Homework 2

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1 Question 1

■ **Notation 1.1** For $I \subseteq E$ and $b \in B$, we will denote $I(b) = \{a \in A \mid (a,b) \in I\}$ and by $I(X) = \{a \in X \mid \exists b \in B, (a,b) \in I\}$.

We then define the matroids $\mathbb{A} = (E, \mathcal{A})$, $\mathbb{B} = (E, \mathcal{B})$ where :

$$\mathcal{A} = \{I \subseteq E \mid |I(a)| \leq 1 \forall a \in A\}$$
$$\mathcal{B} = \{I \subseteq E \mid I(b) \in \mathcal{M}_b \forall b \in B\}$$

We then see that $M \subseteq E$ is a A-perfect matching if and only if $|M| = |A|$ and M is an independent set of $\mathcal{A}$ and $\mathcal{B}$. Thus, we will call sets in $\mathcal{A} \cap \mathcal{B}$ independent matchings.

Then, since $|A| \geq \max_{I \in \mathcal{A}} |I|$, from Edmonds' mini-max formula on matroid intersection, we just need to have $\min_{I \subseteq E} r_{\mathcal{A}}(I) + r_{\mathcal{B}}(E \setminus I) \geq |A|$ to have the existence of a A-perfect matching.

We define $s : 2^E \to \mathbb{N}$ as :

$$s(I) = \sum_{b \in B} rank_{M_b}(I(b) \cap N(b)) \tag{1}$$

We see that the rank set in $\mathcal{B}$ can be seen as the ranks on each component (by separating edges on the $b \in \mathcal{B}$ they are connected to). Indeed, since $\mathcal{B}$ can be seen as a union of matroids (the M_b seen as matroids on the edges connected to b) we have, for $I \subseteq E$:

$$r_{\mathcal{B}}(I) = \min_{T \subseteq I} |I \setminus T| + s(T) = \min_{T \subseteq I} |I| - |T| + s(T)$$

Then plugging this into our main equation :

$$\begin{aligned} r_{\mathcal{A}}(E \setminus I) + r_{\mathcal{B}}(I) =& r_{\mathcal{A}}(E \setminus I) + \min_{T} |I| - |T| + s(T) \\ \geq& \min_{T} |I| - |T| + s(T) \\ =& \min_{T} |A| - |T(A)| + s(T) \end{aligned}$$

But since this should be greater than $|A|$ for all T and all I, it is equivalent to being true for all possible $A' = T(A)$ (and modifying the *type* of s accordingly, which doesn't change anything) and thus :

$$\boxed{\max_{I \in \mathcal{A} \cap \mathcal{B}} |I| = |A| \iff \forall A' \subseteq A, s(A') - |A'| \geq 0}$$

which is the wanted result.

2 Question 2

Let $F = 2^I$ and let us denote by $g : 2^{\mathcal{F}} \to \mathbb{R}^+$ the function that to a family of sets gives their combined profit. Clearly, g is submodular. Furthermore we denote by X_0 the empty set, and by X_i the set of items taken after i knapsacks were filled by our algorithm. Since we apply the FPTAS k times, and since g is submodular, we have :

$$g(X_i) - g(X_{i-1}) \geq (1-\varepsilon)\frac{OPT - g(X_{i-1})}{k} \tag{2}$$

for each i, where OPT is the weight of an optimal solution. Then, we have :

$$g(X_1) - g(X_0) = g(X_1) \geq (1-\varepsilon)\frac{OPT}{k} = OPT(1 - \left(1 - \frac{1}{k}\right) - \varepsilon) = OPT\left(1 - \left(1 - \frac{1}{k}\right) - \mathcal{O}(\varepsilon)\right) \tag{3}$$

and then :

$$\begin{aligned} g(X_2) \geq (1-\varepsilon)\frac{OPT - g(X_1)}{k} =& (1-\varepsilon) OPT\left(1 - \left(1 - \frac{1}{k}\right) - \varepsilon\right) \\ =& OPT\left(1 - \left(1 - \frac{1}{k}\right)^2 - \varepsilon\right) - OPT \times \varepsilon\left(1 - \left(1 - \frac{1}{k}\right) - \varepsilon\right) \\ =& OPT\left(1 - \left(1 - \frac{1}{k}\right)^2 - \mathcal{O}(\varepsilon)\right) \end{aligned}$$

By induction :

$$g(X_i i) \geq OPT\left(1 - \left(1 - \frac{1}{k}\right)^i - \mathcal{O}(\varepsilon)\right)$$

And thus :

$$g(X_k) \geq OPT\left(1 - \left(1 - \frac{1}{k}\right)^k - \mathcal{O}(\varepsilon)\right) \geq OPT\left(1 - \frac{1}{e} - \mathcal{O}(\varepsilon)\right)$$

3 Question 3

3.1 Part 1

I worked on this question with Mateo Torrents.

Let $\Delta_k = \Delta_{i \in [\![1,k]\!]} V_{f_j}$. We will consider increasing sets A_k of vertices to prove by induction : $\Delta_k \cap A_k = U \cap A_k$ or $V \setminus U \cap A_k$ Let $\mathcal{H} = H - (f_i)_{i \in [\![1,t]\!]}$. We will denote by C_v the component containing v in $\in \mathcal{H}$. We always have $C_v \subseteq U$ or $C_v \subseteq V \setminus U$. Let A_k such that :

- $A_1 = C_v$ for a certain $v \in V$
- $A_{k+1} = A_k \cup C_v$ for a certain $v \in \delta(A_k)$. We write $C_{k+1} = C_v$

We order the f_i such that the edge between A_k and C_{k+1} is f_k. We will now show the property by induction. It is clearly true for $k = 1$. Notice that $V_{f_{k+1}}$ contains only one of A_k and C_{k+1} and is disjoint from the other. Let $f_k = (u, v)$: when going from Δ_i to Δ_{i+1}, with $i < k$, then u, v stay in the same state (in or out of Δ_i). Then, only when adding V_{f_k} to the difference do u and v get treated differently. Therefore, $\Delta_k \cap A_k \subseteq U$ if and only if $\Delta_k \cap C_k \subseteq V \setminus U$. But when going to Δ_{k+1}, either u or v changes side, and thus we get $\Delta_{k+1} \cap A_k \subseteq U$ if and only if $\Delta_{k+1} \cap C_{k+1} \subseteq U$, hence keeping the proposition.

3.2 Part 2

Algorithme 1 Minimum Odd Size Cut

- First, we build the Gomory-Hu tree of our graph.
- Then, for each edge in the tree we consider both components formed by removing the edge.
- For every odd-sized such component, we retrieve the cut size (the label of the edge in the Gomory-Hu tree), if it's less than one we return True. If none are of cut size ≤ 1 then we return false.

This algorithm takes :

$$\mathcal{O}\left(\underbrace{(n-1)\times\text{max-flow}}_{\text{Gomory-Hu algorithm}} + \underbrace{n^2}_{\text{Check Sizes}} + \underbrace{n}_{\text{Retrieve Cut-size}}\right)$$

For correctness, we only need to show there is at least one V_f of odd size. Indeed if V_f is the minimum $u-v$ cut (for $u \in U$, $v \in V \setminus U$), then $w(V_f) \leq w(U)$ since U is a $u-v$ cut which gives the result if V_f is odd. Then to show one of the V_f is odd, we only need to see that if all of the V_f are even (as well as their complements), then both V and $\Delta_{i\in[\![1,t]\!]} V_{f_i}$ are even since $|A \Delta B| = |A| + |B| - 2\,|A \cap B|$. But since U is odd and V is even, $V \setminus U$ is odd and we have a contradiction. Thus, at least one of the V_f or $V \setminus V_f$ is odd.