Class Notes

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Contents

Chapter 1

Sequences of Real Numbers

In these notes we closely follow [1].

Definition 1. *A sequence of real numbers is a function from the set of positive integers to the set of all real numbers.*

It is customary to drop the functional notation in favor of the subscript notation to denote sequences of real numbers. So, if $f : \mathbb{N} \to \mathbb{R}$ is a sequence of real numbers, given any $n \in \mathbb{N}$, we write f_n in order to represent the real number $f(n)$. It's also customary to represent such a function with more compact notations such as $(f_n)_{n\in\mathbb{N}}$ or (f_n).

Example 1. *List the first four terms and the tenth term of the sequence whose nth term is as follows:*

(a) $a_n = \dfrac{n}{n+1}$;

(b) $a_n = 2 + (0.1)^n$;

(c) $a_n = (-1)^{n+1}\dfrac{n^2}{3n-1}$;

(d) $a_n = 4$.

Some sequences of real numbers (a_n) have the property that, as n increases its value indefinitely, the term a_n gets closer and closer to some real number L. In other words, the number $|a_n - L|$ decreases to zero as n increases to infinity. For example, if

$$a_n = 2 + \frac{1}{2^n},$$

then, we see that

$$|a_n - L| = \left|2 + \frac{1}{2^n} - 2\right| = \frac{1}{2^n},$$

decreases to zero as n increases to infinity.

Definition 2. *Let (a_n) be a sequence of real numbers. Then, we say that the limit of the a_n as n approaches infinity is a real number L if for every $\epsilon > 0$ there corresponds a real number M such that $|a_n - L| < \epsilon$ whenever $n > M$.*

If $L \in \mathbb{R}$ is the limit of a sequence of real numbers (a_n), then we write

$$\lim_{n\to\infty} a_n = L.$$

Proposition 1. *Let (a_n) be a sequence of real numbers. Then, for any real numbers L and M, if*

$$\lim_{n\to\infty} a_n = L \quad \textit{and} \quad \lim_{n\to\infty} a_n = M,$$

then we must have $L = M$.

Proof of Proposition 1. Suffices to say that

$$|L - M| \leqslant |a_n - L| + |a_n - M|,$$

for every $n \in \mathbb{N}$. □

Proposition 2. *If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$, then:*

(a) $\lim\limits_{n\to\infty} (a_n + b_n) = L + M$;

(b) $\lim\limits_{n\to\infty} (a_n - b_n) = L - M$;

(c) $\lim\limits_{n\to\infty} (a_n b_n) = LM$;

(d) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{M}$, *if* $M \neq 0$.

Proof of Proposition 2. In class exercise. □

Example 2. *Find* $\lim\limits_{n\to\infty} \frac{2n}{5n-3}$.

Theorem 1 (The Sandwich Theorem)**.** *If (a_n), (b_n) and (c_n) are sequences of real numbers such that $a_n \leqslant b_n \leqslant c_n$ for all n, and if $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim\limits_{n\to\infty} b_n = L$.*

Proof of Theorem 1. For every $\epsilon > 0$ there corresponds a real number M such that both $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$, whenever $n > M$. Therefore, we have that

$$L - \epsilon < a_n \leqslant b_n \leqslant c_n < L + \epsilon,$$

from what it follows that $L - \epsilon < b_n < L + \epsilon$ or, equivalently, $|b_n - L| < \epsilon$, for every $n \in \mathbb{N}$ with $n > M$. This completes the proof. $\square$

Proposition 3. *Let (a_n) be a sequence of real numbers. Then, the following are true:*

(a) If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$;

(b) If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

TODO

- Define bounded sequences
- Define monotonic sequences

Theorem 2. *A bounded, monotonic sequence of real numbers has a limit.*

TODO

- State the completeness property of $\mathbb{R}$.

Bibliography

[1] Earl William Swokowski. *Calculus with analytic geometry*. Taylor & Francis, 1979.