Final Project

Michael Daniels

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Introduction

Financial modeling is a complex and continually evolving field requiring sophisticated statistical techniques. In Unit 5 of our course, I was introduced to Metropolis-Hastings (MH) algorithms and Markov Chain Monte Carlo (MCMC) methods, igniting a keen interest in the potential of Bayesian Statistics within financial modeling. Fundamental to Bayesian statistics, these techniques captured my attention for their promising applications in finance, particularly in options pricing.

Traditionally, the Black-Scholes (BS) formula, known for its simplicity and widespread use, has dominated options pricing. However, financial markets' volatile and unpredictable nature calls for more adaptable and robust modeling techniques. This need for a more dynamic approach led me to focus my project on exploring options pricing, explicitly investigating the application of Bayesian analysis, and comparing it with the conventional BS formula.

A critical aspect of this study was the selection of stocks. I chose META, HD (Home Depot), and AAPL (Apple), each representing distinct market behaviors — two showing consistent upward trends over the past year and one exhibiting higher volatility. This selection provides a comprehensive platform for evaluating pricing models under varied market conditions.

To ensure the study aligned with established norms in options pricing and to maintain simplicity in the analysis, I assumed no dividend payouts for META, HD, and AAPL during the study period despite their histories of dividends. This common academic practice in exploring options pricing allows a more focused examination of the core dynamics of the pricing models, free from the complexities introduced by dividends.

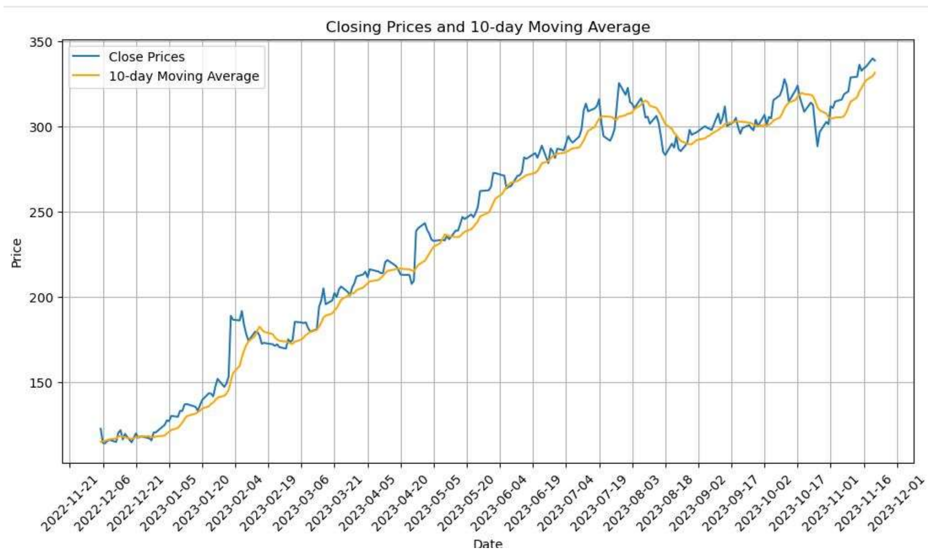
In this project, I aim to demonstrate the applicability and advantages of Bayesian methodologies in options pricing. By contrasting the Bayesian approach with the traditional Black-Scholes model, this study sheds light on the potential of Bayesian methods in navigating the intricate and constantly shifting landscape of options trading.

Data Description

I carefully assembled the dataset for this study from Yahoo Finance. It encompasses an extensive record of daily closing prices for the selected stocks (META, HD, AAPL) over an entire year. This one year was intentionally chosen to comprehensively capture the performance trajectories of these stocks, thus laying a solid foundation for a robust Bayesian options pricing analysis.

Central to this study is the method employed for determining the strike price of each option, which was based on the 10-day moving average of the stock's closing prices. This approach ensures that the strike price accurately mirrors the recent market behavior of each stock, maintaining a dynamic correlation with their short-term trading history. This strategy bolsters the relevance and precision of our option valuations and roots them in a more current and representative market context.

META:



HD:

AAPL

A uniform expiration period of 90 days was set for all options to enable a standardized comparison across different options. This time frame is a deliberate compromise, long enough to meaningfully reflect the influence of market dynamics on the option's value yet short enough to stay pertinent for real-world trading and investment scenarios. To visually articulate this relationship, the study includes graphs delineating the interaction between daily closing prices and the 10-day moving average. These graphical representations are crucial as they vividly illustrate the volatility of the stocks and underscore how the moving average acts as a stabilizing influence amidst market fluctuations.

Additionally, the analysis incorporates a constant annual rate of return of 5%. This rate is a critical element in options pricing models, significantly influencing the valuation of options over their life cycle. The selection of this rate represents a balance between realism and simplicity, facilitating a focused comparative analysis of different pricing models. The careful choice of these parameters in our dataset establishes a comprehensive framework for evaluating the efficacy of Bayesian methods in options pricing. This framework is instrumental in systematically comparing traditional models and the Bayesian approach, opening the door to a more dynamic and potentially insightful perspective in financial modeling.

Bayesian Model Description

I chose Python for my analysis due to its robustness and flexibility. It excels in handling statistical computations and parallel processing. In pursuing a Bayesian approach to options pricing, a model was constructed to use the log-normal properties inherent to the Black-Scholes (BS) formula. This formula displays the traditional method of option pricing, characterized by its log-normal distribution assumption of stock prices.

The Metropolis-Hastings algorithm is a cornerstone of Bayesian computational techniques, particularly in Markov Chain Monte Carlo (MCMC) methods. It eases the sampling from complex probability distributions where direct sampling is challenging. The MH algorithm works as follows:

1. Initialize the process with a starting point $X_1 = x_1$

2. For each iteration *t*, propose a new value *y* drawn from a transition kernel $Q(y|x_t)$
3. Calculate the acceptance probability *A* as the minimum of one and the ratio of the target distributions multiplied by the transition probabilities.
4. Accept the proposed value with probability *A*, updating $x_{t+1}=y$, or keep the current value, setting $x_{t+1} = x_t$

In this model, the transition kernel Q was initially chosen as a normal distribution centered around the current sample, a common choice for random walk Metropolis algorithms. However, it was observed that the symmetry of the normal distribution (i.e., $Q(y|x) = Q(x|y)$) simplifies the acceptance probability calculation, effectively nullifying the need to account for the proposal distribution in the acceptance ratio. This led to an optimized Metropolis-Hastings algorithm focusing solely on the target distributions called mh_call_option_pricing_optimized.

The **mh_call_option_pricing_optimized** function is a sophisticated tool designed for simulating and calculating the pricing of call options using Bayesian methodologies, specifically the Metropolis-Hastings algorithm. It executes the following process:

1. **Input Parameters and Setup**:
    - The function begins by receiving several essential inputs:
        - **S0**: The initial stock price.
        - **K**: The strike price of the option.
        - **T**: The option's expiration time, expressed in years.
        - **r**: The risk-free interest rate.
        - **sigma**: The volatility of the stock.
        - **n_simulations**: The number of simulations to perform.
    - Each simulation's total number of steps (n_steps) is calculated as 365 (days in a year) times **T**.
2. **Initialization of Simulation Array**:
    - An array named **synthetic_call_prices** is initialized to store the outcome of each simulated call option payout. The length of this array is equal to **n_simulations**.
3. **Running Simulations**:
    - The function conducts **n_simulations**, each consisting of a series of steps:
        - **Stock Price Initialization**: Set the initial stock price (**S_current**) to **S0**.
        - **Daily Simulation Loop**:
            - For each day until the option's expiration:

        - A new stock price (**S_new**) is proposed. This value is calculated by adjusting the current stock price based on a normally distributed random value, factoring in the volatility (**sigma**) and the risk-free rate (**r**).
        - **Acceptance Probability Calculation**:
            - The acceptance probability (**A**) is calculated by comparing the target probabilities of the current and new stock prices. This comparison involves estimating the likelihood of each stock price under a normal distribution centered around **S0**.
            - The new stock price is accepted or rejected based on this probability. If received, **S_current** is updated to **S_new**; otherwise, it remains unchanged.
    - **Call Option Payout Calculation**:
        - After the daily simulation steps, the call option payout is calculated based on the final stock price and stored in the **synthetic_call_prices** array.

4. **Output Calculation**:
    - After all simulations are complete, the function calculates and returns the average of the stored call option payouts. This average represents the estimated price of the call option, accounting for the time value of money and the underlying stock's price movements throughout the option's lifespan.

By leveraging the Metropolis-Hastings algorithm, this function offers a nuanced approach to options pricing, simulating a range of outcomes under varying market conditions. This method provides a dynamic alternative to traditional models, capturing the complexities of financial markets with greater fidelity.

This optimized function streamlines the option pricing process by efficiently simulating the underlying stock price paths using a random walk influenced by volatility (**sigma**). It provides a direct and transparent method for option pricing, facilitating comparisons with traditional models like Black-Scholes.

The **mh_call_option_pricing_optimized** function employs the Metropolis-Hastings (MH) algorithm to estimate call option prices by simulating potential future stock price paths. A vital feature of this implementation is the MH algorithm's ability to generate a sequence of sample values, where each value is contingent on its predecessor. In this specific function, the proposed stock price at each time step is determined through a random walk, reflecting the stock's volatility. The decision to accept or reject the suggested stock price hinges on comparing target probabilities, a distinctive aspect of the Metropolis-Hastings algorithm. These probabilities are calculated for current and proposed stock prices, reflecting their likelihood under a normal distribution centered around the initial stock price. The ratio of these target probabilities determines the acceptance probability, ensuring that a new stock price is accepted only if it is more probable or equally likely as the current price. This method effectively balances the exploration of new stock price paths with adherence to the probabilistic model dictated by the Metropolis-Hastings algorithm, embodying the core principle of probabilistic sampling and decision-making inherent in Bayesian statistical methods.

Implementation

The model was designed to assess the predicted call price of each stock option, with the strike price set to the stock's 10-day moving average. This measure provides a dynamic yet stable exercise price reflective of recent market conditions. The first step in the implementation process involved leveraging a Black-Scholes (BS) call formula, a widely accepted analytical tool for determining the theoretical price of European call options. The procedure, defined by the cumulative distribution function of the standard normal distribution, is given by:

```
N = norm.CDF
def vectorized_BS_CALL(S0, K, T, r, sigma):
    d1 = (np.log(S0 / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    return S0 * N(d1) - K * np.exp(-r * T) * N(d2)
```

I used a vectorized version of the above BS call formula to calculate the theoretical prices for an array of options based on the closing prices from the META.csv, HD.csv, and AAPL.csv datasets. I also figured out the historical volatility necessary for the BS formula using a function that applies frequentist principles to estimate volatility from historical price data. For the Bayesian aspect, the Metropolis-Hastings (MH) algorithm was employed to generate a distribution of call option prices. This algorithm focused on the target distributions derived from the log-normal properties of the stock prices, as discussed in the Bayesian Model Description. The execution of the MH algorithm, particularly over a year's worth of data, needed extensive computational resources.

Implementation Outcome and Missed Opportunities

The culmination of this study was a comprehensive dataset that juxtaposed option prices simulated using the Metropolis-Hastings (MH) algorithm against theoretical prices calculated via the Black-Scholes (BS) model. This side-by-side comparison sheds light on the disparities between these two methodologies, offering a unique Bayesian perspective on option valuation.

One notable aspect of the dataset is the treatment of the volatility parameter, which is essential in both the MH and BS models. Due to practical constraints, volatility was calculated using a frequentist approach despite initially conceptualizing a Bayesian method for its estimation. This Bayesian method envisioned using an inverse gamma distribution for the variance parameter of log returns, leveraging its conjugacy with the normal distribution. The plan was to infer the posterior distribution of this variance parameter, setting prior parameters to reflect empirical data points. The alpha and beta parameters of the inverse gamma distribution would be iteratively adjusted to align the theoretical mean and variance with practical values. This Bayesian model, planned to be implemented using PyMC, would integrate prior beliefs with observed data for a sophisticated analysis.

However, updating the sigma parameter and re-running the MH algorithm for each new iteration posed significant computational demands, exceeding this project's scope. The requirement for time-efficient analysis necessitated the continuation of a frequentist approach for volatility computation. Consequently, the final datasets, encapsulating all estimated prices and their variances, have been saved in CSV format for future review and graphical analysis.

Despite these practical limitations, the Bayesian approach to volatility estimation remains a theoretically sound and potentially more flexible alternative for future studies. With sufficient computational resources, this method could provide valuable insights, especially in scenarios requiring extensive simulations and adaptive, real-time analysis.

Results

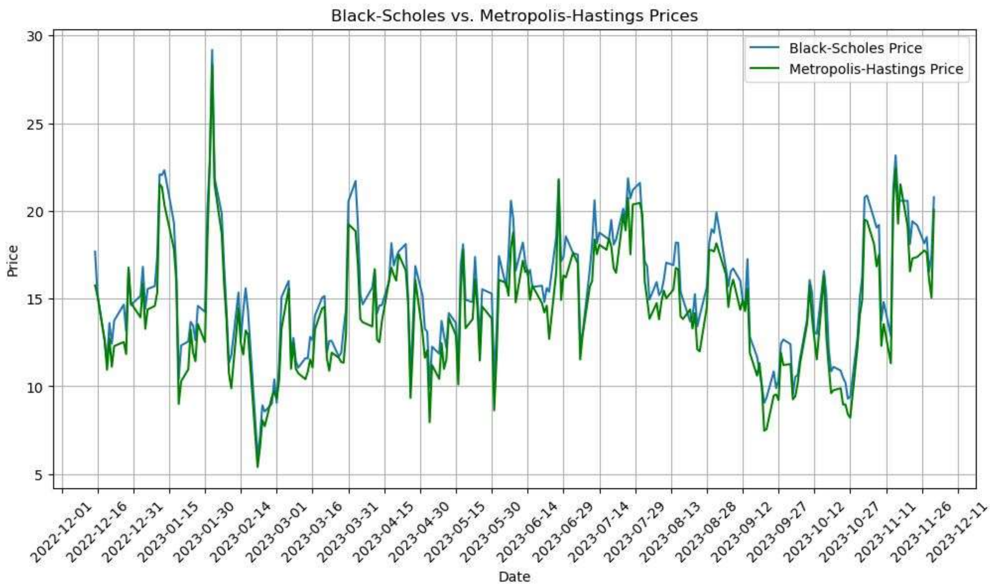
This project's implementation phase involved applying the Metropolis-Hastings (MH) Bayesian and conventional Black-Scholes (BS) models to estimate call option prices for various stocks over the past year. Contrary to initial expectations, the MH model did not exhibit greater price volatility than the BS model. The primary results indicated that the MH model's price estimates were less volatile.

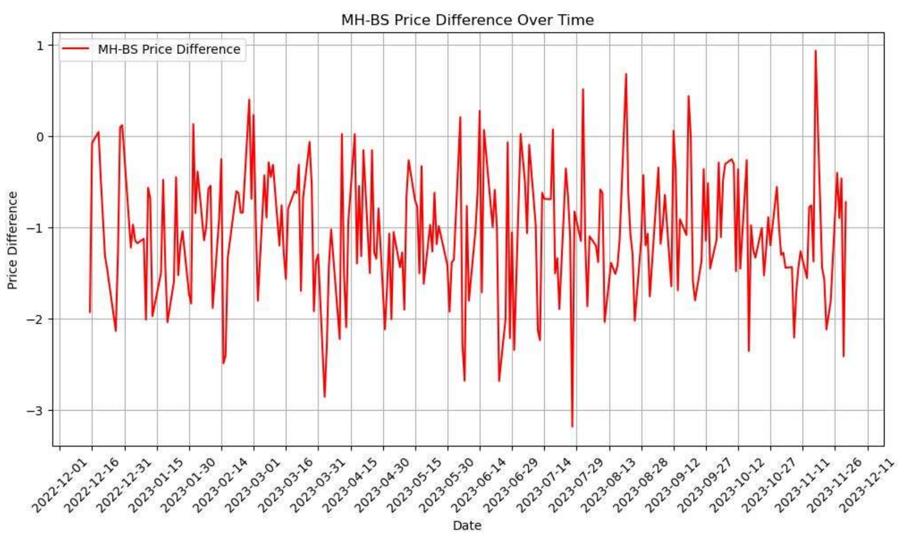
MH vs. BS Price Dynamics Analysis:

A comprehensive analysis using both MH and BS models across three distinct tests revealed a surprising trend. The MH algorithm closely mirrored the BS model's pricing, deviating from the hypothesis that the MH model would exhibit more pronounced volatility. Interestingly, there was no consistent pattern in how the MH model priced the options relative to the BS model. For stocks like META, HD, and AAPL, the MH model neither consistently overpriced nor underpriced the possibilities compared to the BS model. This lack of a consistent trend in the MH model's pricing about the general trends in stock prices or the moving average emerged as a significant observation. It suggests that the MH model may not inherently possess a systematic bias in pricing options as initially hypothesized. This finding is crucial in understanding the dynamics and potential reliability of the MH model in the complex realm of financial modeling.

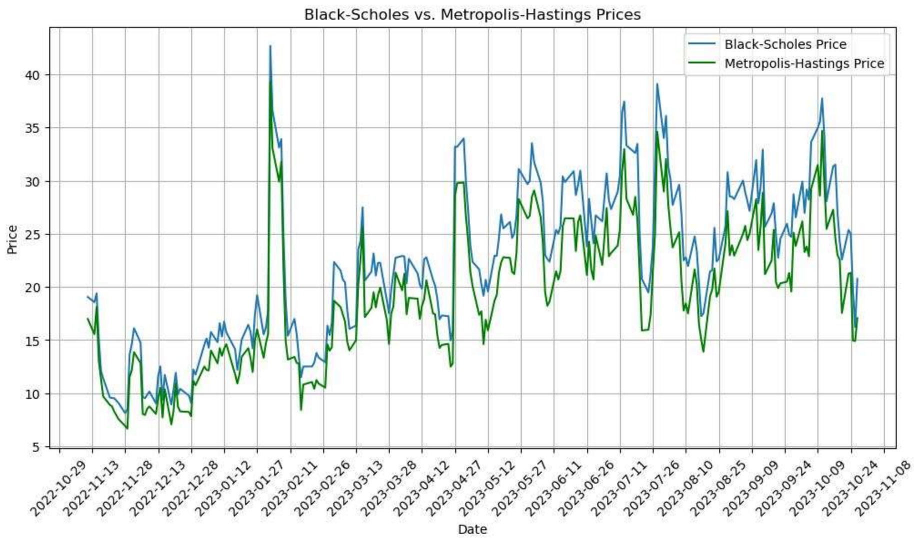
Graphical Illustrations:

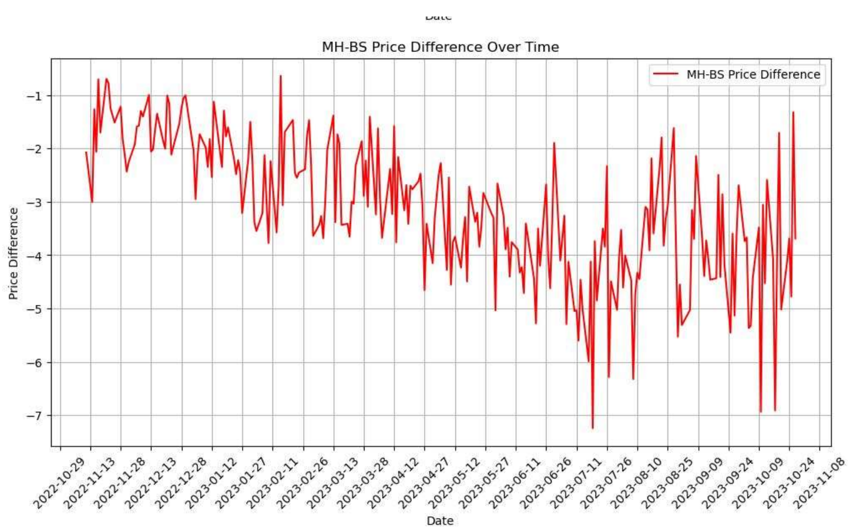
Home Depot:



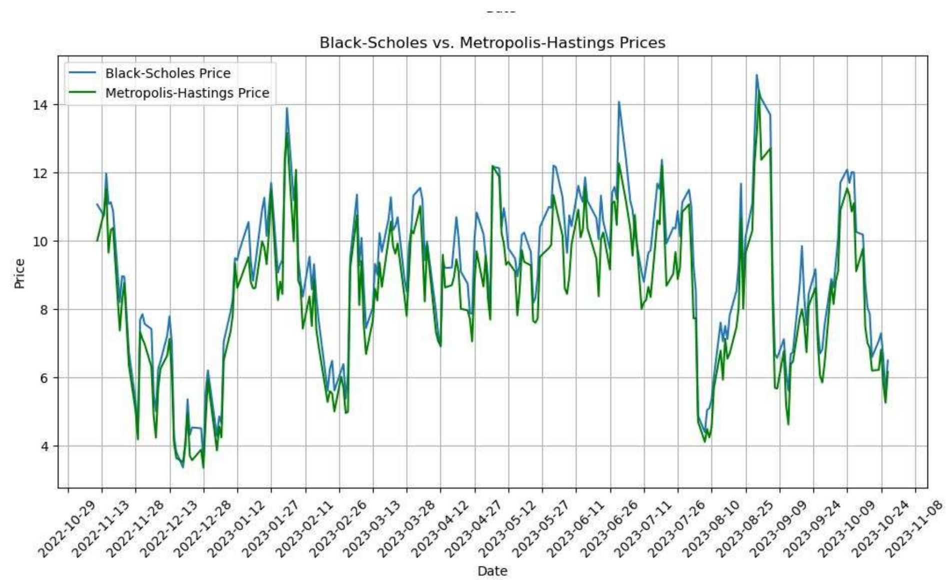


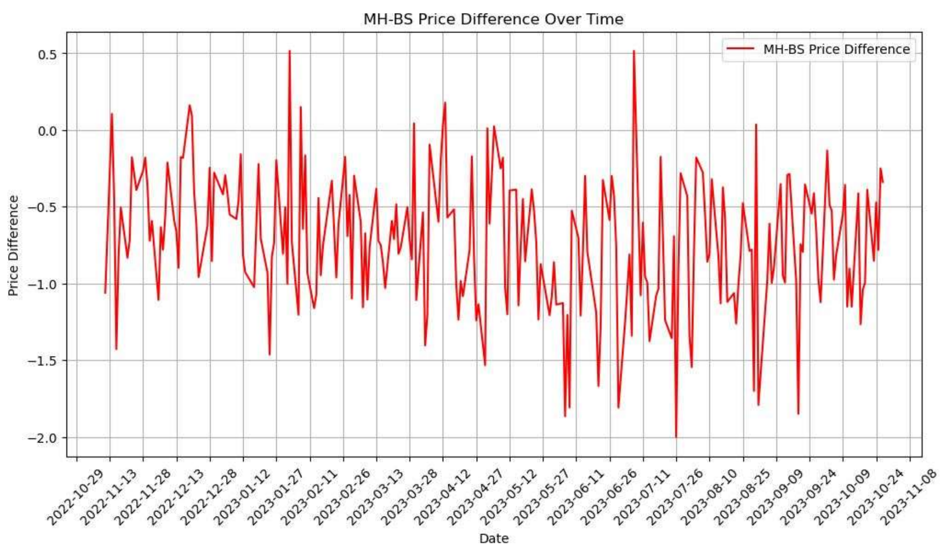
META:


Black-Scholes vs. Metropolis-Hastings Prices
Black-Scholes Price
Metropolis-Hastings Price
Price
Date


MH-BS Price Difference Over Time
MH-BS Price Difference
Price Difference
Date

AAPL:





Future Directions:

While the current study supplies valuable insights, future research might explore integrating a Bayesian approach for estimating volatility, which could offer a more consistent and comprehensive Bayesian framework for option pricing. This could result in a model that captures market sentiments more accurately and adapts to current information more effectively, an advantage in the rapidly changing financial markets.

Conclusion

This study embarked on a comprehensive exploration of Bayesian methodologies in financial modeling, specifically in the context of options pricing. Through a detailed analysis involving META, HD, and AAPL stocks, the research provided a novel perspective by juxtaposing the Metropolis-Hastings (MH) algorithm with the traditional Black-Scholes (BS) model. Contrary to initial hypotheses, the MH model demonstrated notable stability, exhibiting less volatility in price estimates than the BS model. This outcome challenges preconceived notions about Bayesian methods in financial modeling, particularly in their application to options pricing. The findings underscore the potential of the MH model as a viable and reliable alternative in financial forecasting, offering a less volatile approach to pricing options. This is particularly relevant in the rapidly changing and often unpredictable landscape of financial markets, where traditional models may need to catch up in capturing the nuances of market dynamics.

Looking ahead, the research opens several avenues for further exploration and development. One significant opportunity lies in integrating Bayesian methods for estimating volatility, a crucial parameter in the MH and BS models. While this study employed a frequentist approach for practical reasons, the theoretical framework for Bayesian volatility estimation remains compelling. Implementing this framework could lead to a more dynamic and comprehensive Bayesian model for options pricing. Such a model would enhance the accuracy of price predictions and adapt more effectively to real-time market information, aligning closely with the evolving nature of financial markets. This direction holds promise for advancing theoretical understanding and practical applications in finance, where the need for robust, adaptive models is ever-present. This study underscores the journey into Bayesian statistics in finance as a path of continuous discovery and innovation.

References and Citations:

https://stackoverflow.com/questions/8870982/how-do-i-get-a-lognormal-distribution-in-python-with-mu-and-sigma

https://www.codearmo.com/python-tutorial/options-trading-black-scholes-model

https://finance.yahoo.com/quote/META?p=META&.tsrc=fin-srch

https://stackoverflow.com/questions/70958350/implementation-of-the-metropolis-hasting-algorithm-for-solving-gaussian-integral

https://github.com/santoshphilip/eppy/issues/300 https://stackoverflow.com/questions/67715226/ways-to-optimize-independent-n-body-simulations