Homework 1

Convex Optimization

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1 Exercises

Convex Sets

- Let $\mathcal{C} = \left\{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\right\}$. We can see that $\mathcal{C} = \left\{(x_1, x_2) \in \mathbb{R}^2_{+*} \mid x_1 \geq \frac{1}{x_2}\right\}$. Thus, $\mathcal{C}$ is the epigraph of $x \mapsto \frac{1}{x}$ which is convex since its derivative is $x \mapsto \frac{1}{x^4}$.
- In general, this set is not convex. Indeed, for sets in $\mathbb{R}$, take $S = \{-1, 1\}, T = \{0\}$. Then the set of points closer to S than to T is $\{x \in \mathbb{R} \mid x \leq -0.5 \vee x \geq 0.5\}$.
- Let us take $x_1, x_2 \in \{x \mid x + S_2 \subseteq S_1\}$. For $x = \lambda x_1 + (1 - \lambda)\, x_2$ consider $x + y$ for $y \in S_2$:

$$x + y = \lambda x_1 + (1 - \lambda)\, x_2 + y = \lambda(x_1 + y) + (1 - \lambda x_2 + y)$$

  Since $x_i + S_2 \subseteq S_1$ for $i = 1, 2$, and since S_1 is convex, for all $y \in S_2$ the above sum is in S_1 and thus $x + S_2 \subseteq S_2$, and our set is convex.
- Let $x_1, x_2 \in \{x \mid \exists y \in S_2, x + y \in S_1\}$. Then let y_1, y_2 be associated points to x_1, x_2. For $x = \lambda x_1 + (1 - \lambda)\, x_2$ and $y = \lambda y_1 + (1 - \lambda) y_2$. Then, since S_2 is convex, $y \in S_2$. Moreover :

$$x + y = \underbrace{\lambda \underbrace{(x_1 + y_1)}_{\in S_1} + (1 - \lambda) \underbrace{(x_2 + y_2)}_{\in S_1}}_{\in S_1 \text{ since } S_1 \text{ is convex}}$$

  Then, our set is convex.

Convex Functions

- The hessian of f at (x, y) is the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. On $\mathbb{R}^2$ this matrix is not positive nor negative and thus the function is neither convex, or concave. However, the upper level sets are convex (from our first example) and thus the function is quasi-concave.
- On $\mathbb{R}^2_{+*}$, the hessian of the function is positive semidefinite and thus the function is convex.
- The hessian matrix of f is the matrix $\begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$ Since its determinant is < 0 the Hessian is not positive semi-definite and the function is not convex. However, its sublevel sets are defined by the equations $x_1 \leq \alpha x_2$ and are thus convex (since half-planes). Thus, f is quasi-convex.

- Let us define the Löwner order $\preceq$ as the partial order defined by the convex cone of positive semi-definite matrices. We know that :
$$A \preceq B \Rightarrow B^{-1} \preceq A^{-1} \forall A, B \in S_{++}^n$$
From this, we know that for all $t \leq 1$:
$$((1-t)X + tY)^{-1} \preceq (1-t)X^{-1} + tY^{-1}$$
By linearity of the trace, we can now see that $X \mapsto \mathrm{Tr}(X^{-1})$ is convex.

Fenchel Conjugate

- We have :
$$f^*(y) = \sup_x \left({}^{\mathsf{t}}xy - \|x\|\right) = \begin{cases} 0 & \text{if } \|y\|_* \leq 1 \\ \infty & \text{otherwise} \end{cases}$$
To show this, let y such that $\|y\|_* = \sup_{\|x\| \leq 1} \|{}^{\mathsf{t}}xy\| \leq 1$. Then by Cauchy-Schwarz inequality, we know that ${}^{\mathsf{t}}xy \leq \|x\| \|y\|_* \leq \|x\|$ and the term in the supremum is always ≤ 0. If $\|y\|_* > 1$ however, then there exists z such that $\|z\| \leq 1$ such that ${}^{\mathsf{t}}zy > 1$. Taking $x = tz$ in the supremum, we know that $f^*(y) \geq t\left({}^{\mathsf{t}}zy - \|z\|\right)$ which goes to infinity with $t \to \infty$. Thus, the fenchel conjugate of a norm is the convex indicator of the unit ball of the dual norm.
- Let us denote the infimal convolution of g, h by $g \square h$. Then we will show that :
$$(g \square h)^* = (g^* + h^*)$$
Note that :
$$\begin{aligned}(g \square h)^*(\alpha) =& \sup_{x,y} \left\{{}^{\mathsf{t}}x\alpha - g(y) - h(x-y)\right\} \\ =& \sup_{x_1,x_2} \left\{{}^{\mathsf{t}}(x_1 + x_2)\alpha - g(x_1) - h(x_2)\right\} \\ =& \sup_{x_1,x_2} \left\{{}^{\mathsf{t}}x_1\alpha - g(x_1)\right\} + \sup_{x_2} \left\{{}^{\mathsf{t}}x_2\alpha - h(x_2)\right\} \\ =& g^* + h^*\end{aligned}$$
Given $g = \|\cdot\|_1$ and $h = \frac{1}{2\alpha} \|\cdot\|_2^2$. Let $x = u + v$. Alors :
$$f(x) = \inf_v \{g(x-v) + h(v)\}$$
Substituting the expressions, we get :
$$\begin{aligned}f(x) =& \inf_v \left\{\|x-v\|_1 + \frac{1}{2\alpha}\|v\|_2^2\right\} \\ =& \sum_{i=1}^n \inf_{v_i} \left\{|x_i - v_i| + \frac{1}{2\alpha} v_i^2\right\}\end{aligned}$$
We will now compute the infima independently. We have two cases :
  1. $v_i \leq x_i$: Let $\varphi(v_i) = x_i - v_i + \frac{1}{2\alpha} v_i^2$. We want to find a minimum for φ, which is found at $v_i = \alpha$. This solution is valid if $\alpha \leq x_i$.
  2. $v_i > x_i$: Let $\varphi(v_i = \frac{1}{2\alpha} + v_i - x_i$. We want to find a minimum for φ, which is at $v_i = -\alpha$. This solution is valid if $\alpha > x_i$.

Then, we have three possible cases for the optimal v_i :

1. If $x_i \geq \alpha$, $v_i = \alpha$
2. If $x_i \leq -\alpha$, $v_i = -\alpha$
3. If $-\alpha < x_i < \alpha$, $v_i = x_i$.

Plugging this into f :

$$f(x) = \sum_{i=1}^{n} \left(\begin{cases} \frac{1}{2\alpha} x_i^2 & |x_i| \leq \alpha \\ |x_i - \alpha| - \frac{\alpha}{2} & |x_i| > \alpha \end{cases} \right)$$