Effect-Driven Parsing

Formal studies on a categorical approach to semantic parsing

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Context

This work, based on [BC25] aims to provide a categorical formalization of a type and effects system for semantic interpretation of the natural language.

We will develop a graphical formalism for semantic type-driven parsing that explains how to derive the meaning of a sentence from the meaning of its words.

Typed Semantics for Natural Languages

Expression	Type	λ-Term
planet	$\mathtt{e} \to \mathtt{t}$	$\lambda x.\textbf{planet}\, x$
	Generalizes to **common nouns**	
carnivorous	$(\mathtt{e} \to \mathtt{t})$	$\lambda x.\textbf{carnivorous}\, x$
	Generalizes to **predicative adjectives**	
skillful	$(\mathtt{e} \to \mathtt{t}) \to (\mathtt{e} \to \mathtt{t})$	$\lambda p.\lambda x.px \wedge \textbf{skillful}\, x$
	Generalizes to **predicate modifier adjectives**	
Jupiter	$\mathtt{e}$	$\mathbf{j} \in \mathrm{Var}$
	Generalizes to **proper nouns**	
sleep	$\mathtt{e} \to \mathtt{t}$	$\lambda x.\textbf{sleep}\, x$
	Generalizes to **intransitive verbs**	

Syntactic Types and Semantic Types

What should be the type of expressions such as **a cat** or **Jupiter, a planet**?

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Since we should be able to use **a cat** and **the cat** interexchangebly - from a syntactic point of view - they should have the same type. We use *effects* to do the difference between them:

$$\textbf{a cat} = \{c \mid \textbf{cat}\, c\}$$

$$\textbf{the cat} = x \text{ if } \textbf{cat}^{-1}(\top) = \{x\} \text{ else } \#$$

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Effects as Functors

Since [Mog89], we know that monads are a good model for programming side-effects.

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In our case, functors are enough, because might sometimes not want to be able to merge effects together nor create them from nothing (see below).

String Diagrams

String Diagrams are a formalism (see [HM23] for example) that allows to visually represent the different threads of a computation and the possible side-effects that appear.

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They were presented as a categorical object in [JS91] and will be the focus of our study, as a tool introduced for parsing, similarly to [CSC10].

Other Categorical Theories

In [Mar], and [SM25], a mathematical model of parsing trees based on a Hopf algebra is presented.

This allows to integrate morphological features inside the syntactic structures without breaking the model chosen for syntax.

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Notations

Let $\mathcal{L}$ be our language (more on that later). We only suppose that our words can be applied to one another in their denotation system.

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We consider $\bar{\mathcal{C}}$ the categorical closure of $\mathcal{C}$ under the action of $\mathcal{F}(\mathcal{L})^*$. We close it for the cartesian product and exponential of $\mathcal{C}$. $\bar{\mathcal{C}}$ represents all possible combinations of a sequence of effects and a base type, contains functions and products.

Intuitionistic-style Typing Judgements

We then have typing judgements for basic combinations:

$$\frac{\Gamma \vdash x : \tau \qquad \Gamma \vdash F \in \mathcal{F}(\mathcal{L})}{\Gamma \vdash Fx : F\tau}\textsf{Cons}$$

$$\frac{\Gamma \vdash x : F\tau_1 \qquad \Gamma \vdash \varphi : \tau_1 \to \tau_2}{\Gamma \vdash \varphi x : F\tau_2}\texttt{fmap}$$

$$\frac{\Gamma \vdash x : \tau_1 \qquad \Gamma \vdash \varphi : \tau_1 \to \tau_2}{\Gamma \vdash \varphi x : \tau_2}\textsf{App}$$

Intuitionistic-style Typing Judgements

We then have typing judgements for basic combinations:

$$\frac{\Gamma \vdash x : A\tau_1 \qquad \Gamma \vdash \varphi : A\left(\tau_1 \to \tau_2\right)}{\Gamma \vdash \varphi x : A\tau_2}\texttt{<*>}$$

Intuitionistic-style Typing Judgements

Typing judgements for natural transformations:

$$\frac{\Gamma \vdash x : \tau}{\Gamma \vdash x : A\tau}\texttt{pure/return}$$

$$\frac{\Gamma \vdash x : MM\tau}{\Gamma \vdash x : M\tau}\texttt{>>=}$$

More generally:

$$\forall F \overset{\theta}{\Longrightarrow} G, \qquad \frac{\Gamma \vdash x : F\tau \qquad \Gamma \vdash G : S' \subseteq \star \qquad \tau \in S'}{\Gamma \vdash x : G\tau}\texttt{nat}$$

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Information

To present the language, we of course need the syntax of the language, as well as an increased model of our lexicon.

Updated Lexicon

Expression	Type	λ-Term
it	$\texttt{Ge}$	$\lambda g.g_0$
$\cdot$, **a** $\cdot$	$\texttt{e} \to (\texttt{e} \to \texttt{t}) \to \texttt{We}$	$\lambda x.\lambda p.\,\langle x, px\rangle$
which	$(\texttt{e} \to \texttt{t}) \to \texttt{Se}$	$\lambda p.\,\{x \mid px\}$
the	$(\texttt{e} \to \texttt{t}) \to \texttt{Me}$	$\lambda p.x$ if $p^{-1}(\top) = \{x\}$ else $\#$
a	$(\texttt{e} \to \texttt{t}) \to \texttt{De}$	$\lambda p.\lambda s.\,\{\langle x, x \mathbin{+\!\!+} s\rangle \mid px\}$
every	$(\texttt{e} \to \texttt{t}) \to \texttt{Ce}$	$\lambda p.\lambda c.\forall x, px \Rightarrow cx$

Introducing Higher-Order Constructs

Using the notion of functors, we can also implement higher-order semantic constructions in our lexicon, such as the future, without caring about morphological markers:

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Using the notion of functors, we can also implement higher-order semantic constructions in our lexicon, such as the future, without caring about morphological markers:

$$\mathbf{future}\left(\mathbf{be}\left(\mathbf{I}, \mathbf{a\ cat}\right)\right) \xrightarrow{\beta} \mathbf{be}\left(\mathbf{future}\left(\mathbf{I}\right), \mathbf{a\ cat}\right) \xrightarrow{\beta} \mathbf{be}\left(\mathbf{future}\left(\mathbf{I}\right), \mathbf{a\ cat}\right)$$

Those constructs are integrated by using natural transformations explaining their propagation through other effects, as those are purely semantic predicates.

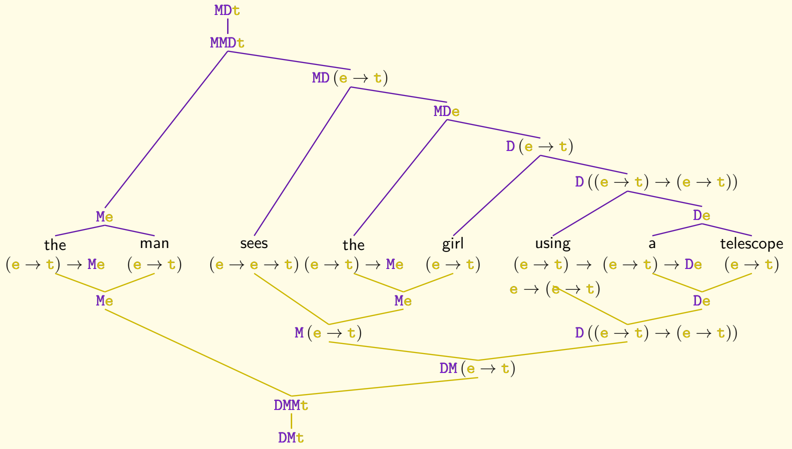
Introducing Higher-Order Constructs

For the plural, this gives:

CN(P)	$\Gamma \vdash p : (\mathtt{e} \to \mathtt{t})$	$\Pi(p) = \lambda x.\,(px \wedge \lvert x\rvert \geq 2)$
ADJ(P)	$\Gamma \vdash p : (\mathtt{e} \to \mathtt{t})$	$\Pi(p) = \lambda x.\,(px \wedge \lvert x\rvert \geq 2)$
	$\Gamma \vdash p : (\mathtt{e} \to \mathtt{t}) \to (\mathtt{e} \to \mathtt{t})$	$\Pi(p) = \lambda \nu.\lambda x.\,(p\,(\nu)\,(x) \wedge \lvert x\rvert \geq 2)$
NP	$\Gamma \vdash p : \mathtt{e}$	$\Pi(p) = p$
	$\Gamma \vdash p : (\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$	$\Pi(p) = \lambda \nu.p\,(\Pi\nu)$
IV(P)/VP	$\Gamma \vdash p : \mathtt{e} \to \mathtt{t}$	$\Pi(p) = \lambda o.\,(po \wedge \lvert x\rvert \geq 2)$
TV(P)	$\Gamma \vdash p : \mathtt{e} \to \mathtt{e} \to \mathtt{t}$	$\Pi(p) = \lambda s.\lambda o.\,(p\,(s)\,(o) \wedge \lvert s\rvert \geq 2)$

Introducing Higher-Order Constructs

Ambiguity


MDt
MMDt
MD (e → t)
MDe
D (e → t)
D ((e → t) → (e → t))
Me
De
the
man
sees
the
girl
using
a
telescope
(e → t) → Me
(e → t)
(e → e → t)
(e → t) → Me
(e → t)
(e → t) → e → (e → t)
(e → t) → De
(e → t)
Me
Me
De
M (e → t)
D ((e → t) → (e → t))
DM (e → t)
DMMt
DMt

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Handlers

A handler for an effect F is a natural transformation $F \Rightarrow \mathrm{Id}$, as proposed in [WSH14].

Handlers should also be exact inverses to monadic and applicative units: this partially justifies semantically why we can remove the usage of the unit rule out of certain situations.

Defining Handlers

There are two main types of handlers that are of interest to us:

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1. Language-Defined Handlers, which are defined with adjunctions and comonads, for example. Those arise from fundamental properties of the considered effects.
2. Speaker-dependant handlers, which are considered when retrieving the denotation from a sentence from under the effects that arose in the computation of its meaning. Those need to be considered dependent on the speaker because for example of the multiple ways to solve non-determinism.

Scope Islands

The notion of handlers allows us to enforce the notion of scope islands. To do so, it would suffice to ask that the words enclosing the island, are not defined on not certain effectful types and make handlers a part of the combination modes, as introduced in [BC25].

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We would for example have:

$$\mathbf{if}\ : (\mathtt{t} \setminus \mathcal{F}\mathcal{L}^{*}\mathtt{C}\mathtt{t}) \to \mathtt{t} \to \mathtt{t}$$

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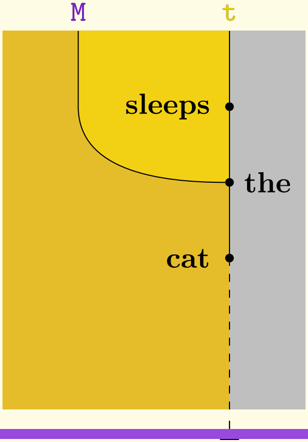
String Diagrams Representation of Sentences I

Here, a string diagram is a representation of the side-effects and types of a sentence across its computation.



This diagram for example represents the sentence *The cat sleeps*. The order of the words and position of the strings will be explained in detail in the next section.

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Deformation of String Diagrams

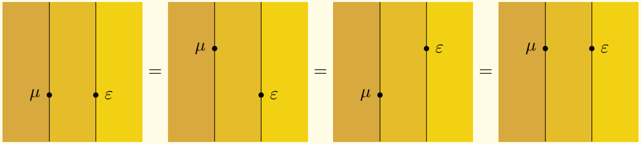
String diagrams will be the formalism we use to compute equality between denotations, and especially handling the denotations.

Theorem 3.1 — **Theorem 3.1 [Sel10], Theorem 1.2 [JS91]** A well-formed equation between morphism terms in the language of monoidal categories follows from the axioms of monoidal categories if and only if it holds, up to planar isotopy, in the graphical language.

Equations on String Diagrams I

Every property of the functors, monads, natural transformations, adjunctions and more can be explained in terms of commutative diagrams, but also as string diagrams.

First, the elevator equations are a consequence of 3.1:



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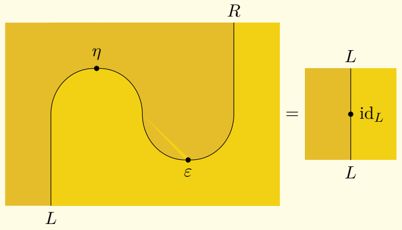
Equations on String Diagrams II

The Snake equations are a rewriting of the properties of an adjunction:



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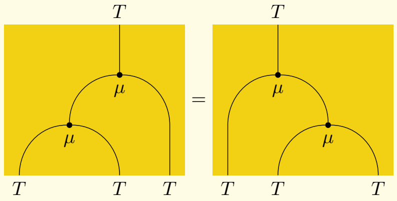
Equations on String Diagrams III

The Monadic equations are a rewriting of the properties of a monad:



(μ)

Confluence of Reductions I

Theorem 3.2 — **Confluence** Our reduction system is confluent and therefore defines normal forms:

1. Right reductions are confluent and therefore define *right* normal forms for diagrams under the equivalence relation induced by exchange.
2. Equational reductions are confluent and therefore define *equational* normal forms for diagrams under the equivalence relation induced by exchange.

Confluence of Reductions II

Theorem 3.3 — Normalization Complexity Reducing a diagram to its normal form is done in quadratic time in the number of natural transformations in it.

This is accomplished using a formalism based on [DV22].

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CFG Model of Parsing I

We use a Context-Free Grammar to model our typing system and take its product with the syntax defining grammar.

$$
\begin{aligned}
>, \beta &::= (\alpha \to \beta), \alpha \\
<, \beta &::= \alpha, (\alpha \to \beta) \\
\mathsf{J}_{\mathtt{F}}\ \mathtt{F}\tau &::= \mathtt{FF}\tau \\
\mathsf{DN}_{\mathtt{C}}\ \tau &::= \mathtt{C}_\tau \tau \\
\mathsf{ML}_{\mathtt{F}}(\alpha, \beta) &::= \mathtt{F}\alpha, \beta \\
\mathsf{MR}_{\mathtt{F}}(\alpha, \beta) &::= \alpha, \mathtt{F}\beta
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{A}_{\mathtt{F}}(\alpha, \beta) &::= \mathtt{F}\alpha, \mathtt{F}\beta \\
\mathsf{UR}_{\mathtt{F}}(\alpha \to \alpha', \beta) &::= \mathtt{F}\alpha \to \alpha', \beta \\
\mathsf{UL}_{\mathtt{F}}(\alpha, \beta \to \beta') &::= \alpha, \mathtt{F}\beta \to \beta' \\
\mathsf{C}_{\mathtt{LR}}(\mathtt{L}\alpha, \mathtt{R}\beta) &::= (\alpha, \beta) \\
\mathsf{ER}_{\mathtt{R}}(\mathtt{R}(\alpha \to \alpha'), \beta) &::= \alpha \to \mathtt{R}\alpha', \beta \\
\mathsf{EL}_{\mathtt{R}}(\alpha, \mathtt{R}(\beta \to \beta')) &::= \alpha, \beta \to \mathtt{R}\beta'
\end{aligned}
$$

CFG Model of Parsing II

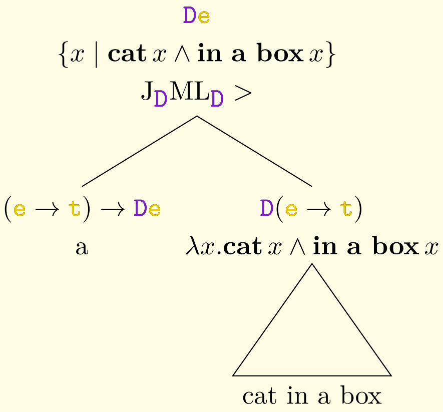
This grammar works in five major sections:

1. We reintroduce the grammar defining the type and effect system.
2. We introduce a structure for the semantic parse trees and their labels, the combination modes from [BC25].
3. We introduce rules for basic type combinations.
4. We introduce rules for higher-order unary type combinators.
5. We introduce rules for higher-order binary type combinators.

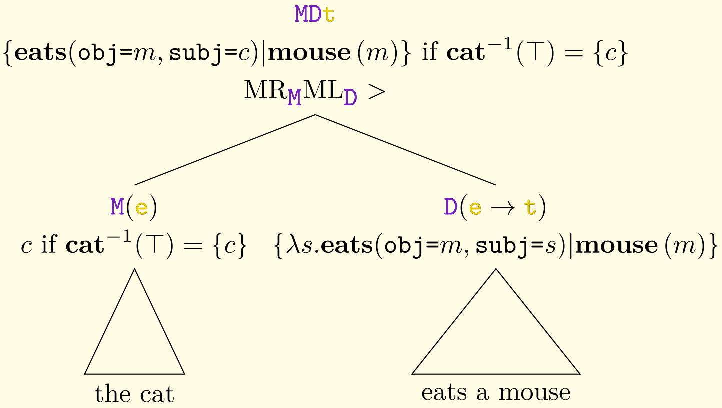
CFG Model of Parsing III

Computationally, the introduction of this labeling system increases the size of the grammar by a factor linear in $|\mathcal{F}(\mathcal{L})|$. The parsing algorithms are then still polynomial, and scaling in sentence size does not change.

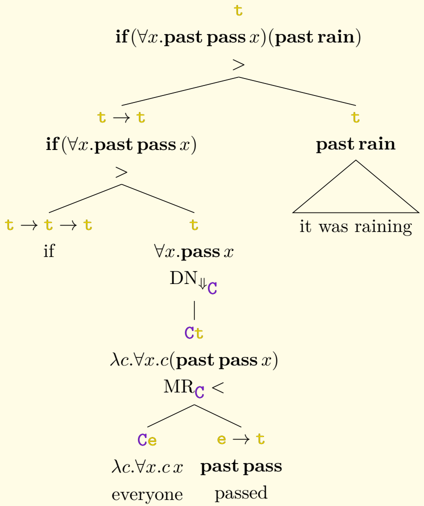
Semantic Parse Trees I


$\mathtt{De}$
$\{x \mid \mathbf{cat}\, x \wedge \mathbf{in\ a\ box}\, x\}$
$\mathrm{J}_{\mathtt{D}}\mathrm{ML}_{\mathtt{D}} >$
$(\mathtt{e} \to \mathtt{t}) \to \mathtt{De}$
a
$\mathtt{D}(\mathtt{e} \to \mathtt{t})$
$\lambda x.\mathbf{cat}\, x \wedge \mathbf{in\ a\ box}\, x$
cat in a box

Semantic Parse Trees II


$\mathtt{MDt}$
$\{\mathbf{eats}(\mathtt{obj=}m, \mathtt{subj=}c) | \mathbf{mouse}\,(m)\}$ if $\mathbf{cat}^{-1}(\top) = \{c\}$
$\mathrm{MR}_{\mathtt{M}}\mathrm{ML}_{\mathtt{D}} >$
$\mathtt{M}(\mathtt{e})$
c if $\mathbf{cat}^{-1}(\top) = \{c\}$
$\mathtt{D}(\mathtt{e} \to \mathtt{t})$
$\{\lambda s.\mathbf{eats}(\mathtt{obj=}m, \mathtt{subj=}s) | \mathbf{mouse}\,(m)\}$
the cat
eats a mouse

Semantic Parse Trees III


$\mathtt{t}$
$\mathbf{if}\,(\forall x.\mathbf{past\ pass}\ x)(\mathbf{past\ rain})$
$>$
$\mathtt{t} \to \mathtt{t}$
$\mathbf{if}\,(\forall x.\mathbf{past\ pass}\ x)$
$>$
$\mathtt{t}$
$\mathbf{past\ rain}$
it was raining
$\mathtt{t} \to \mathtt{t} \to \mathtt{t}$
if
$\mathtt{t}$
$\forall x.\mathbf{pass}\ x$
$\mathrm{DN}_{\Downarrow\mathtt{C}}$
$\mathtt{Ct}$
$\lambda c.\forall x.c(\mathbf{past\ pass}\ x)$
$\mathrm{MR}_{\mathtt{C}} <$
$\mathtt{Ce}$
$\lambda c.\forall x.c\,x$
everyone
$\mathtt{e} \to \mathtt{t}$
$\mathbf{past\ pass}$
passed

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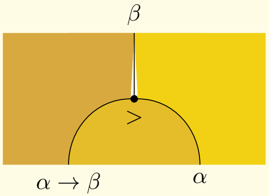
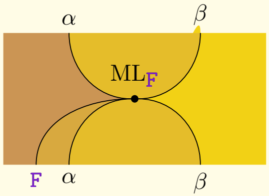
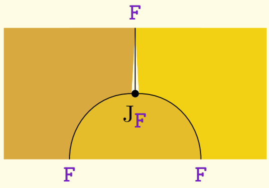
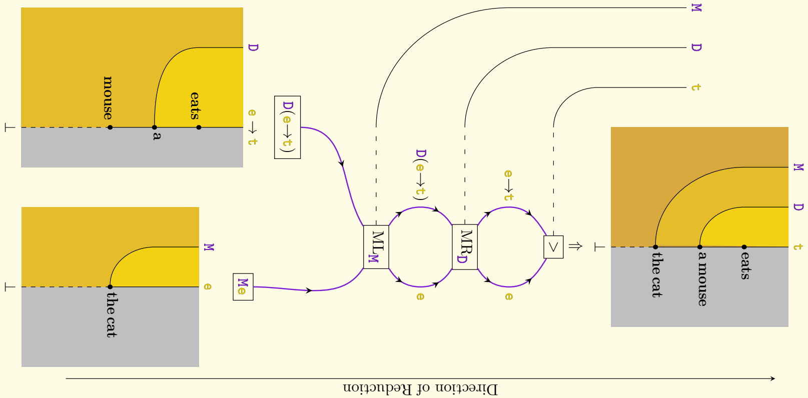
Combinators as String Diagrams



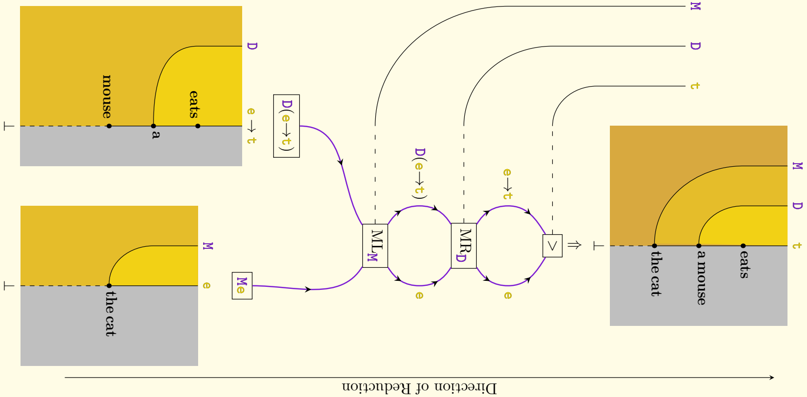




A Parsing Diagram Step

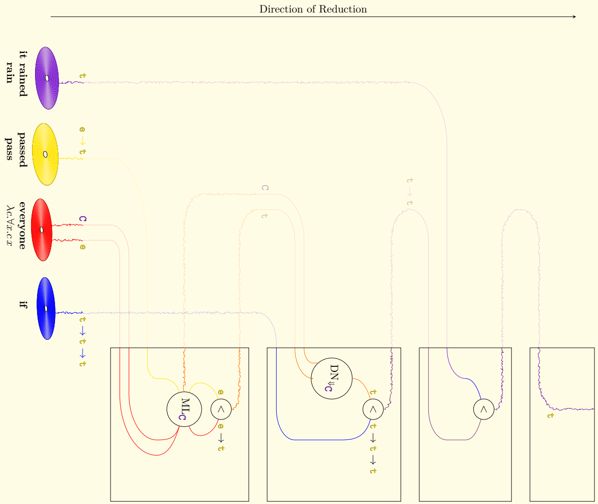


A Parsing Diagram Step



Building an Intuition

A Full Parsing Diagram



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Reducing the grammar

We introduce denotations for our combinators, to allow us to define reductions that relieve a bit the ambiguity of the parsing grammar:

$$>= \lambda\varphi.\lambda x.\varphi x$$

$$\mathsf{ML}_{\mathbb{F}} = \lambda M.\lambda x.\lambda y.(\mathtt{fmap}_{\mathbb{F}}\lambda a.M(a,y))x$$

$$\mathsf{A}_{\mathbb{F}} = \lambda M.\lambda x.\lambda y.(\mathtt{fmap}_{\mathbb{F}}\lambda a.\lambda b.M(a,b))(x)\mathtt{<*>}y$$

$$\mathsf{UL}_{\mathbb{F}} = \lambda M.\lambda x.\lambda\varphi.M(x,\lambda b.\varphi(\eta_{\mathbb{F}}b))$$

$$\mathsf{J}_{\mathbb{F}} = \lambda M.\lambda x.\lambda y.\mu_{\mathbb{F}}M(x,y)$$

Reductions

Reductions include the following:

- When two effects commute, we choose an order to apply them.
- We use UR instead of using MR or DNMR when possible.
- We always use modes J, C and DN as early as possible. The fact this works is a consequence of Theorem 3.1 on the denotation diagrams.
- All the rules provided in the previous section can be rewritten here too.

Conclusion

We have formalised an enhancement of usual type systems for natural language semantics.

Thanks to the introduction of the string diagrams, this did not come at the cost of comprehension of the system nor efficiency.

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I would like to thank Simon Charlow for his advice, my mother for the knitting, Antoine Groudiev for the rotation of the snakes in equation labels, Bella Senturia, Bob Frank and Paul-André Melliès for their suggestions of papers to read about categories and linguistics.

Thank you for your attention.

Do you have any questions?

Functor Denotations

Constructor	`fmap`	Typeclass
$\mathtt{G}\,(\tau) = \mathtt{r} \to \tau$	$\mathtt{G}\varphi\,(x) = \lambda r.\varphi\,(xr)$	Monad
$\mathtt{W}\,(\tau) = \tau \times \mathtt{t}$	$\mathtt{W}\varphi\,(\langle a, p\rangle) = \langle \varphi a, p\rangle$	Monad
$\mathtt{S}\,(\tau) = \{\tau\}$	$\mathtt{S}\varphi\,(\{x\}) = \{\varphi(x)\}$	Monad
$\mathtt{C}\,(\tau) = (\tau \to \mathtt{t}) \to \mathtt{t}$	$\mathtt{C}\varphi\,(x) = \lambda c.x\,(\lambda a.c\,(\varphi a))$	Monad
$\mathtt{T}\,(\tau) = \mathtt{s} \to (\tau \times \mathtt{s})$	$\mathtt{D}\varphi\,(\lambda s.\,\{\langle x, x + s\rangle \mid px\}) = \lambda s.\,\langle \varphi x, \varphi x + s\rangle$	Monad
$\mathtt{F}\,(\tau) = \tau \times \mathtt{S}\tau$	$\mathtt{F}\varphi\,(\langle v, \{x \mid x \in D_e\}\rangle) = \langle \varphi\,(v)\,, \{x \mid x \in D_e\}\rangle$	Monad
$\mathtt{D}\,(\tau) = \mathtt{s} \to \mathtt{S}\,(\tau \times \mathtt{s})$	$\mathtt{D}\varphi\,(\lambda s.\,\{\langle x, x + s\rangle \mid px\}) = \lambda s.\,\{\langle \varphi x, \varphi x + s\rangle \mid px\}$	Monad
$\mathtt{M}\,(\tau) = \tau + \bot$	$\mathtt{M}\varphi\,(x) = \begin{cases} \varphi\,(x) & \text{if } \Gamma \vdash x : \tau \\ \# & \text{if } \Gamma \vdash x : \# \end{cases}$	Monad

CFG of English

$$
\begin{aligned}
\texttt{CP} &::= \texttt{DP, VP} \\
&\;\mid\; \texttt{Cmp, CP} \\
&\;\mid\; \texttt{CP, CBar} \\[1em]
\texttt{CBar} &::= \texttt{Cor, CP} \\[1em]
\texttt{Dbar} &::= \texttt{Cor, DP} \\[1em]
\texttt{DP} &::= \texttt{DP, Dbar} \\
&\;\mid\; \texttt{Dmp, DP} \\
&\;\mid\; \texttt{Det, NP} \\
&\;\mid\; \texttt{Gen, TN} \\[1em]
\texttt{Gen} &::= \texttt{DP, GenD}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{NP} &::= \texttt{AdjP, NP} \\
&\;\mid\; \texttt{NP, AdjP} \\[1em]
\texttt{AdjP} &::= \texttt{TAdj, DP} \\
&\;\mid\; \texttt{Deg, AdjP} \\[1em]
\texttt{VP} &::= \texttt{TV, DP} \\
&\;\mid\; \texttt{AV, CP} \\
&\;\mid\; \texttt{VP, AdvP} \\[1em]
\texttt{TV} &::= \texttt{DV, DP} \\[1em]
\texttt{AdvP} &::= \texttt{TAdv, DP}
\end{aligned}
$$

Combinator Denotations

$$> = \lambda\varphi.\lambda x.\varphi x$$

$$< = \lambda x.\lambda\varphi.\varphi x$$

$$\mathsf{ML}_{\mathtt{F}} = \lambda M.\lambda x.\lambda y.(\mathtt{fmap}_{\mathtt{F}}\lambda a.M(a, y))x$$

$$\mathsf{MR}_{\mathtt{F}} = \lambda M.\lambda x.\lambda y.(\mathtt{fmap}_{\mathtt{F}}\lambda b.M(x, b))y$$

$$\mathsf{A}_{\mathtt{F}} = \lambda M.\lambda x.\lambda y.(\mathtt{fmap}_{\mathtt{F}}\lambda a.\lambda b.M(a, b))(x)\texttt{<*>}y$$

$$\mathsf{UL}_{\mathtt{F}} = \lambda M.\lambda x.\lambda\varphi.M(x, \lambda b.\varphi(\eta_{\mathtt{F}}b))$$

$$\mathsf{UR}_{\mathtt{F}} = \lambda M.\lambda\varphi.\lambda y.M(\lambda a.\varphi(\eta_{\mathtt{F}}a), y)$$

$$\mathsf{J}_{\mathtt{F}} = \lambda M.\lambda x.\lambda y.\mu_{\mathtt{F}}M(x, y)$$

$$\mathsf{C}_{\mathtt{LR}} = \lambda M.\lambda x.\lambda y.\varepsilon_{\mathtt{LR}}(\mathtt{fmap}_{\mathtt{L}}(\lambda l.\mathtt{fmap}_{\mathtt{R}}(\lambda r.M(l, r))(y))(x))$$

$$\mathsf{EL}_{\mathtt{R}} = \lambda M.\lambda\varphi.\lambda y.M(\Upsilon_{\mathtt{R}}\varphi, y)$$

$$\mathsf{ER}_{\mathtt{R}} = \lambda M.\lambda x.\lambda\varphi.M(x, \Upsilon_{\mathtt{R}}\varphi)$$

$$\mathsf{DN}_{\Downarrow} = \lambda M.\lambda x.\lambda y.\Downarrow M(x, y)$$