

A Diagrammatic Calculus for a Functional Model of Natural Language Semantics

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Abstract

In this paper, we study a functional programming approach to natural language semantics, allowing us to increase the expressivity of a more traditional denotation style. We will formalize a category based type and effect system, and construct a diagrammatic calculus to model parsing and handling of effects, and use it to efficiently compute the denotations for sentences.

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1 Introduction

What is *a chair*? How do I know that *Jupiter, a planet*, is *a planet*? To answer those questions, [2] provide a HASKELL based view on the notion of typing in natural language semantics. Their main idea is to include a layer of effects which allows for improvements in the expressiveness of the denotations used. This allows to model complex concepts such as anaphoras, or non-determinism in an easy way, independent of the actual way the words are represented. Indeed, when considering the usual denotations of words as typed lambda-terms, this allows us to solve the issue of meaning getting lost through impossible typing, while still being able to compose meanings properly. When two expressions have the same syntactic distribution, they must also have the same type, which forces quantificational noun phrases to have the same type as proper nouns: the entity type **e**. However, there is no singular entity that is the referent of *every planet*, and so, the type system gets in the way of meaning, instead of being a tool at its service.

Our formalism is inscribed in the contemporary natural language semantic theories which are based on three main elements: a *lexicon*, a *syntactic description* of the language, and a theory of *composition*. More specifically, we explain how to extend the domain of the lexicon and the theory of composition to account for the phenomena described above. We will not be discussing most of the linguistic foundations for the usage of the formalism, nor its usefulness. We refer the reader to [2] to get an overview of the linguistic considerations that are the base of the theory.

In this paper, we will provide a formal definition of an enhanced type and effect system for natural language semantics, based on categorical tools. This will increase the complexity (both in terms of algorithmic operations and in comprehension of the model) of the parsing algorithms, but through the use of string diagrams to model the effect of composition on potential effectful denotations (or more generally computations), we will provide efficient algorithms for computing the set of meanings of a sentence, from the meaning of its components.



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2 Related Work

This is not the first time a categorical representation of compositional semantics of natural language is proposed, [3] already suggested an approach based on monoidal categories using an external model of meaning. What our approach gives more, is additional latitude for the definition of denotations in the lexicon, and a visual explanation of the difference between multiple possible parsing trees. We will go back later on the differences between their Lambek inspired grammars and our more abstract way of looking at the semantic parsing of a sentence.

On a completely different approach, [6] provide a categorical structure based on Hopf algebra and coloured operads to explain their model of syntax. Similarly, [9] provides a modeling of CFGs using coloured operads. Our approach is based on the suggestion that merge in syntax can be done using labels, independent on how it is mathematically modelled.

3 Categorical Semantics of Effects: A Typing System

In this section, we will designate by \mathcal{L} our language, as a set of words (with their semantics) and syntactic rules to combine them semantically. We denote by $\mathcal{O}(\mathcal{L})$ the set of words in the language whose semantic representation is a low-order function and $\mathcal{F}(\mathcal{L})$ the set of words whose semantic representation is a functor or high-order function. Our goals here are to describe more formally, using a categorical vocabulary, the environment in which the typing system for our language will exist, and how we connect words and other linguistic objects to the categorical formulation.

3.1 Typing Category

3.1.1 Types

Let \mathcal{C} be a closed cartesian category, which will be used to represent the domain of types for our domain of denotations. This represents our main typing system, consisting of words $\mathcal{O}(\mathcal{L})$ that can be expressed without effects. Remember that \mathcal{C} contains a terminal object \perp representing the empty type or the lack thereof. We consider $\bar{\mathcal{C}}$ the category closure of $\mathcal{F}(\mathcal{L})(\mathcal{C})$, that is consisting of all the different type constructors (ergo, functors) that could be formed in the language. What this means is that we consider for our category objects any object that can be attained in a finite number from a finite number of functorial applications from an object of \mathcal{C} .

We consider for our types the quotient set $\star = \text{Obj}(\bar{\mathcal{C}}) / \mathcal{F}(\mathcal{L})$. Since $\mathcal{F}(\mathcal{L})$ does not induce an equivalence relation on $\text{Obj}(\bar{\mathcal{C}})$ but a preorder, we consider the chains obtained by the closure of the relation $x \succeq y \Leftrightarrow \exists F, y = F(x)$ (which is seen as a subtyping relation as proposed in [8]). We also define \star_0 to be the set obtained when considering types which have not yet been *affected*, that is $\text{Obj}(\mathcal{C})$. In contexts of polymorphism, we identify \star_0 to the adequate subset of \star . In this paradigm, constant objects (or results of fully done computations) are functions with type $\perp \rightarrow \tau$ which we will denote directly by $\tau \in \star_0$.

3.1.2 Functors, Applicatives and Monads

Our point of view leads us to consider *language functors*¹ as polymorphic functions: for a (possibly restrained, though it seems to always be \star) set of base types S , a functor is a function

$$x : \tau \in S \subseteq \star \mapsto Fx : F\tau$$

This means that if a functor can be applied to a type, it can also be applied to all *affected* versions of that type, i.e. $\mathcal{F}(L)(\tau \in \star)$. This gives us two typing judgements for the functor F :

$$\frac{\Gamma \vdash x : \tau \in \star_0}{\Gamma \vdash Fx : F\tau \notin \star_0} \quad \frac{\Gamma \vdash x : \tau}{\Gamma \vdash Fx : F\tau \preceq \tau}$$

We use the same notation for the *language functor* and the *type functor* in the examples, but it is important to note those are two different objects, although connected. More precisely, the *language functor* is to be seen as a function whose computation yields an effect, while the *type functor* is the endofunctor of $\bar{\mathcal{C}}$ (so a functor from \mathcal{C}) that represents the effect in our typing category.

In that sense, applicatives and monads only provide with more flexibility on the ways to combine functions: they provide intermediate judgements to help with the combination of trees. For example, the multiplication of the monad provides a new *type conversion* judgement:

$$\frac{\Gamma \vdash x : MM\tau}{\Gamma \vdash x : M\tau \succeq MM\tau}$$

This is actually a special case of the natural transformation rule that we define below, which means that, in a way, types $MM\star$ and $M\star$ are equivalent, as there is a canonical way to go from one type to another. Remember however that $M\star$ is still a proper subtype of $MM\star$ and that the objects are not actually equal: they are simply equivalent.

3.1.3 Natural Transformations

We could also add judgements for adjunctions, but the most interesting thing is to add judgements for natural transformations, as adjunctions are particular examples of natural transformations which arise from *natural* settings. While in general we do not want to find natural transformations, we want to be able to express these in three situations:

1. Adjunctions.
2. To deal with the resolution of effects as explained in Section 4
3. To create *higher-order* constructs which transform words from our language into other words, while keeping the functorial aspect. This idea is developed in Section 3.1.3.2.

To see why we want this rule, which is a larger version of the monad multiplication and the monad/applicative unit, it suffices to see that the diagram defining the properties of a natural transformation provides a way to construct the *correct function* on the *correct functor* side of types.

In the Haskell programming language, any polymorphic function is a natural transformation from the first type constructor to the second type constructor, as proved by [12]. This will guarantee for us that given a *Haskell* construction for a polymorphic function, we will get the associated natural transformation.

¹ The elements of our language, not the categorical construct.

120 3.1.3.1 Handlers

121 As introduced by [7], the use of handlers as annotations to the syntactic tree of the sentence
 122 is an appropriate formalism. This could also give us a way to construct handlers for our
 123 effects as per [1], or [10]. As considered by [13] and [11], handlers are to be seen as natural
 124 transformations describing the free monad on an algebraic effect. Considering handlers as so,
 125 allows us to directly handle our computations inside our typing system, by “transporting” our
 126 functors one order higher up without loss of information or generality since all our functors
 127 undergo the same transformation. Using the framework proposed in [11] we simply need
 128 to create handlers for our effects/functors and we will then have in our language the result
 129 needed. The only thing we will require from an algebraic handler h is that for any applicative
 130 functor of unit η , $h \circ \eta = \text{id}$.

131 Note that the choice of the handler being part of the lexicon or the parser over the
 132 other is a philosophical question more than a semantical one, as both options will result
 133 in semantically equivalent models, the only difference will be in the way we consider the
 134 resolution of effects. This choice does not arise in the case of the adjunction-induced handlers.
 135 Indeed here, the choice is caused by the non-uniqueness of the choices for the handlers as
 136 two different speakers may have different ways to resolve the non-determinism effect that
 137 arises from the phrase *A chair*. This is the difference with the adjunctions: adjunctions are
 138 intrinsic properties of the coexistence of the effects, while the handlers are user-defined.

139 3.1.3.2 Higher-Order Constructs

140 Functors may also be used to add plurals, superlatives, tenses, aspects and other similar con-
 141 structs which act as function modifiers. For each of these, we give a functor Π corresponding
 142 to a new class of types along with natural transformations for all other functors F which
 143 allows to propagate down the high-order effect. This transformation will need to be from
 144 $\Pi \circ F$ to $\Pi \circ F \circ \Pi$ or simply $\Pi \circ F \Rightarrow F \circ \Pi$ depending on the situation. This allows us to
 145 add complexity not in the compositional aspects but in the lexicon aspects, by simply saying
 146 that those constructs are predicate modifiers passed down (with or without side effects) to
 147 the arguments of predicates:

$$\begin{aligned}
 &\text{future}(\text{be})(\text{arg}_1, \text{arg}_2) \xrightarrow{\eta} \text{future}(\text{be})(\text{arg}_2)(\text{future}(\text{arg}_1)) \\
 &\xrightarrow{\eta} \text{future}(\text{be})(\text{future}(\text{arg}_2))(\text{future}(\text{arg}_1))
 \end{aligned}$$

149 3.1.3.3 Monad Transformers

150 In [2], the authors present constructions which they call monad transformers or *higher-order*
 151 *constructors* and which take a monad as input and return a monad as output. One way to
 152 type those easily would be to simply create, for each such construct, a monad (the result
 153 of the application to any other monad) and a natural transformation which mimics the
 154 application and can be seen as the constructor.

155 3.2 Typing Judgements

156 To complete this section, Figure 1 gives a simple list of different typing composition judgements
 157 through which we also re-derive the subtyping judgement to allow for its implementation.
 158 Note that here, the syntax is not taken into account: a function is always written left of its
 159 arguments, whether or not they are actually in that order in the sentence.

$$\begin{array}{c}
\frac{\Gamma \vdash x : \tau \quad \Gamma \vdash F : S \subseteq \star \quad \overbrace{\tau \in S}^{\exists \tau' \in S, \tau \preceq \tau'}}{\Gamma \vdash Fx : F\tau \preceq \tau} \text{Cons} \\
\\
\frac{\Gamma \vdash x : \tau \quad \tau \in \star_0}{\Gamma \vdash Fx : F\tau \notin \star_0} FT_0 \\
\\
\frac{\Gamma \vdash x : F\tau_1 \quad \Gamma \vdash \varphi : \tau_1 \rightarrow \tau_2}{\Gamma \vdash \varphi x : F\tau_2} \text{fmap} \\
\\
\frac{\Gamma \vdash x : A\tau_1 \quad \Gamma \vdash \varphi : A(\tau_1 \rightarrow \tau_2)}{\Gamma \vdash \varphi x : A\tau_2} \langle * \rangle \\
\\
\forall F \xRightarrow{\theta} G, \frac{\Gamma \vdash x : F\tau \quad \Gamma \vdash G : S' \subseteq \star \quad \tau \in S'}{\Gamma \vdash x : G\tau} \text{nat}
\end{array}$$

■ **Figure 1** Typing and subtyping judgements for implementation of effects in the type system.

160 Using these typing rules for our semantic parsing steps, it is important to see that our
 161 grammar will still be ambiguous. The next sections will explain how to reduce this ambiguity
 162 in quick enough time.

163 Moreover, the observant reader might have noticed that our typing system is not decidable,
 164 because of the **nat/pure/return** rules which may allow for unbounded derivations. This is
 165 not actually an issue because of considerations on handling, as semantically void units will
 166 get removed at that time. This leads to derivations of sentences to be of bounded height,
 167 linear in the length of the sentence.

168 4 Handling Ambiguity

169 The typing judgements proposed in Section 3.2 lead to ambiguity. In this section we propose
 170 ways to get our derivations to a certain normal form, by deriving an equivalence relation on
 171 our derivation and parsing trees, based on string diagrams.

172 4.1 String Diagram Modelisation of Sentences

173 String diagrams are the Poincaré duals of the usual categorical diagrams when considered
 174 in the 2-category of categories and functors. This means that we represent categories as
 175 regions of the plane, functors as lines separating regions and natural transformations as the
 176 intersection points between two lines.

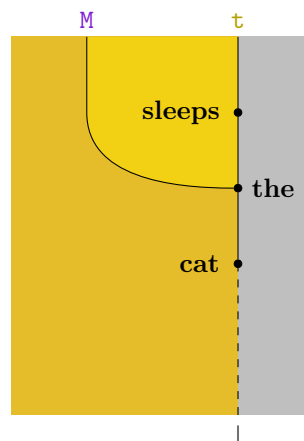
177 We will always consider application as applying to the right of the line so that composition
 178 is written in the same way as in equations. This gives us a new graphical formalism to
 179 represent our effects using a few equality rules between diagrams. The commutative aspect of
 180 functional diagrams is now replaced by an equality of string diagrams, which will be detailed
 181 in the following section.

182 We get a way to visually see the meaning get reduced from effectful composition to
 183 propositional values, without the need to specify what the handler does. This delimits our
 184 usage of string diagrams as ways to look at computations and a tool to provide equality rules
 185 to reduce ambiguity.

Let us define the category $\mathbb{1}$ with exactly one object and one arrow: the identity on that object. It will be shown in grey in the string diagrams below. A functor of type $\mathbb{1} \rightarrow \mathcal{C}$ is equivalent to choosing an object in \mathcal{C} , and a natural transformation between two such functors τ_1, τ_2 is exactly an arrow in \mathcal{C} of type $\tau_1 \rightarrow \tau_2$. Knowing that allows us to represent the type resulting from a sequence of computations as a sequence of strings whose farthest right represents an object in \mathcal{C} , that is, a base type.

The question of providing rules to compose the string diagrams for parts of the sentences will be discussed in the next section, as it is related to parsing.

In the end, we will have the need to go from a certain set of strings (the effects that applied) to a single one, through a sequence of handlers, monadic and comonadic rules and so on. Notice that we never reference the zero-cells and that in particular their colors are purely an artistic touch.



4.2 Achieving Normal Forms

We will now provide a set of rewriting rules on string diagrams (written as equations) which define the set of different possible reductions.

First, Theorem 1 reminds the main result by [5] about string diagrams which shows that our artistic representation of diagrams is correct and does not modify the equation or the rule we are presenting.

► **Theorem 1** (Theorem 1.2 [5]). *A well-formed equation between morphism terms in the language of monoidal categories follows from the axioms of monoidal categories if and only if it holds, up to planar isotopy, in the graphical language.*

Secondly, let us now look at a few of the equations that arise from the commutation of certain class of diagrams:

The Snake Equations are a rewriting of the categorical diagrams which are the defining properties of an adjunction.

The (co-)Monadic Equations are the string diagrammatic translation of the properties of unitality and associativity of the monad. Similarly, there are co-monadic equations which are the categorical dual of the previous equations.

This set of equations, when added to our reduction rules from the Section 5 explain all the different reductions that can be made to limit non-determinism in our parsing strategies. Indeed, considering the equivalence relation \mathcal{R} freely generated from the equations defined above and the equivalence relationship \mathcal{R}' of planar isotopy from Theorem 1, we get a set of normal forms \mathcal{N} from the set of all well-formed parsing diagrams \mathcal{D} defined by:

$$\mathcal{N} = (\mathcal{D}/\mathcal{R})/\mathcal{R}'$$

4.3 Computing Normal Forms

Now that we have a set of rules telling us what we can and cannot do in our model while preserving the equality of the diagrams, we provide a combinatorial description of our

diagrams to help compute the possible equalities between multiple reductions of a sentence meaning.

[4] proposed a combinatorial description to check in linear time for equality under Theorem 1. However, this model does not suffice to account for all of our equations, especially as labelling will influence the equations for monads, comonads and adjunctions. To provide with more flexibility (in exchange for a bit of time complexity) we use the description provided and change the description of inputs and outputs of each 2-cell by adding types and enforcing types. In this section we formally describe the data structure we propose, as well as algorithms for validity of diagrams and a system of rewriting that allows us to compute the normal forms for our system of effects.

4.3.1 Representing String Diagrams

We follow [4] in their combinatorial description of string diagrams. We describe a diagram by an ordered set of its 2-cells (the natural transformations) along with the number of input strings, for each 2-cell the following information:

- Its horizontal position: the number of strings that are right of it.
- Its type: an array of effects that are the inputs to the natural transformation and an array of effects that are the outputs to the natural transformation.

We will then write a diagram D as a tuple of 3 elements: $(D.N, D.S, D.L)$ where $D.N$ is a positive integers representing the height (or number of nodes) of D , $D.S$ is an array for the input strings of D and where $D.L$ is a function which takes a natural number smaller than $D.N - 1$ and returns its type as a tuple of arrays $nat = (nat.h, nat.in, nat.out)$. From this, we can derive a naive algorithm to check if a string diagram is valid or not.

Since our representation contains strictly more information (without slowing access by a non-constant factor) than the one it is based on, our datastructure supports the linear and polynomial time algorithms proposed with the structure by [4]. In particular our structure can be normalized in time $\mathcal{O}(n \times \sup_i |D.L(i).in| + |D.L(i).out|)$, which depends on our lexicon but most of the times will be linear time.

4.3.2 Equational Reductions

We are faced a problem when computing reductions using the equations for our diagrams which is that by definition, an equation is symmetric. To solve this issue, we only use equations from left to right to reduce as much as possible our result instead. This also means that trivially our reduction system computes normal forms: it suffices to re-apply the algorithm for recumbent equivalence after the rest of equational reduction is done. Moreover, note that all our reductions are either incompatible or commutative, which leads to a confluent reduction system, and the well definition of our normal forms:

► **Theorem 2 (Confluence).** *Our reduction system is confluent and therefore defines normal forms:*

1. *Right reductions are confluent and therefore define right normal forms for diagrams under the equivalence relation induced by exchange.*
2. *Equational reductions are confluent and therefore define equational normal forms for diagrams under the equivalence relation induced by exchange.*

Before proving the theorem, let us first provide the reductions for the different equations for our description of string diagrams.

273 **The Snake Equations** First, let's see when we can apply the equation for id_L to a diagram
 274 D which is in *right* normal form, meaning it's been right reduced as much as possible.
 275 Suppose we have an adjunction $L \dashv R$. Then we can reduce D along the equation at i if,
 276 and only if:

- 277 ■ $D.L(i).h = D.L(i+1).h - 1$
- 278 ■ $D.L(i) = \eta_{L,R}$
- 279 ■ $D.L(i+1) = \varepsilon_{L,R}$

280 This comes from the fact that we can't send either ε above η using right reductions
 281 and that there cannot be any natural transformations between the two. Obviously the
 282 equation for id_R works the same. Then, the reduction is easy: we simply delete both
 283 strings, removing i and $i+1$ from D and reindexing the other nodes.

284 **The Monadic Equations** For the monadic equations, we only use the unitality equation as
 285 a way to reduce the number of natural transformations, since the goal here is to attain
 286 normal forms and not find all possible reductions. We ask that associativity is always
 287 used in the direct sense $\mu(\mu(TT), T) \rightarrow \mu(T\mu(TT))$ so that the algorithm terminates.
 288 We use the same convention for the comonadic equations. The validity conditions are
 289 as easy to define for the monadic equations as for the *snake* equations when considering
 290 diagrams in *right* normal forms. Then, for unitality we simply delete the nodes and for
 291 associativity we switch the horizontal positions for i and $i+1$.

292 **Proof of the Confluence Theorem.** The first point of this theorem is exactly Theorem 4.2
 293 in [4]. To prove the second part, note that the reduction process terminates as we strictly
 294 decrease the number of 2-cells with each reduction. Moreover, our claim that the reduction
 295 process is confluent is obvious from the associativity equation and the fact the other equations
 296 delete nodes. Since right reductions do not modify the equational reductions, and thus right
 297 reducing an equational normal form yields an equational normal form, combining the two
 298 systems is done without issue, completing our proof of Theorem 2. ■

299 ► **Theorem 3 (Normalization Complexity).** *Reducing a diagram to its normal form is done in*
 300 *polynomial time in the number of natural transformations in it.*

301 **Proof.** Let's now give an upper bound on the number of reductions. Since each reductions
 302 either reduces the number of 2-cells or applies the associativity of a monad, we know the
 303 number of reductions is linear in the number of natural transformations. Moreover, since
 304 checking if a reduction is possible at height i is done in constant time, checking if a reduction
 305 is possible at a step is done in linear time, rendering the reduction algorithm quadratic in
 306 the number of natural transformations. Since we need to *right* normalize before and after
 307 this method, and that this is done in linear time, our theorem is proved. ■

308 5 Efficient Semantic Parsing

309 In this section we explain our algorithms and heuristics for efficient semantic parsing with as
 310 little non-determinism as possible, and reducing time complexity of our parsing strategies.

311 5.1 Syntactic-Semantic Parsing

312 Using a naïve strategy on the trees yields an exponential algorithm, the best way to improve
 313 the algorithm, is to directly leave the naïve strategy. To do so, we could simply extend the
 314 grammar system used to do the syntactic part of the parsing. In this section, we will take
 315 the example of a CFG since it suffices to create our typing combinators, In Figure 2, we

$>, \beta$	$::= (\alpha \rightarrow \beta), \alpha$	$ML_F(\alpha, \beta)$	$::= F\alpha, \beta$
$<, \beta$	$::= \alpha, (\alpha \rightarrow \beta)$	$MR_F(\alpha, \beta)$	$::= \alpha, F\beta$
$\wedge, \alpha \rightarrow \mathbf{t}$	$::= (\alpha \rightarrow \mathbf{t}), (\alpha \rightarrow \mathbf{t})$	$A_F(\alpha, \beta)$	$::= F\alpha, F\beta$
$\vee, \alpha \rightarrow \mathbf{t}$	$::= (\alpha \rightarrow \mathbf{t}), (\alpha \rightarrow \mathbf{t})$	$UR_F(\alpha \rightarrow \alpha', \beta)$	$::= F\alpha \rightarrow \alpha', \beta$
		$UL_F(\alpha, \beta \rightarrow \beta')$	$::= \alpha, F\beta \rightarrow \beta'$
$J_F F\tau$	$::= FF\tau$	$C_{LR}(L\alpha, R\beta)$	$::= (\alpha, \beta)$
$DN_C \tau$	$::= C_\tau \tau$	$ER_R(R(\alpha \rightarrow \alpha'), \beta)$	$::= \alpha \rightarrow R\alpha', \beta$
		$EL_R(\alpha, R(\beta \rightarrow \beta'))$	$::= \alpha, \beta \rightarrow R\beta'$

■ **Figure 2** Possible type combinations in the form of a near CFG

$\geq = \lambda\varphi.\lambda x.\varphi x$	$UR_F = \lambda M.\lambda\varphi.\lambda y.M(\lambda a.\varphi(\eta_F a), y)$
$\leq = \lambda x.\lambda\varphi.\varphi x$	$J_F = \lambda M.\lambda x.\lambda y.\mu_F M(x, y)$
$ML_F = \lambda M.\lambda x.\lambda y.(\mathbf{fmap}_F \lambda a.M(a, y))x$	$C_{LR} =$
$MR_F = \lambda M.\lambda x.\lambda y.(\mathbf{fmap}_F \lambda b.M(x, b))y$	$\lambda M.\lambda x.\lambda y.\varepsilon_{LR}(\mathbf{fmap}_L(\lambda l.\mathbf{fmap}_R(\lambda r.M(l, r))(y))(x))$
$A_F = \lambda M.\lambda x.\lambda y.(\mathbf{fmap}_F \lambda a.\lambda b.M(a, b))(x) < * > y$	$EL_R = \lambda M.\lambda\varphi.\lambda y.M(\Upsilon_R \varphi, y)$
$UL_F = \lambda M.\lambda x.\lambda\varphi.M(x, \lambda b.\varphi(\eta_F b))$	$ER_R = \lambda M.\lambda x.\lambda\varphi.M(x, \Upsilon_R \varphi)$
	$DN_\Downarrow = \lambda M.\lambda x.\lambda y.\Downarrow M(x, y)$

■ **Figure 3** Denotations describing the effect of the combinators used in the grammar describing our combination modes presented in Figure 2

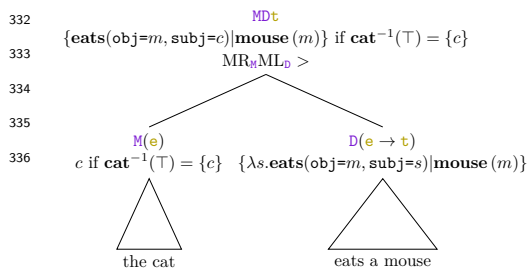
explicit a grammar of combination modes, based on [2] as it simplifies the rewriting of our typing judgements in a CFG.

This grammar works in five major sections:

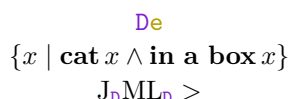
1. We reintroduce the grammar defining the type and effect system.
2. We introduce a structure for the semantic parse trees and their labels, the combination modes from [2].
3. We introduce rules for basic type combinations.
4. We introduce rules for higher-order unary type combinators.
5. We introduce rules for higher-order binary type combinators.

We do not prove here that these typing rules cannot be written in a simpler syntactic structure, but it is easy to see why a regular expression would not work, and since we need trees, to express our full system, the best we can do would be to disambiguate the grammar.

Each of these combinators can be, up to order, associated with a inference rule, and, as such, with a higher-order denotation, which explains the actual effect of the combinator, and are described in Figure 3.



(a) Labelled tree representing the equivalent parsing diagram to 5



The main reason we need to get denotations associated to combinators, is to properly define how they actually do the combination of denotations. The important thing on those derivation is that they're a direct translation of the rules defining the notions of functors, applicatives, monads

and thus are not specific to any denotation system, even though we will use lambda-calculus styled denotations to describe them. This makes us able to compute the actual denotations associated to a sentence using our formalism, as presented in figure 4.

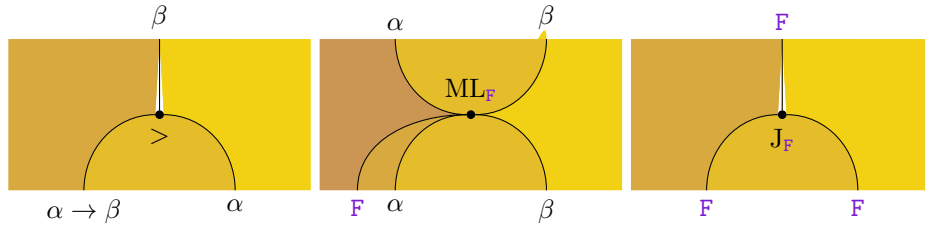
Note that the order of combination modes is not actually the same as the one that would come from the grammar. The reason why will become more apparent when string diagrams for parsing are introduced in the next section, but simply, this comes from the fact that while we think of ML and MR as reducing the number of effects on each side (and this is the correct way to think about those), this is not actually how its denotation works, but there is no real issue with that. This formalism gives us the following theorems:

► **Theorem 4.** *Semantic parsing of a sentence is polynomial in the length of the sentence and the size of the type system and syntax system.*

Proof. Suppose we are given a syntactic generating structure G_s along with our type combination grammar G_τ . The system G that can be constructed from the (tensor) product of G_s and G_τ has size $|G_s| \times |G_\tau|$. Indeed, we think of its rules as a rule for the syntactic categories and a rule for the type combinations. Its terminals and non terminals are also in the cartesian products of the adequate sets in the original grammars. What this in turn means, is that if we can syntactically parse a sentence in polynomial time in the size of the input and in $|G_s|$, then we can syntactico-semantically parse it in polynomial time in the size of the input, $|G_s|$ and $|G_\tau|$. ■

► **Theorem 5.** *Retrieving a pure denotation for a sentence can be done in polynomial time in the length of the sentence, given a polynomial time syntactic parsing algorithm and polynomial time combinators.*

Proof. We have proved in Theorem 4 that we can retrieve a semantic parse tree from a sentence in polynomial time in the input. Since we also have shown that the length of a semantic parse tree is quadratic in the length of the sentence it represents, being linear in the length of a syntactic parse tree linear in the length of the sentence. We have already seen that given a denotation, handling all effects and reducing effect handling to normal forms can be done in polynomial time. The superposition of these steps yields a polynomial-time algorithm in the length of the input sentence. ■



■ **Figure 6** String Diagrammatic Representation of Combinator Modes $>$, ML and J

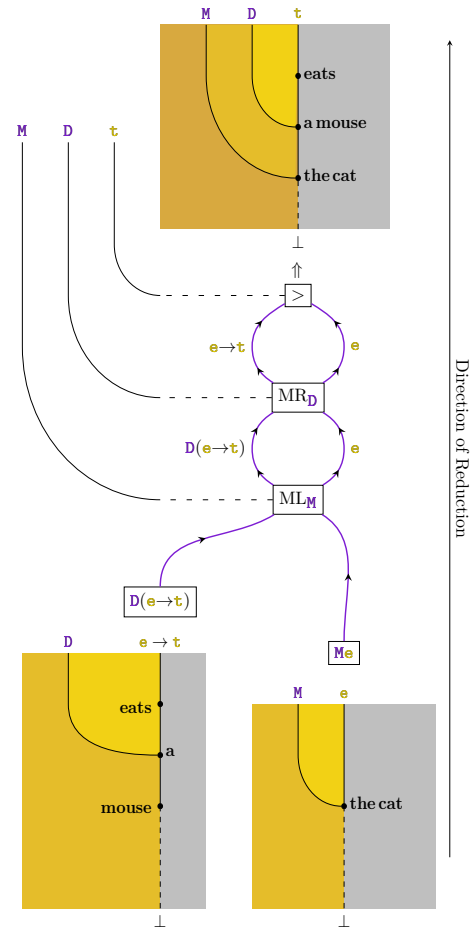
While we have gone with the hypothesis that we have a CFG for our language, any type of polynomial-time structure could work, as long as it is more expressive than a CFG.

The *polynomial time combinators* assumption in the following theorem is not a complex assumption, this is for example true for denotations based on lambda-calculus, with function application being linear in the number of uses of variables in the function term, which in turn is linear in the number of terms used to construct the function term and thus of words, and the different \mathbf{fmap} being in constant time for the same reason.

5.2 Diagrammatical Parsing

When considering [3] way of using string diagrams for syntactic parsing/reductions, we can see them as (yet) another way of writing our parsing rules. In our typed category, we can make see our combinators as natural transformations (2-cells): then we can see the different sets of combinators as different arity natural transformations. $>$, ML_F and J_F are represented in Figure 6, up to the coloring of the regions, because that is purely for an artistic rendition.

Now, a way to see this would be to think of this as an orthogonal set of diagrams to the ones of Section 4: we could use the syntactic version of the diagrams to model our parsing, according to the rules in Figure 2, and then combine the diagrams as shown in Figure 5, which highlights why we can consider the diagrams orthogonal. In this figure we exactly see the sequence of reductions play out on the types of the words, and thus we also see what exact *quasi-braiding* would be needed to construct the effect diagram. Here we talk about *quasi-braiding* because, in a sense, we use 2-cells to do braiding-like operations on the strings, and don't actually allow for braiding inside the diagrammatic computation. To better understand what happens in those parsing diagrams, Figure 4 provides the translations in labelled trees of the parsing



■ **Figure 5** Representation of a parsing diagram for the sentence *the cat eats a mouse*. See Figure 4b for translation in a parse tree.

Suppose you're knitting a rainbow scarf. You have multiple threads (the different words) of the different colours (their types and effects) you're using to knit the scarf. When you decide to change the color you take the different threads you have been using, and mix them up. You can create a new colour² thread from two (that's the base combinators) create a thicker one from two of the same colour (the applicative mode and the monadic join), put aside a thread until a later step (that's the `fmap`), add a new thread to the pattern (that's the unit), or cut a thread you will not be using anymore (that's the co-units and closure operators). Changing a thread by cutting it and making a knot at another point is basically what the `eject` combinators do. This more tangible representation can be seen in a larger diagram in Figure 9.



Figure 8 Example of a *Jacquard* knitwork. Photography and work courtesy of the author's mother.

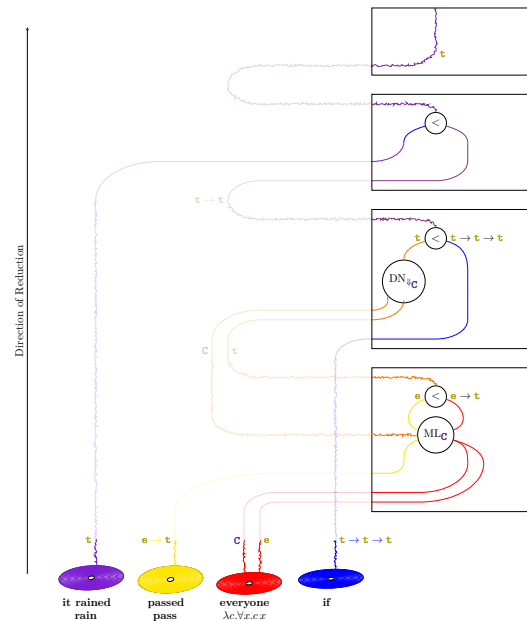


Figure 9 Knitting-like representation of the diagrammatic parsing of a sentence. See Figure 4c for the translation in a parse tree

get a result with effects `FG` or with effects `GF` (or possibly more). In general those effects are not the same, but if they happen to be, which trivially will be the case when one of the effects is external (the plural or islands functors for example), the order of application does not matter and thus we choose to get the outer effect the one of the left side of the combinator.

The sections in the rectangle represent what happen when considering our combination step as implementing patterns inside a knitwork, as seen in Figure 8. The different patterns provide, in order, a visual representation of the different ways one can combine two strings, i.e., two types and thus two denotations. The sections outside of the rectangle are the strings of yarn not currently being used to make a pattern.

5.3 Rewriting Rules

Here we provide a rewriting system on that allows us to improve our time complexity by reducing the size of the grammar. In the worst case, in big o notation there is no improvement in the size of the sentence, but there is no loss.

First consider the case where we have presuppositions (or sub-trees) for the two arguments of our parsing step, that are of type `F τ` and `G τ'` . In that case we could either

² This is not how wool works, but if you prefer you can imagine a pointillist-like way of drawing using multiple coloured lines that superimpose on each other, or a marching band's multiple instruments playing either in harmony or in disharmony and changing that during a score.

Then there are sequence of modes that clearly encompass other ones the grammar notation for ease of explanation. One should not use the unit of a functor after using ML or MR, as that adds void semantics. Same things can be said for certain other derivations containing the lowering and co-unit combinators:
We use DN when we have not used any of the following, in all derivations:

- | | | | | | |
|-----|---|---|-----|---|-----------------|
| 513 | ■ | m_F, DN, m_F where $m \in \{MR, ML\}$ | 516 | ■ | ML_F, DN, A_F |
| 514 | ■ | ML_F, DN, MR_F | | | |
| 515 | ■ | A_F, DN, MR_F | 517 | ■ | C |

We use J if we have not used any of the following, for $j \in \{\varepsilon, J_F\}$

- | | | | | | |
|-----|---|--|-----|---|-----------------------------------|
| 519 | ■ | $\{m_F, j, m_F\}$ where $m \in \{MR, ML\}$ | 524 | ■ | If F is commutative as a monad: |
| 520 | ■ | ML_F, j, MR_F | 525 | ■ | MR_F, A_F |
| 521 | ■ | A_F, j, MR_F , | 526 | ■ | A_F, ML_F |
| 522 | ■ | ML_F, j, A_F | 527 | ■ | MR_F, j, ML_F |
| 523 | ■ | k, C for $k \in \{\varepsilon, A_F\}$ | 528 | ■ | A_F, j, A_F |

► **Theorem 6.** *The rules proposed above yield equivalent results.*

Proof. For the first point, it is obvious since based on an equality. For the second point: The rules about not using combinators UL and UR come from the notion of handling and granting termination and decidability to our system. The rules about adding J and DN after moving two of the same effect from the same side (i.e. MLML or MRMR) are normalization of Theorem 1. Indeed, in the denotation, the only reason to keep two of the same effects and not join them is to at some point have something get in between the two. Joining and closure should then be done at earliest point in parsing where t can be done, and that is equivalent to later points because of the elevator equations, or Theorem 1. The last set of rules follows from the following: we should not use JMLMR instead of A, as those are equivalent because of the equation defining them. The same thing goes for the other two, as we should use the units of monads over applicative rules and **fmap**. ■

When using our diagrammatic approach to parsing, we can write all the reductions described above to our paradigm: it simply amounts to constructing a set of equational reductions for the string diagrams. This leads to the same algorithms developed in Section 4 being usable here: we just have a new improved version of Theorem 2 which adds the normal forms specified in this section to the newly added *orthogonal* axis of diagrammatic computations. What we're actually doing is computing two different normal forms along the tensor product of our reduction schemes, which amounts to computing a larger normal form. Moreover, considering the possible normal forms of syntactic reductions or denotational reductions adds ways to reduce our diagrams to normal forms. Of course, this does not mean that our system is complete, as there might still be many possible reductions to be found, although this would not reduce the complexity of the algorithm and is more of a thought experiment.

6 Conclusion

The functional programming approach developed in [2] allows for increased expressiveness in the choice of denotations, especially from a purely theoretical point of view. In this paper

we have successfully prove that it is well-founded theoretically, but also that it doesn't come at the cost of comprehensibility or efficiency.

Moreover, while our methods for implementing a type and effect system have been applied to natural language semantics, it could be applied in any language with purely compositional semantics. Of course, there are still improvements that can be made, in particular around the unorthodox use of effects to define what we have called higher-order constructs and scope islands, but also in integrating the theory in more complicated models of denotations, such as the ones learned through a neural network for example. In that case, it would suffice to understand what base combinators exist for the model to implement the formalism we described.

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601 A Presenting a Language

602 In this appendix, we provide tables (10a and 10b) describing the modeling of a subset of the
 603 English language in our formalism.

Expression	Type	λ -Term
planet	$e \rightarrow t$	$\lambda x. \mathbf{planet} \ x$
Generalizes to common nouns		
carnivorous	$(e \rightarrow t)$	$\lambda x. \mathbf{carnivorous} \ x$
Generalizes to predicative adjectives		
skillful	$(e \rightarrow t) \rightarrow (e \rightarrow t)$	$\lambda p. \lambda x. px \wedge \mathbf{skillful} \ x$
Generalizes to predicate modifier adjectives		
Jupiter	e	$\mathbf{j} \in \text{Var}$
sleep	$e \rightarrow t$	$\lambda x. \mathbf{sleep} \ x$
chase	$e \rightarrow e \rightarrow t$	$\lambda o. \lambda s. \mathbf{chase} \ (o) \ (s)$
be	$(e \rightarrow t) \rightarrow e \rightarrow t$	$\lambda p. \lambda x. px$
it	$\mathbf{G}e$	$\lambda g. g_0$
the	$(e \rightarrow t) \rightarrow \mathbf{M}e$	$\lambda p. x \text{ if } p^{-1}(\top) = \{x\} \text{ else } \#$
a	$(e \rightarrow t) \rightarrow \mathbf{D}e$	$\lambda p. \lambda s. \{ \langle x, x \# s \rangle \mid px \}$
no	$(e \rightarrow t) \rightarrow \mathbf{C}e$	$\lambda p. \lambda c. \neg \exists x. px \wedge c \ x$
$\cdot, \mathbf{a} \cdot$	$e \rightarrow (e \rightarrow t) \rightarrow \mathbf{W}e$	$\lambda x. \lambda p. \langle x, px \rangle$

(a) Lexicon for a subset of the English language

Constructor	fmap	Typeclass
$\mathbf{G}(\tau) = \mathbf{r} \rightarrow \tau$	$\mathbf{G}\varphi(x) = \lambda r. \varphi(xr)$	Monad
$\mathbf{W}(\tau) = \tau \times t$	$\mathbf{W}\varphi(\langle a, p \rangle) = \langle \varphi a, p \rangle$	Monad
$\mathbf{S}(\tau) = \{\tau\}$	$\mathbf{S}\varphi(\{x\}) = \{\varphi(x)\}$	Monad
$\mathbf{C}(\tau) = (\tau \rightarrow t) \rightarrow t$	$\mathbf{C}\varphi(x) = \lambda c. x (\lambda a. c(\varphi a))$	Monad
$\mathbf{D}(\tau) = s \rightarrow \mathbf{S}(\tau \times s)$	$\mathbf{D}\varphi(\lambda s. \{ \langle x, x \# s \rangle \mid px \}) = \lambda s. \{ \langle \varphi x, \varphi x \# s \rangle \mid px \}$	Monad
$\mathbf{M}(\tau) = \tau + \perp$	$\mathbf{M}\varphi(x) = \begin{cases} \varphi(x) & \text{if } \Gamma \vdash x : \tau \\ \# & \text{if } \Gamma \vdash x : \# \end{cases}$	Monad

(b) Definition of a few functors, with their map on functions

■ **Figure 10** Presentation of a lambda-calculus lexicon for the English language