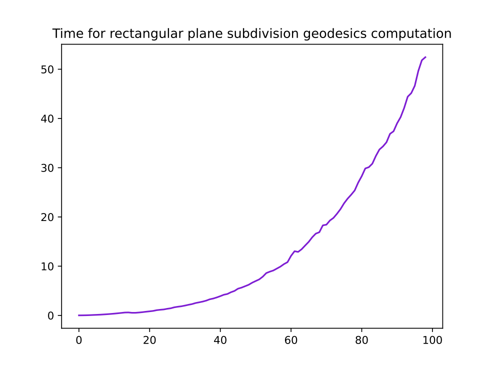

Figure – Computation times for evenly spaced triangular mesh of step $1/i$ on the unit square.

Results in time Complexity II

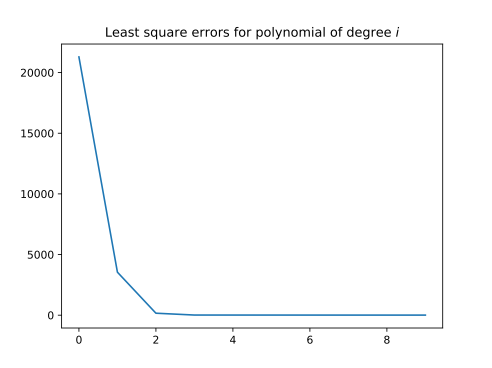

Figure – Errors of fit for polynomials of degree up to 10 for the computation time presented before, using the ℓ^2 norm.

Theoretical Accuracy I

Let us take our plane mesh defined by ε. The geodesic distance to 0 can be directly computed :

$d(x) = \|x\|$ and on indices $d_\varepsilon(i) = \varepsilon\sqrt{\left(i \mod \frac{1}{\varepsilon}\right)^2 + \left(i // \frac{1}{\varepsilon}\right)^2}$

Theoretical Accuracy II

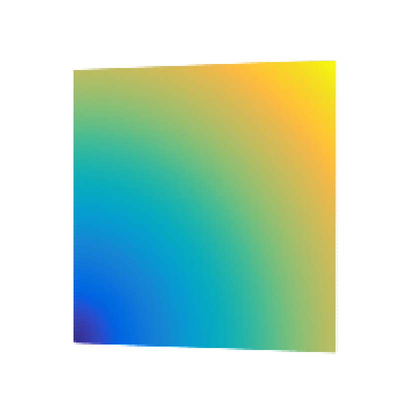

Figure – Illustration of the geodesic distance from 0 on the evenly spaced mesh of step $1/70$ on the plane

Theoretical Accuracy III

Then in theory :

$$d\left(i\right) - \varepsilon\varphi\left(i\right) \leq \frac{\varepsilon}{2}\left(\left(i \mod \frac{1}{\varepsilon}\right) + \left(i // \frac{1}{\varepsilon}\right)\right)$$

Practical difference to theoretical difference

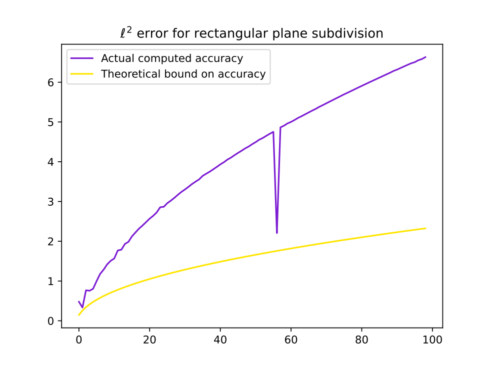

Figure – Theoretical bound on accuracy of the heat methods for regular meshes for the plane of step $1/i$

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