Contents

Chapter 1

Groups

Definition 1. *A right action of a group G on a nonempty set X is a function*

$$X \times G \to X, \quad (x, g) \mapsto xg,$$

such that:

i. $x(gh) = (xg)h$ *for all* $g, h \in G$ *and* $x \in X$*;*

ii. $x1 = x$ *for all* $x \in X$*.*

The set X is called a G-set. A left action is defined in a similar fashion.

Example 1. *Let S_n be the symmetric group of degree n. Then, S_n acts on the the set $\{1, \ldots, n\}$ from the right in a rather natural way:*

$$\{1, \ldots, n\} \times S_n \to \{1, \ldots, n\}, \quad (x, \alpha) \mapsto x^\alpha.$$

Example 2. *Let G be a group. Then, G acts on itself from the right by conjugation:*

$$G \times G \to G, \quad (x, g) \mapsto x^g = g^{-1}xg.$$

Definition 2. *Let X be a G-set. Then, following common terminology, the set:*

i. $xG = \{y \in G : y = gx \text{ for some } g \in G\}$ *is called the G-orbit of X;*

ii. $G_x = \{g \in G : xg = x\}$ *is called the G-stabilizer of x,*

for any $x \in X$.

Proposition 1. *Let X be a G-set. Then, the binary relation given by*

$$\forall x, y \in X: \quad x \equiv y \pmod{G} \iff \exists g \in G: \; xg = y, \tag{1.1}$$

is an equivalence relation on X. Moreover, the equivalence class

$$\{y \in X : x \equiv y \pmod{G}\},$$

equals xG, that is, the G-orbit of x, for any point $x \in X$.

Proof. For any given x, y and z in X, we have that:

1. $x \equiv x \pmod{G}$ for every $x \in X$, since $x1 = x$;
2. If $x \equiv y \pmod{G}$, then $xg = y$ for some $g \in G$. But, then $yg^{-1} = x$ and so $y \equiv x \pmod{G}$;
3. If $x \equiv y \pmod{G}$ and $y \equiv z \pmod{G}$, then we have that $xg = y$ and $yh = z$ for certain $g, h \in G$. Therefore, $x(gh) = (xg)h = yh = z$ and so $x \equiv z \pmod{G}$.

Now, notice that if $y \in \{y \in X : x \equiv y \pmod{G}\}$, then $y = gx$ for some $g \in G$. Conversely, for any $g \in G$, $gx \equiv x \pmod{G}$ because $g^{-1} \in G$ and $g^{-1}(gx) = (g^{-1}g)x = 1x = x$. Therefore, we conclude that

$$\{y \in X : x \equiv y \pmod{G}\} = \{gx : g \in G\} = xG.$$

□

Proposition 2. *Let X be a G-set. Then, we have:*

i. $G_x \leqslant G$;

ii. $|xG| = (G : G_x)$,

for every $x \in X$.

Proof. Let $x \in X$ be given. The identity element of G obviously belongs to G_x and, for any pair of elements $g, h \in G_x$, we have that

$$x(gh^{-1}) = (xg)h^{-1} = xh^{-1} = (xh)h^{-1} = x(hh^{-1}) = x1 = x,$$

and as such $gh^{-1} \in G_x$. Therefore, G_x is a subgroup of G. Now, regarding the function

$$G/G_x \to xG, \quad G_x g \mapsto xg.$$

it's true that

$$xg = xh \iff x(gh^{-1}) = x \iff gh^{-1} \in G_x \iff G_x g = G_x h,$$

for every pair of elements $g, h \in G$, from what it follows that $G_x g \mapsto xg$ is an injective function, as well as

$$y \in xG \iff \exists g \in G:\ y = xg \implies G_x g \mapsto y = xg,$$

which shows that $G_x g \mapsto xg$ is also onto. Henceforth, $|xG| = (G : G_x)$ as claimed. This completes the proof. $\square$