Homework 3

ISyE 6420

Fall 2023

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Fall23 HW3.1. Marietta Traffic Authority is concerned about the repeated accidents at the intersection of Canton and Piedmont Roads. Bayes-inclined city-engineer would like to estimate the accident rate, even better, find a credible set.

A well known model for modeling the number of road accidents in a particular location/time window is the Poisson distribution. Assume that X represents the number of accidents in a 3 month period at the intersection od Canton and Piedmont Roads.

Assume that $[X|\theta] \sim \mathcal{P}oi(\theta)$. Nothing is known a priori about θ, so it is reasonable to assume the Jeffreys' prior

$$\pi(\theta) = \frac{1}{\sqrt{\theta}} \mathbf{1}(0 < \theta < \infty).$$

In the four most recent three-month periods the following realizations for X are observed: 1, 2, 0, and 2.

(a) Compare the Bayes estimator for θ with the MLE (For Poisson, recall, $\hat{\theta}_{MLE} = \bar{X}$).

(b) Compute (numerically) a 95% equitailed credible set.

(c) Compute (numerically) a 95% HPD credible set.

(d) Numerically find the mode of the posterior, that is, MAP estimator of θ.

(e) If you test the hypotheses

$$H_0 : \theta \geq 1 \qquad vs \qquad H_1 : \theta < 1,$$

based on the posterior, which hypothesis will be favored?

Fall23 HW3.2. Find the Jeffreys' prior for the parameter α of the Maxwell distribution

$$p(x|\alpha) = \sqrt{\frac{2}{\pi}} \alpha^{3/2} x^2 \exp(-\frac{1}{2}\alpha x^2)$$

and find a transformation of this parameter in which the corresponding prior is uniform.

Fall23 HW3.3. Let

$$
\begin{aligned}
y_i|\theta_i &\sim^{ind.} Exp(\theta_i), \\
\theta_i &\sim^{iid} Inv-Gamma(\alpha, 1),
\end{aligned}
$$

for $i = 1, \ldots, n$. Find the empirical Bayes estimator of θ_i, $i = 1, \ldots, n$ (Note: If $x \sim Exp(\theta)$, then $p(x) = 1/\theta e^{-x/\theta}$ and if $x \sim Inv{-}Gamma(\alpha, \beta)$, then $p(x) = 1/\{\beta^\alpha \Gamma(\alpha)\} x^{-\alpha-1} e^{-1/(\beta x)}$.).