Formalizing Typing Rules for Natural Languages using Effects

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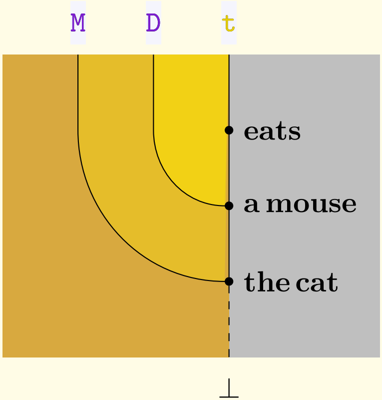
17th April 2025

Plan

What the hell am I doing?



Defining definites and indefinites

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Introduction

Mathematical Background

Effects, Semantics

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Definition 2.1 — Category. A (small) *category* is described by the following data:

0 A class of objects (nodes of a graph).

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1 For a pair A, B of objects, a set $\mathrm{Hom}(A, B)$ of functions from A to B called *morphisms*, *maps* or *arrows*. We denote it by $f : A \to B$ or $A \xrightarrow{f} B$.

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2 For all triplets A, B, C, a composition law $\circ_{A,B,C}$:

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2 For all objects A, an identity map $\mathrm{id}_A \in \mathrm{Hom}(A, A)$.

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- **Set** whose objects are Sets and arrows are functions between sets.
- **Top** whose objects are Topological Spaces and arrows are continuous maps between those.
- **Grp** whose objects are Groups and arrows are Group Homomorphisms.
- **Vec** whose objects are Vector Spaces on a field k and arrows are Linear Maps.

Commuter Rail, basically

The right language for categories is the commutative diagram one.

The associativity rewrites as:

$$\begin{tikzcd}
 & B \arrow[r, "g"] & C \arrow[dr, "h"] & \\
A \arrow[ur, "f"] \arrow[urr, "g\circ f"'] \arrow[rrr, "h\circ(g\circ f)"'] & & & D
\end{tikzcd}$$

Commuter Rail, basically

In Set we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\times 2"] \arrow[d, ".^2"'] & \mathbb{R} \arrow[d, ".^2"] \\
\mathbb{R}^+ \arrow[r, "\times 4"] & \mathbb{R}^+
\end{tikzcd}$$

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$$

It simply states that:

$$\forall x \in \mathbb{R}, \left(2x^2\right) = 4x^2$$

Of wolf, and man.

Definition 2.7 Let $\mathcal{A}, \mathcal{B}$ be two categories. A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is:

0 An object $F(A) \in \mathcal{B}$ for each object A of $\mathcal{A}$.

1 For each pair $A_1, A_2 \in \mathcal{A}$, a function:

$$\begin{array}{rrcl} F_{A_1,A_2} : & \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) & \to & \mathrm{Hom}_{\mathcal{B}}(FA_1, FA_2) \\ & f & \mapsto & F(f) \end{array}$$

Of wolf, and man.

We ask the following equations to be satisfied:

$F(g \circ f) = F(g) \circ F(f)$, that is

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "F"] & B \arrow[r, "f"] \arrow[d, "F"] & C \arrow[d, "F"] \\ FA \arrow[r, "Fg"'] & FB \arrow[r, "Ff"'] & FC \end{tikzcd}$$

and $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$

O.K. Corral

Definition 2.8 A natural transformation is a functor in the category of small categories and functors. If $F, G : \mathcal{A} \Rightarrow \mathcal{B}$ are functors, a natural transformation θ from F to G is, for each object of $\mathcal{A}$ a function θ_A such that the following diagram commutes for all $f : A \to B$.

$$
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] & FB \arrow[d, "\theta_B"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}
$$

O.K. Corral

Definition 2.9 An adjunction $L \dashv R$ between two functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ is a pair of natural transformations $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow R \circ L$ and $\varepsilon : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{B}}$ verifying the zigzag equations:

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{id}_{\mathcal{A}}"] \arrow[dr, "L"'] \arrow[dr, phantom, shift left=3ex, "\boxed{\eta}"] & \mathcal{A} \arrow[dr, "L"] \arrow[dr, phantom, shift right=2ex, "\varepsilon"] & & & \\
& \mathcal{B} \arrow[u, "R"] \arrow[r, "\mathrm{id}_B"'] & \mathcal{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left, "L"] \arrow[rr, bend right, "L"'] \arrow[rr, phantom, "\Downarrow \mathrm{id}_L"] & & \mathcal{B}
\end{tikzcd}
$$

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Yu-Gi-Oh

Definition 2.10 A product of two objects A and B in a category φ is a triplet

$$(A \times B, \pi_1 : A \times B \to A, \pi_2 : A \times B \to B)$$

$$\begin{tikzcd} A & & B \\ & A \times B \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

such that for all pair of arrows $X \xrightarrow{f} A$ et $X \xrightarrow{g} B$, there is a unique $h : X \to A \times B$ such that $f = \pi_1 \circ h, g = \pi_2 \circ h$.

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Definition 2.11 A terminal object $\mathbb{1}$ in a category $\Gamma \vdash$ is an object such that for all A in $\Gamma \vdash$ there is one and only one arrow $A \to \mathbb{1}$.

An initial object is the same thing with the arrows reversed.

Definition 2.12 A category is cartesian if all products exist and it has a terminal object $\mathbb{1}$.

Cartesian products and terminal objects are unique, up to isomorphism.

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Definition 2.13 A cartesian closed category is a cartesian category where we define for each object A a functor $A \Rightarrow \Gamma \vdash\!\rightarrow \Gamma \vdash$ right adjunct to $A \times \Gamma \vdash\!\rightarrow \Gamma \vdash$. This means we have bijections $\Phi_{X,Y} : \mathrm{Hom}\left(A \times X, Y\right) \rightarrow \mathrm{Hom}\left(X, A \Rightarrow Y\right)$ called currification bijections..

Yu-Gi-Oh 3

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Set is a cartesian closed category, where currification is defined by the partial application of high arity functions to a subset of their arguments.

Use the Types, Luke

Definition 2.15 The λ-calculus is defined by the following basic grammar:

$$\begin{array}{rcll} \mathsf{E} & ::= & x \in Var & \textit{(Variables)} \\ & \mid & App(E, E) & \textit{(Application)} \\ & \mid & \lambda x.E & \textit{(Evaluation)} \end{array}$$

Use the Types, Luke

Definition 2.16 We give ourselves a set of type variables $TyVar$ and define types by:

$$\begin{array}{rcl} A, B & ::= & \alpha \in TyVar \\ & | & A \times B \\ & | & \mathbb{1} \\ & | & A \Rightarrow B \end{array}$$

A context $M = x_1 : A_1, \ldots, x_n : A_n$ is a list of pairs $x_i : A_i$ with a variable x_i and a type A_i, all the variables being different.

Fudge Supreme

Definition 2.17 A typing $\Gamma \vdash M : A$ is a triplet composed of a context Γ, a λ-term M and a type A, such that all free variables of M are in Γ. A proof tree for a typing judgement is constructed inductively from a set of rules of the form:

$$\frac{}{x : A \vdash x : A} \text{ Var}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B} \text{ Lam}$$

$$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Delta \vdash N : A}{\Gamma, \Delta \vdash App(M, N) : B} \text{ App}$$

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