Mathematical Boat Design Report

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Abstract

The purpose of this project is to mathematically design and construct a boat. The boat must satisfy a number of performance requirements: the boat must float when loaded, the deck of the boat should be parallel to the surface of the water ($\leq 10°$ between the two) when loaded, the angle of vanishing stability of the boat should between 120° and 140°, the boat should have a maximum righting moment of ≥ 0.2 N·m, and the boat should be as fast as possible given an applied force of 0.1 N. Additionally, the boat's construction was constrained by a number of rules: two 24" x 18" pieces of hardboard are used to construct the boat, the boat must be created from an equation-driven CAD model, between 700g and 1000g of cargo must be held in the boat, the boat must be a monohull displacement hull, a mast (0.5m by 3/8" aluminum dowel) must be included in the boat design, and an eyelet must be attached to the hull to enable speed testing.

In order to design and build a boat that met the design specifications, we began by creating several mathematically-defined regions and computationally analyzing them in order to determine their center of mass and angle of vanishing stability. Then, we incorporated the cargo and mast into our calculations, refining the parameters of our design to keep everything within the required limits. Finally, we converted the mathematical region of our boat's hull into a number of slices that were modeled in Solidworks, then laser cut and constructed. Our final boat design fulfilled all of the requirements, although some overlooked elements of the design affected the stability of the boat when floating upright.

1 Boat Terminology

Knowledge of basic boat terminology is useful when designing a boat. Some basic terms used to refer to parts of a boat are addressed below. The front and rear of the boat are referred to as the *bow* and *stern*, while the left and right sides are referred to as *port* and *starboard*, respectively.

The various measurements that define a boat have specific names as well. The *beam* is the distance across the widest part of the boat from port to starboard, the *length* is the distance across the longest part of the boat from bow to stern, and the *draft* is the distance from the deepest point on the hull to the deck.

Aside from these terms, the most important terms relating to boats are the center of mass (COM) and the center of buoyancy (COB). The COM is the position at which the gravitational forces on the hull are abstracted to be acting upon, the mass centroid of the hull; similarly, the COB is a position at which the buoyant forces on the hull are abstracted to be acting. Additionally, the COB can be easily found; it is the center of mass of the volume of water displaced by the boat.

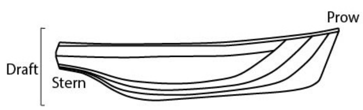

Figure 1: Buttocks of the boat.

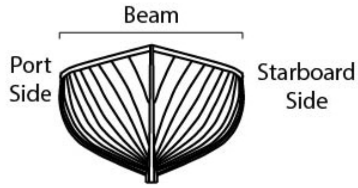

Figure 2: Sections of the boat at various points along its length.

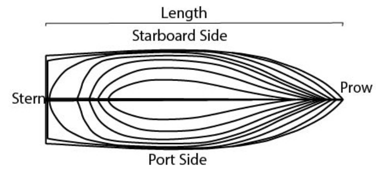

Figure 3: Various waterlines of the boat at different heights.

In order to define the boat mathematically, it's important to define ways of measuring the boat's size and orientation. As shown in Figure 1, *buttocks* are slices of the boat taken lengthwise from top to bottom; *sections*, shown in Figure 2, are slices taken from port to starboard and top to bottom; and *waterlines*, in Figure 3 are slices taken along the plane where the water meets the hull at a particular height.

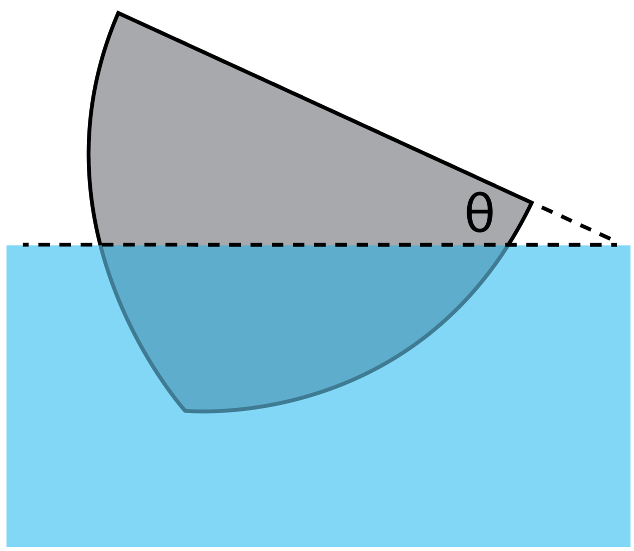

Figure 4: The heel angle of a boat, as indicated by θ.

When the boat is rotated along an axis parallel to its length - as shown in Figure 4 - the angle between the plane of the water and the deck of the boat is the *heel angle*. The boat is upright when the angle is equal to 0°, and completely capsized when the angle is equal to 180°. The Angle of Vanishing Stability (AVS) is the heel angle at which the boat no longer rights itself and turns over. The *righting moment* of the boat is the torque pushing it back toward stable, or toward capsizing.

2 Design Considerations

2.1 Floating

The most basic consideration of boat design is determining whether the boat floats. The boat floats at the point where:

$$\vec{F_G} = \vec{F_B} \tag{1}$$

where $\vec{F_G}$ is the force of gravity and $\vec{F_B}$ is the buoyant force. Because $\vec{F_G}$ is proportional to the mass of the boat and $\vec{F_B}$ is proportional to the mass of the water displaced, this is also the point where the mass of the boat, m_{boat}, equals the mass of the water displaced, m_{water}. The mass of the boat is defined mathematically by the integral of the boat density function over the region of the boat:

$$m_{boat} = \iiint_{boat} \rho_{boat}\ dV \tag{2}$$

where V_{boat} is the mathematically-defined region of the boat, and ρ_{boat} is the density of the boat; this can be either a function or a constant.
In order to find the water displaced, a mathematical region is defined for the water. In order to simplify calculations to determine the waterline, the boat is assumed to be stationary relative to the coordinate system, and the waterline is rotated as the heel angle changes. The waterline is defined mathematically by the following function:

$$water = \begin{cases} z \leq tan(\theta)y + b & \theta \leq \pi/2 \\ 0 \leq y \leq 2 & \theta = \pi/2 \\ z \geq tan(\theta)y + b & \theta \geq \pi/2 \end{cases} \tag{3}$$

where θ is the heel angle of the boat. However, for a given θ, b must be calculated to find the level at which the boat will actually float. To do this, a region is defined for the submerged region of the boat, which is the same as the water displaced:

$$sub = boat \cap water \tag{4}$$

It's important to note that *boat*, *water*, and *sub* are mathematical regions which define the boat, water, and submerged region, respectively, *not* the volumes of these regions. Next, the mass of the water displaced is calculated by integrating ρ_{water} ($1000\ \frac{kg}{m^3}$) over the submerged region, similarly to how the mass of the boat is calculated:

$$m_{disp} = \iiint_{sub} \rho_{water}\ dV \tag{5}$$

Thus, the equation for finding where the boat floats is also the solution for finding b, since m_{disp} is a function of b:

$$m_{boat} = m_{disp} \tag{6}$$

Solving this equation gives the waterline for where the boat will float.

2.2 Floating flat: COM and COB

In order to determine the boat will float flat, the sum of torques, also known as the *righting moment*, must be zero at $\theta = 0$. In the model, only two forces are assumed to be acting on the boat: $\vec{F_G}$ and $\vec{F_B}$. Thus, the only torque on the boat is equal to the torque about the COM caused by the COB. By summing the moments of the boat over its mass, the position of the center of the mass can be calculated:

$$\overrightarrow{COM} = \frac{\iint_{boat} \rho_{boat}\vec{r}\ dV}{m_{boat}} \tag{7}$$

where $\vec{r}$ is the three-dimensional position vector. Similarly, the COB can be calculated using the same method on the submerged region:

$$\overrightarrow{COB} = \frac{\iint_{sub} \rho_{water} \vec{r}\, dV}{m_{water}} \tag{8}$$

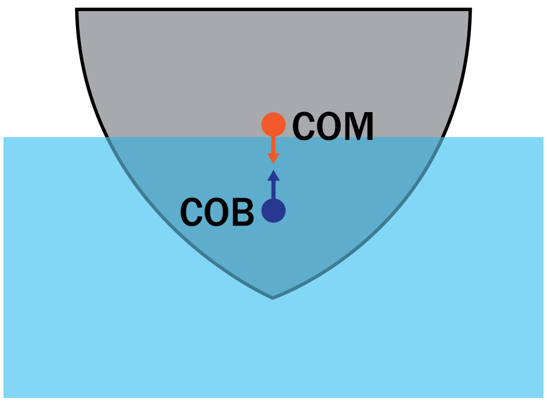

Figure 5: A boat floating with $\vec{F_G}$ and $\vec{F_B}$ aligned; the total torque on the boat is zero, and thus it floats flat.

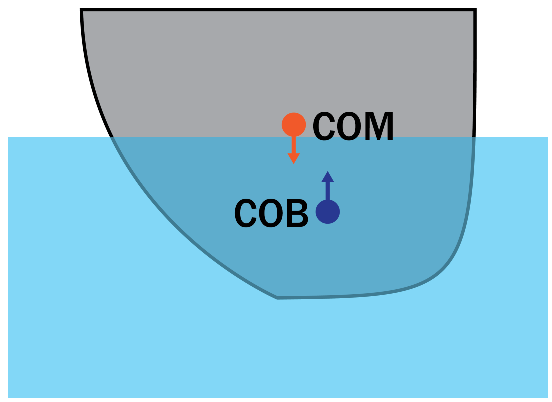

Figure 6: $\vec{F_G}$ and $\vec{F_B}$ being misaligned vertically results in a torque on this boat.

Figures 5 and 6 illustrate how a different COM and COB affect how the boat floats. When the COM and COB are aligned, there is no torque produced, and the boat floats flat; when they aren't aligned, the resulting torque causes the boat to tilt.

2.3 Righting Moment

All the components to compute the righting moment are now calculated. Because the F_B is acting about the COM from the COB, the moment arm is vector from the COM to the COB:

$$a\vec{r}m = C\vec{O}B - C\vec{O}M \tag{9}$$

Next, $\vec{F_B}$ is computed. In the coordinate system used, $\vec{F_B}$ doesn't simply point upward; instead, it acts in a direction normal to the waterline, with magnitude proportional to m_{disp}:

$$\vec{F_B} = \langle -m_{boat} g sin(\theta), m_{boat} g cos(\theta) \rangle \tag{10}$$

Then, the righting moment can be computed by taking the cross product of the moment arm and the buoyant force.

$$\vec{\tau} = \overrightarrow{arm} \times \vec{F_B} \tag{11}$$

In order for the total torque on the boat to be zero, $\vec{F_B}$ must be pointing in the same direction as $\overrightarrow{arm}$ - which means that the COM and COB must be aligned vertically. Note that this doesn't necessarily mean that the boat will always float flat in practice - if τ is sufficiently low at small values of θ, the boat may tend to stay at an angle if it is disturbed slightly or placed at an angle.

2.4 AVS

The righting moment plays a critical part in the design of the boat, because it determines the heel angles for which the boat continues to right itself.

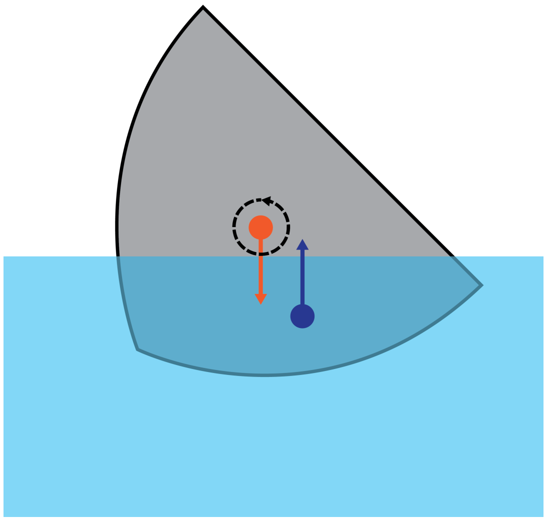

Figure 7: The $\vec{F_B}$, in blue, causes a positive righting moment around the COM, in red.

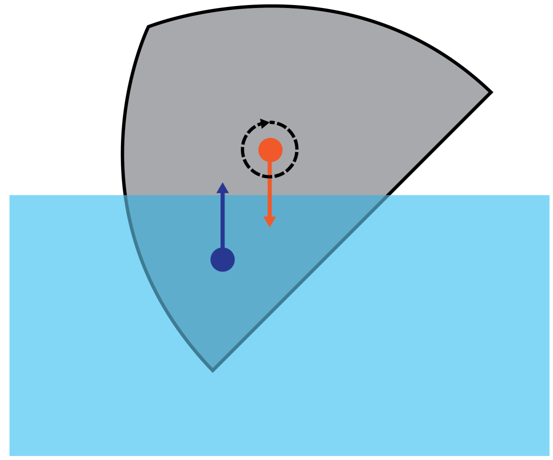

Figure 8: With a large heel angle, the $\vec{F_B}$ causes a negative righting moment that capsizes the boat.

The righting moment of the boat depends on the relative positions of the COM and COB. In Figure 7, the COB is to the right of the COM, which results in a positive moment that pushes the boat back toward equilibrium. When the COB is to the left of the COM, as in Figure 8, the righting moment is negative, and the boat will capsize. The heel angle at which the moment switches from positive to negative is the AVS; thus, having an AVS within the required window requires the COM and COB to change in a specific manner relative to the heel angle.

3 Proposed Design/Justification

3.1 Mathematical Hull Definition

The design parameters of the boat required it to float flat and have an AVS of 120° - 140°. Speed was also a consideration, but it was less critical to the success of the design as there was no set requirement how fast the boat had to go. To begin, a function was selected to define the boat. A fairly complex exponential function was used initially, but it proved slow and difficult to work with. Eventually, the function was simplified to a straightforward parabolic function. Because a parabolic function is symmetric, the resulting COM of the boat would be centered at 0 along the y-axis. The structure of the code began with a 2D equation for the boat, using the example equation that was used for class exercises shown in Equation 12.

$$2\left|\frac{y}{2}\right|^n \leq z \leq 2 \tag{12}$$

$$2\left|\frac{y}{w}\right|^n + \frac{hx^2}{\frac{l}{2}^2} \leq z \leq h \tag{13}$$

Equation 12 was then converted into a 3D region, as shown in Equation 13. Certain parameters, shown in Table 1, define the size and shape of the boat.

Table 1: Variables Controlling Boat Region

Variable	**Meaning**
n	Power of boat function. Controls how flat the bottom of the boat is.
w	Multiplier that controls the width of the boat.
l	Boat length (m)
h	Boat height (m)

3.2 Center of Mass

A interesting problem encountered when transitioning between modeling a solid boat of uniform density and modeling a hollow boat with ribs to retain shape was that of density. If changes weren't made, the predictions would be inaccurate because a real boat would not have the uniform density modeled in the code. The boat had ribs every 2", and the fiberboard ribs were each 1/8" in thickness; It was therefore assumed that for every cubic meter of space in the hull, 1/8 of the volume would be hardboard and the rest would be air. The equation for calculating the revised density is shown in Equation 14 , where $1.225kg/m^3$ is the density of air, and $892kg/m^3$ is the density of fiberboard.

$$\rho_{boat} = 1.225kg/m^3 * \frac{1.875}{2} + 892kg/m^3 * \frac{0.125}{2} \tag{14}$$

The next step in the model was to take into account the mass of the 0.5m long aluminum mast and the chosen cargo for the ship, which was a cylindrical piece of steel weighing 800 grams. The equation of the resulting COM is shown in Equation15. The cargo was relatively uniform, which meant that it could be mounted directly in the middle of the ship with its COM lined up along the z-axis, in line with the boat and mast's COMs. Originally, the cargo was to be cut in half and positioned further away from the middle of the boat for distribution of weight. However, due to the curvature of the boat's keel, moving the cargo away from the midpoint of the boat moved the cargo's COM upwards as well, making the boat less stable. It was decided that the cargo would remain at the very bottom of the ship.

$$\vec{COM}_{total} = \frac{COM_{boat} * m_{boat} + COM_{cargo} + m_{cargo} + COM_{mast} + m_{mast}}{m_{boat} + m_{cargo} + m_{mast}} \tag{15}$$

After defining the values for the masses and COMs of each component of the boat, we calculated the total COM. Based on our mathematical calculations, we found the location of the COM to be $\langle 0, 0, 0.0927m\rangle$.

3.3 Boat Size Parameters

The boat design was modified to also be symmetric across the y axis to ensure that the only critical dimension of the position of the COM would be its position along the z-axis. For the region of the hull, the coordinate system is defined with the sections in the yz-plane and the length in the x-direction.

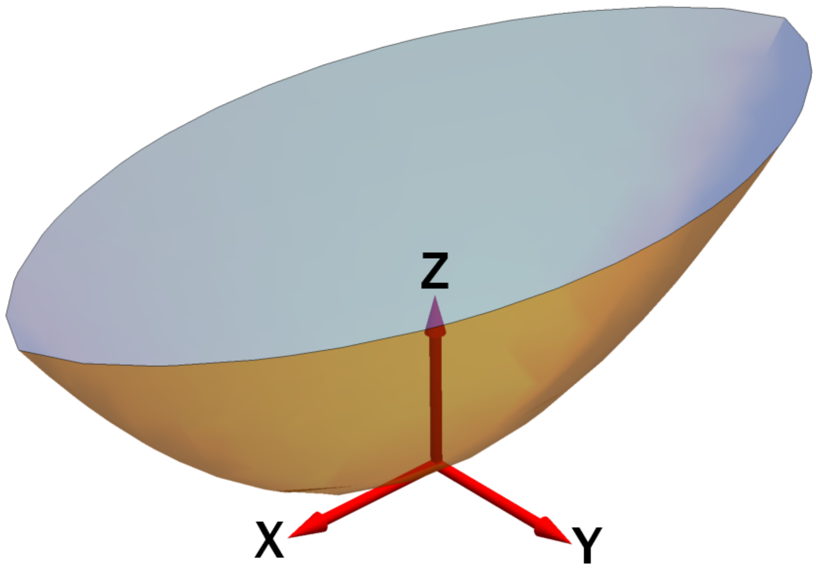

Figure 9: The coordinate system and origin of the boat as defined in the hull equation.

As shown in Figure 9, the origin is located at the lowest point of the boat at the middle section, with the positive z-axis pointing toward the deck. The variables that control the size/curvature of the boat are defined earlier in Table 1.

- n was set to 1.5 to make the boat slimmer to minimize drag, but still wide enough at its base to accommodate ballast.
- w was set to 0.65 to set the AVS; thinner boats are less stable, while wider boats are more stable, so w provided direct control over the AVS.
- l was set to 0.5 m to provide a good proportion between the length and width of the boat.
- h was set to 0.17 m to get the ballast reasonably deep and set the AVS.

During the construction of the boat, it was found that the cargo had been measured as 100g too heavy and, as such, the boat's true AVS was too low. To fix the mistake, the value of w was decreased and the value of h was increased, creating a slightly thinner and deeper boat and returning the AVS to within acceptable bounds.

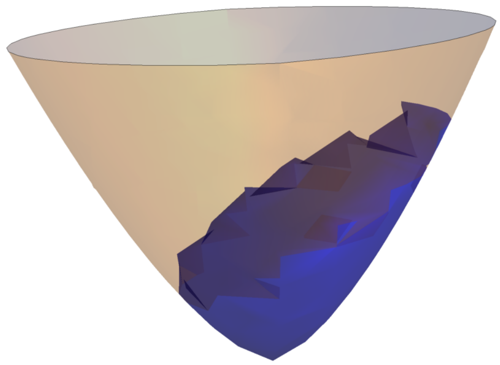

Figure 10: The submerged region of the boat in 3D at a heel angle of 30°.

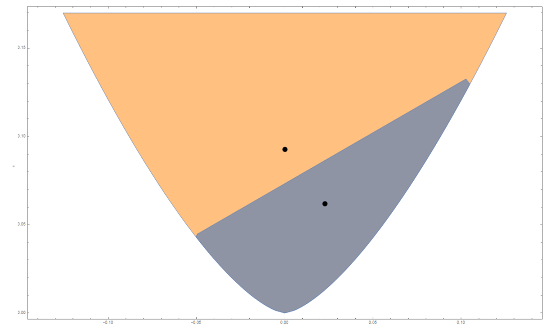

Figure 11: A 2D slice of the boat at x = 0 with the COM and COB of the boat plotted on the image.

An example of the submerged region of the boat at a heel angle of 30° is shown in Figure 10, and a 2D slice at x=0, as well as the COM and COB, is shown in Figure 11.

Because the boat is symmetric with respect to both the xz and yz-planes, the aspect of the boat's COM that most affected the AVS was it's position along the z-axis. Each parameter affected this position in a unique manner.

3.4 AVS Calculation

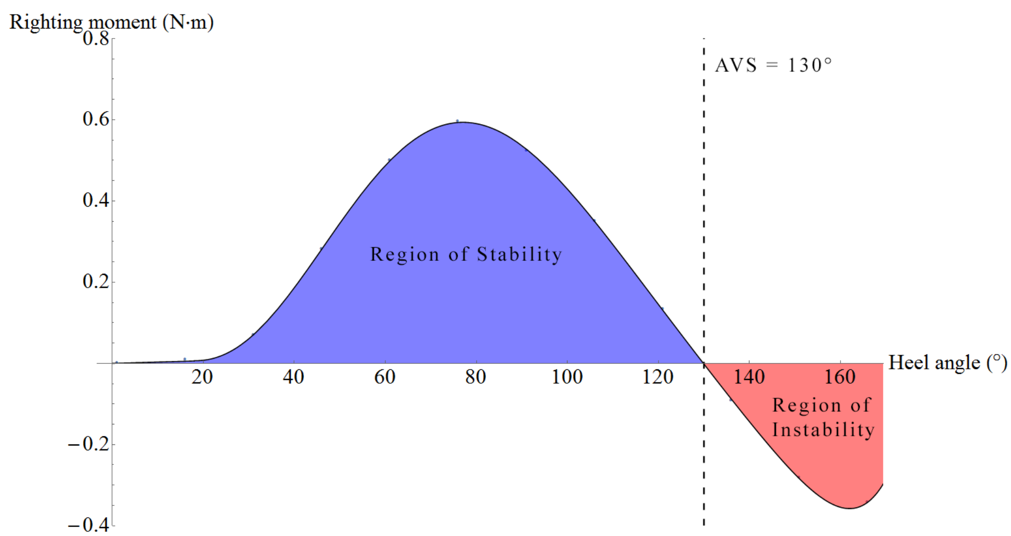

Figure 12: The stability curve of the boat: righting moment as a function of heel angle. Note the AVS is approximately 130°

Functional values for parameters were chosen by guessing nominal values, then refining the values to produce the required AVS of approximately 130°. The righting moment begins at zero when the boat is level, reaches a maximum of approximately 0.6 N·m, and crosses from positive negative at the point of the AVS. This AVS, in the middle in the acceptable range, allows for maximum error in the construction and calculations of the boat. However, as a consequence of this design, the boat has a very low righting moment at low heel angles which causes the boat to not float completely flat, particularly when disturbed slightly.

3.5 CAD Model

Figure 13: A full assembly of the boat in Solidworks. Pictured with mast and ballast installed, but without outer skin.

With a full parametrization complete, the boat was modeled in Solidworks, as shown in Figure 13. The model consisted of the individual ribs, the solid deck, the cargo mass, and the mast. The model did not include the vinyl shrink wrap hull, the figurehead and tail 3D printed for aesthetic, or the tape and glue used on the ship. The resulting center of mass location calculated by Solidworks was $\langle 0, 0, 0.09258m \rangle$, which was very close to our mathematically computed value, indicating that our calculations were fairly accurate.

4 Performance

The final examination of the boat tested buoyancy, stability, precision of AVS, and speed. The requirements were that the boat had to float with a level deck, the AVS had to be between 120°and 140°, and the maximum righting moment had to be greater than 0.2 N·m. The boats were then ranked on speed. The final boat, as pictured in Figure 14, finished with the following statistics:

Table 2: Statistics and measurements of the boat after the final demonstration and test.

Floats (y/n)	**Floats Level (y/n)**	**AVS (degrees)**	**Speed (m/s)**
y	y	130	0.431

As compared to the predicted AVS shown in Figure 12, The boat's actual AVS of 130 °is close, if not exactly, the value predicted by the calculations. Additionally referring to Figure 12, the maximum moments acting on the boat were substantially above 0.2 N·m, allowing to boat to right itself quickly upon perturbation. The boat performed well in the speed test, although a patch job made necessary during the sealing of the hull allowed some water to leak into the hull. Overall, the boat's performance validated the calculations.

Figure 14: The finished boat with hull sealed and decorations affixed.

5 Conclusion

In this project,mathematical analysis of various functions was performed in order to determine an optimal design for the actual boat. Several different functions to define the mathematical region of the boat were tried; the final function was fast and simple to evaluate while still leaving room for optimization. The size parameters were refined in order to create a boat design with an AVS that is in the middle of the required range, allowing maximum room for error. Our boat design worked successfully, with the expected AVS, despite some slight instability in floating flat.